The Annals of Probability
2008, Vol. 36, No. 1, 319–330
DOI: 10.1214/009117907000000132

HITTING TIMES FOR GAUSSIAN PROCESSES

BY LAURENT DECREUSEFOND AND DAVID NUALART[1]

GET/Telecom Paris and University of Kansas

We establish a general formula for the Laplace transform of the hitting times of a Gaussian process. Some consequences are derived, and particular cases like the fractional Brownian motion are discussed.

1. Introduction. Consider a zero mean continuous Gaussian process $(X_t, t \geq 0)$, and for any $a > 0$, we denote by τ_a the hitting time of the level a defined by

$$\tau_a = \inf\{t \geq 0 : X_t = a\} = \inf\{t \geq 0 : X_t \geq a\}. \tag{1.1}$$

Thus, the map $(a \mapsto \tau_a)$ is left-continuous and increasing, hence, with right limits. The map $(a \mapsto \tau_{a^+})$ is right continuous where

$$\tau_{a^+} = \lim_{b \downarrow a, b > a} \tau_a = \inf\{t \geq 0 : X_t > a\}.$$

Little is known about the distribution of τ_a. It is explicitly known in particular cases like the Brownian motion. If X is a fractional Brownian motion with Hurst parameter H, there is a result by Molchan [5] which stands that

$$P(\tau_a > t) = t^{-(1-H)+o(1)}$$

as t goes to infinity.

When X is a standard Brownian motion, it is well known that

$$E(\exp(-\alpha\tau_a)) = \exp\big(-a\sqrt{2\alpha}\big) \tag{1.2}$$

for all $\alpha > 0$. This result is easily proved using the exponential martingale

$$M_t = \exp\big(\lambda B_t - \tfrac{1}{2}\lambda^2 t\big).$$

By Doob's optional stopping theorem applied at time $t \wedge \tau_a$ and letting $t \to \infty$, one gets $1 = E(M_{\tau_a}) = E(\exp(\lambda B_{\tau_a} - \lambda^2\tau_a/2))$. Since $B_{\tau_a} = a$, we thus obtain (1.2). If we consider a general Gaussian process X_t, the exponential process

$$M_t = \exp\big(\lambda X_t - \tfrac{1}{2}\lambda^2 V_t\big),$$

where $V_t = E(X_t^2)$ is no longer a martingale. However, it is equal to 1 plus a divergence integral in the sense of Malliavin calculus. The aim of this paper is to

Received September 2006; revised January 2007.
[1]Supported in part by the NSF Grant DMS-06-04207.
AMS 2000 subject classifications. Primary 60H05; secondary 60G15, 60H07.
Key words and phrases. Fractional Brownian motion, hitting times.

take advantage of this fact in order to derive a formula for $E(\exp(-\frac{1}{2}\lambda^2 V_{\tau_a}))$. We derive an equation involving this expectation in Theorem 3.4, under rather general conditions on the covariance of the process. As a consequence, we show that if the partial derivative of the covariance is nonnegative, then $E(\exp(-\frac{1}{2}\lambda^2 V_{\tau_a})) \le 1$, which implies that V_{τ_a} has infinite moments of order p for all $p \ge \frac{1}{2}$ and finite negative moments of all orders. In particular, for the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, we have the inequality

$$E(\exp -\alpha\tau_a^{2H}) \le \exp\bigl(-a\sqrt{2\alpha}\bigr)$$

for all $\alpha, a > 0$.

The paper is organized as follows. In Section 2 we present some preliminaries on Malliavin calculus, and the main results are proved in Section 3.

2. Preliminaries on Malliavin calculus. Let $(X_t, t \ge 0)$ be a zero mean Gaussian process such that $X_0 = 0$ and with covariance function

$$R(s,t) = E(X_t X_s).$$

We denote by $\mathcal{E}$ the set of step functions on $[0, +\infty)$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}} = R(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \longrightarrow X_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(X)$ associated with X. We will denote this isometry by $\varphi \longrightarrow X(\varphi)$.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(X(\phi_1), \dots, X(\phi_n)), \tag{2.1}$$

where $n \ge 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}$.

