The Annals of Applied Probability
2010, Vol. 20, No. 3, 785–805
DOI: 10.1214/09-AAP639

INTERMEDIATE RANGE MIGRATION IN THE TWO-DIMENSIONAL STEPPING STONE MODEL

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We consider the stepping stone model on the torus of side L in $\mathbb{Z}^2$ in the limit $L \to \infty$, and study the time it takes two lineages tracing backward in time to coalesce. Our work fills a gap between the finite range migration case of [*Ann. Appl. Probab.* **15** (2005) 671–699] and the long range case of [*Genetics* **172** (2006) 701–708], where the migration range is a positive fraction of L. We obtain limit theorems for the intermediate case, and verify a conjecture in [*Probability Models for DNA Sequence Evolution* (2008) Springer] that the model is homogeneously mixing if and only if the migration range is of larger order than $(\log L)^{1/2}$.

1. Introduction. The subject of this paper is the stepping stone model of population genetics, and in particular the contrast between recent results of [14] and [18] in the two-dimensional setting. There is a vast literature on the many variants of the stepping stone model dating back to the seminal work of Malècot [13] and Kimura [11]. (A few sources for background and references are [5, 8, 15] and [16].) We will begin by describing the version of the model we consider here, generally following the setup in [18].

Let $\mathbb{Z}^2$ be the two-dimensional integer lattice, and fix $\nu > 0$ and $q : \mathbb{Z}^2 \to [0, 1]$ with $q(0) = 0$ and $\sum_x q(x) = 1$. We suppose that at each site x in

$$\mathbb{T}_L = (-L/2, L/2]^2 \cap \mathbb{Z}^2$$

there is a colony of $2N$ haploid individuals. We think of $\mathbb{T}_L$ as a torus, and assume a continuous-time Moran model of reproduction. In this model, a given individual in colony x dies at rate one, independently of all other individuals, and is replaced by a copy of an individual chosen at random from the same colony with probability $1 - \nu$ or colony y with probability $\nu q(y - x)$ computed modulo L. In this way, we treat $\mathbb{T}_L$ as a torus. The genealogical structure of a sample of n individuals is determined by tracing their lineages backward in time.

We will focus on the case of $n = 2$ lineages, where one is interested in T_0, the time it takes the lineages to enter the same colony, and t_0, the time to coalescence of

Received July 2009; revised January 2010.
[1]Supported in part by NSF Grant 0803517.
AMS 2000 subject classifications. Primary 60K35, 60G50, 92D10; secondary 82C41.
Key words and phrases. Stepping stone model, torus random walk, hitting times, coalescence times.

the lineages. There are many limit theorems for T_0 and t_0 in the literature. (A small sampling can be found [2, 3, 10, 14, 16, 17] and [18].) One may allow $N \to \infty$, $\nu \to 0$ and q to vary as $L \to \infty$. To understand the asymptotic behavior of t_0, one must first understand the behavior of T_0 so we will concentrate on the latter. Furthermore, the question we want to consider is already of interest in the simplest case of one individual per colony, so we will assume from now on that $\nu = 2N = 1$, but allow q to vary.

The *meanfield* or *homogeneous mixing* case is obtained by taking q to be uniform over $\mathbb{T}_L \setminus \{0\}$. Suppose the two lineages start at $0, x \in \mathbb{T}_L$, $x \neq 0$. The law of T_0 is exponential with mean $(L^2-1)/2$ and is independent of x, and so T_0/L^2 converges in law, uniformly in $x \neq 0$, to the exponential distribution with mean $1/2$. Matsen and Wakeley show in [14] that the same limiting behavior of T_0/L^2 holds uniformly in $x \neq 0$ assuming that q is uniform on only a positive fraction of the torus. By contrast, if q is kept fixed as $L \to \infty$, then the right normalization for T_0 is $L^2 \log L$, and the limiting law depends on the starting positions $0, x$. (See [2, 3] and [18] for results of this type.) The purpose of this paper is to fill the gap between these two situations.

Following two lineages backward in time amounts to following two random walks until they meet. The difference between the lineage locations is also a random walk, and T_0 is just the time it takes this difference walk to hit 0. On account of this, we will now focus on the following random walk setting. For $k > 0$, let

$$\Lambda_k = [-k/2, k/2]^2 \cap \mathbb{Z}^2$$

and for any $A \subset \mathbb{R}^2$ let

$$A' = A \setminus \{0\}.$$

For $r > 0$, let $B(r) = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq r\}$. Let $M_1, M_2, \ldots$ be a sequence of positive integers and assume that $q_{M_L} : \mathbb{Z}^2 \to [0, 1]$ satisfies

$$q_{M_L}(x) = 0 \qquad \text{for } x \notin \Lambda'_{M_L},$$

(P0)
$$\sum_x q_{M_L}(x) = 1, q_{M_L} \text{ is symmetric} \quad \text{and}$$

$$\sigma^2_{M_L} \equiv \sum_x x_1^2 q_{M_L}(x) = \sum_x x_2^2 q_{M_L}(x) > 0.$$

The uniform distributions $u_{M_L}(x) = 1_{\Lambda'_{M_L}}(x)/|\Lambda'_{M_L}|$ clearly satisfy (P0).

Let Y_t^L be a rate one random walk on $\mathbb{Z}^2$ with jump distribution q_{M_L}, and let X_t^L be the corresponding walk on $\mathbb{T}_L$ viewed as a torus. Given Y_t^L we construct X_t^L by setting $X_t^L = Y_t \bmod L$. Let H_L be the hitting time for X_t^L of the origin,

$$H_L = \inf\{t \geq 0 : X_t^L = 0\}.$$

Then H_L has the same law as $2T_0$, so we will study H_L. Let P_x and E_x denote probability law and expectation for the walk starting at x.

With the above notation, the Matsen and Wakely result is as follows. Fix $0 < c < 1$ and let $M_L = cL$ and $q_{M_L} = u_{M_L}$. Then as $L \to \infty$,

$$H_L/L^2 \Rightarrow \mathcal{E}(1) \qquad \text{uniformly in } X_0^L \in \mathbb{T}'_L, \tag{1.1}$$

where $\Rightarrow$ indicates the law of the left-hand side converges weakly to the distribution on right-hand side, and $\mathcal{E}(\beta)$ is the exponential distribution with mean β. On the other hand, if $M_L \equiv M$ is fixed, so there is a single jump distribution q_M, then by Theorem 1 of [18], if $0 < \alpha < 1$ and $|X_0^L| \approx L^\alpha$ as $L \to \infty$, then

$$\frac{H_L}{L^2 \log L} \Rightarrow (1-\alpha)\delta_0 + \alpha \mathcal{E}(1/\pi\sigma_M^2). \tag{1.2}$$

Here, $x_L \approx L^\alpha$ means $x_L \in \mathbb{T}_{L^\alpha \log L} \setminus \mathbb{T}_{L^\alpha/\log L}$.

It seems clear that the homogeneous mixing behavior of (1.1) should hold if $M_L \to \infty$ at a sufficiently fast rate, and it is natural to ask what this rate might be. Durrett (see Section 5.6 and Theorem 5.18 of [8]) conjectured that it should be quite slow, only of greater order than $\sqrt{\log L}$ as $L \to \infty$, meaning that (1.1) should hold exactly when $M_L/\sqrt{\log L} \to \infty$. We verify this conjecture for a large class of jump distributions in Theorems 1.2 and 1.3 below, and obtain a slightly improved version of (1.2) when $M_L = O(\sqrt{\log L})$. The proof of (1.1) in [14] makes use of Markov chain techniques from [1] and [6]. The proof of (1.2) relies heavily on local central limit theorem estimates for $P_0(Y_t^L = 0)$ to then estimate $P_0(X_t^L = x)$ [for use in (2.2) below]. Here, we will use a more direct Fourier-type approach that seems simpler, and works for both (1.1) and (1.2) as well.

