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SOME PARABOLIC PDEs WHOSE DRIFT IS AN IRREGULAR RANDOM NOISE IN SPACE

BY FRANCESCO RUSSO AND GERALD TRUTNAU

Université Paris 13 and Universität Bielefeld

A new class of random partial differential equations of parabolic type is considered, where the stochastic term consists of an irregular noisy drift, not necessarily Gaussian, for which a suitable interpretation is provided. After freezing a realization of the drift (stochastic process), we study existence and uniqueness (in some appropriate sense) of the associated parabolic equation and a probabilistic interpretation is investigated.

1. Introduction. This paper focuses on a random partial differential equation consisting of a parabolic PDE with irregular noise in the drift. Formulation, existence (with uniqueness in a certain sense) and double probabilistic representation are discussed. The equation itself is motivated by *random irregular media models*.

Let $T > 0$, $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\dot{\eta}(x)$ a generalized random field playing the role of a noise. Let $u^0 : \mathbb{R} \to \mathbb{R}$, $\lambda : [0, T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Consider the problem

$$\begin{aligned} -\partial_t v(t,x) + \frac{\sigma^2(x)}{2}\,\partial_{xx}^2 v(t,x) + \dot{\eta}(x)\,\partial_x v(t,x) &= \lambda(T-t,x), \\ v(0,x) &= u^0(x), \end{aligned} \tag{1.1}$$

where $\dot{\eta}$ is the derivative in the sense of distributions of a continuous process. Among examples of possible η, we have in mind not only different possibilities of continuous processes as classical Wiener process and (multi) fractional Brownian motion, but also non-Gaussian processes. The derivative in the sense of distributions $\dot{\eta}(x)$ will be the associated noise. (1.1) is a new type of SPDE, not yet studied in any real depth even when η is a classical Brownian motion. For the situation where $\dot{\eta}(x)$ is replaced by a space-time white noise $\dot{\eta}(t,x)$, some relevant work was done by Nualart and Viens (see, e.g., [17]). In this article, time dependence is useful for the corresponding stochastic integration.

Equation (1.1) is equivalent to the following *dual* problem:

$$\begin{aligned} \partial_t u(t,x) + \frac{\sigma^2(x)}{2}\,\partial_{xx}^2 u(t,x) + \dot{\eta}(x)\,\partial_x u(t,x) &= \lambda(t,x), \\ u(T,x) &= u^0(x). \end{aligned} \tag{1.2}$$

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Formally speaking, setting $u(t,x) = v(T-t,x)$, v solves (1.1) if and only if u solves (1.2). This is rigorously confirmed in Section 9 so that at this stage, the choice of whether to work with equation (1.1) or (1.2) is arbitrary. We have decided to concentrate on equation (1.2) because it corresponds to the standard form for probabilistic representation.

The idea of this paper is to first freeze the realization ω, to set $b(x) = \eta(x)(\omega)$ and then to consider the deterministic Cauchy problem associated with (1.2),

$$\begin{aligned} \partial_t u(t,x) + \frac{\sigma^2(x)}{2}\,\partial_{xx}^2 u(t,x) + b'(x)\,\partial_x u(t,x) &= \lambda(t,x), \\ u(T,x) &= u^0(x), \end{aligned} \tag{1.3}$$

where b' is the derivative of the continuous function b.

Since the product of a distribution and a continuous function is not defined in the theory of Schwarz distributions, we must develop some substitution tools. Ideally, we would like to represent the parabolic PDE probabilistically through a diffusion which is the solution of the stochastic differential equation (SDE)

$$dX_t = \sigma(X_t)\,dW_t + b'(X_t)\,dt \tag{1.4}$$

with generalized drift. We will give a meaning to (1.4) at three different levels:

- the level of a martingale problem;
- the level of a stochastic differential equation in the sense of probability laws;
- the level of a stochastic differential equation in the strong sense.

For each of these levels, we shall provide conditions for equation (1.4), with given initial data, to be well posed. Later, the notion of a C_b^0-solution to the generalized parabolic PDE (1.3) will be defined; related to this, existence, uniqueness and probabilistic representation will be shown.

When η is a strong finite cubic variation process and $\sigma = 1$, the solutions to (1.3) obtained for $b = \eta(\omega)$ provide solutions to the SPDE (1.1). This is shown in the last part of the paper. A typical example of a strong zero cubic variation process is the fractional Brownian motion with Hurst index $H \geq \frac{1}{3}$. Equation (1.3) will be understood in some *weak distributional* sense that we can formally reconstruct as follows. We freeze $b = \eta(\omega)$ as a realization and formally integrate equation (1.1) from 0 to t in time against a smooth test function α with compact support in space. The result is

$$\begin{aligned} -\int_{\mathbb{R}} dx\,\alpha(x)u(t,x) + \int_{\mathbb{R}} dx\,\alpha(x)u^0(x) - \int_0^t ds\,\tfrac{1}{2}\int_{\mathbb{R}} dx\,\alpha'(x)\,\partial_x u(s,x) \\ + \int_0^t ds \int_{\mathbb{R}} b(dx)\alpha(x)\,\partial_x u(s,x) \\ = \int_0^t ds \int_{\mathbb{R}} dx\,\alpha(x)\lambda(T-s,x). \end{aligned} \tag{1.5}$$

The integral $\int_{\mathbb{R}} \alpha(x)\,\partial_x u(s,x) b(dx)$ needs interpretation since b is not generally of bounded variation and it involves the product of the distribution b' and the function $\partial_x u(s,\cdot)$; in general, this function is, unfortunately, only continuous. As expected this operation is deterministically undefined, unless one uses a generalized functions theory. However, since b is a frozen realization of a stochastic process η, we can hope to justify the integral in a stochastic sense. Note that it cannot be of Itô type, even if η were a semimartingale, since $\partial_x u(s,\cdot)$ is not necessarily adapted to some corresponding filtration. We will, in fact, interpret the stochastic integral element $b(dx)$ or $\eta(dx)$ as a symmetric (Stratonovich) integral $d^0\eta$ of regularization type; see Section 3.

DEFINITION 1.1. A continuous random field $(v(t,x), t \in [0,T], x \in \mathbb{R})$, a.s. in $C^{0,1}(]0,T[\times\mathbb{R})$, is said to be a (*weak*) *solution* to the SPDE (1.1) if

$$\begin{aligned}
&-\int_{\mathbb{R}} dx\,\alpha(x) v(t,x) + \int_{\mathbb{R}} dx\,\alpha(x) v^0(x) - \int_0^t ds\,\tfrac{1}{2}\int_{\mathbb{R}} dx\,\alpha'(x)\,\partial_x v(s,x)\\
&\qquad + \int_{\mathbb{R}} d^\circ\eta(x)\alpha(x)\left(\int_0^t ds\,\partial_x v(s,x)\right)\\
&\qquad = \int_0^t ds \int_{\mathbb{R}} dx\,\alpha(x)\lambda(T-s,x)
\end{aligned} \tag{1.6}$$

for every smooth function with compact support α.

If we integrate equation (1.2) from t to T in time against a smooth test function α with compact support in space, we are naturally led to the following.

DEFINITION 1.2. A continuous random field $(u(t,x), t \in [0,T], x \in \mathbb{R})$, a.s. in $C^{0,1}(]0,T[\times\mathbb{R})$, is said to be a (*weak*) *solution* to the SPDE (1.2) if

$$\begin{aligned}
&-\int_{\mathbb{R}} dx\,\alpha(x) u(t,x) + \int_{\mathbb{R}} dx\,\alpha(x) u^0(x) - \int_t^T ds\,\tfrac{1}{2}\int_{\mathbb{R}} dx\,\alpha'(x)\,\partial_x u(s,x)\\
&\qquad + \int_{\mathbb{R}} d^\circ\eta(x)\alpha(x)\left(\int_t^T ds\,\partial_x u(s,x)\right)\\
&\qquad = \int_t^T ds \int_{\mathbb{R}} dx\,\alpha(x)\lambda(s,x)
\end{aligned} \tag{1.7}$$

for every smooth function with compact support α.

We will show that the *probabilistic* solutions that we construct through stochastic equation (1.4) will, in fact, solve (1.5).

Diffusions in the generalized sense were studied by several authors beginning with (at least to our knowledge) [19]. Later, many authors considered special cases of stochastic differential equations with generalized coefficients. It is difficult to quote them all. In particular, we refer to the case when b is a measure [7, 16, 18]. In all of these cases, solutions were semimartingales. More recently, [8] considered special cases of nonsemimartingales solving stochastic differential equations with generalized drift; those cases include examples coming from Bessel processes.

[10] and [11] treated well-posedness of the martingale problem, Itô's formula under weak conditions, semimartingale characterization and the Lyons–Zheng decomposition. The only assumption was the strict positivity of σ and the existence of the function $\Sigma(x) = 2\int_0^x \frac{b'}{\sigma^2}\,dy$ with appropriate regularizations. Bass and Chen [2] were also interested in (1.4) and provided a well-stated framework when σ is $\frac{1}{2}$-Hölder continuous and b is γ-Hölder continuous, $\gamma > \frac{1}{2}$.

Beside the martingale problem, in the present paper, we shall emphasize the formulation of (1.4) as a stochastic differential equation which can be solved by introducing more assumptions on the coefficients. Several examples are provided for the case of weak and strong solutions of (1.4).

The paper is organized as follows. Section 2 is devoted to basic preliminaries, including definitions and properties related to Young integrals. Section 3 is devoted to some useful remainder in stochastic calculus via regularization. In Section 4, we introduce the formal *elliptic* operator L and recall the concept of a C^1-generalized solution of $Lf = \dot{\ell}$ for continuous real functions $\dot{\ell}$. We further introduce a fundamental hypothesis on L for the sequel, called Technical Assumption $\mathcal{A}(\nu_0)$, and we illustrate several examples where it is verified. In Section 5, we discuss different notions of martingale problems. Section 6 provides notions of solutions to *stochastic differential equations with distributional drift* and their connections with martingale problems. The notion of solution is coupled with a property of *extended local time regularity*. This concept of solution is new, even when the drift is an ordinary function. Section 7 presents the notion of a C_b^0-solution for a parabolic equation $\mathcal{L}u = \lambda$, where λ is bounded and continuous with $\mathcal{L} = \partial_t + L$. We also provide existence, uniqueness and probabilistic representations of C_b^0-solutions to $\mathcal{L}u = \lambda$. Section 8 discusses *mild* solutions to the previous parabolic PDE and useful integrability properties for its solutions. In Section 9, we finally show that the C_b^0-solutions provide, in fact, true *weak* solutions to the SPDE (1.1) if $\sigma = 1$.

2. Preliminaries. In this paper, T will be a fixed horizon time, unless otherwise specified. A function f defined on $[0, T]$ (resp., $\mathbb{R}_+$) will be extended, without mention, by setting $f(t) = f(0)$ for $t \le 0$ and $f(T)$ for $t \ge T$ [resp., $f(0)$ for $t \le 0$].

$C^0(\mathbb{R})$ will indicate the set of continuous functions defined on $\mathbb{R}$, $C^p(\mathbb{R})$, the space of real functions with differentiability class C^p. We denote by $C_0^0(\mathbb{R})$ [resp., $C_0^1(\mathbb{R})$] the space of continuous (continuous differentiable) functions vanishing at zero. When there is no confusion, we will also simply use the symbols C^0, C^p, C_0^0, C_0^1. We denote by $C_b^0([0, T] \times \mathbb{R})$ the space of real continuous bounded functions defined on $[0, T] \times \mathbb{R}$. $C_b^0(\mathbb{R})$, or simply C_b^0, indicates the space of continuous bounded functions defined on $\mathbb{R}$.

The vector spaces $C^0(\mathbb{R})$ and $C^p(\mathbb{R})$ are topological Fréchet spaces, or F-spaces, according to the terminology of [5], Chapter 1.2. They are equipped with the following natural topology. A sequence f_n belonging to $C^0(\mathbb{R})$ [resp., $C^p(\mathbb{R})$] is said to converge to f in the $C^0(\mathbb{R})$ [resp., $C^p(\mathbb{R})$] sense if f_n (resp., f_n and all derivatives up to order p) converges (resp., converge) to f (resp., to f and all its derivatives) uniformly on each compact of $\mathbb{R}$.

We will consider functions $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ which are bounded and continuous. A sequence (u_n) in $C_b^0([0, T] \times \mathbb{R})$ will be said to converge *in a bounded way* to u if:

- $\lim_{n\to\infty} u_n(t, x) = u(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}$;
- there exists a constant $c > 0$, independent of the sequence, such that

$$\sup_{t\le T, x\in\mathbb{R}} |u_n(t, x)| \le c \qquad \forall n \in \mathbb{N}. \tag{2.1}$$

If the sequence (u_n) does not depend on t, we similarly define the convergence of $(u_n) \in C_b^0(\mathbb{R})$ to $u \in C_b^0(\mathbb{R})$ in a bounded way.

Given two functions $u_1, u_2 : [0, T] \times \mathbb{R} \to \mathbb{R}$, the composition notation $u_1 \circ u_2$ means $(u_1 \circ u_2)(t, x) = u_1(t, u_2(t, x))$.

For positive integers m, k, $C^{m,k}$ will indicate functions in the corresponding differentiability class. For instance, $C^{1,2}([0, T[\times\mathbb{R})$ will be the space of $(t, x) \mapsto u(t, x)$ functions which are C^1 on $[0, T[\times\mathbb{R}$ (i.e., once continuously differentiable) and such that $\partial_{xx}^2 u$ exists and is continuous.

$C_b^{m,k}$ will indicate the set of functions $C^{m,k}$ such that the partial derivatives of all orders are bounded.

If I is a real compact interval and $\gamma \in]0, 1[$, we denote by $C^\gamma(I)$ the vector space of real functions defined on I which are Hölder with parameter γ. We denote by $C^\gamma(\mathbb{R})$, or simply C^γ, the space of locally Hölder functions, that is, Hölder on each real compact interval.

Suppose $I = [\tau, T]$, τ, T being two real numbers such that $\tau < T$. Here, T does not necessarily need to be positive. Recall that $f : I \mapsto \mathbb{R}$ belongs to $C^\gamma(I)$ if

$$N_\gamma(f) := \sup_{\tau\le s,t\le T} \frac{|f(t) - f(s)|}{|t - s|^\gamma} < \infty.$$

Clearly, $f \mapsto |f(\tau)| + N_\gamma(f)$ defines a norm on $C^\gamma(I)$ which makes it a Banach space. $C^\gamma(\mathbb{R})$ is an F-space if equipped with the topology of convergence related to $C^\gamma(I)$ for each compact interval I. A sequence (f_n) in $C^\gamma(\mathbb{R})$ converges to f if it converges according to $C^\gamma(I)$ for every compact interval I.