The *derivative operator* D of a smooth and cylindrical random variable F of the form (2.1) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\phi_1), \dots, X(\phi_n))\phi_i.$$

The derivative operator D is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. The Sobolev space $\mathbb{D}^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2}^2 = E(F^2) + E(\|DF\|_{\mathcal{H}}^2).$$

The *divergence operator* δ is the adjoint of the derivative operator. We say that a random variable u in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator, denoted by Dom δ, if

$$|E(\langle DF, u\rangle_{\mathcal{H}})| \le c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{S}$. In this case $\delta(u)$ is defined by the duality relationship

$$E(F\delta(u)) = E(\langle DF, u\rangle_{\mathcal{H}}), \tag{2.2}$$

for any $F \in \mathbb{D}^{1,2}$.

Set $V_t = R(t,t)$. For any $\lambda > 0$, we define

$$M_t = \exp\bigl(\lambda X_t - \tfrac{1}{2}\lambda^2 V_t\bigr).$$

Formally, the Itô formula for the divergence integral, proved, for instance, in [1], implies that

$$M_t = 1 + \lambda\delta\bigl(M\mathbf{1}_{[0,t]}\bigr), \tag{2.3}$$

where $M\mathbf{1}_{[0,t]}$ represents the process $(s \mapsto M_s\mathbf{1}_{[0,t]}(s), s \geq 0)$. However, the process $M\mathbf{1}_{[0,t]}$ does not belong, in general, to the domain of the divergence operator. This happens, for instance, in the following basic example.

EXAMPLE 1. Fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process $(B_t^H, t \geq 0)$ with the covariance

$$R_H(t,s) = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{2.4}$$

In this case, the processes $(B_s^H\mathbf{1}_{[0,t]}(s), s \geq 0)$ and $(\exp(\lambda B_s^H - \frac{1}{2}\lambda^2 s^{2H})\mathbf{1}_{[0,t]}(s), s \geq 0)$ do not belong to $L^2(\Omega; \mathcal{H})$ if $H \leq \frac{1}{4}$ (see [2]).

In order to define the divergence of $M\mathbf{1}_{[0,t]}$ and to establish formula (2.3), we introduce the following additional property on the covariance function of the process X.

(H0) The covariance function $R(t,s)$ is continuous, the partial derivative $\frac{\partial R}{\partial s}(s,t)$ exists in the region $\{0 < s, t, s \neq t\}$, and for all $T > 0$,

$$\sup_{t\in[0,T]} \int_0^T \left|\frac{\partial R}{\partial s}(s,t)\right| ds < \infty.$$

Notice that this property is satisfied by the covariance (2.4) for all $H \in (0,1)$.

Define

$$\delta_t M = \frac{1}{\lambda}(M_t - 1). \tag{2.5}$$

The following proposition asserts that $\delta_t M$ satisfies an integration by parts formula, and in this sense, it coincides with an extension of the divergence of $M\mathbf{1}_{[0,t]}$.

PROPOSITION 2.1. *Suppose that* (H0) *holds. Then, for any $t > 0$, and for any smooth and cylindrical random variable of the form $F = f(X_{t_1}, \dots, X_{t_n})$, we have*

$$E(F\delta_t M) = E\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) \int_0^t M_s \frac{\partial R}{\partial s}(s, t_i)\, ds\right). \tag{2.6}$$

PROOF. First notice that condition (H0) implies that the right-hand side of equation (2.6) is well defined. Then, it suffices to show equation (2.6) for a function of the form

$$f(x_1, \dots, x_n) = \exp\left(\sum_{i=1}^n \lambda_i x_i\right),$$

where $\lambda_i \in \mathbb{R}$. In this case we have, for all $0 < t_1 < \cdots < t_n$,

$$\begin{aligned}
&\frac{1}{\lambda} E\big(F(M_t - 1)\big)\\
&\qquad = \frac{1}{\lambda} \exp\left\{\frac{1}{2}\sum_{i=1}^n \lambda_i \lambda_j R(t_i, t_j)\right\}\left(\exp\left\{\sum_{i=1}^n \lambda\lambda_i R(t, t_i)\right\} - 1\right)\\
&\qquad = \sum_{i=1}^n \int_0^t \exp\left\{\frac{1}{2}\sum_{i=1}^n \lambda_i\lambda_j R(t_i, t_j) + \lambda \sum_{i=1}^n \lambda_i R(s, t_i)\right\}\lambda_i \frac{\partial R}{\partial s}(s, t_i)\, ds\\
&\qquad = \int_0^t E\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) M_s \frac{\partial R}{\partial s}(s, t_i)\right) ds,
\end{aligned}$$