For a jump distribution q_M, define the characteristic function

$$\phi_M(\theta) = \sum_{x \in \Lambda_M} e^{i\theta x} q_M(x), \qquad \theta \in \mathbb{R}^2,$$

where $\theta x = \theta \cdot x$. We will assume that the jump distributions q_{M_L} have characteristic functions ϕ_{M_L} which satisfy the conditions (P1)–(P3) listed below. These conditions are satisfied for the uniform distributions u_{M_L} (see the Appendix of [4], where M^2 in (P2) there should be M). Proposition 1.1 below shows that they are satisfied in some generality. Note that the symmetry condition in (P0) implies each ϕ_M is real-valued. The conditions we need are the following.

(P1) There is a $\sigma^2 > 0$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all large L,

$$\frac{1 - \phi_{M_L}(\theta)}{\sigma^2 M_L^2 |\theta|^2/2} \in (1-\varepsilon, 1+\varepsilon) \qquad \text{for all } \theta \in B'(\delta/M_L).$$

(P2) For all $\delta > 0$, there exists $\delta' > 0$ and $\zeta > 0$ such that for all large L,

$$1 - \phi_{M_L}(\theta) > \zeta \qquad \text{for all } \theta \in B(\delta') \setminus B(\delta/M_L).$$

(P3) For all $\varepsilon > 0$ and $a > 0$,

$$|\phi_{M_L}(\theta)| < \varepsilon \qquad \text{for all } \theta \in B(\pi) \setminus B(a) \text{ and all large } L.$$

PROPOSITION 1.1. *Let f be a positive, continuous function on $B(1/2)$ such that $f(x_1, x_2) = f(x_2, x_1) = f(-x_1, x_2)$. Define $c_M > 0$ and $q_M(x) = c_M f(x/M) u_M(x)$ so that $\sum_x q_M(x) = 1$. Then for any $M_L \to \infty$ as $L \to \infty$, the corresponding sequence of characteristic functions ϕ_{M_L} satisfies properties (P1)–(P3).*

In addition to (P1)–(P3), we impose the mild regularity condition

$$\lim_{L\to\infty} \frac{M_L^2}{\log L} = \rho \in [0, \infty]. \tag{P4}$$

Our first result shows that homogeneous mixing occurs if $M_L^2/\log L \to \infty$.

THEOREM 1.2. *Assume conditions (P0)–(P4) hold with $\rho = \infty$. Then for all $\lambda > 0$,*

$$\lim_{L\to\infty} \sup_{x\in\mathbb{T}'_L} |E_x(e^{-\lambda H_L/L^2}) - (1+\lambda)^{-1}| = 0 \tag{1.3}$$

and

$$\lim_{L\to\infty} \sup_{x\in\mathbb{T}'_L} |E_x(H_L/L^2) - 1| = 0. \tag{1.4}$$

Our next result shows that homogeneous mixing does *not* occur if $\rho < \infty$, and that H_L can grow at any rate between L^2 and $L^2 \log L$. We will use the following notation. For $v > 0$, define

$$\mathcal{A}_L(\alpha, v) = \begin{cases} \mathbb{T}'_v, & \text{if } \alpha = 0, \\ \mathbb{T}_{L^\alpha v} \setminus \mathbb{T}_{L^\alpha/v}, & \text{if } 0 < \alpha < 1, \\ \mathbb{T}_L \setminus \mathbb{T}_{L/v}, & \text{if } \alpha = 1, \end{cases}$$

and let

$$t_L = \frac{\log L}{M_L^2} \quad \text{and} \quad \beta = \rho + \frac{1}{\pi\sigma^2}. \tag{1.5}$$

THEOREM 1.3. *Assume $M_L \to \infty$ and the conditions (P0)–(P4) hold with $\rho < \infty$. Fix $0 \le \alpha \le 1$ and $k > 0$, and put $v_L = (\log L)^k$. Then for all $\lambda > 0$,*

$$\lim_{L\to\infty} \sup_{x\in\mathcal{A}_L(\alpha,v_L)} |E_x(e^{-\lambda H_L/L^2 t_L}) - [(1-\alpha') + \alpha'(1+\beta\lambda)^{-1}]| = 0 \tag{1.6}$$

where $\alpha' = (\alpha + \rho\pi\sigma^2)/(1 + \rho\pi\sigma^2)$. Furthermore,

$$\lim_{L\to\infty} \sup_{x\in\mathcal{A}_L(\alpha,v_L)} |E_x(H_L/L^2 t_L) - \alpha'\beta| = 0. \tag{1.7}$$

REMARK 1.4. If we set $\rho = 0$ in (1.6), then we recover the form (1.2). The proof of (1.6) is easily adapted to handle the case of a fixed q_M satisfying (P0), providing a slight strengthening of (1.2). One can also see that (1.6) is consistent with (1.3) by setting $t_L \equiv 1$, rephrasing (1.6) appropriately, and then setting $\rho = \infty$.

A one-dimensional stepping stone model was considered in [9], where exponential limit laws for H_L were obtained under rather general assumptions on the jump distributions. We will not state their results, but note that in analogy with Theorem 3 there, one might hope in our two-dimensional setting that with $M_L^2 = \log L$ some version of (1.2) would hold with (P1)–(P3) replaced by the simpler conditions

$$\begin{aligned} &\text{(i)} \quad \lim_{L\to\infty} \sigma_{M_L}^2 / M_L^2 = \sigma^2 \quad \text{and} \\ &\text{(ii)} \quad \text{for some } c > 0, \qquad q_{M_L} \geq c u_{M_L}. \end{aligned} \tag{1.8}$$

More precisely, the desired result would be that (1.8) implies $H_L/L^2 \Rightarrow \mathcal{E}(\beta_0)$ for X_0^L large, where the limiting mean β_0 depends only on σ^2 and c. This is not the case, as the following example shows.

EXAMPLE 1.5. Fix $0 < c < 1$ and $q_0 : \mathbb{Z}^2 \to [0, 1]$ satisfying (P0) for some fixed M_0, and let $\hat{q}_0(\theta) = \sum_x q_0(x) e^{i\theta x}$. Put $q_{M_L}(x) = c u_{M_L}(x) + (1 - c) q_0(x)$, assume that $\lim_{L\to\infty} M_L^2 / \log L = 1$, and define

$$\beta_0 = \frac{12}{c\pi} + \frac{1}{(2\pi)^2} \int_{B(\pi)} \frac{d\theta}{1 - (1 - c)\hat{q}_0(\theta)}. \tag{1.9}$$

Then q_{M_L} satisfies (1.8) with $\sigma^2 = c/12$. If $L/\log L < \ell_L < L$, then for all $\lambda > 0$,

$$\sup_{x \in \mathbb{T}_L \setminus \mathbb{T}_{\ell_L}} |E_x(e^{-\lambda H_L / L^2}) - (1 + \beta_0 \lambda)^{-1}| \to 0 \qquad \text{as } L \to \infty. \tag{1.10}$$

REMARK 1.6. The influence of the short range jumps is reflected in the dependence of β_0 on $\hat{q}_0$. Other mixtures of jump distributions could also be considered, e.g., $\sum_i c_i u_{M_L^i}$ where $M_L^1, M_L^2, \ldots$ tend to infinity at different rates.

The proofs in [2, 3] and [18] for the fixed jump distribution case use the fact that X_t^L becomes uniformly distributed over the torus by times of larger order than L^2. The analogous fact in our setting is given below, it will be used in the proof of (1.7).

THEOREM 1.7. *Assume* (P1)–(P4) *hold. If* $s_L / [(L^2/M_L^2) \vee \log L] \to \infty$ *as* $L \to \infty$, *then*

$$\lim_{L\to\infty} \sup_{t \geq s_L} \sup_{x \in \mathbb{T}_L} L^2 |P_0(X_t^L = x) - L^{-2}| = 0. \tag{1.11}$$

Returning to the stepping stone model, we could now consider the genealogy of a sample of $n > 2$ individuals. Let ζ_t^L be a system of rate one coalescing random walks on $\mathbb{T}_L$ with jump distribution q_{M_L}. If we consider lineages starting at $x_i \in \mathbb{T}_L$, $1 \le i \le n$, and put $\zeta_0^L = \{x_1, \ldots, x_n\}$, then $|\zeta_t^L|$ is the number of distinct lineages left at time t. Under the assumptions of Theorem 1.3, and assuming $|x_i - x_j| \ge L/\log L$ for $i \ne j$, the analog of Theorem 2 of [18] would be

$$\lim_{L\to\infty} P(|\zeta^L_{sL^2 t_L}| = k) = P(D_{\pi\sigma^2 s} = k), \qquad k = 1, \ldots, n, \tag{1.12}$$

where D_t is the pure death process on the positive integers which makes transition $k \to k-1$ at rate $\binom{k}{2}$. In fact, the genealogy of the lineages (on this time scale) converges to the genealogy described by Kingman's coalescent (see [12]). We will not pursue these matters here, since with the results developed the methods of [2, 3] and [18] could be adapted to prove such limit laws.