We will also provide some reminders about the so-called *Young integrals* (see [27]) but will remain, however, in a simplified framework, as in [9] or [23]. We recall the essential inequality, stated, for instance, in [9]:

Let $\gamma, \beta > 0$ be such that $\gamma + \beta > 1$. If $f, g \in C^1(I)$, then

$$\left| \int_a^b (f(x) - f(a))\, dg(x) \right| \le C_\rho (b - a)^{1+\rho} N_\gamma(f) N_\beta(g) \tag{2.2}$$

for any $[a,b] \subset I$ and $\rho \in]0, \gamma+\beta-1[$, where C_ρ is a constant not depending on f, g. The bilinear map sending (f,g) to $\int_0^{\cdot} f\,dg$ can be continuously extended to $C^\gamma(I) \times C^\beta(I)$ with values in $C^0(I)$. By definition, that object will be called the *Young integral* of f with respect to g on I. We also denote it $\int_\tau^{\cdot} f\,d^{(y)}g$.

By additivity, we set, for $a, b \in [\tau, T]$,

$$\int_a^b f\,d^{(y)}g = \int_\tau^b f\,d^{(y)}g - \int_\tau^a f\,d^{(y)}g.$$

Moreover, the bilinear map defined on $C^1(\mathbb{R}) \times C^1(\mathbb{R})$ by $(f,g) \to \int_0^{\cdot} f\,dg$ extends continuously to $C^\gamma(\mathbb{R}) \times C^\beta(\mathbb{R})$ onto $C^0(\mathbb{R})$. Again, that object, defined on the whole real line, will be called *Young integral* of f with respect to g and will again be denoted by $\int_0^{\cdot} f\,d^{(y)}g$.

REMARK 2.1. Inequality (2.2) remains true for $f \in C^\gamma(I)$, $g \in C^\beta(I)$. In particular, $t \mapsto \int_\tau^t f\,d^{(y)}g$ belongs to $C^\beta(I)$. In fact,

$$\left|\int_a^b f\,dg\right| \le \left|\int_a^b \bigl(f - f(a)\bigr)\,dg\right| + \bigl|f(a)\bigl(g(b) - g(a)\bigr)\bigr|.$$

Through the extension of the bilinear operator sending (f,g) to $\int_0^{\cdot} f\,dg$, it is possible to get the following chain rule for Young integrals.

PROPOSITION 2.2. *Let $f, g, F : I \to \mathbb{R}$, $I = [\tau, T]$. We suppose that $g \in C^\beta(I)$, $f \in C^\gamma(I)$, $F \in C^\delta(I)$ with $\gamma + \beta > 1$, $\delta + \beta > 1$. We define $G(t) = \int_\tau^t f\,d^{(y)}g$. Then*

$$\int_\tau^t F\,d^{(y)}G = \int_\tau^t Ff\,d^{(y)}g.$$

PROOF. If $g \in C^1(I)$, then the result is obvious. We remark that $G \in C^\gamma(I)$. Repeatedly using inequality (2.2), one can show that the two linear maps $g \mapsto \int_\tau^{\cdot} F\,d^{(y)}G$ and $g \mapsto \int_\tau^{\cdot} Ff\,d^{(y)}g$ are continuous from $C^\delta(I)$ to $C^0(I)$. This concludes the proof of the proposition. $\square$

By a *mollifier*, we mean a function $\Phi \in \mathcal{S}(\mathbb{R})$ (i.e., a C^∞-function such that itself and all its derivatives decrease to zero faster than any power of $|x|^{-1}$ as $|x| \to \infty$) with $\int \Phi(x)\,dx = 1$. We set $\Phi_n(x) := n\Phi(nx)$.

The result below shows that mollifications of a Hölder function f converge to f with respect to the Hölder topology.

PROPOSITION 2.3. *Let Φ be a mollifier and let $f \in C^{\gamma'}(I)$. We write $f_n = \Phi_n * f$. Then $f_n \to f$ in the $C^\gamma(I)$ topology for any $0 < \gamma < \gamma'$.*

PROOF. We need to show that $N_\gamma(f - f_n)$ converges to zero. We set $\Delta_n(t) = (f - f_n)(t)$. Let $a, b \in I$. We will establish that

$$|\Delta_n(b) - \Delta_n(a)| \le \text{const}\,|b-a|^\gamma \left(\frac{1}{n}\right)^{\gamma'-\gamma}. \tag{2.3}$$

Without loss of generality, we can suppose that $a < b$. We distinguish between two cases.

Case $a < a + \frac{1}{n} < b$.

We have

$$\begin{aligned}
|\Delta_n(b) - \Delta_n(a)| &\le \left| \int \left(f\left(b - \frac{y}{n}\right) - f(b) \right) \Phi(y)\,dy \right| \\
&\quad + \left| \int \left(f\left(a - \frac{y}{n}\right) - f(a) \right) \Phi(y)\,dy \right| \\
&\le 2 \int \left| \frac{y}{n} \right|^{\gamma'} |\Phi(y)|\,dy \\
&\le 2 \int |\Phi(y)||y|^{\gamma'}\,dy (b-a)^\gamma \left(\frac{1}{n}\right)^{\gamma'-\gamma}.
\end{aligned}$$

Case $a < b \le a + \frac{1}{n}$.

In this case, we have

$$\begin{aligned}
&|\Delta_n(b) - \Delta_n(a)| \\
&\quad \le \int |f(b) - f(a)||\Phi(y)|\,dy + \int \left| f\left(b + \frac{y}{n}\right) - f\left(a + \frac{y}{n}\right) \right| |\Phi(y)|\,dy \\
&\quad \le 2(b-a)^{\gamma'} \int |\Phi(y)|\,dy \le 2 \int |\Phi(y)|\,dy (b-a)^\gamma \left(\frac{1}{n}\right)^{\gamma'-\gamma}.
\end{aligned}$$

Therefore, (2.3) is verified with $\text{const} = 2 \int |\Phi(y)|(1 + |y|^{\gamma'})\,dy$. This implies that

$$N_\gamma(f - f_n) \le \text{const}\left(\frac{1}{n}\right)^{\gamma'-\gamma},$$

which allows us to conclude. $\square$

For convenience, we introduce the topological vector space defined by

$$D^\gamma = \bigcup_{\gamma' > \gamma} C^{\gamma'}(\mathbb{R}).$$

It is also a *vector algebra*, that is, D^γ is a vector space and an algebra with respect to the *sum and product of functions*.

The next corollary is a consequence of the definition of the Young integral and Remark 2.1.

COROLLARY 2.4. *Let $f \in D^\gamma$, $g \in D^\beta$ with $\gamma + \beta \geq 1$. Then $t \mapsto \int_0^t f\, d^{(y)} g$ is well defined and belongs to D^β.*

D^γ is not a metric space, but an inductive limit of the F-spaces C^γ; the weak version of the Banach–Steinhaus theorem for F-spaces can be adapted.

In fact, a direct consequence of the Banach–Steinhaus theorem of [5], Section 2.1, is the following.

THEOREM 2.5. *Let $E = \bigcup_n E_n$ be an inductive limit of F-spaces E_n and F another F-space. Let (T_n) be a sequence of continuous linear operators $T_n : E \to F$. Suppose that $Tf := \lim_{n\to\infty} T_n f$ exists for any $f \in E$. Then $T : E \to F$ is again a continuous (linear) operator.*

3. Previous results in stochastic calculus via regularization. We recall here a few notions related to stochastic calculus via regularization, a theory which began with [21]. We refer to a recent survey paper [23].

The stochastic processes considered may be defined on $[0, T], \mathbb{R}_+$ or $\mathbb{R}$. Let $X = (X_t, t \in \mathbb{R})$ be a continuous process and $Y = (Y_t, t \in \mathbb{R})$ be a process with paths in L^1_{loc}. For the paths of process Y with parameter on $[0, T]$ (resp., $\mathbb{R}_+$), we apply the same convention as was applied at the beginning of previous section for functions. So we extend them without further mention, setting Y_0 for $t \leq 0$ and Y_T for $t \geq T$ (resp., Y_0 for $t \leq 0$). $\mathcal{C}$ will denote the vector algebra of continuous processes. It is an F-space if equipped with the topology of u.c.p. (uniform convergence in probability) convergence.

In the sequel, we recall the most useful rules of calculus; see, for instance, [23] or [22].

The forward symmetric integrals and the covariation process are defined by the following limits in the u.c.p. sense, whenever they exist:

$$\int_0^t Y\, d^- X := \lim_{\varepsilon\to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\, ds, \tag{3.1}$$

$$\int_0^t Y_s\, d^\circ X_s := \lim_{\varepsilon\to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_{s-\varepsilon}}{2\varepsilon}\, ds, \tag{3.2}$$

$$[X, Y]_t := \lim_{\varepsilon\to 0+} C^\varepsilon(X, Y)_t, \tag{3.3}$$

where

$$C^\varepsilon(X, Y)_t := \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)\, ds.$$

All stochastic integrals and covariation processes will of course be elements of $\mathcal{C}$. If $[X, Y]$, $[X, X]$ and $[Y, Y]$ exist, we say that (X, Y) *has all of its mutual covariations*.

REMARK 3.1. If X is (locally) of bounded variation, we have:

- $\int_0^t X\, d^- Y = \int_0^t X_s\, d^\circ Y_s = \int_0^t X_s\, dY_s$, where the third integral is meant in the Lebesgue–Stieltjes sense;
- $[X, Y] \equiv 0$.

REMARK 3.2. (a) $\int_0^t Y_s\, d^\circ X_s = \int_0^t Y_s\, d^- X_s + \frac{1}{2}[X, Y]$ provided that two of the three integrals or covariations exist.

(b) $X_t Y_t = X_0 Y_0 + \int_0^t Y_s\, d^- X_s + \int_0^t X_s\, d^- Y_s + [X, Y]_t$ provided that two of the three integrals or covariations exist.

(c) $X_t Y_t = X_0 Y_0 + \int_0^t Y\, d^\circ X + \int_0^t X_s\, d^\circ Y_s$ provided that one of the two integrals exists.

REMARK 3.3. (a) If $[X, X]$ exists, then it is always an increasing process and X is called a *finite quadratic variation process*. If $[X, X] = 0$, then X is said to be a *zero quadratic variation process*.

(b) Let X, Y be continuous processes such that (X, Y) has all of its mutual covariations. Then $[X, Y]$ has locally bounded variation. If $f, g \in C^1$, then

$$[f(X), g(Y)]_t = \int_0^t f'(X) g'(Y)\, d[X, Y].$$

(c) If A is a zero quadratic variation process and X is a finite quadratic variation process, then $[X, A] \equiv 0$.

(d) A bounded variation process is a zero quadratic variation process.

(e) (*Classical Itô formula.*) If $f \in C^2$, then $\int_0^\cdot f'(X)\, d^- X$ exists and is equal to

$$f(X) - f(X_0) - \frac{1}{2} \int_0^\cdot f''(X)\, d[X, X].$$

(f) If $g \in C^1$ and $f \in C^2$, then the forward integral $\int_0^\cdot g(X)\, d^- f(X)$ is well defined.

In this paper, all filtrations are supposed to fulfill the usual conditions. If $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a filtration, X an $\mathbb{F}$-semimartingale and Y is an $\mathbb{F}$-adapted cadlag process, then $\int_0^\cdot Y\, d^- X$ is the usual Itô integral. If Y is an $\mathbb{F}$-semimartingale, then $\int_0^\cdot Y\, d^\circ X$ is the classical Fisk–Stratonovich integral and $[X, Y]$ is the usual covariation process $\langle X, Y \rangle$.

We now introduce the notion of Dirichlet process, which was essentially introduced by Föllmer [12] and has been considered by many authors; see, for instance, [3, 24] for classical properties.

In the present section, (W_t) will denote a classical $(\mathcal{F}_t)$-Brownian motion.

DEFINITION 3.4. An $(\mathcal{F}_t)$-adapted (continuous) process is said to be a *$(\mathcal{F}_t)$-Dirichlet process* if it is the sum of an $(\mathcal{F}_t)$-local martingale M and a zero quadratic variation process A. For simplicity, we will suppose that $A_0 = 0$ a.s.

REMARK 3.5. (i) Process (A_t) in the previous decomposition is an $(\mathcal{F}_t)$-adapted process.

(ii) An $(\mathcal{F}_t)$-semimartingale is an $(\mathcal{F}_t)$-Dirichlet process.

(iii) The decomposition $M + A$ is unique.

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be of class C^1 and let X be an $(\mathcal{F}_t)$-Dirichlet process. Then $f(X)$ is again an $(\mathcal{F}_t)$-Dirichlet process with local martingale part $M_t^f = f(X_0) + \int_0^t f'(X)\,dM$.

The class of semimartingales with respect to a given filtration is known to be stable with respect to C^2 transformations. Remark 3.3(b) says that finite quadratic variation processes are stable through C^1 transformations. The last point of the previous remark states that C^1 stability also holds for Dirichlet processes.

Young integrals introduced in Section 2 can be connected with the forward and symmetric integrals via the regularization appearing before Remark 3.1. The next proposition was proven in [23].

PROPOSITION 3.6. *Let X, Y be processes whose paths are respectively in C^γ and C^β, with $\gamma > 0$, $\beta > 0$ and $\gamma + \beta > 1$.*

For any symbol $\star \in \{-, \circ\}$, the integral $\int_0^\cdot Y\,d^\star X$ coincides with the Young integral $\int_0^\cdot Y\,d^{(y)} X$.

REMARK 3.7. Suppose that X and Y satisfy the conditions of Proposition 3.6. Then Remark 3.2(a) implies that $[X, Y] = 0$.

We need an extension of stochastic calculus via regularization in the direction of higher n-variation. The properties concerning variation higher than 2 can be found, for instance, in [6].

We set

$$[X, X, X]_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)^3\,ds.$$

We also define

$$\|[X, X, X]^\varepsilon\|_t = \frac{1}{\varepsilon} \int_0^t |X_{s+\varepsilon} - X_s|^3\,ds.$$

If the limit in probability of $[X, X, X]_t^\varepsilon$ when $\varepsilon \to 0$ exists for any t, we denote it by $[X, X, X]_t$. If the limiting process $[X, X, X]$ has a continuous version, we say that X is a *finite cubic variation process*.

If, moreover, there is a positive sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ converging to zero such that

$$\sup_{\varepsilon_n} \|[X, X, X]^{\varepsilon_n}\|_T < +\infty, \tag{3.4}$$

then we say that X is a (*strong*) *finite cubic variation process*. If X is a (strong) finite cubic variation process such that $[X, X, X] = 0$, then X will be said to be a (strong) zero *finite cubic variation process*.