which completes the proof of the proposition. $\square$

In many cases like in Example 1 with $H > \frac{1}{4}$, the process $M\mathbf{1}_{[0,t]}$ belongs to the space $L^2(\Omega; \mathcal{H})$, and then the right-hand side of equation (2.6) equals

$$E\langle DF, M\mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}.$$

In this situation, taking into account the duality formula (2.2), equation (2.6) says that $M\mathbf{1}_{[0,t]}$ belongs to the domain of the divergence and $\delta(M\mathbf{1}_{[0,t]}) = \delta_t M$.

3. Hitting times. In this section we will assume the following conditions:

(H1) The partial derivative $\frac{\partial R}{\partial s}(s, t)$ exists and it is continuous in $[0, +\infty)^2$.
(H2) $\limsup_{t\to\infty} X_t = +\infty$ almost surely.
(H3) For any $0 \le s < t$, we have $E(|X_t - X_s|^2) > 0$.

Under these conditions, the process X has a continuous version because

$$\begin{aligned}
E(|X_t - X_s|^2) &= R(t, t) + R(s, s) - 2R(s, t)\\
&= \int_s^t \left[\frac{\partial R}{\partial u}(u, t) - \frac{\partial R}{\partial u}(u, s)\right] du\\
&\le 2|t - s| \sup_{s \le u \le t} \left|\frac{\partial R}{\partial u}(u, t)\right|.
\end{aligned}$$

For any $a > 0$, we define the hitting time τ_a by (1.1). We know that $P(\tau_a < \infty) = 1$ by condition (H2). Set

$$S_t = \sup_{s \in [0,t]} X_s. \tag{3.1}$$

From the results of [6], it follows that, for all $t > 0$, the random variable S_t belongs to the space $\mathbb{D}^{1,2}$. Furthermore, condition (H3) allows us to compute the derivative of this random variable.

LEMMA 3.1. *For all $t > 0$, with probability one, the maximum of the process X in the interval $[0, t]$ is attained in a unique point, that is, $\tau_{S_t} = \tau_{S_t^+}$ and $DS_t = \mathbf{1}_{[0,\tau_{S_t}]}$.*

PROOF. The fact that the maximum is attained in a unique point follows from condition (H3) and Lemma 2.6 in Kim and Pollard [4]. The formula for the derivative of S_t follows easily by an approximation argument. □

We need the following regularization of the stopping time τ_a. Suppose that φ is a nonnegative smooth function with compact support in $(0, +\infty)$ and define for any $T > 0$

$$Y = \int_0^\infty \varphi(a)(\tau_a \wedge T)\, da. \tag{3.2}$$

The next result states the differentiability of the random variable Y in the sense of Malliavin calculus and provides an explicit formula for its derivative.

LEMMA 3.2. *The random variable Y defined in (3.2) belongs to the space $\mathbb{D}^{1,2}$, and*

$$D_r Y = -\int_0^{S_T} \varphi(y)\mathbf{1}_{[0,\tau_y]}(r)\, d\tau_y. \tag{3.3}$$

PROOF. Clearly, Y is bounded. On the other hand, for any $r > 0$, we have

$$\{\tau_a > r\} = \{S_r < a\}.$$

Therefore, we can write using Fubini's theorem

$$Y = \int_0^\infty \varphi(a)\left(\int_0^{\tau_a \wedge T} d\theta\right) da = \int_0^T \left(\int_{S_\theta}^\infty \varphi(a)\, da\right) d\theta,$$

which implies that $Y \in \mathbb{D}^{1,2}$ because $S_\theta \in \mathbb{D}^{1,2}$, and

$$D_r Y = -\int_0^T \varphi(S_\theta) D_r S_\theta\, d\theta = -\int_0^T \varphi(S_\theta)\mathbf{1}_{[0,\tau_{S_\theta}]}(r)\, d\theta.$$

Finally, making the change of variable $S_\theta = y$ yields

$$D_r Y = -\int_0^{S_T} \varphi(y)\mathbf{1}_{[0,\tau_y]}(r)\,d\tau_y. \qquad \square$$

Notice that $M_Y = \exp(\lambda X_Y - \frac{1}{2}\lambda^2 V_Y)$. Hence, letting $t = Y$ in equation (2.5) and taking the mathematical expectation of both members of the equality yields

$$E(M_Y) = 1 + \lambda E(\delta_t M|_{t=Y}). \tag{3.4}$$

We are going to show the following result which provides a formula for the left-hand side of equation (3.4).