The outline of the rest of the paper is as follows. In Section 2, we develop some simple Fourier analytic tools. Proposition 1.1 is proved in Section 3, Theorem 1.7 is proved in Section 4, Theorem 1.2 is proved in Section 5, and Theorem 1.3 is proved in Section 6. Finally, we verify the claims for Example 1.5 in Section 7. For simplicity, we will assume throughout the rest of the paper that $L, M, M_L, \ldots$ are positive even integers.

2. Preliminaries. For a jump distribution q_M satisfying (P0) with characteristic function ϕ_M, define the transforms

$$\begin{aligned} \phi_M^t(\theta) &= E_0(e^{i\theta Y_t^L}) = \exp\bigl(-t\bigl(1 - \phi_M(\theta)\bigr)\bigr), \\ F_L(x,\lambda) &= E_x(e^{-\lambda H_L}) \quad \text{and} \\ G_L(x,\lambda) &= \int_0^\infty e^{-\lambda s} P_x(X_s^L = 0)\, ds, \end{aligned} \tag{2.1}$$

where $\theta \in \mathbb{R}^2$, $t \ge 0$, $x \in \mathbb{T}_L$ and $\lambda \ge 0$. The reason for our interest in $G_L(x,\lambda)$ is the formula

$$F_L(x,\lambda) = \frac{G_L(x,\lambda)}{G_L(0,\lambda)}, \tag{2.2}$$

a simple consequence of the strong Markov property. We will also make use of the well-known Fourier inversion formula

$$P_0(X_t^L = x) = \frac{1}{L^2} \sum_{y\in\mathbb{T}_L} \phi_M^t(2\pi y/L) e^{2\pi i x y/L}, \qquad x \in \mathbb{T}_L, \tag{2.3}$$

from which it is easy to derive

$$G_L(x,\lambda) = \frac{1}{L^2} \sum_{y\in\mathbb{T}_L} \frac{e^{2\pi i x y/L}}{1 + \lambda - \phi_M(2\pi i y/L)}. \tag{2.4}$$

In order to obtain useful bounds on the above, we will need to estimate sums of complex exponentials over various regions, including

$$D_k = \{x \in \mathbb{Z}^2 : |x| \le k/2\},$$

where $|x| = \|x\|_2$.

LEMMA 2.1. (a) *For $K \ge 1$ and $\theta \in B(\pi)$,*

$$\begin{aligned} &\left| \sum_{x \in \mathbb{T}_K} e^{i\theta x} \right| \le 4(K+1)(1 + \|\theta\|_\infty^{-1}) \quad \textit{and} \\ &\left| \sum_{x \in D_K} e^{i\theta x} \right| \le 4(K+1)\|\theta\|_\infty^{-1}. \end{aligned} \tag{2.5}$$

(b) *There is a constant C_0 such that for all $J \ge 1$ and $\theta \in B'(\pi)$,*

$$\sup_{K>J} \left| \sum_{y \in D_K \setminus D_J} \frac{e^{i\theta y}}{|y|^2} \right| \le \frac{C_0}{1 \wedge (J\|\theta\|_\infty)}. \tag{2.6}$$

(c)

$$\lim_{K\to\infty} \frac{1}{\log K} \sum_{y \in \mathbb{T}'_K} \frac{1}{|y|^2} = \lim_{K\to\infty} \frac{1}{\log K} \sum_{y \in D'_K} \frac{1}{|y|^2} = 2\pi. \tag{2.7}$$

PROOF. Combining the two elementary facts $\sin u \ge u/2$ for $|u| \le \pi/2$ and $\sum_{j=-k}^{k} e^{iju} = \sin((k+\frac{1}{2})u)/\sin\frac{u}{2}$ for any positive integer k and real u we obtain

$$\left| \sum_{j=-k}^{k} e^{iju} \right| \le 4/|u| \qquad \text{for all } k \in \mathbb{Z}^+, u \in B(\pi).$$

Consequently,

$$\left| \sum_{x \in \Lambda_K} e^{i\theta x} \right| \le \sum_{k=-K/2}^{K/2} \left| \sum_{j=-K/2}^{K/2} e^{i\theta_2 j} \right| \le \frac{4(K+1)}{|\theta_2|}.$$

This bound holds with θ_1 replacing θ_2, and therefore

$$\left| \sum_{x \in \Lambda_K} e^{i\theta x} \right| \le 4(K+1)\|\theta\|_\infty^{-1} \qquad \text{for all } \theta \in B(\pi). \tag{2.8}$$

The first bound in (2.5) follows from this inequality and the fact that $|\Lambda_K \setminus \mathbb{T}_K| = 2K+1$. The second bound in (2.5) is derived using the argument for (2.8).

For (b), if $1 \le k \le |y| \le k+1$, then

$$0 \le \frac{1}{k^2} - \frac{1}{|y|^2} \le \frac{1}{k^2} - \frac{1}{(k+1)^2} \le \frac{6}{|y|^3}.$$

Let $\gamma_k(\theta) = k^{-2}\sum_{x\in D_k} e^{i\theta x}$ and $C = 6\sum_{y\in\mathbb{Z}^2\setminus\{0\}} |y|^{-3} < \infty$. Then

$$\left|\sum_{y\in D_K\setminus D_J} \frac{e^{i\theta y}}{|y|^2}\right| \le C + \left|\sum_{k=J}^{K-1}\sum_{y\in D_{k+1}\setminus D_k} \frac{e^{i\theta y}}{k^2}\right|.$$

We can rewrite the sum on the right-hand side above, obtaining

$$\begin{aligned}\left|\sum_{k=J}^{K-1}\sum_{y\in D_{k+1}\setminus D_k} \frac{e^{i\theta y}}{k^2}\right| &= \left|\sum_{k=J}^{K-1}\left(\frac{(k+1)^2}{k^2}\gamma_{k+1}(\theta) - \gamma_k(\theta)\right)\right| \\ &= \left|\sum_{k=J}^{K-1}\left(\gamma_{k+1}(\theta) - \gamma_k(\theta) + \frac{2k+1}{k^2}\gamma_{k+1}(\theta)\right)\right| \\ &\le |\gamma_K(\theta)| + |\gamma_J(\theta)| + 3\sum_{k=J}^{K-1}\frac{|\gamma_{k+1}(\theta)|}{k}.\end{aligned}$$

By the bound (2.5),

$$3\sum_{k=J}^{K-1}\frac{|\gamma_{k+1}(\theta)|}{k} \le \frac{18}{\|\theta\|_\infty}\sum_{k=J}^{K-1}\frac{1}{k(k+1)} \le \frac{18}{J\|\theta\|_\infty}.$$

Making use of the trivial bound $|\gamma_k(\theta)| \le (k+1)^2/k^2 \le 4$ for $|\gamma_K(\theta)|$ and $|\gamma_J(\theta)|$, we therefore have

$$\left|\sum_{y\in D_K\setminus D_J}\frac{e^{i\theta}}{|y|^2}\right| \le C + 8 + \frac{18}{J\|\theta\|},$$

proving (2.6).