For instance, if $X = B^H$, a fractional Brownian motion with Hurst index H, then X is a finite quadratic variation process if and only if $H \geq \frac{1}{2}$; see [22]. It is a strong zero cubic variation process if and only if $H \geq \frac{1}{3}$; see [6]. On the other hand, B^H is a zero cubic variation process if and only if $H > \frac{1}{6}$; see [13].

It is clear that a finite quadratic variation process is a strong zero cubic variation process. On the other hand, processes whose paths are Hölder continuous with parameter greater than $\frac{1}{3}$ are strong zero cubic variation processes.

As for finite quadratic variation and Dirichlet processes, the C^1-stability also holds for finite cubic variation processes. The next proposition is a particular case of a result contained in [6].

PROPOSITION 3.8. *Let X be a strong finite cubic variation process, V a locally bounded variation process and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of class C^1. Then $Z = f(V, X)$ is again a strong finite cubic variation process and*

$$[Z, Z, Z]_t = \int_0^t \partial_x f(V_s, X_s)^3 \, d[X, X, X]_s.$$

Moreover, an Itô chain rule property holds, as follows.

PROPOSITION 3.9. *Let X be a strong finite cubic variation process, V a bounded variation process and Y a cadlag process. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be of class $C^{1,3}$. Then*

$$\int_0^t Y \, d^\circ f(V, X) = \int_0^t Y \, \partial_v f(V_s, X_s) \, dV_s + \int_0^t Y \, \partial_x f(V_s, X_s) \, d^\circ X_s - \frac{1}{12} \int_0^t Y \, \partial_{xxx}^3 f(V_s, X_s) \, d[X, X, X]_s.$$

We deduce, in particular, that a C^1 transformation of a strong zero cubic variation process is again a strong zero cubic variation process.

We conclude the section by introducing a concept of *definite integral* via regularization. If processes X, Y are indexed by the whole real line, a.s. with compact support, we define

$$\int_{\mathbb{R}} Y \, d^- X := \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon} \, ds, \tag{3.5}$$

$$\int_{\mathbb{R}} Y_s \, d^\circ X_s := \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} Y_s \frac{X_{s+\varepsilon} - X_{s-\varepsilon}}{2\varepsilon} \, ds, \tag{3.6}$$

where the limit is understood in probability. Integration by parts [Remark 3.2(c)], Proposition 3.6 and the chain rule property (Proposition 3.9) can all be immediately adapted to these definite integrals.

4. The PDE operator L. Let $\sigma, b \in C^0(\mathbb{R})$ be such that $\sigma > 0$. Without loss of generality, we will suppose that $b(0) = 0$.

We consider a formal PDE operator of the following type:

$$Lg = \frac{\sigma^2}{2} g'' + b' g'. \tag{4.1}$$

If b is of class C^1, so that b' is continuous, we will say that L is a *classical* PDE operator.

For a given mollifier Φ, we denote

$$\sigma_n^2 := (\sigma^2 \wedge n) * \Phi_n, \qquad b_n := \big(-n \wedge (b \vee n)\big) * \Phi_n.$$

We then consider

$$\begin{aligned} L_n g &= \frac{\sigma_n^2}{2} g'' + b_n' g' \qquad \text{for } g \in C^2(\mathbb{R}), \\ \mathcal{L}_n u &= \partial_t u + L_n u \qquad \text{for } u \in C^{1,2}([0, T[\times\mathbb{R}), \end{aligned} \tag{4.2}$$

where L_n acts on x. A priori, σ_n^2, b_n and the operator L_n depend on the mollifier Φ.

Previous definitions are slightly different from those in papers [10, 11], but a considerable part of the analysis of L and the study of the martingale problem can be adapted. In those papers, there was only regularization but no truncation; here, truncation is used to study the associated parabolic equations.

DEFINITION 4.1. A function $f \in C^1(\mathbb{R})$ is said to be a C^1-*generalized solution* to

$$Lf = \dot{\ell}, \tag{4.3}$$

where $\dot{\ell} \in C^0$ if for any mollifier Φ, there are sequences (f_n) in C^2 and $(\dot{\ell}_n)$ in C^0 such that

$$L_n f_n = \dot{\ell}_n, \qquad f_n \to f \text{ in } C^1, \qquad \dot{\ell}_n \to \dot{\ell} \text{ in } C^0. \tag{4.4}$$

PROPOSITION 4.2. *There is a solution $h \in C^1$ to $Lh = 0$ such that $h'(x) \neq 0$ for every $x \in \mathbb{R}$ if and only if*

$$\Sigma(x) := \lim_{n\to\infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy$$

exists in C^0, independently of the mollifier. Moreover, in this case, any solution f to $Lf = 0$ fulfills

$$f'(x) = e^{-\Sigma(x)} f'(0). \tag{4.5}$$

PROOF. This result follows in a very similar way to the proof of Proposition 2.3 in [10]—first at the level of regularization and then passing to the limit. □

For the remainder of this paper, we will suppose the existence of this function Σ. We will consider $h \in C^1$ such that

$$h'(x) := \exp(-\Sigma(x)), \qquad h(0) = 0. \tag{4.6}$$

In particular, $h'(0) = 1$ holds. Even though we discuss the general case with related nonexplosion conditions in [10], here, in order to ensure conservativeness, we suppose that

$$\begin{aligned} \int_{-\infty}^{0} e^{-\Sigma(x)}\,dx &= \int_0^{\infty} e^{-\Sigma(x)}\,dx = +\infty, \\ \int_{-\infty}^{0} \frac{e^{\Sigma(x)}}{\sigma^2}\,dx &= \int_0^{\infty} \frac{e^{\Sigma(x)}}{\sigma^2}\,dx = +\infty. \end{aligned} \tag{4.7}$$

Previous assumptions are of course satisfied if σ is lower bounded by a positive constant and b is constant outside a compact interval.

Condition (4.7) implies that the image set of h is $\mathbb{R}$.

REMARK 4.3. Proposition 4.2 implies uniqueness of the problem

$$Lf = \dot{\ell}, \qquad f \in C^1, \qquad f(0) = x_0, \qquad f'(0) = x_1 \tag{4.8}$$

for every $\dot{\ell} \in C^0$, $x_0, x_1 \in \mathbb{R}$.

REMARK 4.4. We present four important examples where Σ exists:

(a) If $b(x) = \alpha(\frac{\sigma^2(x)}{2} - \frac{\sigma^2(0)}{2})$ for some $\alpha \in]0, 1]$, then

$$\Sigma(x) = \alpha \log\left(\frac{\sigma^2(x)}{\sigma^2(0)}\right)$$

and

$$h'(x) = \frac{\sigma^{2\alpha}(0)}{\sigma^{2\alpha}(x)}.$$

If $\alpha = 1$, the operator L can be formally expressed in divergence form as $Lf = (\frac{\sigma^2}{2} f')'$.

(b) Suppose that b is locally of bounded variation. We then get

$$\int_0^x \frac{b_n'}{\sigma_n^2}(y)\,dy = \int_0^x \frac{db_n(y)}{\sigma_n^2(y)} \to \int_0^x \frac{db}{\sigma^2}$$

since $db_n \to db$ in the weak-$*$ topology and $\frac{1}{\sigma^2}$ is continuous.

(c) If σ has bounded variation, then we have

$$\Sigma(x) = -2\int_0^x b\, d\left(\frac{1}{\sigma^2}\right) + \frac{2b}{\sigma^2}(x) - \frac{2b}{\sigma^2}(0).$$

In particular, this example contains the case where $\sigma = 1$ for any b.

(d) Suppose that σ is locally Hölder continuous with parameter γ and that b is locally Hölder continuous with parameter β such that $\beta + \gamma > 1$. Since σ is locally bounded, σ^2 is also locally Hölder continuous with parameter γ. Proposition 2.3 implies that $\sigma_n^2 \to \sigma^2$ in $C^{\gamma'}$ and $b_n \to b$ in $C^{\beta'}$ for every $\gamma' < \gamma$ and $\beta' < \beta$. Since σ is strictly positive on each compact, $\frac{1}{\sigma_n^2} \to \frac{1}{\sigma^2}$ in $C^{\gamma'}$. By Remark 2.1, Σ is well defined and locally Hölder continuous with parameter β'.

Again, the following lemma can be proven at the level of regularizations; see also Lemma 2.6 in [10].

LEMMA 4.5. *The unique solution to problem* (4.8) *is given by*

$$f(0) = x_0,$$
$$f'(x) = h'(x)\left(2\int_0^x \frac{\dot{\ell}(y)}{(\sigma^2 h')(y)}\, dy + x_1\right).$$

REMARK 4.6. If $b' \in C^0(\mathbb{R})$ and $f \in C^2(\mathbb{R})$ is a classical solution to $Lf = \dot{\ell}$, then f is clearly also a C^1-generalized solution.

REMARK 4.7. Given $\ell \in C^1$, we denote by $T\ell$ the unique C^1-generalized solution f to problem (4.8) with $\dot{\ell} = \ell'$, $x_0 = 0$, $x_1 = 0$. The unique solution to the general problem (4.8) is given by

$$f = x_0 + x_1 h + T\ell.$$

We write $T^{x_1}\ell = T\ell + x_1 h$, that is, the solution with $x_0 = 0$.

REMARK 4.8. Let $f \in C^1$. There is at most one $\dot{\ell} \in C^0$ such that $Lf = \dot{\ell}$. In fact, to see this, it is enough to suppose that $f = 0$. Lemma 4.5 implies that

$$2\int_0^x \frac{\dot{\ell}}{\sigma^2 h'}(y)\, dy \equiv 0.$$

Consequently, $\dot{\ell}$ is forced to be zero.

This consideration allows us to define without ambiguity $L : \mathcal{D}_L \to C^0$, where $\mathcal{D}_L$ is the set of all $f \in C^1(\mathbb{R})$ which are C^1-generalized solution to $Lf = \dot{\ell}$ for some $\dot{\ell} \in C^0$. In particular, $T\ell \in \mathcal{D}_L$.

A direct consequence of Lemma 4.5 is the following useful result.

LEMMA 4.9. *$\mathcal{D}_L$ is the set of $f \in C^1$ such that there exists $\psi \in C^1$ with $f' = e^{-\Sigma}\psi$.*

In particular, it gives us the following density proposition.

PROPOSITION 4.10. *$\mathcal{D}_L$ is dense in C^1.*

PROOF. It is enough to show that every C^2-function is the C^1-limit of a sequence of functions in $\mathcal{D}_L$. Let (ψ_n) be a sequence in C^1 converging to $f'e^{\Sigma}$ in C^0. It follows that

$$f_n(x) := f(0) + \int_0^x e^{-\Sigma}(y)\psi_n(y)\,dy, \qquad x \in \mathbb{R},$$

converges to $f \in C^1$ and $f_n \in \mathcal{D}_L$. □

We must now discuss technical aspects of the way L and its domain $\mathcal{D}_L$ are transformed by h. We recall that $Lh = 0$ and that h' is strictly positive. Condition (4.7) implies that the image set of h is $\mathbb{R}$.

Let L^0 be the classical PDE operator

$$L^0\phi = \frac{\tilde{\sigma}_h^2}{2}\phi'', \qquad \phi \in C^2, \tag{4.9}$$

where

$$\tilde{\sigma}_h(y) = (\tilde{\sigma}h')(h^{-1}(y)), \qquad y \in \mathbb{R}.$$

L^0 is a classical PDE map; however, we can also consider it at the formal level and introduce $\mathcal{D}_{L^0}$.

PROPOSITION 4.11. (a) $h^2 \in \mathcal{D}_L$, $Lh^2 = h'^2\sigma^2$.

(b) $\mathcal{D}_{L^0} = C^2$.

(c) *$\phi \in \mathcal{D}_{L^0}$ holds if and only if $\phi \circ h \in \mathcal{D}_L$. Moreover, we have*

$$L(\phi \circ h) = (L^0\phi) \circ h \tag{4.10}$$

for every $\phi \in C^2$.

PROOF. This follows similarly as for Proposition 2.13 of [10]. □

We will now discuss another operator related to L. Given a function f, we need to provide a suitable definition of $f \mapsto \int_0^x Lf(y)\,dy$, that is, some primitive of Lf.

- One possibility is to define that map, through previous expression, for $f \in \mathcal{D}_L$.

- Otherwise, we try to define it as linear map on C^2. For this, first suppose that b' is continuous. Then integrating by parts, we obtain

$$\int_0^x Lf(y)\,dy = \int_0^x \left(\frac{\sigma^2}{2} - b\right) f''(y)\,dy + (bf')(x) - (bf')(0). \tag{4.11}$$

  We remark that the right-hand side of this expression makes sense for any $f \in C^2$ and continuous b. We will thus define $\hat{L} : C^2 \to C_0^0$ as follows:

$$\hat{L}f := \int_0^x \left(\frac{\sigma^2}{2} - b\right) f''(y)\,dy + (bf')(x) - (bf')(0). \tag{4.12}$$

One may ask if, in the general case, the two definitions $f \to \int_0^x Lf(y)\,dy$ on $\mathcal{D}_L$ and $\hat{L}$ on C^2 are compatible. We will later see that under Assumption $\mathcal{A}(\nu_0)$, this will be the case. However, in general, $\mathcal{D}_L \cap C^2$ may be empty.

Thus far, we have learned how to eliminate the first-order term in a formal PDE operator through the transformation h introduced at (4.6); when L is classical, this was performed by Zvonkin (see [28]). We would now like to introduce a transformation which puts the PDE operator in a divergence form.

Let L be a PDE operator which is formally of type (4.1):

$$Lg = \frac{\sigma^2}{2} g'' + b' g'.$$

We consider a function of class C^1, namely $k : \mathbb{R} \to \mathbb{R}$ such that

$$k(0) = 0 \quad \text{and} \quad k'(x) = \sigma^{-2}(x) \exp(\Sigma(x)). \tag{4.13}$$

According to assumptions (4.7), k is bijective on $\mathbb{R}$.

REMARK 4.12. If there is no drift term, that is, $b = 0$, then we have $k'(x) = \sigma^{-2}(x)$.