LEMMA 3.3. *Assume conditions* (H1), (H2) *and* (H3). *Then*, *we have*

$$E(M_Y) = 1 - \lambda E\biggl(M_Y \int_0^{S_T} \varphi(y)\frac{\partial R}{\partial s}(Y, \tau_y)\,d\tau_y\biggr). \tag{3.5}$$

PROOF. The proof will be done in two steps.

Step 1. We claim that for any function $p(x)$ in $\mathcal{C}_0^\infty(\mathbb{R})$ we have

$$E(\delta_t M p(Y)) = -E\biggl(\int_0^t M_s p'(Y) \int_0^{S_T} \varphi(y)\frac{\partial R}{\partial s}(s, \tau_y)\,d\tau_y\,ds\biggr). \tag{3.6}$$

We can write $Y = \int_0^T \psi(S_\theta)\,d\theta$, where $\psi(x) = \int_x^\infty \varphi(a)\,da$. Consider an increasing sequence D_n of finite subsets of $[0, T]$ such that their union is dense in $[0, T]$. Set $Y_n = \int_0^T \psi(S_\theta^n)\,d\theta$, and $S_\theta^n = \max\{X_t, t \in D_n \cap [0, \theta]\}$. Then, Y_n is a Lipschitz function of $\{X_t, t \in D_n\}$. Hence, formula (2.6), which holds for Lipschitz functions, implies that

$$E(\delta_t M p(Y_n)) = -E\biggl(p'(Y_n)\int_0^T \varphi(S_\theta^n)\biggl(\int_0^t M_s \frac{\partial R}{\partial s}(s, \tau_{S_\theta^n})\,ds\biggr)d\theta\biggr).$$

The function $r \to \int_0^t M_s \frac{\partial R}{\partial s}(s, r)\,ds$ is continuous and bounded by condition (H1). As a consequence, we can take the limit of the above expression as n tends to infinity and we get

$$E(\delta_t M p(Y)) = -E\biggl(p'(Y)\int_0^T \varphi(S_\theta)\biggl(\int_0^t M_s \frac{\partial R}{\partial s}(s, \tau_{S_\theta})\,ds\biggr)d\theta\biggr).$$

Finally, making the change of variable $S_\theta = y$ yields (3.6).

Step 2. We write

$$E(\delta_t M|_{t=Y}) = E\biggl(\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \delta_t M p_\varepsilon(Y - t)\,dt\biggr),$$

where $p_\varepsilon(x)$ is an approximation of the identity, and by convention, we assume that $\delta_t M = 0$ if t is negative. We can commute the expectation with the above limit by the dominated convergence theorem because

$$\int_{-\infty}^{\infty} |\delta_t M| p_\varepsilon(Y-t)\,dt = \int_{-\infty}^{\infty} \frac{1}{\lambda}|M_t - 1| p_\varepsilon(Y-t)\,dt$$
$$\leq \frac{1}{\lambda} \sup_{0\leq t\leq T+1} (|M_t|+1),$$

if the support of $p_\varepsilon(x)$ is included in $[-\varepsilon, \varepsilon]$, and $\varepsilon \leq 1$. Hence,

$$E(\delta_t M|_{t=Y}) = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} E\big(\delta_t M p_\varepsilon(Y-t)\big)\,dt. \tag{3.7}$$

Using formula (3.6) yields

$$\begin{aligned} &E\big(\delta_t M p_\varepsilon(Y-t)\big) \\ &\quad = -\int_0^t E\bigg(p'_\varepsilon(Y-t) M_s \bigg(\int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y)\,d\tau_y\bigg)\bigg)\,ds. \end{aligned} \tag{3.8}$$