The second limit in (c) follows from a simple comparison with an integral. The first follows from a second comparison showing that

$$\lim_{K\to\infty}\sum_{D_{2K}\setminus D_K}\frac{1}{|y|^2} = 2\pi\log 2. \tag{2.9}$$

□

We close this section by recording the fact

$$\sum_{y\in\mathbb{T}_L} e^{2\pi i x y/L} = 0 \qquad \text{for all } x \in \mathbb{T}'_L. \tag{2.10}$$

3. Proof of Proposition 1.1. Throughout this section, we will write M for M_L. It is straightforward to check that the assumptions of Proposition 1.1 imply the following. As $M\to\infty$:

(i) $c_M \to c_0 = 1/\int_{B(1/2)} f(x)\,dx$,
(ii) $\sigma_M^2/M^2 \to \sigma^2 = c_0\int_{B(1/2)} x_1^2 f(x)\,dx$ and

(iii) $\phi_M(\theta/M) \to \tilde{\phi}(\theta) = c_0 \int_{B(1/2)} e^{i\theta x} f(x)\,dx$, $\theta \in B(\pi)$.

Let Z_M have distribution q_M. By a standard inequality (see (2.3.6) in [7]) and the fact that $|Z_M| \le M/2$,

$$\begin{aligned} |1 - \phi_M(\theta) - (\sigma_M^2|\theta|^2/2)| &\le E\big((|\theta Z_M|^3/6) \wedge |\theta Z_M|^2\big) \\ &\le \frac{|\theta|^2 M^2}{4}\big((|\theta|M/12) \wedge 1\big). \end{aligned} \tag{3.1}$$

Using (ii), this implies that for any $\delta > 0$,

$$\begin{aligned} \sup_{\theta \in B'(\delta/M)} \left| \frac{1 - \phi_M(\theta)}{\sigma_M^2|\theta|^2/2} - 1 \right| &\le \frac{2M^2}{4\sigma_M^2}\big((\delta/12) \wedge 1\big) \\ &\to \frac{1}{2\sigma^2}\big((\delta/12) \wedge 1\big) \qquad \text{as } M \to \infty. \end{aligned}$$

Using (ii) again, this is enough to establish (P1).

Fix $\varepsilon > 0$ and put $\bar{c} = \sup\{c_M\}$. We will prove that there exists a finite constant A depending on ε such that

$$\limsup_{M\to\infty} \sup_{\theta \in B(\pi)\setminus B(A/M)} \sum_{x \in \Lambda_M} e^{i\theta x} q_M(x) \le \varepsilon \bar{c}(1 + 20\|f\|_\infty), \tag{3.2}$$

which is stronger than (P3). First, we replace the sum over Λ_M with one over $\mathbb{T}_M$ at the cost of a small error,

$$\left| \sum_{x \in \Lambda_M} e^{i\theta x} q_M(x) - \frac{c_M}{|\Lambda'_M|} \sum_{x \in \mathbb{T}_M} e^{i\theta x} f(x/M) \right| \le \frac{(2M+2)\bar{c}\|f\|_\infty}{|\Lambda'_M|}. \tag{3.3}$$

The idea now is to break the sum over $\mathbb{T}_M$ into sums over disjoint translates of $\mathbb{T}_K$, where $K < M$ is chosen so that $f(x/M)$ is essentially constant on the translates, and then apply (2.5).

To do this, let $\Gamma_{M,K} = \{z \in K\mathbb{Z}^2 : z + \mathbb{T}_K \subset \mathbb{T}_M\}$, and choose $\varepsilon' \in (0, \varepsilon)$ small enough so that $|f(x) - f(x')| < \varepsilon$ if $\|x - x'\|_\infty < \varepsilon'$. Choose A large enough so that $A\varepsilon' > \varepsilon^{-1}$ and suppose $\|\theta\|_\infty > A/M$. Since $|\mathbb{T}_M \setminus \bigcup_{z \in \Gamma_{M,K}} (z + \mathbb{T}_K)| \le 4MK$,

$$\left| \sum_{x \in \mathbb{T}_M} e^{i\theta x} f(x/M) - \sum_{z \in \Gamma_{M,K}} \sum_{x \in \mathbb{T}_K} e^{i\theta(z+x)} f\big((z+x)/M\big) \right| \le 4\|f\|_\infty KM. \tag{3.4}$$

For large M, we can choose K to satisfy $\varepsilon'/2 < K/M < \varepsilon'$. By our choice of ε', $|f((z+x)/M) - f(z/M)| < \varepsilon$ for all $z \in \Gamma_{M,K}$ and $x \in \mathbb{T}_K$. Applying this bound gives

$$\begin{aligned} &\left| \sum_{z \in \Gamma_{M,K}} \sum_{x \in \mathbb{T}_K} e^{i\theta(z+x)} f\big((z+x)/M\big) \right. \\ &\qquad \left. - \sum_{z \in \Gamma_{M,K}} e^{i\theta z} f(z/M) \sum_{x \in \mathbb{T}_K} e^{i\theta x} \right| \le \varepsilon M^2. \end{aligned} \tag{3.5}$$

By (2.5) and the bound $|\Gamma_{M,K}| \le M^2/K^2$,

$$\begin{aligned}\Big|\sum_{z\in\Gamma_{M,K}} e^{i\theta z} f(z/M) \sum_{x\in\mathbb{T}_K} e^{i\theta x}\Big| &\le \frac{M^2\|f\|_\infty}{K^2} 4(K+1)(1+\|\theta\|^{-1}) \\ &\le \frac{8M^2\|f\|_\infty}{K}(1+\|\theta\|_\infty^{-1}).\end{aligned} \tag{3.6}$$

By combining (3.3)–(3.6), the bounds $\varepsilon'/2 < K/M < \varepsilon'$ and then using $\|\theta\|_\infty > A/M$, we obtain

$$\begin{aligned}|\phi_M(\theta)| &\le \frac{\bar c}{|\Lambda'_M|}\Big[(2M+2)\|f\|_\infty + 4KM\|f\|_\infty \\ &\qquad + \varepsilon|M^2| + \frac{8M^2}{K}\|f\|_\infty(1+\|\theta\|_\infty^{-1})\Big] \\ &\le \frac{\bar c}{|\Lambda'_M|}\big[\varepsilon|M^2| + \|f\|_\infty\big((2M+2) + 4\varepsilon' M^2 + 16(M/\varepsilon')(1+M/A)\big)\big] \\ &\to \bar c\Big[\varepsilon + 4\varepsilon'\|f\|_\infty + \frac{16\|f\|_\infty}{\varepsilon' A}\Big] \qquad \text{as } M\to\infty.\end{aligned}$$

Since $A\varepsilon' > \varepsilon^{-1}$ and $\varepsilon' < \varepsilon$, the right-hand side above is no larger than $\varepsilon\bar c(1 + 20\|f\|_\infty)$, which establishes (3.2).

To prove (P2), it now suffices to prove that for all $0 < \delta < A < \infty$ there exists $\zeta > 0$ such that

$$\limsup_{M\to\infty} \sup_{\theta\in B(A/M)\setminus B(\delta/M)} \phi_M(\theta) \le 1-\zeta. \tag{3.7}$$

Let $\tilde\phi_M(\theta) = \phi_M(\theta/M)$. By (iii), $\tilde\phi_M(\theta) \to \tilde\phi(\theta)$ as $M\to\infty$, and the convergence is uniform on compact sets. Since the probability distribution with density $c_0 f(x)$ on $B(1/2)$ is not degenerate or of lattice type, $|\tilde\phi(\theta)|$ must be bounded away from 1 on any compact set not containing 0. For $0 < \delta < A$, we may choose $\zeta > 0$ such that $\tilde\phi(\theta) < 1-\zeta$ for all $\theta \in B(A)\setminus B(\delta)$. The uniform convergence $\tilde\phi_M \to \tilde\phi$ on $B(A)\setminus B(\delta)$ now implies (3.7).

4. Proof of Theorem 1.7. We continue to write M for M_L. It suffices to prove that

$$\lim_{L\to\infty} \sup_{x\in\mathbb{T}_L} L^2|P_0(X^L_{s_L} = x) - L^{-2}| = 0. \tag{4.1}$$

By pulling out the $y = 0$ term from (2.3), we see that

$$L^2|P_0(X^L_{s_L} = x) - L^{-2}| = \Big|\sum_{y\in\mathbb{T}'_L} \phi_M^{s_L}(2\pi y/L) e^{2\pi i x y/L}\Big| \le \sum_{y\in\mathbb{T}'_L} \phi_M^{s_L}(2\pi y/L).$$

The limit (4.1) will follow from showing the last sum tends to zero as $L \to \infty$.