LEMMA 4.13. *We consider the formal PDE operator given by*

$$L^1 g = \frac{\bar{\sigma}_k^2}{2} g'' + \left(\frac{\bar{\sigma}_k^2}{2}\right)' g' = \left(\frac{\bar{\sigma}_k^2}{2} g'\right)', \tag{4.14}$$

where

$$\bar{\sigma}_k(z) = (\sigma k') \circ k^{-1}(z), \qquad z \in \mathbb{R}.$$

Then:

(i) $g \in \mathcal{D}_{L^1}$ *if and only if* $g \circ k \in \mathcal{D}_L$;
(ii) *for every* $g \in \mathcal{D}_{L^1}$, *we have* $L^1 g = L(g \circ k) \circ k^{-1}$.

PROOF. It is practically the same as in Lemma 2.16 of [10]. $\square$

We now give a lemma whose proof can be easily established by investigation. Suppose that L is a classical PDE operator. Then $\mathcal{L} = \partial_t + L$ is well defined for $C^{1,2}([0,T[\times\mathbb{R})$ functions where L acts on the second variable. Given a function $\varphi \in C([0,T] \times \mathbb{R})$, we will hereafter set $\tilde{\varphi}: [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\tilde{\varphi}(t,y) = \varphi(t, h^{-1}(y))$.

LEMMA 4.14. *Let us suppose that $h \in C^2(\mathbb{R})$. We set $\sigma_h = \sigma h'$.*
We define the PDE operator $\mathcal{L}^0$ by $\mathcal{L}^0\varphi = \partial_t\varphi + L^0\varphi$, where L^0 is a classical operator acting on the space variable x and

$$L^0 f = \frac{\tilde{\sigma}_h^2}{2} f''.$$

If $f \in C^{1,2}([0,T[\times\mathbb{R})$ and $\mathcal{L}f = \gamma$ in the classical sense, then $\mathcal{L}^0 \tilde{f} = \tilde{\gamma}$.

We will now formulate a supplementary assumption which will be useful when we study singular stochastic differential equations in the proper sense and not only in the form of a martingale problem.

TECHNICAL ASSUMPTION $\mathcal{A}(\nu_0)$. *Let ν_0 be a topological F-space which is a linear topological subspace of $C^0(\mathbb{R})$ (or, eventually, an inductive limit of sub-F-spaces). The ν_0-convergence implies convergence in C^0 and, therefore, pointwise convergence.*

We say that L fulfills Assumption $\mathcal{A}(\nu_0)$ if the following conditions hold:

(i) *$C^1 \subset \nu_0$, which is dense.*

(ii) *For every $g \in C^1(\mathbb{R})$, the multiplicative operator $\phi \to g\phi$ maps ν_0 into itself.*

(iii) *Let $T: C^1(\mathbb{R}) \subset \nu_0 \to C^1(\mathbb{R})$ as defined in Lemma 4.5, that is, $f = T\ell$ is such that*

$$f(0) = 0,$$
$$f'(x) = e^{-\Sigma}(x)\left(2\int_0^x \frac{e^{\Sigma}(y)\ell'(y)}{\sigma^2(y)}\,dy\right).$$

We recall that $f = T\ell$ solves problem $Lf = \ell'$ with $f(0) = f'(0) = 0$. We suppose that T admits a continuous extension to ν_0.

(iv) *Let $x_1 \in \mathbb{R}$. For every $f \in C^2$ with $f(0) = 0$ and $f'(0) = x_1$ so that $]\hat{L}f = \ell$, we have $\ell \in \nu_0$ and $T^{x_1}\ell = f$, where T^{x_1} denotes the continuous extension of T^{x_1} (see Remark 4.7) to ν_0, which exists by* (iii).

(v) *The set $\hat{L}C^2$ is dense in $\{\ell \in \nu_0 | \ell(0) = 0\}$.*

REMARK 4.15. Let $x_1 \in \mathbb{R}$.

(i) Remark 4.7 and point (iii) above together imply that $T^{x_1}: C^1(\mathbb{R}) \subset \nu_0 \to C^1(\mathbb{R})$ extends continuously to ν_0. Moreover,

$$\{f \in C^2 | f(0) = 0, f'(0) = x_1\} \subset ImT^{x_1}.$$

(ii) Point (iv) above shows that $b \in \nu_0$ and $T^1 b = id$, where $id(x) = x$; in fact, $id(0) = 0$, $id'(1) = 1$ and (4.12) implies that $\hat{L} id = b$.

(iii) Point (i) above is satisfied if, for instance, the map T is closable as a map from C^0 to C^1. In that case, ν_0 may be defined as the domain of the closure of C^1, equipped with the graph topology related to $C^0 \times C^1$.

Below, we give some sufficient conditions for points (iv) and (v) of the Technical Assumption to be satisfied.

We define by $C^1_{\nu_0}$ the vector space of functions $f \in C^1$ such that $f' \in \nu_0$. This will be an F-space if equipped with the following topology. A sequence (f_n) will be said to converge to f in $C^1_{\nu_0}$ if $f_n(0) \to f(0)$ and (f'_n) converges to f' in ν_0. In particular, a sequence converging according to $C^1_{\nu_0}$ also converges with respect to C^1. On the other hand, $C^2 \subset C^1_{\nu_0}$ and a sequence converging in C^2 also converges with respect to $C^1_{\nu_0}$. Moreover, C^2 is dense in $C^1_{\nu_0}$ because C^1 is dense in ν_0.

LEMMA 4.16. *Suppose that points* (i) *to* (iii) *of the Technical Assumption are fulfilled. We suppose, moreover, that*:

(a) $h \in C^1_{\nu_0}$.

(b) *For every* $f \in C^2$, $f(0) = 0$, $f'(0) = 0$, $\hat{L} f = \ell$, *we have* $\ell \in \nu_0$ *and* $T\ell = f$.

(c) $\hat{L}: C^2 \to \nu_0$ *is well defined and has a continuous extension to* $C^1_{\nu_0}$, *still denoted by* $\hat{L}$, *such that* $\hat{L} h = 0$.

(d) $ImT \subset C^1_{\nu_0}$.

(e) $\hat{L} T$ *is the identity map on* $\{\ell \in \nu_0 | \ell(0) = 0\}$.

Then T, T^{x_1} *for every* $x_1 \in \mathbb{R}$ *are injective and points* (iv) *and* (v) *of the Technical Assumption are satisfied.*

PROOF. The injectivity of T follows from point (e). The injectivity of T^{x_1} is a consequence of Remark 4.7.

We prove point (iv). Point (c) says that $\hat{L} h = 0$. We set $\hat{f} = f - x_1 h$, $f \in C^2$, where $f(0) = 0$, $f'(0) = x_1$. Clearly, $\hat{L} \hat{f} = \hat{L} f = \ell$ and $\hat{f}(0) = 0$, $\hat{f}'(0) = 0$. Point (b) implies that $T\ell = \hat{f}$. Hence, $T^{x_1}\ell = T\ell + x_1 h = f$ and (iv) is satisfied.

Concerning point (v), let $\ell \in \nu_0$ with $\ell(0) = 0$ and set $f = T\ell$. Since f belongs to $C^1_{\nu_0}$ by (c), f' belongs to ν_0. Point (i) of the technical assumption implies that there exists a sequence (f'_n) of C^1 functions converging to f' in the ν_0 sense and thus also in C^0. Let (f_n) be the sequence of primitives of (f_n') (which are of class C^2) such that $f_n(0) = 0$. In particular, we have that (f_n) converges to f in the $C^1_{\nu_0}$-sense. By (c), there exists λ in ν_0 which is the limit of $\hat{L} f_n$ in the ν_0-sense. Observe that because of (b), $T(\hat{L} f_n) = f_n$. On the other hand, $\lim_{n\to+\infty} f_n = f$ in C^1. Applying T and using (iii) of the Technical Assumption, we obtain

$$T\lambda = \lim_{n\to+\infty} T(\hat{L} f_n) = \lim_{n\to+\infty} f_n = f = T\ell.$$

The injectivity of T allows us to conclude that $\ell = \lambda$. □

REMARK 4.17. Under the assumptions of Lemma 4.16, we have:

- $\mathcal{D}_L \subset C^1_{\nu_0}$;
- $\hat{L} f = \int_0^x Lf(y)\,dy$, $f \in \mathcal{D}_L$.

In fact, let $f \in \mathcal{D}_L$. Without loss of generality, we can suppose that $f(0) = 0$. Let $x_1 = f'(0)$ and set $\hat{f} = f + x_1 h$ so that $\hat{f}(0) = \hat{f}'(0) = 0$. Setting $\dot{\ell} = L\hat{f}$, Lemma 4.5 implies that $\hat{f} = T\ell$, where $\ell = \int_0^x \dot{\ell}(y)\,dy$. So $\hat{f} \in ImT \subset C^1_{\nu_0}$. Since $h \in C^1_{\nu_0}$, it follows that $f \in C^1_{\nu_0}$, by additivity.

On the other hand,

$$\begin{aligned} Lf &= L\hat{f} + x_1 Lh = \hat{L} f = \dot{\ell}, \\ \hat{L} f &= \hat{L}\hat{f} + x_1 \hat{L} h = \hat{L} T\ell = \ell, \end{aligned}$$

by point (e) of Lemma 4.16.

EXAMPLE 4.18. We provide here a series of four significant examples when Technical Assumption $\mathcal{A}(\nu_0)$ is verified. We only comment on the points which are not easy to verify.

(i) The first example is simple. It concerns the case when the drift b' is continuous. This problem, to be studied later, corresponds to an ordinary SDE where

$$\nu_0 = C^1, \qquad C^1_{\nu_0} = C^2, \qquad \hat{L} f = \int_0^{\cdot} Lf(y)\,dy.$$

(ii) L is *close to divergence type*, that is, $b = \frac{\sigma^2 - \sigma^2(0)}{2} + \beta$ and where β is a locally bounded variation function vanishing at zero. The operator is of divergence type with an additional Radon measure term, that is, we have $\Sigma = \ln \sigma^2 + 2\int_0^x \frac{d\beta}{\sigma^2}$. In this case, we have $\nu_0 = C^0$. Points (i) and (ii) of the Technical Assumption are trivial.

We have, in fact,

$$h'(x) = e^{-\Sigma} = \frac{1}{\sigma^2(x)} \exp\left(-2\int_0^x \frac{d\beta}{\sigma^2}\right).$$

T defined at point (iii) of the Technical Assumption is such that $T\ell = f$, where $f(0) = 0$ and

$$f'(x) = \frac{2\sigma^2(0)}{\sigma^2(x)} \exp\left(-2\int_0^x \frac{d\beta}{\sigma^2}\right) \int_0^x \ell'(y) \exp\left(2\int_0^y \frac{d\beta}{\sigma^2}\right) dy. \tag{4.15}$$

Consequently, the extension of T to $\nu_0 = C^0$, always still denoted by the same letter T, is given by $f = T\ell$ with $f(0) = 0$ and

$$\begin{aligned} f'(x) = \frac{2}{\sigma^2(x)} \biggl\{ \ell(x) - 2\exp\left(-2\int_0^x \frac{d\beta}{\sigma^2}\right) \\ \times \left(\ell(0) + \int_0^x \ell(y) \exp\left(2\int_0^y \frac{d\beta}{\sigma^2}\right) \frac{1}{\sigma^2(y)}\, d\beta(y)\right)\biggr\}. \end{aligned} \tag{4.16}$$

Points (iv) and (v) are seen to be satisfied via Lemma 4.16. We have $C^1_{\nu_0} = C^1$. Point (a) is obvious since $h' \in C^0$ and so $h \in C^1_{\nu_0}$. Let $f \in C^2$. Using Lebesgue–Stieltjes calculus, we can easily show that

$$\ell(x) = \hat{L} f(x) = \frac{\sigma^2(x)}{2} f'(x) - \frac{\sigma^2(0)}{2} f'(0) + \int_0^x f'\, d\beta. \tag{4.17}$$

This shows that $\ell \in C^0 = \nu_0$ and therefore the first part of (b). We remark that we can, in fact, consider $\hat{L} : C^2 \to \nu_0$ because

$$\hat{L} f = \hat{L}(f - x_1 h) + x_1 \hat{L} h = \hat{L}(f - x_1 h) \in \nu_0.$$

The expression of $\hat{L} f$ extends continuously to $f \in C^1$, which yields the first part of point (c). Moreover, inserting the expression for h' into f' in (4.17), one shows that $\hat{L} h = 0$.

Suppose, now, that in expression (4.17), $f \in C^2$, $f(0) = 0$, $f'(0) = 0$. A simple investigation shows that $T\ell = f$, so the second part of point (b) is fulfilled; point (d) is also clear because of (4.16). Finally point (d) holds because one can prove by inspection that $\hat{L} T$ is the identity on C^0_0.

(iii) We recall the notation $D^\gamma(\mathbb{R})$ which indicates the topological vector space of locally Hölder continuous functions defined on $\mathbb{R}$ with parameter $\alpha > \gamma$. We recall that $D^\gamma(\mathbb{R})$ is a vector algebra.

Suppose that $\sigma \in D^{1/2}$ and $b \in C^{1/2}$ (or $\sigma \in C^{1/2}$ and $b \in D^{1/2}$). Remark 4.4(d) implies that Σ also belongs to $D^{1/2}$. We set $\nu_0 = D^{1/2}$.

Technical Assumption $\mathcal{A}(\nu_0)$ is verified for the following reasons.

Since $\Sigma \in D^{1/2}$, $h' = e^{-\Sigma}$ belongs to the same space.

Point (i) follows because of Proposition 2.3 and point (ii) follows because $D^{1/2}$ is an algebra. Corollary 2.4 yields that for every $\ell \in D^{1/2}$, the function

$$f'(x) = e^{-\Sigma(x)} \int_0^x 2\frac{e^\Sigma}{\sigma^2}(y)\, d^{(y)}\ell(y) \tag{4.18}$$

is well defined and belongs to $D^{1/2}$. This shows that T can be continuously extended to ν_0 and point (iii) is established.

Concerning points (iv) and (v), we again use Lemma 4.16. We observe that

$$C^1_{\nu_0} = \{f \in C^1 | f' \in D^{1/2}\}.$$

Point (a) is obvious since $h' = e^{-\Sigma} \in D^{1/2}$. Let $f \in C^2$. Considering b as a deterministic process and recalling the definition of $\hat{L}$ as in (4.12), integration by parts in Remark 3.2(c) and Proposition 3.6 together imply that

$$\ell(x) = \int_0^x \frac{\sigma^2}{2} d^0 f' + \int_0^x f' d^\circ b, \tag{4.19}$$

$$\ell(x) = \int_0^x \frac{\sigma^2}{2} d^{(y)} f' + \int_0^x f' d^{(y)} b. \tag{4.20}$$

The first part of point (b) follows because of Proposition 2.2. Of course, the previous expression can be extended to $f \in C^1_{\nu_0}$ and this shows the first part of point (c).