Hence, substituting (3.8) into (3.7) and integrating by parts, we obtain

$$\begin{aligned} &E(\delta_t M|_{t=Y}) \\ &\quad = -\lim_{\varepsilon\to 0} E\bigg(\int_{-\infty}^{\infty} p'_\varepsilon(Y-t)\bigg(\int_0^t M_s \bigg(\int_0^{S_T} \varphi(y)\frac{\partial R}{\partial s}(s,\tau_y)\,d\tau_y\bigg)\,ds\bigg)\,dt\bigg) \\ &\quad = -\lim_{\varepsilon\to 0} E\bigg(\int_{-\infty}^{\infty} p_\varepsilon(Y-t)\bigg(M_t \int_0^{S_T} \varphi(y)\frac{\partial R}{\partial t}(t,\tau_y)\,d\tau_y\bigg)\,dt\bigg). \end{aligned}$$

Notice that

$$\bigg|\int_0^{S_T} \varphi(y)\frac{\partial R}{\partial s}(s,\tau_y)\,d\tau_y\bigg| \leq T \sup_{0\leq s,u\leq T}\bigg|\frac{\partial R}{\partial s}(s,u)\bigg| \|\varphi\|_\infty.$$

Hence, applying the dominated convergence theorem, we get

$$\begin{aligned} E(M_Y) &= 1 + \lambda E(\delta_t M|_{t=Y}) \\ &= 1 - \lambda \lim_{\varepsilon\to 0} E\bigg(\int_{-\infty}^{\infty} p_\varepsilon(Y-t)\bigg(M_t\int_0^{S_T}\varphi(y)\frac{\partial R}{\partial s}(t,\tau_y)\,d\tau_y\bigg)\,dt\bigg) \\ &= 1 - \lambda E\bigg(M_Y \int_0^{S_T}\varphi(y)\frac{\partial R}{\partial s}(Y,\tau_y)\,d\tau_y\bigg). \end{aligned}$$

□

The next step will be to replace the function $\varphi(x)$ by an approximation of the identity and let T tend to infinity. Notice that (3.5) still holds for $\varphi(x) = \mathbf{1}_{[0,b]}(x)$ for any $b \geq 0$. In this way we can establish the following result.

THEOREM 3.4. *Assume conditions* (H1), (H2) *and* (H3). *For any* $a > 0$ *and* $\lambda \in \mathbb{R}$, *we have*

$$\int_0^a E(M_{\tau_y})\,dy = a - \lambda E\biggl(\int_0^a \int_0^1 M_{z\tau_{y^+}+(1-z)\tau_y} \frac{\partial R}{\partial s}(z\tau_{y^+} + (1-z)\tau_y, \tau_y)\,dz\,d\tau_y\biggr). \tag{3.9}$$

Notice that we are not able to differentiate with respect to a, the integral in the rightmost expectation of (3.9), because the (random) measure $d\tau_y$, in general, is not absolutely continuous with respect to the Lebesgue measure.

PROOF OF THEOREM 3.4. Fix $a > 0$. We first replace the function $\varphi(x)$ by an approximation of the identity of the form $\varphi_\varepsilon(x) = \varepsilon^{-1}\mathbf{1}_{[0,1]}(x/\varepsilon)$ in formula (3.5). We will make use of the following notation:

$$Y_{\varepsilon,a} = \int_0^\infty \varphi_\varepsilon(x-a)(\tau_x \wedge T)\,dx.$$

At the same time we fix a nonnegative smooth function $\psi(x)$ with compact support such that $\int_{\mathbb{R}} \psi(a)\,da = c$ and we set

$$\int_{\mathbb{R}} E(M_{Y_{\varepsilon,a}})\psi(a)\,da = c - \lambda \int_{\mathbb{R}} E\biggl(M_{Y_{\varepsilon,a}} \int_0^{S_T} \varphi_\varepsilon(y-a)\frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y)\,d\tau_y\biggr)\psi(a)\,da.$$

The increasing property of the function $x \to \tau_x$ implies that $\tau_{a^+} \wedge T \le Y_{\varepsilon,a} \le \tau_{a+\varepsilon} \wedge T$. Hence, Y_ε converges to $\tau_{a^+} \wedge T$ as ε tends to zero. Thus, almost surely, we have

$$\lim_{\varepsilon\to 0} M_{Y_{\varepsilon,a}} = \exp\bigl(\lambda X_{\tau_{a^+}\wedge T} - \tfrac{1}{2}\lambda^2 V_{\tau_{a^+}\wedge T}\bigr).$$