By (P1) there exists $\delta > 0$ such that for large L,

$$1 - \phi_M(2\pi y/L) \geq \pi^2\sigma^2 M^2|y|^2/L^2 \qquad \text{for all } y \in \mathbb{T}'_{\delta L/M}. \tag{4.2}$$

This implies [recall (2.1)] that

$$\sum_{y\in\mathbb{T}'_{\delta L/M}} \phi_M^{s_L}(2\pi y/L) \leq \sum_{y\in\mathbb{T}'_{\delta L/M}} \exp(-s_L\pi^2\sigma^2 M^2|y|^2/L^2).$$

This last sum tends to 0 as $L \to \infty$ by comparison with

$$\int_0^\infty e^{-\pi^2\sigma^2 s_L(M^2/L^2)r^2} r\,dr = \frac{1}{2\pi^2\sigma^2 s_L M^2/L^2} \to 0$$

since $s_L M^2/L^2 \to \infty$ by assumption.

By (P2) and (P3), there exists $\zeta > 0$ such that for all large L,

$$1 - \phi_M(2\pi y/L) \geq \zeta \qquad \text{for all } y \in \mathbb{T}_L \setminus \mathbb{T}_{\delta M/L}.$$

This bound implies

$$\sum_{y\in\mathbb{T}_L\setminus\mathbb{T}_{\delta L/M}} \phi_M^{s_L}(2\pi y/L) \leq L^2\exp(-\zeta s_L) \to 0$$

since $s_L/\log L \to \infty$ by assumption. This completes the proof of (4.1).

5. Proof of Theorem 1.2. We continue to write M for M_L. To prove (1.3), it suffices in view of (2.2) to establish the following facts:

$$\lim_{L\to\infty} G_L(0, \lambda/L^2) = \lambda^{-1} + 1 \tag{5.1}$$

and

$$\lim_{L\to\infty} \sup_{x\in\mathbb{T}'_L} |G_L(x, \lambda/L^2) - \lambda^{-1}| = 0. \tag{5.2}$$

PROOF OF (6.1). By (2.4),

$$G_L(0, \lambda/L^2) = \lambda^{-1} + \frac{1}{L^2}\sum_{y\in\mathbb{T}'_L} \frac{1}{1 + \lambda/L^2 - \phi_M(2\pi y/L)} \tag{5.3}$$

and thus (5.1) will follow from

$$\lim_{L\to\infty} \frac{1}{L^2}\sum_{y\in\mathbb{T}'_L} \frac{1}{1 - \phi_M(2\pi y/L)} = 1. \tag{5.4}$$

We will prove (5.4) by breaking $\mathbb{T}'_L$ into regions appropriate for utilizing (P1)–(P3). To prepare for this, fix $\varepsilon > 0$. By (P1), there exists $\delta > 0$ such that for all large L,

$$\frac{1}{1 - \phi_M(2\pi y/L)} \leq \frac{1}{\pi^2\sigma^2 M^2|y|^2/L^2} \qquad \text{for } y \in \mathbb{T}'_{\delta L/M}. \tag{5.5}$$

By (P2) there exists $\delta' > 0$ and $\zeta > 0$ such that for all large L,

$$\frac{1}{1-\phi_M(2\pi y/L)} < 1/\zeta \qquad \text{for } y \in \mathbb{T}_{\delta' L} \setminus \mathbb{T}_{\delta L/M}. \tag{5.6}$$

By (P3), for any $0 < a < \delta'$ and all large L,

$$\left|\frac{1}{1-\phi_M(2\pi y/L)} - 1\right| < \varepsilon \qquad \text{for } y \in \mathbb{T}_L \setminus \mathbb{T}_{aL}. \tag{5.7}$$

We claim that

$$\lim_{L\to\infty} \frac{1}{L^2} \sum_{y\in\mathbb{T}'_{\delta L/M}} \frac{1}{1-\phi_M(2\pi y/L)} = 0, \tag{5.8}$$

$$\limsup_{L\to\infty} \frac{1}{L^2} \sum_{y\in\mathbb{T}_{aL}\setminus\mathbb{T}_{\delta L/M}} \frac{1}{1-\phi_M(2\pi y/L)} \le a^2/\zeta \tag{5.9}$$

and

$$\limsup_{L\to\infty} \left| \frac{1}{L^2} \sum_{y\in\mathbb{T}_L\setminus\mathbb{T}_{aL}} \frac{1}{1-\phi_M(2\pi y/L)} - (1-a^2) \right| \le \varepsilon. \tag{5.10}$$

The bounds (5.9) and (5.10) are immediate from (5.6) and (5.7). For (5.8), we note that since $M^2/\log L \to \infty$, (2.7) implies that

$$\lim_{L\to\infty} \frac{1}{M^2} \sum_{y\in\mathbb{T}'_L} \frac{1}{|y|^2} = 0. \tag{5.11}$$

This fact and (5.5) easily imply (5.8). We note for later use that neither (5.9) nor (5.10) require $M^2/\log L \to \infty$, they hold for any $M \to \infty$ and ϕ_M satisfying (P2) and (P3).

Having established (5.8)–(5.10), we combine them to obtain

$$\limsup_{L\to\infty} \left| \frac{1}{L^2} \sum_{y\in\mathbb{T}'_L} \frac{1}{1-\phi_M(2\pi y/L)} - 1 \right| \le a^2/\zeta + a^2 + \varepsilon.$$

Let $a \downarrow 0$ and then $\varepsilon \downarrow 0$ to complete the proof of (5.4). □

PROOF OF (5.2). After separating out the $y = 0$ term as before, it suffices to prove that

$$\lim_{L\to\infty} \sup_{x\in\mathbb{T}'_L} \frac{1}{L^2} \left| \sum_{y\in\mathbb{T}'_L} \frac{e^{2\pi i x y/L}}{1-\phi_M(2\pi y/L)} \right| = 0. \tag{5.12}$$

In view of (5.8) and (5.9), we may concentrate on the region $\mathbb{T}_L \setminus \mathbb{T}_{aL}$. By (5.7), uniformly in $x \in \mathbb{T}'_L$,

$$\limsup_{L\to\infty} \frac{1}{L^2} \left| \sum_{y\in\mathbb{T}_L\setminus\mathbb{T}_{aL}} e^{2\pi i x y/L} \left(\frac{1}{1-\phi_M(2\pi y/L)} - 1 \right) \right| \le \varepsilon.$$

It is here we make use of (2.10). It implies that for all $x \in \mathbb{T}'_L$,

$$\left|\frac{1}{L^2}\sum_{y\in\mathbb{T}_L\setminus\mathbb{T}_{aL}} e^{2\pi i x y/L}\right| = \left|-\frac{1}{L^2}\sum_{y\in\mathbb{T}_{aL}} e^{2\pi i x y/L}\right| \le a^2.$$

By the last two facts,

$$\limsup_{L\to\infty}\sup_{x\in\mathbb{T}'_L}\frac{1}{L^2}\left|\sum_{y\in\mathbb{T}_L\setminus\mathbb{T}_{aL}}\frac{e^{2\pi i x y/L}}{1-\phi_M(2\pi y/L)}\right| \le \varepsilon + a^2 \tag{5.13}$$

and we note here that (5.13) does not require that $M^2/\log L \to \infty$. Taken together, (5.8), (5.9) and (5.13) imply

$$\limsup_{L\to\infty}\sup_{x\in\mathbb{T}'_L}\frac{1}{L^2}\left|\sum_{y\in\mathbb{T}'_L}\frac{e^{2\pi i x y/L}}{1-\phi_M(2\pi y/L)}\right| \le \varepsilon + a^2(1+1/\zeta). \tag{5.14}$$