Showing that the second part of point (c) of Lemma 4.16 holds consists of verifying that $\hat{L}h = 0$. Substituting $h' = e^{-\Sigma}$ into the previous expression, through Proposition 2.2, we obtain

$$\ell(x) = -\int_0^x \frac{\sigma^2}{2} e^{-\Sigma} d^{(y)} \Sigma + \int_0^x e^{-\Sigma} d^{(y)} b = 0.$$

Concerning the second part of point (b), let $f \in C^2$ so that $f(0) = f'(0) = 0$. We want to show that $\varphi = T\ell$ coincides with f.

Since $\varphi(0) = 0$, it remains to check that $\varphi' = f'$. We recall that

$$\varphi'(x) = e^{-\Sigma}(x)\left(2\int_0^x \frac{e^\Sigma}{\sigma^2}(y)\, d^{(y)}\ell(y)\right).$$

Twice applying the chain rule of Proposition 2.2 and using (4.19), the fact that

$$e^\Sigma(x) = \int_0^x e^\Sigma \frac{2\, d^{(y)} b}{\sigma^2} + 1$$

and integration by parts, we obtain

$$\begin{aligned}
\varphi'(x) &= e^{-\Sigma}(x)\left\{\int_0^x e^\Sigma d^0 f' + \int_0^x 2\frac{e^\Sigma}{\sigma^2} f' d^{(y)} b\right\} \\
&= e^{-\Sigma}(x)\left\{\int_0^x e^\Sigma d^0 f' + \int_0^x f' d^{(y)} e^\Sigma\right\} \\
&= e^{-\Sigma}(x)\left\{\int_0^x e^\Sigma d^0 f' + \int_0^x f' d^0 e^\Sigma\right\} \\
&= e^{-\Sigma}(x)\{(f'e^\Sigma)(x) - (f'e^\Sigma)(0)\} \\
&= f'(x).
\end{aligned}$$

Point (b) is therefore completely established.

Point (d) follows because in (4.18), when $\ell \in \nu_0$, it follows that $f' \in \nu_0$.

Clearly, as for the previous example, $ImT \subset C^1_{\nu_0}$. It remains to show that $\hat{L}T$ is the identity map $\{f \in D^{1/2} | f(0) = 0\}$.

For this, we first remark that

$$\hat{L} f(x) = \int_0^x \frac{\sigma^2}{2} e^{-\Sigma} d^{(y)}(f' e^{\Sigma}). \tag{4.21}$$

In fact, by Proposition 3.6 and integration by parts contained in Remark 3.2(c), we obtain

$$f'(x) e^{\Sigma(x)} = f'(0) + \int_0^x e^{\Sigma} d^{(y)} f' + \int_0^x f' d^{(y)} e^{\Sigma}.$$

By the chain rule of Proposition 2.2, we obtain the right-hand side of (4.21).

At this point, by definition, if $f = T\ell$, we have

$$f'(x) e^{\Sigma(x)} = \int_0^x 2 \frac{e^{\Sigma}}{\sigma^2} d^{(y)} \ell.$$

Therefore, (4.21) and Proposition 2.2 allow us to conclude that

$$\hat{L} f(x) = \int_0^x \frac{\sigma^2}{2} e^{-\Sigma} 2 \frac{e^{\Sigma}}{\sigma^2} d^{(y)} \ell = \ell(x) - \ell(0).$$

(iv) Suppose b is locally with bounded variation. Then the Technical Assumption is satisfied for $\nu_0 = BV$, where BV is the space of continuous real functions, locally with bounded variation v, equipped with the following topology. A sequence (v_n) in BV converges to v if

$$v_n(0) \to v(0),$$
$$dv_n \to dv \qquad \text{in the weak-} * \text{ topology}.$$

The arguments for proving that the Technical Assumption is satisfied are similar, but easier, than those for the previous point. Young-type calculus is replaced by classical Lebesgue–Stieltjes calculus.

5. Martingale problem. In this section, we consider a PDE operator satisfying the same properties as in previous section, that is,

$$Lg = \frac{\sigma^2}{2} g'' + b' g', \tag{5.1}$$

where $\sigma > 0$ and b are continuous. In particular, we assume that

$$\Sigma(x) = \lim_{n \to \infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy \tag{5.2}$$

exists in C^0, independently of the chosen mollifier. Then h defined by $h'(x) := \exp(-\Sigma(x))$ and $h(0) = 0$ is a solution to $Lh = 0$ with $h' \neq 0$.

Here, we aim to introduce different notions of martingale problem, trying, when possible, to also clarify the classical notion. For the next two definitions, we consider the following convention. Let $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ fulfill the *usual conditions*; see, for instance, [14], Definition 2.25, Chapter 1.

DEFINITION 5.1. A process X is said to solve *the martingale problem* related to L (with respect to the aforementioned filtered probability space) with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}$, if

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s)\,ds$$

is an $(\mathcal{F}_t)_{t\geq s}$-local martingale for $f \in \mathcal{D}_L$ and $X_0 = x_0$.

More generally, for $s \geq 0$, $x \in \mathbb{R}$, we say that $(X_t^{s,x}, t \geq 0)$ solves the martingale problem related to L with initial value x at time s if for every $f \in \mathcal{D}_L$,

$$f(X_t^{s,x}) - f(x) - \int_s^t Lf(X_r^{s,x})\,dr, \qquad t \geq s,$$

is an $(\mathcal{F}_t)_{t\geq s}$-local martingale.

We remark that $X^{s,x}$ solves the martingale problem at time s if and only if $X_t := X_{t+s}^{s,x}$ solves the martingale problem at time 0.

DEFINITION 5.2. Let (W_t) be an $(\mathcal{F}_t)$-classical Wiener process. An $(\mathcal{F}_t)$-progressively measurable process $X = (X_t)$ is said to solve *the sharp martingale problem* related to L (on the given filtered probability space) with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}$, if

$$f(X_t) - f(x_0) - \int_0^t Lf(X_r)\,dr = \int_0^t f'(X_r)\sigma(X_r)\,dW_r$$

for every $f \in \mathcal{D}_L$.

More generally, for $s \geq 0$, $x \in \mathbb{R}$, we say that $(X_t^{s,x}, t \geq s)$ solves the sharp martingale problem related to L with initial value x at time s if for every $f \in \mathcal{D}_L$,

$$f(X_t^{s,x}) - f(x) - \int_s^t Lf(X_r^{s,x})\,dr = \int_s^t f'(X_r^{s,x})\sigma(X_r^{s,x})\,dW_r, \qquad t \geq s.$$

REMARK 5.3. Let (W_t) be an $(\mathcal{F}_t)$-Wiener process. If b' is continuous, then a process X solves the (corresponding) sharp martingale problem with respect to L if and only if it is a classical solution of the SDE

$$X_t = x_0 + \int_0^t b'(X_r)\,dr + \int_0^t \sigma(X_r)\,dW_r.$$

For this, a simple application of the classical Itô formula gives the result.

REMARK 5.4. (i) In general, $f(x) = x$ does not belong to $\mathcal{D}_L$, otherwise a solution to the martingale problem with respect to L would be a semimartingale. According to Remark 5.18, this is generally not the case. In [11], we gave necessary and sufficient conditions on b so that X is a semimartingale.

(ii) Given a solution X to the martingale problem related to L, we are interested in the operators

$$\mathcal{A}: \mathcal{D}_L \to \mathcal{C}, \qquad \text{given by } \mathcal{A}(f) = \int_0^{\cdot} Lf(X_s)\, ds,$$

and

$$A: C^1 \to \mathcal{C}, \qquad \text{given by } A(\ell) = \int_0^{\cdot} \ell'(X_s)\, ds,$$

where $\mathcal{C}$ is the vector algebra of continuous processes.

We may ask whether $\mathcal{A}$ and A are closable in C^1 and C^0, respectively. We will see that $\mathcal{A}$ admits a continuous extension to C^1. However, A can be extended continuously to some topological vector subspace ν_0 of C^0, where ν_0 includes the drift, only when Assumption $\mathcal{A}(\nu_0)$ is satisfied.

Similarly, as in the case of classical stochastic differential equations, it is possible to distinguish two types of existence and uniqueness for the martingale problem. Even if we could treat initial conditions which are random $\mathcal{F}_0$-measurable solutions, here we will only discuss deterministic ones. We will denote by $MP(L, x_0)$ [resp. $MP(L, x_0)$] the martingale problem (resp. sharp martingale problem) related to L with initial condition x_0. The notions will only be formulated with respect to the initial condition at time 0.

DEFINITION 5.5 (*Strong existence*). We will say that $SMP(L, x_0)$ admits *strong existence* if the following holds. Given any probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ and an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(W_t)_{t\geq 0}, x_0 \in \mathbb{R}$, there is a process $(X_t)_{t\geq 0}$ which solves the sharp martingale problem with respect to L and initial condition x_0.

DEFINITION 5.6 (*Pathwise uniqueness*). We will say that $SMP(L, x_0)$ admits *pathwise uniqueness* if the following property is fulfilled.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(W_t)_{t\geq 0}$. If two processes $X, \tilde{X}$ are two solutions of the sharp martingale problem with respect to L and x_0, such that $X_0 = \tilde{X}_0$ a.s., then X and $\tilde{X}$ coincide.

DEFINITION 5.7 (*Existence in law* or *weak existence*). We will say that $MP(L; x_0)$ *admits weak existence* if there is a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a process $(X_t)_{t\geq 0}$ which is a solution of the corresponding martingale problem.

We say that $MP(L)$ admits weak existence if $MP(L; x_0)$ admits weak existence for every x_0.

DEFINITION 5.8 (*Uniqueness in law*). We say that $MP(L; x_0)$ has a *unique solution in law* if the following holds. We consider an arbitrary probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a solution X of the corresponding martingale problem. We also consider another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ equipped with another filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and a solution $\tilde{X}$. We suppose that $X_0 = x_0$, P-a.s. and $\tilde{X}_0 = x_0$, $\tilde{P}$-a.s. Then X and $\tilde{X}$ must have the same law as a r.v.'s with values in $E = C(\mathbb{R}_+)$ (or $C[0, T]$).

REMARK 5.9. Let us suppose b' to be a continuous function. We do not suppose σ to be strictly positive (only continuous).

(i) The $SMP(L, x_0)$ then admits strong existence and pathwise uniqueness if the corresponding classical SDE

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b'(X_s)\, ds$$

admits strong existence and pathwise uniqueness. In this case, $\mathcal{D}_L = C^2$ and to establish this, it is enough to use the classical Itô formula.

(ii) It is well known (see [14, 26]) that weak existence (resp., uniqueness in law) of the martingale problem is equivalent to weak existence (resp., uniqueness in law) of the corresponding SDE.

For the rest of the section let $s \in [0, T]$, $x_0 \in \mathbb{R}$. Moreover, let $(\Omega, (\mathcal{F}_t), P)$ be a fixed filtered probability space fulfilling the usual conditions.

The first result concerning solutions to the martingale problem related to L is the following.

PROPOSITION 5.10. *Let* $y_0 = h(x_0)$.

(i) *A process X solves the martingale problem related to L with initial condition x at time s if and only if $Y = h(X)$ is a local martingale which solves*, *on the same probability space*,

$$Y_t = y_0 + \int_s^t \tilde{\sigma}_h(Y_s)\, dW_s, \tag{5.3}$$

where $\tilde{\sigma}_h(y) = (\sigma h')(h^{-1}(y))$ and where (W_t) is an $(\mathcal{F}_t)$-classical Brownian motion.

(ii) *Let (W_t) be an $(\mathcal{F}_t)$-classical Brownian motion. If Y is a solution to equation* (5.3), *then $X = h^{-1}(Y)$ is a solution to the sharp martingale problem with respect to L with initial condition x at time s.*

REMARK 5.11. Let X be a solution to the martingale problem with respect to L and set $Y = h(X)$ as in point (i) above. Since Y is a local martingale, we know from Remark 3.5(iv) that $X = h^{-1}(Y)$ is an $(\mathcal{F}_t)$-Dirichlet process with martingale part

$$M_t^X = \int_0^t (h^{-1})'(Y_s)\, dY_s.$$

In particular, X is a finite quadratic variation process with

$$[X, X] = [M^X, M^X]_t = \int_0^t \sigma^2(X_s)\, ds.$$

PROOF OF PROPOSITION 5.10. For simplicity, we will set $s = 0$.

First, let X be a solution to the martingale problem related to L. Since $h \in \mathcal{D}_L$ and $Lh = 0$, we know that $Y = h(X)$ is an $(\mathcal{F}_t)$-local martingale. In order to calculate its bracket, we recall that $h^2 \in \mathcal{D}_L$ and $Lh^2 = \sigma^2(h')^2$ hold by Proposition 4.11(a). Thus,

$$h^2(X_t) - \int_0^t (\sigma h')^2(X_s)\, ds$$

is an $(\mathcal{F}_t)$-local martingale. This implies that

$$[Y, Y]_t = \int_0^t (\sigma h')^2(h^{-1}(Y_s))\, ds = \int_0^t \tilde{\sigma}_h^2(Y_s)\, ds.$$

Finally, Y is a solution to the SDE (5.3) with respect to the standard $\mathcal{F}_Y$-Brownian motion W given by

$$W_t = \int_0^t \frac{1}{\tilde{\sigma}_h(Y_s)}\, dY_s,$$

where $\mathcal{F}_Y$ is the canonical filtration generated by Y.

Now, let $Y = h(X)$ be a solution to (5.3) and let $f \in \mathcal{D}_L$. Proposition 4.11(c) says that $\phi := f \circ h^{-1} \in \mathcal{D}_{L^0} \equiv C^2$, where

$$L^0\phi = \frac{\tilde{\sigma}_h^2}{2}\phi'' = (Lf) \circ h^{-1}. \tag{5.4}$$

We can therefore apply Itô's formula to evaluate $\phi(Y)$, which coincides with $f(X)$. This gives

$$\phi(Y_t) = \phi(Y_0) + \int_0^t \phi'(Y_s)\, dY_s + \tfrac{1}{2}\int_0^t \phi''(Y_s)\, d[Y, Y]_s.$$

Using $d[Y, Y]_s = \tilde{\sigma}_h^2(Y_s)\, ds$ and taking into account (5.4), we conclude that

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + \int_0^t Lf(X_s)\, ds. \tag{5.5}$$

This establishes the proposition. $\square$

REMARK 5.12. From Proposition 5.10 in particular, we have the following.

Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space fulfilling the usual conditions. Let $x_0 \in \mathbb{R}$ and X be a solution to the martingale problem related to L with initial condition x_0. Then there exists a classical Brownian motion (W_t) such that X is a solution to the sharp martingale problem related to L with initial condition x_0.