By the dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} E(M_{Y_{\varepsilon,a}})\psi(a)\,da = \int_{\mathbb{R}} E\bigl(\exp\bigl(\lambda X_{\tau_{a^+}\wedge T} - \tfrac{1}{2}\lambda^2 V_{\tau_{a^+}\wedge T}\bigr)\bigr)\psi(a)\,da.$$

Now, set $F(t) = M_t \frac{\partial R}{\partial s}(t, \tau_y)$. Then, assuming that $\varphi_\varepsilon(x) = \varepsilon^{-1}\mathbf{1}_{[0,1]}(x/\varepsilon)$, we have

$$\begin{aligned}
&\int_{y-\varepsilon}^{y} \varphi_\varepsilon(y-a) M_{Y_{\varepsilon,a}} \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y)\psi(a)\,da \\
&\quad = \frac{1}{\varepsilon^2}\int_{y-\varepsilon}^{y} \mathbf{1}_{[0,1]}\Bigl(\frac{y-a}{\varepsilon}\Bigr) F\biggl(\int_a^{a+\varepsilon} \mathbf{1}_{[0,1]}\Bigl(\frac{x-a}{\varepsilon}\Bigr)(\tau_x \wedge T)\,dx\biggr)\psi(a)\,da \\
&\quad = \int_0^1 F\biggl(\int_0^1 (\tau_{y+\varepsilon\xi-\varepsilon\eta} \wedge T)\,d\xi\biggr)\psi(y-\varepsilon\eta)\,d\eta
\end{aligned}$$

$$= \int_0^1 F\left(\int_0^\eta (\tau_{y+\varepsilon\xi-\varepsilon\eta} \wedge T)\, d\xi + \int_\eta^1 (\tau_{y+\varepsilon\xi-\varepsilon\eta} \wedge T)\, d\xi\right) \psi(y-\varepsilon\eta)\, d\eta.$$

As ε tends to zero, this expression clearly converges to

$$\psi(y) \int_0^1 F\big(\eta(\tau_y \wedge T) + (1-\eta)(\tau_{y^+} \wedge T)\big)\, d\eta.$$

So, we have proved that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\mathbb{R}} M_{Y_{\varepsilon,a}} \varphi_\varepsilon(y-a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a)\, da \\ &\qquad = \psi(y) \int_0^1 M_{z\tau_{y^+}+(1-z)\tau_y} \frac{\partial R}{\partial s}\big(z\tau_{y^+} + (1-z)\tau_y, \tau_y\big)\, dz. \end{aligned} \tag{3.10}$$

In order to complete the proof of the theorem, we will apply the dominated convergence theorem. We have the following estimate for $y \le S_T$:

$$\left| \int_{\mathbb{R}} M_{Y_{\varepsilon,a}} \varphi_\varepsilon(y-a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a)\, da \right| \le \|\psi\|_\infty \sup_{s,t\le T} \left| \frac{\partial R}{\partial s}(s,t) \right| \sup_{t \le T} |M_t|,$$

which allows us to commute the limit (3.10) with the integral with respect to the measure $P \times d\tau_y$ on the set $\{(\omega, y) : y \le S_T(\omega)\}$. In this way we get

$$\begin{aligned} &\int_{\mathbb{R}} E(M_{\tau_y}) \psi(y)\, dy \\ &\quad = \int_{\mathbb{R}} \psi(y)\, dy \\ &\qquad - \lambda E\left(\int_0^{S_T} \psi(y) \int_0^1 M_{z\tau_{y^+}+(1-z)\tau_y} \frac{\partial R}{\partial s}\big(z\tau_{y^+} + (1-z)\tau_y, \tau_y\big)\, dz\, d\tau_y \right). \end{aligned}$$

Approximating $\mathbf{1}_{[0,a]}$ by a sequence of smooth functions $(\psi_n, n \ge 1)$ and letting T tend to infinity completes the proof. $\square$

If we assume that the partial derivative $\frac{\partial R}{\partial t}(t,s)$ is nonnegative, then we can derive the following result.