Let $a \to 0$ and then $\varepsilon \to 0$ to complete the proof of (5.12). □

PROOF OF (1.4). By standard monotonicity arguments,

$$P_x(H_L > uL^2) \to e^{-u} \qquad \text{uniformly in } x \in \mathbb{T}'_L,\ u \ge 0 \tag{5.15}$$

as $L \to \infty$. In particular, for all large L,

$$P_x(H_L > L^2) \le e^{-1/2} \qquad \text{for all } x \in \mathbb{T}'_L.$$

By this bound and the Markov property,

$$\begin{aligned} P_x(H_L > kL^2) &= \sum_{y\in\mathbb{T}'_L} P_x\big(X^L_{(k-1)L} = y,\ H_L > (k-1)L^2\big)P_y(H_L > L^2) \\ &\le e^{-1/2}P_x\big(H_L > (k-1)L^2\big). \end{aligned}$$

Consequently, for all large L, $P_x(H_L > kL^2) \le e^{-k/2}$ for $k \ge 1$ and $x \in \mathbb{T}'_L$. This fact and (5.15) easily imply (1.4). □

6. Proof of Theorem 1.3. We continue to write M for M_L. The limit (1.6) follows easily from a little algebra and the following analogues of (5.1), (5.2):

$$\lim_{L\to\infty}\frac{G_L(0,\lambda/L^2 t_L)}{t_L} = \lambda^{-1} + \rho + \frac{1}{\pi\sigma^2} \tag{6.1}$$

and

$$\lim_{L\to\infty}\sup_{x\in\mathcal{A}(\alpha,v_L)}\left|\frac{G_L(x,\lambda/L^2 t_L)}{t_L} - \left[\lambda^{-1} + \left(\frac{1-\alpha}{\pi\sigma^2}\right)\right]\right| = 0. \tag{6.2}$$

The proofs of (6.1) and (6.2) are similar to the proofs of (5.1) and (5.2), but require a bit more care.

Fix $\varepsilon > 0$. By (P1) there exist $\delta > 0$ and functions ψ_L such that $\|\psi_L\|_\infty < \varepsilon$ and for all large L,

$$\frac{1}{1-\phi_M(2\pi y/L)} = \frac{1+\psi_L(y)}{2\pi^2\sigma^2 M^2|y|^2/L^2} \qquad \text{for } y \in \mathbb{T}'_{\delta L/M}. \tag{6.3}$$

As before, we assume $\delta', \zeta > 0$ are such that for all $0 < a < \delta'$, (5.6) and (5.7) hold. Recall that we are now assuming $M^2/\log L \to \rho < \infty$.

PROOF OF (6.1). The $y = 0$ term in the sum for $G_L(0, \lambda/t_L)$ yields λ^{-1}, so it suffices to prove that

$$\lim_{L\to\infty} \frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}'_L} \frac{1}{1-\phi_M(2\pi y/L)} = \rho + \frac{1}{\pi\sigma^2}. \tag{6.4}$$

We claim that:

$$\limsup_{L\to\infty} \left| \frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}'_{\delta L/M}} \frac{1}{1-\phi_M(2\pi y/L)} - \frac{1}{\pi\sigma^2} \right| \le \frac{\varepsilon}{\pi\sigma^2}, \tag{6.5}$$

$$\limsup_{L\to\infty} \frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}_{aL}\setminus\mathbb{T}_{\delta L/M}} \frac{1}{1-\phi_M(2\pi y/L)} \le \rho a^2/\zeta \tag{6.6}$$

and

$$\limsup_{L\to\infty} \left| \frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}_L\setminus\mathbb{T}_{aL}} \frac{1}{1-\phi_M(2\pi y/L)} - (1-a^2)\rho \right| \le \varepsilon\rho. \tag{6.7}$$

The limits (5.9) and (5.10) and the fact that $1/t_L \to \rho$ imply (6.6) and (6.7), so consider the region the region $\mathbb{T}'_{\delta L/M}$. By (6.3),

$$\frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}'_{\delta L/M}} \frac{1}{1-\phi_M(2\pi y/L)} = \frac{1}{\log L} \sum_{y\in\mathbb{T}'_{\delta L/M}} \frac{1+\psi_L(y)}{2\pi^2\sigma^2|y|^2}.$$

By using (2.7) above, we obtain (6.5).

Combining (6.5)–(6.7) gives

$$\limsup_{L\to\infty} \left| \frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}'_L} \frac{1}{1-\phi_M(2\pi y/L)} - \beta \right| \le \frac{\varepsilon}{\pi\sigma^2} + \rho a^2/\eta + \rho(a^2+\varepsilon).$$

Let $a \to 0$ and then $\varepsilon \to 0$ to complete the proof of (6.4). □

PROOF OF (6.2). Fix $0 < \alpha < 1$. (We will not give the slight changes in proof needed to handle the cases $\alpha = 0, 1$.) It suffices to prove that uniformly in $x \in \mathcal{A}(\alpha, v_L)$,

$$\frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}'_L} \frac{e^{2\pi i x y/L}}{1-\phi_M(2\pi y/L)} \to \frac{1-\alpha}{\pi\sigma^2} \qquad \text{as } L \to \infty. \tag{6.8}$$

With ε, δ as before, we claim that

$$\limsup_{L\to\infty} \sup_{x\in\mathcal{A}(\alpha,v_L)} \Big| \frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}'_{\delta L/M}} \frac{e^{2\pi i xy/L}}{1-\phi_M(2\pi y/L)} - \frac{1-\alpha}{\pi\sigma^2} \Big| \le \frac{\varepsilon}{\pi\sigma^2}. \tag{6.9}$$

Given this, (5.13) and (6.6) imply

$$\begin{aligned} &\limsup_{L\to\infty} \sup_{x\in\mathcal{A}(\alpha,v_L)} \Big| \frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}'_L} \frac{e^{2\pi i xy/L}}{1-\phi_M(2\pi y/L)} - \frac{1-\alpha}{\pi\sigma^2} \Big| \\ &\qquad \le \frac{\varepsilon}{\pi\sigma^2} + \rho(\varepsilon + a^2 + a^2/\zeta), \end{aligned}$$

which is enough to establish (6.8).

The first step in proving (6.9) is to use (6.3) to obtain

$$\frac{1}{L^2 t_L} \sum_{y\in\mathbb{T}'_{\delta L/M}} \frac{e^{2\pi i xy/L}}{1-\phi_M(2\pi y/L)} = \frac{1}{\log L} \sum_{y\in\mathbb{T}'_{\delta L/M}} \frac{e^{2\pi i xy/L}}{2\pi\sigma^2|y|^2}\big(1+\psi_L(y)\big). \tag{6.10}$$

Next, we may replace $\mathbb{T}'_{\delta L/M}$ in the right-hand side above with $D'_{\delta L/M}$ because

$$\lim_{L\to\infty} \frac{1}{\log L} \sum_{y\in\mathbb{T}'_{\delta L/M}\setminus D'_{\delta L/M}} \frac{1}{|y|^2} = 0 \tag{6.11}$$

by (2.9). Now, we break $\mathcal{A}_L(\alpha, v_L)$ into the union of the smaller regions

$$\mathcal{D}_L(\alpha, m) = D_{L^\alpha(\log L)^{m+1}} \setminus D_{L^\alpha(\log L)^m}, \qquad m \in [-k, k) \cap \mathbb{Z}.$$

We will prove that for each fixed m,

$$\lim_{L\to\infty} \sup_{x\in\mathcal{D}_L(\alpha,m)} \Big| \frac{1}{\log L} \sum_{y\in D'_{\delta M/L}} \frac{e^{2\pi i xyy/L}}{2\pi^2\sigma^2|y|^2} - \frac{1-\alpha}{\pi\sigma^2} \Big| = 0. \tag{6.12}$$

Since (6.9) will follow from (6.10)–(6.12), the problem now is to prove (6.12).