COROLLARY 5.13. *Let X be a solution to the martingale problem related to L with initial condition x_0. Then map $\mathcal{A}$ admits a continuous extension from $\mathcal{D}_L$ to C^1 with values in $\mathcal{C}$ which we will again denote by $\mathcal{A}$. Moreover, $\mathcal{A}(f)$ is a zero quadratic variation process for every $f \in C^1$.*

PROOF. $\mathcal{A}$ has a continuous extension because of (5.5). $\mathcal{A}(f)$ is a zero quadratic variation process because X is a Dirichlet process with martingale part $\int_0^\cdot \sigma(X_s)\,dW_s$ and because of Remark 3.5. □

REMARK 5.14. The extension of (5.5) to C^1 gives

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\,dW_s + \mathcal{A}(f). \tag{5.6}$$

Choosing $f = id$ in (5.6), we get

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dW_s + \mathcal{A}(id).$$

We will see that if there is a subspace ν_0 of C^0 such that Technical Assumption $\mathcal{A}(\nu_0)$ is verified, then the operator A will be extended to ν_0. If b is an element of that space, then it will be possible to write $\hat{L}id = b$ and $\mathcal{A}(id) = A(b)$. In that case, we will be able to indicate that X is a solution of the generalized SDE with diffusion coefficient σ and distributional drift b'.

A similar result to Proposition 5.10 can be deduced for the case of a transformation through function k and the divergence-type operator introduced at (4.13).

PROPOSITION 5.15. *We consider the transformation k and the PDE operator L^1 introduced at* (4.13) *and in Lemma* 4.13, *respectively.*

A process X solves the martingale problem related to L with initial condition x_0 at time s if and only if $Z = k(X)$ solves the martingale problem related to L^1 with initial condition $k(x_0)$ at time s.

PROOF. This is an easy consequence of Lemma 4.13. □

Let $x_0 \in \mathbb{R}$, $y_0 = h(x_0)$. Let σ, b, Σ, h be as in Section 4.

We set $\tilde{\sigma}_h = (\sigma e^{-\Sigma}) \circ h^{-1}$.

From Proposition 5.10, we have the following.

COROLLARY 5.16. (i) *Strong existence* (*resp*., *pathwise uniqueness*) *holds for* $SMP(L, x_0)$ *if and only if strong existence* (*resp*., *pathwise uniqueness*) *holds for the SDE*

$$dY_t = \tilde{\sigma}_h(Y_r)\, dW_r$$

with initial condition $Y_0 = h(x_0)$.

(ii) *An analogous equivalence holds for weak existence* (*resp*., *uniqueness in law*).

From Proposition 5.10, we can deduce two other corollaries concerning the well-posedness of our martingale problem.

COROLLARY 5.17. *Under the same assumptions as the previous corollary,* $MP(L, x_0)$ *admits weak existence and uniqueness in law.*

PROOF. The statement follows from point (i) of Corollary 5.16 and from the fact that the SDE (5.3) admits weak existence and uniqueness in law because $\tilde{\sigma}_h > 0$; see Theorem 5.7, Chapter 5 of [14], or [7]. □

REMARK 5.18. By Corollary 5.11 of [11], it is immediate to see that the solution is a semimartingale for each initial condition if and only if Σ is locally of bounded variation.

If L is in divergence form [see Remark 4.4(a) with $\alpha = 1$], then the solution corresponds to the process constructed and studied by, for instance, Stroock [25].

COROLLARY 5.19. *Suppose that either* $(\sigma, b) \in (D^{1/2}, C^{1/2})$ *or* $(b, \sigma) \in (D^{1/2}, C^{1/2})$ *and, moreover, that* (4.7) *is satisfied. Then* $MP(L, x_0)$ *admits strong existence and pathwise uniqueness.*

PROOF. In this case, Σ is well defined [see Remark 4.4(d)] and σ belongs to $D^{1/2}$. Since h^{-1} is of class C^1, $\tilde{\sigma}_h$ is Hölder continuous with parameter $\frac{1}{2}$. The SDE (5.3) admits pathwise uniqueness because of Theorem 3.5(ii) of [20] and weak existence, again through Theorem 5.7 of [14]. The Yamada–Watanabe theorem (see [14], Corollary 3.23, Chapter 5) also implies strong existence for (5.3). The result follows from point (i) of Corollary 5.16. □

6. A significant stochastic differential equation with distributional drift. In this section, we will discuss the case where the martingale problem is equivalent to a stochastic differential equation to be specified. First, one would need to give a precise sense to the generalized drift $\int_0^\cdot b'(X_s)\, ds$, b being a continuous function.

We will introduce a property related to a general process X. First, we consider the linear map $A^X : \ell \to \int_0^\cdot \ell'(X_s)\, ds$ defined on $C^1(\mathbb{R})$ with values in $\mathcal{C}$.

DEFINITION 6.1. Let ν_1 be a topological F-space (or, eventually, an inductive limit of F-spaces) which is a topological linear subspace of $C^0(\mathbb{R})$ and such that $\nu_1 \supset C^1(\mathbb{R})$. We will say that X has *extended local time regularity with respect to* ν_1 if:

- A^X admits a continuous extension to ν_1, which will still be denoted by the same symbol;
- $\int_0^{\cdot} g(X)\, d^- A^X(\ell)$ exists for every $g \in C^2$ and every $\ell \in \nu_1$.

REMARK 6.2. The terminology related to *local time* is natural in this context. To illustrate this, we consider a general continuous process X having a local time $(L_t(a), t \in [0, T], a \in \mathbb{R})$ with respect to Lebesgue measure, that is, fulfilling the density occupation identity

$$\int_0^t \varphi(X_s)\, ds = \int_{\mathbb{R}} \varphi(a) L_t(a)\, da, \qquad t \in [0, T],$$

for every positive Borel function φ. X trivially has extended local time regularity, at least with respect to $\nu_1 = C^1$.

Let $\ell \in C^1$. Suppose for a moment that $(L_t(a))$ is a semimartingale in a, as is the case, for instance, if X is a classical Brownian motion. In that case, one would have

$$\int_0^t \ell'(X_s)\, ds = \int_0^t \ell'(a) L_t(a)\, da = -\int_{\mathbb{R}} \ell(a) L_t(da).$$

Clearly, the rightmost integral can be extended continuously in probability to any $\ell \in C^0$, which implies that X also has extended local time regularity related to $\nu_1 = C^0$. We remark that [4] gives general conditions on semimartingales X under which $L_t(da)$ is a good integrator, even if $(L_t(a))$ is not necessarily a semimartingale in a.

DEFINITION 6.3. Let $(\Omega, (\mathcal{F}_t), P)$ a filtered probability space, (W_t) a classical $(\mathcal{F}_t)$-Brownian motion and Z an $\mathcal{F}_0$-measurable random variable. A process X will be called a ν_1-*solution* of the SDE

$$\begin{aligned} dX_t &= b'(X_t)\, dt + \sigma(X_t)\, dW_t, \\ X_0 &= Z, \end{aligned}$$

if:

- X has the extended local time regularity with respect to ν_1;
- $X_t = Z + \int_0^t \sigma(X_s)\, dW_s + A^X(b)_t$;
- X is a finite quadratic variation process.

REMARK 6.4. Suppose that $b \in \nu_1$. If $\nu_1 \subset \nu_1'$, then a ν_1'-solution is also a ν_1-solution.

The previous definition is also new in the classical case, that is, when b' is a continuous function. A ν_1-solution with $\nu_1 = C^1$ corresponds to a solution to the SDE in the classical sense. On the other hand, a ν_1-solution with ν_1 strictly including C^1 is a solution whose local time has a certain additional regularity.

Even in this generalized framework, it is possible to introduce the notions of *strong* ν_1-*existence*, *weak* ν_1-*existence*, *pathwise* ν_1-*uniqueness* and ν_1-*uniqueness in law*. This can be done similarly as in Definition 5.8 according to whether or not the filtered probability space with the classical Brownian motion is fixed a priori.

LEMMA 6.5. *We suppose that Technical Assumption $\mathcal{A}(\nu_0)$ is satisfied. If X is a solution to a martingale problem related to a PDE operator L, then it has extended local time regularity with respect to $\nu_1 = \nu_0$.*

PROOF. Let $\ell \in C^1$. Since X solves the martingale problem with respect to L, setting $f = T\ell$, it follows that

$$\begin{aligned} A^X(\ell)_t &= \int_0^t \ell'(X_s)\,ds = \int_0^t Lf(X_s)\,ds \\ &= f(X_t) - f(X_0) - \int_0^t f'(X_s)\sigma(X_s)\,dW_s. \end{aligned}$$

Continuity of T on ν_0 implies that A^X can be extended to ν_0.

Now, let $\ell \in \nu_0$ and $f = T\ell \in C^1$. Since $f(X)$ equals a local martingale plus $A^X(\ell)$, it remains to show that

$$\int_0^{\cdot} g(X)\,d^- f(X) \tag{6.1}$$

exists for any $g \in C^2$. Integrating by parts, the previous integral (6.1) equals

$$(gf)(X_\cdot) - (gf)(X_0) - \int_0^{\cdot} f(X)\,d^- g(X) - [f(X), g(X)].$$

Remark 3.3(b), (f) shows that the rightmost term member is well defined. □

LEMMA 6.6. *Let X be a process having extended local time regularity with respect to some F-space (or inductive limit) ν_1. Suppose that for fixed $g \in C^1$, the application $\ell \to g\ell$ is continuous from ν_1 to ν_1. Then for every $g \in C^2$ and every $\ell \in \nu_1$, we have*

$$\int_0^{\cdot} g(X)\,d^- A^X(\ell) = A^X(\Phi(g,\ell)), \tag{6.2}$$

where

$$\Phi(g,\ell)(x) = (g\ell)(x) - (g\ell)(0) - \int_0^x (\ell g')(y)\,dy. \tag{6.3}$$

PROOF. The Banach–Steinhaus-type Theorem 2.5 implies that for every $g \in C^2$,

$$\ell \mapsto \int_0^{\cdot} g(X)\, d^- A^X(\ell) \tag{6.4}$$

is continuous from ν_1 to $\mathcal{C}$. In fact, expression (6.4) is the u.c.p. limit of

$$\lim_{\varepsilon\to 0+} \int_0^{\cdot} g(X_s) \frac{A^X(\ell)_{s+\varepsilon} - A^X(\ell)_s}{\varepsilon}\, ds.$$

Note that Φ is a continuous bilinear map from $C^1 \times \nu_1$ to ν_1. Since $A^X : \nu_1 \to \mathcal{C}$ is continuous, the mapping $\ell \to A^X(\Phi(g,\ell))$ is also continuous from ν_1 to $\mathcal{C}$. In order to conclude the proof, we need to check identity (6.2) for $\ell \in C^1$. In that case, since

$$\Phi(g,\ell)(x) = \int_0^x (g\ell')(y)\, dy,$$

both sides of (6.2) equal

$$\int_0^{\cdot} (g\ell')(X_s)\, ds. \qquad \square$$

We will now explore the relation between the martingale problem associated with L and the stochastic differential equations with distributional drift.

PROPOSITION 6.7. *Let $x_0 \in \mathbb{R}$. Suppose that L fulfills Technical Assumption $\mathcal{A}(\nu_0)$. Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space fulfilling the usual conditions and let (W_t) be a classical $(\mathcal{F}_t)$-Brownian motion.*

If X solves the sharp martingale problem with respect to L with initial condition x_0, then X is a ν_0-solution to the stochastic differential equation

$$\begin{aligned} dX_t &= b'(X_t)\, dt + \sigma(X_t)\, dW_t, \\ X_0 &= x_0. \end{aligned} \tag{6.5}$$

REMARK 6.8. In particular, if L is close to divergence type, as in Example 4.18(ii), then X is a C^0-solution to the previous equation with $b = \frac{\sigma^2}{2} + \beta - \frac{\sigma^2(0)}{2}$.

PROOF. Let X be a solution to the martingale problem related to L. We know, by Lemma 6.5, that X has extended local time regularity with respect to ν_1. On the other hand, by Remark 5.11, X is a finite quadratic variation process. It remains to show that

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + A^X(b)_t. \tag{6.6}$$

Let $\ell \in C^1$ and set $f = T^1\ell$. By definition of a sharp martingale problem, we have

$$T^1\ell(X_t) = T^1\ell(X_0) + \int_0^t ((T^1\ell)'\sigma)(X_s)\,dW_s + A^X(\ell)_t. \tag{6.7}$$

According to Remark 4.15(i) concerning the continuity of the map $T^1 : \nu_0 \to C^1$, previous expression can be extended to any $\ell \in \nu_0$.

By Remark 4.15(ii), $\ell = b \in \nu_0$ and $f = T^1\ell = id$. Replacing this in (6.7), we obtain

$$X_t = x_0 + \int_0^t \sigma(X_s)\,dW_s + A^X(b).$$

Since $X_0 = Z$, the proof is complete. $\square$

COROLLARY 6.9. *Let $x_0 \in \mathbb{R}$. Suppose that L fulfills Technical Assumption $\mathcal{A}(\nu_0)$. If $MP(L, x_0)$ [resp. $SMP(L, x_0)$] admits weak (resp., strong) existence, then the SDE (6.5) also admits weak (resp., strong) existence.*

PROOF. The statement concerning strong solutions is obvious. Concerning weak solutions, let us admit the existence of a filtered probability space, where there is a solution to the martingale problem with respect to L with initial condition x_0. Then according to Remark 5.12, this solution is also a solution to a sharp martingale problem and the result follows. $\square$

If X is some ν_1-solution to (6.6), is it a solution to the (sharp) martingale problem related to some operator L? This is a delicate question. In the following proposition, we only provide the converse of Proposition 6.7 as a partial answer.