PROPOSITION 3.5. *Assume that X satisfies hypotheses* (H1), (H2) *and* (H3). *If $\frac{\partial R}{\partial s}(s,t) \ge 0$, then, for all $\alpha, a > 0$, we have*

$$E(\exp(-\alpha V_{\tau_a})) \le e^{-a\sqrt{2\alpha}}. \tag{3.11}$$

PROOF. Since $\frac{\partial R}{\partial t}(t,s) \ge 0$, we obtain

$$E(M_{\tau_a}) \le 1,$$

that is,

$$E\big(\exp\big(\lambda a - \tfrac{1}{2}\lambda^2 V_{\tau_a}\big)\big) \le 1,$$

or

$$E(\exp(-\alpha V_{\tau_a})) \le e^{-a\sqrt{2\alpha}}.$$

The result follows. $\square$

The above proposition means that the Laplace transform of the random variable V_{τ_a} is dominated by the Laplace transform of τ_a, where τ_a is the hitting time of the level a for the ordinary Brownian motion. This domination implies some consequences on the moments of V_{τ_a}. In fact, for any $r > 0$, we have, multiplying (3.11) by α^r,

$$\begin{aligned} E(V_{\tau_a}^{-r}) &= \frac{1}{\Gamma(r)} \int_0^\infty E(e^{-\alpha V_{\tau_a}})\alpha^{r-1}\, d\alpha \\ &\le \frac{1}{\Gamma(r)} \int_0^\infty e^{-a\sqrt{2\alpha}}\alpha^{r-1}\, d\alpha \\ &= \frac{2^r\Gamma(r+1/2)}{\sqrt{\pi}} a^{-2r}. \end{aligned} \tag{3.12}$$

On the other hand, for $0 < r < 1$,

$$\begin{aligned} E(V_{\tau_a}^{r}) &= \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - E(e^{-\alpha V_{\tau_a}}))\alpha^{-r-1}\, d\alpha \\ &\ge \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - e^{-a\sqrt{2\alpha}})\alpha^{-r-1}\, d\alpha. \end{aligned} \tag{3.13}$$

In particular, for $r \in [1/2, 1)$, $E(V_{\tau_a}^r) = +\infty$.

REMARK 3.6. If X is the standard Brownian motion, its covariance $s \wedge t$ does not satisfy condition (H1), but we still can apply our approach. It is known from [3] that $d\tau_a$ has no absolutely continuous part and that $\{a, \tau_a = \tau_a^+\}$ is a Cantor set, hence, of zero Lebesgue measure. It follows from this observation and from (3.10) that

$$\int E(M_{\tau_y})\psi(y)\, dy = \int \psi(y)\, dy.$$

Choosing $\psi = \mathbf{1}_{[0,a]}$ yields to the expected result:

$$E\left(\int_0^a e^{\lambda y - (\lambda^2/2)V(\tau_y)}\, dy\right) = a.$$

If X has independent increments and satisfies (H3), then

$$E\big(e^{-(\lambda^2/2)V(\tau_a)}\big) = e^{-\lambda a}.$$

This follows easily from the fact that X can be written as a time-changed Brownian motion.

REMARK 3.7. Consider that X is a fractional Brownian motion of Hurst index $H = 1$. Then $R(s,t) = st$, and consequently, $X_t = Yt$, where Y is a one-dimensional standard Gaussian random variable. Then, $\tau_a = \tau_{a^+} = a/Y^+$. It is then easy to compute the Laplace transform of τ_a and we obtain

$$E(\exp(-\alpha\tau_a^2)) = \tfrac{1}{2}e^{-a\sqrt{2\alpha}}. \tag{3.14}$$

We show now that our formula also yields to the right answer. We just note that $(y \mapsto \tau_y)$ is continuous. This entails that

$$\frac{\partial R}{\partial s}(z\tau_{y^+} + (1-z)\tau_y, \tau_y) = \frac{\partial R}{\partial s}(\tau_y, \tau_y) = \frac{1}{2}V'(\tau_y)$$

and

$$\begin{aligned} &\int_0^a E\biggl(\exp\biggl(\lambda y - \frac{\lambda^2}{2}V(\tau_y)\biggr)\biggr)\,dy \\ &\quad = a - \frac{\lambda}{2}E\biggl(\int_0^a \exp\biggl(\lambda y - \frac{\lambda^2}{2}V(\tau_y)\biggr)V'(\tau_y)\,d\tau_y\biggr). \end{aligned} \tag{3.15}$$