To do this, fix $m \in \mathbb{Z}$, let $K_L = L^{1-\alpha}(\log L)^{-(m+1/2)}$, and consider the regions $D_{\delta L/M} \setminus D_{K_L}$ and D'_{K_L}. The bound (2.6) implies that for all $x \in \mathcal{D}_L(\alpha, m)$,

$$\begin{aligned} \frac{1}{\log L} \Big| \sum_{y\in D_{\delta L/M}\setminus D_{K_L}} \frac{e^{2\pi i xy/L}}{|y|^2} \Big| &\le \frac{C_0}{(\log L)(1 \wedge K_L|2\pi x/L|)} \\ &\le \frac{C_0}{\log L} \vee \frac{C_0}{2\pi K_L(\log L)^{m+1} L^{\alpha-1}} \to 0 \\ &\qquad \text{as } L \to \infty. \end{aligned} \tag{6.13}$$

To handle the sum over D'_{K_L}, we make use of the fact that $e^{2\pi ixy/L} \approx 1$ there. More precisely, for $x \in \mathcal{D}_L(\alpha, m)$,

$$\frac{1}{\log L}\Big|\sum_{y\in D'_{K_L}} \frac{e^{2\pi ixy/L}-1}{|y|^2}\Big| \le \frac{1}{\log L}\sum_{y\in D'_{K_L}} \frac{2\pi|x|/L}{|y|}$$
$$\le 2\pi L^{\alpha-1}(\log L)^m \sum_{y\in D'_{K_L}} \frac{1}{|y|}.$$

Comparison with an integral shows there is a constant $C < \infty$ such that $\sum_{y\in D'_{K_L}} |y|^{-1} \le CK_L$, so it follows that

$$\lim_{L\to\infty} \sup_{x\in\mathcal{D}_L(\alpha,m)} \frac{1}{\log L}\Big|\sum_{y\in D'_{K_L}} \frac{e^{2\pi ixy/L}-1}{|y|^2}\Big| = 0. \tag{6.14}$$

Coming to the main term at last, by (2.7) we see that

$$\frac{1}{2\pi^2\sigma^2\log L}\sum_{y\in D'_{K_L}} \frac{1}{|y|^2} = \frac{\log K_L}{2\pi^2\sigma^2\log L}\frac{1}{\log K_L}\sum_{y\in D'_{K_L}} \frac{1}{|y|^2} \to \frac{1-\alpha}{\pi\sigma^2}$$
$$\text{as } L\to\infty. \tag{6.15}$$

Taken together, (6.13)–(6.15) establish (6.12), as required. □

PROOF OF (1.7). We proceed as in the proof of (1.4) with just a few changes. First, by (1.6) with $\alpha = m = 1$, there exists a finite L_0 such that for all $L \ge L_0$, $P_y(H_L > L^2 t_L) \le e^{-1/2\beta}$ for all $y \in \mathbb{T}_L \setminus \mathbb{T}_{L/\log L}$. Next, by Theorem 1.7, there exists finite $L_1 \ge L_0$ such that for $L \ge L_1$ and all $x, y \in \mathbb{T}_L$, $P_x(X^L_{L^2t_L} = y) \le 2/L^2$. Therefore, for all $L \ge L_1$ and $x \in \mathbb{T}'_L$,

$$P_x(H_L > 2L^2t_L) \le P_x(X^L_{L^2t_L}\in\mathbb{T}_{L/\log L}) + \sup_{y\in\mathbb{T}_L\setminus T_{L/\log L}} P_y(H_L > L^2t_L)$$
$$\le 2|\mathbb{T}_{L/\log L}|/L^2 + e^{-1/2\beta} \le 2/(\log L)^2 + e^{-1/2\beta}.$$

It follows that for some finite $L_2 \ge L_1$, if $L \ge L_2$ then

$$\sup_{x\in\mathbb{T}'_L} P_x(H_L > 2L^2t_L) \le e^{-1/3\beta}.$$

Iterating as in the proof of (1.4), we obtain

$$\sup_{x\in\mathbb{T}'_L} P_x(H_L > 2kL^2t_L) \le e^{-k/3\beta} \tag{6.16}$$

for all $L \ge L_2$.

Now for a fixed $0 \le \alpha \le 1$ and $k > 0$, (1.6) implies

$$P_x(H_L > uL^2 t_L) \to (1-q)e^{-u/\beta} \quad \text{uniformly in } x \in \mathcal{A}(\alpha, v_L), u \ge 0, \tag{6.17}$$

as $L \to \infty$. The limit (1.7) is a consequence of this fact and (6.16). □

7. Example 1.5. In this section, we verify the claims made in Example 1.5. We first check that

$$\begin{aligned}\frac{\sigma_{M_L}^2}{M_L^2} &= \frac{1}{M_L^2}\sum_x x_1^2 q_{M_L}(x)\\ &= \frac{c}{M_L^2}\sum_x x_1^2 u_{M_L}(x) + \frac{1-c}{M_L^2}\sum_x x_1^2 q_0(x) \to c\int_{B(1/2)} x_1^2\,dx\\ &= \frac{c}{12} \qquad \text{as } L \to \infty,\end{aligned}$$

so (1.8) holds with $\sigma^2 = c/12$. We turn now to the proof of (1.10).

Let $\hat{u}_{M_L}(\theta) = \sum_x u_{M_L}(x)e^{i\theta x}$. Our first step is to establish the analogues of (P1)–(P3) for $\phi_{M_L}(\theta) = c\hat{u}_{M_L}(\theta) + (1-c)\hat{q}_0(\theta)$. By Proposition 1.1, $\hat{u}_{M_L}$ satisfies (P1)–(P3) with $\sigma^2 = 1/12$. Furthermore, it is easy to check that $\hat{q}_{M_0}$ satisfies: for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1-\hat{q}_0(\theta)}{\sigma_0^2|\theta|^2/2} \in (1-\varepsilon, 1+\varepsilon) \qquad \text{for all } \theta \in B'(\delta).$$

With this it is easy to see that the following versions of (P1)–(P3) hold for ϕ_{M_L}.

(P1)′ For $\varepsilon > 0$ there exists $\delta > 0$ such that for all large L,

$$\frac{1}{1-\phi_{M_L}(2\pi y/L)} = \frac{1+\psi_L(y)}{cM_L^2\pi^2|y|^2/6L^2} \qquad \text{for all } y \in \mathbb{T}'_{\delta L/M_L},$$

where $\|\psi_L\|_\infty \le \varepsilon$.

(P2)′ For $\delta > 0$ there exists $\delta' > 0$ and $\zeta > 0$ such that for all large L,

$$1-\phi_{M_L}(2\pi y/L) \ge c\zeta \qquad \text{for all } y \in \mathbb{T}_{\delta' L} \setminus \mathbb{T}_{\delta L/M_L}.$$

(P3)′ For fixed $0 < a < 1$,

$$\lim_{L\to\infty} \sup_{y \in \mathbb{T}_L \setminus \mathbb{T}_{aL}} \left| \frac{1}{1-\phi_{M_L}(2\pi y/L)} - \frac{1}{c + (1-c)(1-\hat{q}_0(2\pi y/L))} \right| = 0.$$

With the above in place, the next step is to prove that

$$\lim_{L\to\infty} G_L(0, \lambda/L^2) = \lambda^{-1} + \beta_0$$

or equivalently

$$\lim_{L\to\infty}\frac{1}{L^2}\sum_{y\in\mathbb{T}'_L}\frac{1}{1-\phi_{M_L}(2\pi y/L)}=\beta_0. \tag{7.1}$$

To do this fix $\varepsilon>0$, choose δ,δ' as in (P1)′ and (P2)′, and break $\mathbb{T}'_L$ into the usual subregions.

Applying (P1)′, we have

$$\frac{1}{L^2}\sum_{y\in\mathbb{T}'_{\delta L/M_L}}\frac{1}{1-\phi_{M_L}(2\pi y/L)}=\frac{6}{cM_L^2\pi^2}\sum_{y\in\mathbb{T}'_{\delta L/M_L}}\frac{1+\psi_L(y)}{|y|^2}.$$

This implies, using (2.7),

$$\limsup_{L\to\infty}\left|\frac{1}{L^2}\sum_{y\in\mathbb{T}'_{\delta L/M_L}}\frac{1}{1-\phi_{M_L}(2\pi y/L)}-\frac{12}{c\pi}\right|\le\frac{12\varepsilon}{c\pi}, \tag{7.2}$$

where we have used $M_L^2/\log L\to 1$. Next, for $0<a<\delta'$, (P2)′ implies

$$\limsup_{L\to\infty}\frac{1}{L^2}\sum_{y\in\mathbb{T}_{aL}\setminus\mathbb{T}_{\delta L/M_L}}\frac{1}{1-\phi_{M_L}(2\pi y/L)}\le\frac{a^2}{c\zeta}. \tag{7.3}$$

By (P3)′ and continuity,

$$\begin{aligned}\frac{1}{L^2}\sum_{y\in\mathbb{T}_L\setminus\mathbb{T}_{aL}}\frac{1}{1-\phi_{M_L}(2\pi y/L)}&\to\int_{B(1/2)\setminus B(a/2)}\frac{d\theta}{c+(1-c)(1-\hat{q}_0(2\pi\theta))}\\&=\frac{1}{(2\pi)^2}\int_{B(\pi)\setminus B(a\pi)}\frac{d\theta}{1-(1-c)\hat{q}_0(\theta)}.\end{aligned}$$

Let $a\downarrow 0$ and then $\varepsilon\downarrow 0$ in (7.2) and (7.3) to complete the proof of (7.1).