PROPOSITION 6.10. *Suppose that the PDE operator L fulfills Technical Assumption $\mathcal{A}(\nu_0)$. Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space fulfilling the usual conditions and let (W_t) be a classical $(\mathcal{F}_t)$-Brownian motion. Let X be a progressively measurable process.*

X solves the sharp martingale problem related to L with respect to some initial condition x_0 if and only if it is a ν_0-solution to the stochastic differential equation

$$\begin{aligned} dX_t &= b'(X_t)\,dt + \sigma(X_t)\,dW_t, \\ X_0 &= x_0. \end{aligned} \tag{6.8}$$

COROLLARY 6.11. *Let $x_0 \in \mathbb{R}$. Suppose that L fulfills Technical Assumption $\mathcal{A}(\nu_0)$. Then weak existence and uniqueness in law (resp., strong existence and pathwise uniqueness) hold for equation (6.8) if and only if the same holds for $MP(L, x_0)$ [resp. $SMP(L, x_0)$].*

PROOF OF PROPOSITION 6.10. Suppose that X is a ν_0-solution to (6.8). Then it is a finite quadratic variation process. Let $f \in C^3$. Since X solves (6.6) and $\int_0^\cdot f'(X_s)\, d^- X_s$ always exists by the classical Itô formula [see Remark 3.3(e) of Section 1], we know that $\int_0^\cdot f'(X)\, d^- A^X(b)$ also exists and is equal to $\int_0^\cdot f'(X)\, d^- X - \int_0^\cdot (f'\sigma)(X)\, dW$. Therefore, this Itô formula says that

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\sigma(X_s)\, dW_s + \int_0^t f'(X)\, d^- A^X(b) + \tfrac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)\, ds$$

holds.

By Lemma 6.6, the linearity of mapping A^X and (4.12), we obtain

$$\begin{aligned}\int_0^t f'(X)\, d^- A^X(b) &+ \frac{1}{2}\int_0^t (f''\sigma^2)(X_s)\, ds \\ &= A^X(\Phi(f', b))_t + \frac{1}{2}\int_0^t (f''\sigma^2)(X_s)\, ds \\ &= \int_0^t \left(\frac{\sigma^2}{2} - b\right)(X_s) f''(X_s)\, ds + A^X(bf') = A^X(\hat{L}f).\end{aligned}$$

This shows that

$$f(X_t) - f(X_0) - \int_0^t (f'\sigma)(X_s)\, dW_s = A^X(\hat{L}f) \tag{6.9}$$

for every $f \in C^3$. In reality, it is possible to show the previous equality for any $f \in C^2$. In fact, the left-hand side extends continuously to C^2 and even to C^1. The right-hand side is also allowed to be extended to C^2 for the following reason. For $f \in C^2$, let (f_n) be a sequence of functions in C^3 converging to f when $n \to \infty$, according to the C^2 topology. In particular, the convergence also holds in $C^1_{\nu_0}$. Since $\hat{L}$ is continuous with respect to the $C^1_{\nu_0}$ topology with values in ν_0, we have $\hat{L}f_n \to \hat{L}f$ in ν_0. Finally, $A^X(\hat{L}f_n) \to A^X(\hat{L}f)$ u.c.p. because of the extended local time regularity with respect to ν_0.

We will, in fact, use the validity of (6.9) for $f \in C^2$ with $f(0) = 0$ and $x_1 = f'(0)$ and $\ell = \hat{L}f$. According to Technical Assumption $\mathcal{A}(\nu_0)$(iv), we have $f = T^{x_1}\ell$. Therefore, (6.9) gives

$$T^{x_1}\ell(X_t) = T^{x_1}\ell(X_0) + \int_0^t ((T^{x_1}\ell)'\sigma)(X_s)\, dW_s + A^X(\ell).$$

Again using extended local time regularity with respect to ν_0 and the continuity of T^{x_1}, we can state the validity of the previous expression for each $\ell \in \nu_0$ with $\ell(0) = 0$, in particular, for $\ell \in C^1$ with $\ell(0) = 0$. But in this case, for any $f \in \mathcal{D}_L$ with $f(0) = 0$ and $\ell' = Lf$, we obtain

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + \int_0^t Lf(X_s)\, ds.$$

This shows the validity of the identity in Definition 5.2 for $f \in \mathcal{D}_L$ and that $f(0) = x_0$ and $x_0 = 0$. If $x_0 \neq 0$, we replace f by $f - x_0$ in the previous identity and use the fact that $L(f - x_0) = Lf$ for any $f \in \mathcal{D}_L$.

It follows that X fulfills a sharp martingale problem with respect to L.

This shows the reversed sense of the statement. The direct implication was proven in Proposition 6.7. $\square$

COROLLARY 6.12. *We suppose that $\sigma \in D^{1/2}$ and $b \in C^{1/2}$, or $\sigma \in C^{1/2}$ and $b \in D^{1/2}$, with conditions* (4.7). *We set $\nu_0 = D^{1/2}$.*

Then equation (6.8) *admits ν_0-strong existence and pathwise uniqueness.*

PROOF. The result follows from Corollaries 6.11 and 5.19. $\square$

7. About C_b^0-generalized solutions of parabolic equations. In this section, we want to discuss the related parabolic Cauchy problem with final condition, which is associated with our stochastic differential equations with distributional drift.

We will adopt the same assumptions and conventions as in Section 4. We consider the formal operator $\mathcal{L} = \partial_t + L$, where L will hereafter act on the second variable.

DEFINITION 7.1. Let λ be an element of $C_b^0([0, T] \times \mathbb{R})$ and let $u^0 \in C_b^0(\mathbb{R})$. A function $u \in C_b^0([0, T] \times \mathbb{R})$ will be said to be a C_b^0-*generalized solution* to

$$\begin{aligned} &\mathcal{L}u = \lambda, \\ &u(T, \cdot) = u^0, \end{aligned} \tag{7.1}$$

if the following are satisfied:

(i) for any sequence (λ_n) in $C_b^0([0, T] \times \mathbb{R})$ converging to λ in a bounded way,

(ii) for any sequence (u_n^0) in $C_b^0(\mathbb{R})$ converging in a bounded way to u^0,

(iii) such there are classical solutions (u_n) in $C_b^0([0, T] \times \mathbb{R})$ of class $C^{1,2}([0, T[\times\mathbb{R})$ to $\mathcal{L}_n u_n = \lambda_n$, $u_n(T, \cdot) = u_n^0$,

then (u_n) converges in a bounded way to u.

REMARK 7.2. (a) u is said to solve $\mathcal{L}u = \lambda$ if there exists $u^0 \in C_b^0(\mathbb{R})$ such that (7.1) holds.

(b) The previous definition depends in principle on the mollifier, but it could be easily adapted so as not to depend on it.

(c) The regularized problem admits a solution: if $u_n^0 \in C_b^3(\mathbb{R})$ and $\lambda_n \in C_b^{0,1}([0,T]\times\mathbb{R})$, then there is a classical solution u_n in $C^{1,2}([0,T]\times\mathbb{R})$ of

$$\mathcal{L}_n v = \lambda_n,$$
$$v(T,\cdot) = u_n^0.$$

For this, it suffices to apply Theorem 5.19 of [15].

We now state a result concerning the case when the operator L is classical. Even if the next proposition could be stated when the drift b' is a continuous function, we will suppose it to be zero. In fact, it will later be applied to $L = L^0$.

PROPOSITION 7.3. *We suppose that* $b = 0$. *Let* $\varphi, \varphi_n \in C_b^0(\mathbb{R})$, $g, g_n \in C_b^0([0,T]\times\mathbb{R})$, $n \in \mathbb{N}$, *such that* $\varphi_n \longrightarrow \varphi$, $g_n \longrightarrow g$ *in a bounded way on* $\mathbb{R}$ *and* $[0,T]\times\mathbb{R}$.

Let σ *be a strictly positive real continuous function.*

Suppose that there exist $u_n \in C^{1,2}([0,T[\times\mathbb{R}) \cap C_b^0([0,T]\times\mathbb{R})$ *such that*

$$\mathcal{L}_n u_n = g_n,$$
$$u_n(T,\cdot) = \varphi_n.$$

Then (u_n) *will converge to* $u \in C_b^0([0,T]\times\mathbb{R})$ *in a bounded way, where the function* u *is defined by*

$$u(s,x) = \mathbb{E}\left(\varphi(Y_T^{s,x}) + \int_s^T g(r, Y_T^{r,x})\,dr\right), \tag{7.2}$$

where $Y = Y^{s,x}$ *is the unique solution (in law) to*

$$Y_t = x + \int_s^t \sigma(X_r)\,dW_r \tag{7.3}$$

and where (W_t) *is a classical Brownian motion on some suitable filtered probability space.*

REMARK 7.4. Usual Itô calculus implies that

$$u_n(s,x) = \mathbb{E}\left(\varphi_n(Y_T^{s,x}(n)) + \int_s^T g_n(r, Y_T^{r,x}(n))\,dr\right), \tag{7.4}$$

where $Y(n) = Y^{s,x}(n)$ is the unique solution in law to the problem

$$Y_t(n) = x + \int_s^t \sigma_n(Y_r(n))\,dW_r. \tag{7.5}$$

Theorem 5.4 (Chapter 5 of [14]) affirms that it is possible to construct a solution (unique in law) $Y = Y^{s,x}$ to the SDE (7.3) [resp., $Y(n) = Y^{s,x}(n)$ to (7.5)].

Suppose that L is a classical PDE operator. Let $u \in C^{1,2}([0,T[\times\mathbb{R})$ be bounded and continuous on $[0,T]\times\mathbb{R}$. Again, Itô calculus shows that u can be represented by (7.2) and (7.3). In particular, a classical solution u to $\mathcal{L}u = g$ is also a C_b^0-generalized solution.

PROOF OF PROPOSITION 7.3. We fix $s \in [0,T]$, $x \in \mathbb{R}$. Using the Engelbert–Schmidt construction (see, e.g., the proof of Theorem 5.4, Chapter 5 and 5.7 of [14]), it is possible to construct a solution $Y = Y^{s,x}$ of the SDE on some fixed probability space which solves (7.3) with respect to some classical Wiener process (W_t). We set $s = 0$ for simplicity. The procedure is as follows. We fix a standard Brownian motion (B_t) on some fixed probability space one set

$$R_t := \int_0^t \frac{du}{\sigma^2(x + B_u)}.$$

R is a.s. a homeomorphism on $\mathbb{R}_+$ and we define A as the inverse of R. A solution Y will be then given by $Y_t = x + B_{A_t}$; in fact, it is possible to show that the quadratic variation of the local martingale Y is

$$\langle Y, Y\rangle_t = \int_0^t \sigma^2(Y_s)\,ds.$$

The Brownian motion W is constructed a posteriori and is adapted to the natural filtration of Y by setting $W_t = \int_0^t \frac{dY_s}{\sigma(Y_s)}$.

So, on the same probability space, we can set $Y_t(n) = x + B_{A_t(n)}$, $A(n)$ being the inverse of $R(n)$, where $R(n)_t := \int_0^t \frac{du}{\sigma_n^2(x+B_u)}$.

Consequently, on the same probability space, we construct $Y_t(n) = x + B_{A_t(n)}$, where $A(n)$ is the inverse of $R(n)$ and $R(n)_t := \int_0^t \frac{du}{\sigma_n^2(x+B_u)}$. $Y(n)$ solves equation (7.5) with respect to a Brownian motion depending on n.

By construction, the family $Y_T^{s,x}(n)$ converges a.s. to $Y_T^{s,x}$. Using Lebesgue dominated convergence theorems and the bounded convergence of (φ_n) and (g_n), we can take the limit when $n \to \infty$ in expression (7.4) and obtain the desired result. □

REMARK 7.5. In particular, the corresponding laws of random variables $(Y^{s,x}(n))$ are tight.

Again, we will adopt the same conventions as in Section 4.

We set $\sigma_h = \sigma h'$. L^0 is the classical operator defined at (4.9). Let us consider $\mathcal{L}^0 = \partial_t + L^0$ as a formal operator.

COROLLARY 7.6. *Let $g \in C_b^0([0,T]\times\mathbb{R})$, $\varphi \in C_b^0(\mathbb{R})$. There is a C_b^0-generalized solution u to $\mathcal{L}^0 u = g$, $u(T,\cdot) = \varphi$. This solution is unique and is given by* (7.2).

We now return to the original PDE operator $\mathcal{L}$ with distributional drift. We again denote by h the same application defined in Section 5 and discuss existence and uniqueness of C_b^0-generalized solutions of related parabolic Cauchy problems.

A useful consequence of Proposition 7.3 is the following.

THEOREM 7.7. *For $\varphi \in C^0([0,T] \times \mathbb{R})$ or $C^0(\mathbb{R})$, we again set $\tilde{\varphi} = \varphi \circ h^{-1}$ according to the conventions of Section 2. Again, we consider $\mathcal{L}^0 = \partial_t + L^0$ as a formal operator.*

Let $\lambda \in C_b^0([0,T] \times \mathbb{R})$, $u^0 \in C_b^0(\mathbb{R})$.

There is a unique solution $u \in C_b^0([0,T] \times \mathbb{R})$ to

$$\begin{aligned} \mathcal{L}u &= \lambda, \\ u(T,\cdot) &= u^0. \end{aligned} \tag{7.6}$$

Moreover, $\tilde{u}$ solves

$$\begin{aligned} \mathcal{L}^0\tilde{u} &= \tilde{\lambda}, \\ \tilde{u}(T,\cdot) &= \tilde{u}^0. \end{aligned} \tag{7.7}$$

PROOF. In accordance with Section 4, let $(h_n)_{n\in\mathbb{N}}$ be an approximating sequence which is related to $Lh = 0$. Let us consider the PDE operators $\mathcal{L}_n$ defined at (4.2). Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $C_b^0([0,T) \times \mathbb{R})$ such that $\lambda_n \to \lambda$, $u_n^0 \to u^0$ in a bounded way and for which there are classical solutions u_n of

$$\begin{aligned} \mathcal{L}_n u_n &= \lambda_n, \\ u_n(T,\cdot) &= u_n^0. \end{aligned}$$

We recall that those sequences always exist because of Remark 7.2(c).

We set

$$g_n = \lambda_n \circ h_n^{-1}, \qquad \varphi_n = \varphi \circ h_n^{-1}, \qquad v_n = u_n \circ h_n^{-1}.$$

By Lemma 4.14, we have

$$\begin{aligned} \mathcal{L}_n^0 v_n &= g_n, \\ v_n(T,\cdot) &= \varphi_n, \end{aligned}$$

where

$$\mathcal{L}_n^0\varphi(t,y) = \partial_t\varphi(t,y) + \sigma_{h_n}^2 \circ h_n^{-1}(t,y)\,\partial_{xx}^2\varphi(t,y).$$

By Proposition 7.3, and Corollary 7.6, $v_n \to \tilde{u}$ in a bounded way, where

$$\begin{aligned} \mathcal{L}^0\tilde{u} &= \tilde{\lambda}, \\ \tilde{u}(T,\cdot) &= \tilde{u}_0. \end{aligned}$$

This concluses the proof of the proposition. $\square$

We now discuss how C_b^0-generalized solutions are transformed under the action of the function k introduced at (4.13). A similar result to Lemma 4.13 for the elliptic case is the following.