Set

$$\Psi(a,\lambda) = E\biggl(\exp\biggl(\lambda a - \frac{\lambda^2}{2}V(\tau_a)\biggr)\biggr),$$

then

$$\frac{\partial\Psi}{\partial a}(a,\lambda) = \lambda\Psi(a,\lambda) - \frac{\lambda^2}{2}E\biggl(M_{\tau_a}\frac{\partial V(\tau_a)}{\partial a}\biggr). \tag{3.16}$$

Substitute (3.15) into (3.16) to obtain

$$\frac{\partial\Psi}{\partial a} = 2\lambda\Psi - \lambda.$$

Then, there exists a function $C(\lambda)$ such that

$$\Psi(a,\lambda) = \frac{1}{2} + C(\lambda)e^{2\lambda a} \qquad \text{so that } E\biggl(\exp\biggl(-\frac{\lambda^2}{2}\tau_a^2\biggr)\biggr) = \frac{1}{2}e^{-\lambda a} + C(\lambda)e^{\lambda a}.$$

By dominated convergence, it is clear that, for any λ,

$$E\biggl(\exp\biggl(-\frac{\lambda^2}{2}\tau_a^2\biggr)\biggr) \overset{a\to\infty}{\longrightarrow} 0,$$

thus, $C(\lambda) = 0$ and

$$E\biggl(\exp\biggl(-\frac{\lambda^2}{2}\tau_a^2\biggr)\biggr) = \frac{1}{2}e^{-\lambda a}.$$

Changing $\lambda^2/2$ into α gives (3.14).

REMARK 3.8. Consider the case of a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Conditions (H1), (H2) and (H3) are satisfied and we obtain

$$\begin{aligned}\int_0^a E(M_{\tau_y})\,dy \\ &= a - \lambda H E\biggl(\int_0^a \int_0^1 M_{z\tau_{y^+}+(1-z)\tau_y}\bigl([z\tau_{y^+} + (1-z)\tau_y]^{2H-1} \\ &\qquad - |z(\tau_{y^+} - \tau_y)|^{2H-1}\bigr)\,dz\,d\tau_y\biggr).\end{aligned}$$

Moreover, $E(e^{-\alpha\tau_a^{2H}}) \le e^{-a\sqrt{2\alpha}}$, and therefore, $E(\tau_a^p) < \infty$ if $p < H$. According to (3.13), $E(\tau_a^p)$ is infinite if $pH > 1/4$ and (3.12) entails that τ_a has finite negative moments of all orders.

Acknowledgments. This work was carried out during a stay of Laurent Decreusefond at Kansas University, Lawrence, KS. He would like to thank KU for warm hospitality and generous support.

REFERENCES

[1] ALÒS, E., MAZET, O. and NUALART, D. (2001). Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* **29** 766–801. MR1849177

[2] CHERIDITIO, P. and NUALART, D. (2005). Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$. *Ann. Inst. H. Poincaré Probab. Statist.* **41** 1049–1081. MR2172209

[3] ITÔ, K. and MCKEAN, H. P., JR (1974). *Diffusion Processes and Their Sample Paths*. Springer, Berlin. MR0345224

[4] KIM, J. and POLLARD, D. (1990). Cube root asymptotics. *Ann. Probab.* **18** 191–219. MR1041391

[5] MOLCHAN, G. M. (2000). On the maximum of fractional Brownian motion. *Theory Probab. Appl.* **44** 97–102. MR1751192

[6] NUALART, D. and VIVES, J. (1988). Continuité absolue de la loi du maximum d'un processus continu. *C. R. Acad. Sci. Paris Sér. I Math.* **307** 349–354. MR0958796

DEPARTMENT OF MATHEMATICS OF INFORMATION
COMMUNICATIONS AND CALCULUS
TELECOM PARIS
46, RUE BARRAULT
75634 PARIS
FRANCE
E-MAIL: Laurent.Decreusefond@enst.fr

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
405 SNOW HALL
1460 JAYHAWK BLVD
LAWRENCE, KANSAS 66045-7523
USA
E-MAIL: nualart@math.ku.edu