The final task is to prove that

$$\lim_{L\to\infty}\sup_{x\in\mathbb{T}_L\setminus\mathbb{T}_{\ell_L}}|G_L(x,\lambda/L^2)-\lambda^{-1}|=0$$

or equivalently

$$\lim_{L\to\infty}\sup_{x\in\mathbb{T}_L\setminus\mathbb{T}_{\ell_L}}\left|\frac{1}{L^2}\sum_{y\in\mathbb{T}'_L}\frac{e^{2\pi ixy/L}}{1-\phi_{M_L}(2\pi y/L)}\right|=0. \tag{7.4}$$

Consider the region $\mathbb{T}'_{\delta L/M}$. By (2.9), we may replace $\mathbb{T}'_{\delta L/M}$ with $D'_{\delta L/M}$, at the cost of a negligible error. We break $D'_{\delta L/M}$ into two pieces. By (2.7),

$$\lim_{L\to\infty}\frac{1}{\log L}\sum_{y\in D'_{L^\varepsilon}}\frac{1}{|y|^2}=2\pi\varepsilon. \tag{7.5}$$

By (2.5), for all $x \in \mathbb{T}_L \setminus \mathbb{T}_\ell$

$$\begin{aligned}\left|\frac{1}{\log L}\sum_{y\in D_{\delta L/M}\setminus D_{L^\varepsilon}}\frac{e^{2\pi ixy/L}}{|y|^2}\right| &\le \frac{C_0}{\log L}\vee\frac{C_0 L}{K_L 2\pi|x|}\\ &\le \frac{C_0}{\log L}\vee\frac{C_0 L^{1-\varepsilon}}{2\pi\ell_L}\to 0 \qquad \text{as } L\to\infty.\end{aligned} \tag{7.6}$$

By (P1)′ and the above,

$$\limsup_{L\to\infty}\sup_{x\in\mathbb{T}_L\setminus\mathbb{T}_{\ell_L}}\left|\frac{1}{L^2}\sum_{y\in\mathbb{T}'_{\delta M_L/L}}\frac{e^{2\pi ixy/L}}{1-\phi_{M_L}(2\pi y/L)}\right|\le\frac{12\varepsilon}{c\pi} \tag{7.7}$$

and combining this with (7.3) gives

$$\limsup_{L\to\infty}\sup_{x\in\mathbb{T}_L\setminus\mathbb{T}_{\ell_L}}\left|\frac{1}{L^2}\sum_{y\in\mathbb{T}'_{aL}}\frac{e^{2\pi ixy/L}}{1-\phi_{M_L}(2\pi y/L)}\right|\le\frac{12\varepsilon}{c\pi}+\frac{a^2}{c\zeta}. \tag{7.8}$$

Now consider the region $\mathbb{T}_L \setminus \mathbb{T}_{aL}$. By (P3)′, for all large L and $x \in \mathbb{T}_L$,

$$\frac{1}{L^2}\sum_{y\in\mathbb{T}_L\setminus\mathbb{T}_{aL}}\left|\frac{e^{2\pi ixy/L}}{1-\phi_{M_L}(2\pi y/L)}-\frac{e^{2\pi ixy/L}}{1-(1-c)\hat{q}_0(2\pi y/L)}\right|\le\varepsilon. \tag{7.9}$$

For integers $K > 0$ define $\Gamma_{L,K} = \{z \in K\mathbb{Z}^2 : z + \mathbb{T}_K \subset \mathbb{T}_L \setminus \mathbb{T}_{aL}\}$, and note that $|\Gamma_{L,K}| \le L^2/K^2$ and $|(\mathbb{T}_L \setminus \mathbb{T}_{aL}) \setminus \bigcup_{z\in\Gamma_{L,K}}(z+\mathbb{T}_K)| \le 8LK$. By the trivial bound $1-(1-c)\hat{q}_0(\theta) \ge c$ and (7.9),

$$\begin{aligned}&\left|\frac{1}{L^2}\sum_{y\in\mathbb{T}_L\setminus\mathbb{T}_{aL}}\frac{e^{2\pi ixy/L}}{1-\phi_{M_L}(2\pi y/L)}\right.\\ &\qquad\left.-\frac{1}{L^2}\sum_{z\in\Gamma_{L,K}}\sum_{y\in z+\mathbb{T}_K}\frac{e^{2\pi ixy/L}}{1-(1-c)\hat{q}_0(2\pi y/L)}\right|\le\varepsilon+\frac{8K}{cL}.\end{aligned} \tag{7.10}$$

By the continuity of $\hat{q}_0$, there exists $\delta'' > 0$ such that if $\theta, \theta' \in B(\pi)$ and $|\theta - \theta'| < \delta''$ then

$$\left|\frac{1}{1-(1-c)\hat{q}_0(\theta)}-\frac{1}{1-(1-c)\hat{q}_0(\theta')}\right|<\varepsilon.$$

Assuming $K < \delta'' L$, this implies

$$\begin{aligned}&\left|\frac{1}{L^2}\sum_{z\in\Gamma_{L,K}}\sum_{y\in z+\mathbb{T}_K}\frac{e^{2\pi ixy/L}}{1-(1-c)\hat{q}_0(2\pi y/L)}\right.\\ &\qquad\left.-\frac{1}{L^2}\sum_{z\in\Gamma_{L,K}}\frac{e^{2\pi ixz/L}}{1-(1-c)\hat{q}_0(2\pi z/L)}\sum_{y\in\mathbb{T}_K}e^{2\pi ixy/L}\right|<\varepsilon.\end{aligned} \tag{7.11}$$

Now (2.5) can be applied, giving

$$\begin{aligned}(7.12)\qquad &\left|\frac{1}{L^2}\sum_{z\in\Gamma_{L,K}}\frac{e^{2\pi ixz/L}}{1-(1-c)\hat{q}_0(2\pi z/L)}\sum_{y\in\mathbb{T}_K}e^{2\pi ixy/L}\right| \\ &\quad\le\frac{|\Gamma_{L,K}|}{cL^2}\left|\sum_{y\in\mathbb{T}_K}e^{2\pi ixy/L}\right|\le\frac{(4(K+1)(1+L/2\pi\ell_L))}{cK^2}\end{aligned}$$

for all $x\in\mathbb{T}_L\setminus\mathbb{T}_{\ell_L}$. Taken together (7.8) and (7.10)–(7.12) yield

$$\begin{aligned}&\limsup_{L\to\infty}\sup_{x\in\mathbb{T}_L\setminus\mathbb{T}_{\ell_L}}\left|\frac{1}{L^2}\sum_{y\in\mathbb{T}'_L}\frac{e^{2\pi ixy/L}}{1-\phi_{M_L}(2\pi y/L)}\right| \\ &\quad\le\frac{12\varepsilon}{c\pi}+\frac{a^2}{c\zeta}+2\varepsilon+\limsup_{L\to\infty}\left(\frac{8K}{cL}+\frac{(4(K+1)(1+L/2\pi\ell_L))}{K^2}\right).\end{aligned}$$

If set $K=L/\sqrt{\ell}$, then the limsup above is 0. Let $a\downarrow 0$ and the $\varepsilon\downarrow 0$ to finish the proof.

Acknowledgment. It is a pleasure to thank Rick Durrett for suggesting this problem.

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