PROPOSITION 7.8. *For $\varphi \in C^0([0,T]\times\mathbb{R})$ or $C^0(\mathbb{R})$, we set $\bar\varphi = \varphi \circ k^{-1}$. We set $\sigma_k = \sigma k'$ and consider the formal operator*

$$\mathcal{L}^1 f = \partial_t f + \tfrac{1}{2}\bar\sigma_k^2\, \partial_{xx}^2 f + \tfrac{1}{2}(\bar\sigma_k^2)'\, \partial_x f.$$

Informally, we can write

$$\mathcal{L}^1 f = \partial_t f + \tfrac{1}{2}\partial_x(\bar\sigma_k^2\, \partial_x f).$$

Let $\lambda \in C_b^0([0,T]\times\mathbb{R})$, $u^0 \in C_b^0(\mathbb{R})$.

Let u be the unique C_b^0-generalized solution in $C_b^0([0,T]\times\mathbb{R})$ to

$$\begin{aligned} \mathcal{L}u &= \lambda, \\ u(T,\cdot) &= u^0. \end{aligned} \tag{7.8}$$

Then $\bar u$ solves

$$\begin{aligned} \mathcal{L}^1\bar u &= \bar\lambda, \\ \bar u(T,\cdot) &= \bar u^0. \end{aligned}$$

PROOF. Let v be the unique solution to

$$\begin{aligned} \mathcal{L}^1 v &= \bar\lambda, \\ v(T,\cdot) &= \bar u^0, \end{aligned}$$

which exists because of Theorem 7.7, taking $\mathcal{L} = \mathcal{L}^1$.

We define $H:\mathbb{R}\to\mathbb{R}$ such that

$$H(0) = 0, \qquad H'(z) = \frac{1}{\sigma_k^2}(z).$$

Again, (4.7) implies that H is bijective on $\mathbb{R}$. This case corresponds to example (a) in Remark 4.4 with $\alpha = 1$.

We set $\tilde v = v \circ H^{-1}$. Again, by Theorem 7.7, we have

$$\begin{aligned} \mathcal{L}^{0,1}\tilde v &= \bar\lambda \circ H^{-1}, \\ \tilde v(T,\cdot) &= u^0 \circ (k^{-1}\circ H^{-1}), \end{aligned}$$

where $\mathcal{L}^{0,1} f = \frac{a^2}{2}\,\partial_{xx}^2 f$ and

$$a = (\sigma_k H') \circ H^{-1} = \frac{1}{\sigma_k} \circ H^{-1}.$$

Since

$$\sigma_k = (\sigma k') \circ k^{-1} = \frac{e^{\Sigma}}{\sigma} \circ k^{-1},$$

this yields

$$a = (\sigma e^{-\Sigma}) \circ (H \circ k)^{-1}.$$

On the other hand, $H \circ k = h$ since

$$H \circ k(0) = 0 = h(0),$$
$$(H \circ k(x))' = H'(k(x))k'(x) = \frac{1}{\sigma_k^2} k'(x) = \frac{1}{\sigma^2 k'} = e^{-\Sigma} = h'.$$

We can therefore conclude that $\mathcal{L}^{0,1} \equiv \mathcal{L}^0$. Since problem (7.7) has a unique solution, $\tilde{v} = \tilde{u}$, where u solves (7.6) and $\tilde{u} = u \circ h^{-1}$. Finally,

$$v = \tilde{v} \circ H = \tilde{u} \circ H = u \circ H \circ h^{-1} = u \circ k^{-1} = \bar{u}. \qquad \square$$

PROPOSITION 7.9. *The unique C_b^0-generalized solution to* (7.6) *admits a probabilistic representation in the sense that*

$$u(s, x) = \mathbb{E}\left(u^0(X_T^{s,x}) + \int_s^T \lambda(r, X_T^{r,x})\, dr\right), \tag{7.9}$$

where $X^{s,x}$ is the solution to the martingale problem related to L at time s and point x.

PROOF. The result follows from Theorem 7.7, Corollary 7.6 and Proposition 5.10, which collectively imply the following. If X is a solution to the martingale problem related to L at point x at time s, then $Y = h(X)$ solves the stochastic differential equation (5.3) with initial condition $h(x)$ at time s. $\square$

8. Density of the associated semigroups. We now discuss the existence of a density law for the solutions $X^{s,x}$ of the martingale problem related to L. First, we suppose that L is an operator in divergence form with $Lf = (\frac{\sigma^2}{2} f')'$ and that there are positive constants such that $c \le \sigma^2 \le C$. We will say, in this case, that L *has the Aronson form*. This terminology refers to the fundamental paper [1] concerning exponential estimates of fundamental solutions of nondegenerate parabolic equations. We begin with some properties (partly classical) stated in [11]. We observe that point (ix) is slightly modified with respect to [11], but this new configuration can be immediately deduced from the proof in [11]. This preparatory work will be applied to the operator L^1 introduced in (4.14).

LEMMA 8.1. *We suppose that* $0 < c \le \sigma^2 \le C$. *Let* σ_n, $n \in \mathbb{N}$, *be smooth functions such that* $0 < c \le \sigma_n^2 \le C$ *and* $\sigma_n^2 \to \sigma^2$ *in* C^0, *as at the beginning of Section* 4. *We set* $L_n g = (\frac{\sigma_n^2}{2} g')'$. *There exists a family of probability measures* $(\nu_t(dx, y), t \ge 0, y \in \mathbb{R})$ [*resp.*, $(\nu_t^n(dx, y), t \ge 0, y \in \mathbb{R})$] *enjoying the following properties*:

(i) $\nu_t(dx, y) = p_t(x, y)\,dx$, $\nu_t^n(dx, y) = p_t^n(x, y)\,dy$;

(ii) (*Aronson estimates*) *there exists* $M > 0$, *depending only on constants* c, C, *with*

$$\frac{1}{M\sqrt{t}} \exp\left(-\frac{M|x-y|^2}{t}\right) \le p_t(x, y) \le \frac{M}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{Mt}\right);$$

(iii) *we have*

$$\partial_t \nu_t(\cdot, y) = L\nu_t(\cdot, y), \qquad \nu_0(\cdot, y) = \delta_y \tag{8.1}$$

and

$$\partial_t \nu_t^n(\cdot, y) = L_n \nu_t^n(\cdot, y), \qquad \nu_0^n(\cdot, y) = \delta_y,$$

where ν (*resp.*, ν^n) *is called the* fundamental solution *related to the previous parabolic linear equation*;

(iv) *we have*

$$\partial_t \nu_t(x, \cdot) = L\nu(x, \cdot),$$
$$\partial_t \nu_t^n(x, \cdot) = L_n \nu^n(x, \cdot);$$

(v) *the map* $(t, x, y) \mapsto p_t(x, y)$ *is continuous from* $]0, \infty[\times\mathbb{R}^2$ *to* $\mathbb{R}$;

(vi) *the* p^n *are smooth on* $]0, \infty[\times\mathbb{R}^2$;

(vii) *we have* $\lim_{n\to\infty} p_t^n(x, y) = p_t(x, y)$ *uniformly on each compact subset of* $]0, \infty[\times\mathbb{R}^2$;

(viii) $p_t(x, y) = p_t(y, x)$ *holds for every* $t > 0$ *and every* $x, y \in \mathbb{R}$;

(ix) $\int_0^T \sup_y (\int_{\mathbb{R}} |\partial_x p_t(x, y)|^2\,dx)^{1/2}\,dt < \infty$.

The previous lemma allows us to establish the following.

THEOREM 8.2. *Let* $Z^{s,x}$ *be the solution to the martingale problem related to* L *at time* s *and point* x. *Suppose that* L *to be of divergence type*, *having the Aronson form. Then there is fundamental solution* $\nu_t = r_t(x, y)$ *of*

$$\partial_t \nu_t(\cdot, y) = L\nu_t(\cdot, y), \qquad \nu_0(\cdot, y) = \delta_y,$$

with the following properties:

(i) *letting* $g \in C_b^0([0, T] \times \mathbb{R})$, $\varphi \in C_b(\mathbb{R})$, *the* C_b^0-*generalized solution* u *to* $\mathcal{L}u = g$, $u(T, \cdot) = \varphi$, *is given by*

$$u(s, x) = \int_{\mathbb{R}} \varphi(y) r_{T-s}(x, y)\,dy + \int_s^T dr \int_{\mathbb{R}} g(r, y) r_{T-r}(x, y)\,dy; \tag{8.2}$$

(ii) *the law of* $Z_T^{s,x}$ *has* $r_{T-s}(x,\cdot)$ *as density with respect to Lebesgue measure.*

PROOF. Let $(r_t^n(x,y))$ be the fundamental solution corresponding to the parabolic equation associated with $L_n f(x) = (\frac{\sigma_n^2 f'}{2})'$, as introduced in Section 4. We observe that (σ_n^2) converges in a bounded way to σ^2.

(i) We define

$$u_n(s,x) = \int_{\mathbb{R}} \varphi(y) r_{T-s}^n(x,y)\,dy + \int_s^T dr \int_{\mathbb{R}} g(r,y) r_{T-r}^n(x,y)\,dy. \tag{8.3}$$

Points (vi) and (ii) of Lemma 8.1 imply that functions u_n belong to $C^{1,2}([0,T[\times\mathbb{R})$, so they are classical solutions to

$$\mathcal{L}_n u_n = g,$$
$$u_n(T,\cdot) = u^0.$$

According to points (ii) and (vii) of the same lemma, one can prove that u_n converges in a bounded way to u defined by (8.2). In fact, the coefficients σ_n^2 are lower and upper bounded with a common constant, related to c and C. Therefore, this u is the C_b^0-generalized solution of the Cauchy problem being considered, which is known to exist. By uniqueness, point (i) is established.

(ii) Setting $g = 0$, point (i) implies that $u(s,x) = \int_{\mathbb{R}} \varphi(y) r_{T-s}(x,y)\,dy$ is the C_b^0-generalized solution to $\mathcal{L}u = 0$ with $u(T,x) = \varphi(x)$. By Proposition 7.9, in particular, using the probabilistic representation, we get $\mathbb{E}(\varphi(Z_T^{s,x})) = \int_{\mathbb{R}} \varphi(y) r_{T-s}(x,y)\,dy$. $\square$

REMARK 8.3. If L is in the divergence form, as before, then $\mathcal{D}_L = \{f \in C^1$ such that there exists $g \in C^1$ with $f' = \frac{g}{\sigma^2}\}$. This is a consequence of Lemma 4.9 and the fact that $e^{-\Sigma} = \frac{1}{\sigma^2}$.

Hereafter, we will consider a general PDE operator L with distributional drift, as in Section 4, for which the assumption (Aronson) below holds.

$$c \le \frac{e^{\Sigma}}{\sigma^2} \le C. \tag{Aronson}$$

We observe that the PDE operator in divergence form of the type $L^1 f = (\frac{\sigma_k^2 f'}{2})'$, where $\sigma_k = (\sigma k') \circ k^{-1}$, has the Aronson form, so the previous theorem can be applied.

THEOREM 8.4. *Let* $X^{s,x}$ *be the solution to the martingale problem related to* L *at time* s *and point* x. *Suppose that* L *fulfills assumption* (Aronson). *Then there exists a kernel* $p_t(x,y)$ *such that*:

(i) *the law of* $X_t^{s,x}$ *has* $p_{t-s}(x,\cdot)$ *as density with respect to Lebesgue measure for each* $t \in]s,T]$;

(ii) *letting* $g \in C_b^0([0,T]\times\mathbb{R})$, $\varphi \in C_b^0(\mathbb{R})$, *the* C_b^0*-generalized solution* u *to* $\mathcal{L}u = g$, $u(T,\cdot)=\varphi$, *is given by*

$$u(s,x) = \int_{\mathbb{R}} \varphi(y) p_{T-s}(x,y)\,dy + \int_s^T dr \int_{\mathbb{R}} g(r,y) p_{T-r}(x,y)\,dy. \tag{8.4}$$

PROOF. (i) Proposition 5.15 says that $Z^{s,x} = k(X^{s,x})$ solves the martingale problem with respect to L^1. Let $r_t(x,y)$ be the fundamental solution associated with the parabolic PDE $\mathcal{L}^1 = \partial_t + L^1$. The first point then follows from the next observation.

REMARK 8.5. By means of a change of variable, it is easy to see that the density law of $X_t^{s,x}$ equals

$$p_t(x,x_1) = r_t(k(x),k(x_1))k'(x_1) = r_t(k(x),k(x_1))\frac{e^{\Sigma}}{\sigma^2}(x_1).$$

(ii) This is a consequence of point (i), Fubini's theorem and Proposition 7.9. □

At this point, we need a lemma which extends to the kernel $p_t(x,x_1)$ the integrability property of the kernel $r_t(x,x_1)$ stated in (8.3) concerning the divergence case.

LEMMA 8.6. *Let* $p_t(x,x_1)$ *be the kernel introduced in Theorem* 8.4. *Then*:

(i) *it is continuous in all variables* $(t,x,x_1) \in]0,T[\times\mathbb{R}^2$;
(ii) *it fulfills Aronson estimates*;
(iii) $\int_0^T (\sup_{x_1} \int_{\mathbb{R}} \partial_x p_t(x,x_1)^2\,dx)^{1/2}\,dt < \infty$.

PROOF. We recall, by Remark 8.5, that

$$p_t(x,x_1) = r_t(k(x),k(x_1))k'(x_1),$$

where $r_t(z,z_1)$ is the fundamental solution associated with the operator $L^1 f = (\frac{\sigma_k^2}{2} f')'$, $k' = \frac{e^{\Sigma}}{\sigma^2}$. This, and point (v) of Lemma 8.1, directly imply the validity of the first point.

Taking into account assumption (Aronson), Aronson estimates for $(r_t(z,z_1))$ and the fact that

$$|k(x) - k(x_1)| = \int_0^1 k'\big(\alpha x + (1-\alpha)x_1\big)\,d\alpha\,|x - x_1|,$$

result (ii) follows easily.

With the same conventions as before, we have

$$\partial_x p_t(x, x_1) = \partial_z r_t(k(x), k(x_1))k'(x)k'(x_1).$$

So, for $x \in \mathbb{R}$,

$$\begin{aligned}\left(\int_{\mathbb{R}} (\partial_x p_t(x, x_1))^2\, dx\right)^{1/2} &= \left(k'(x_1) \int_{\mathbb{R}} (\partial_z r_t(z, k(x_1)))^2\, dz\right)^{1/2} \\ &\le \sqrt{C} \sup_{z_1} \left(\int_{\mathbb{R}} dz (\partial_z r_t(z, z_1))^2\right)^{1/2}.\end{aligned}$$

(iii) Follows after integration with respect to t and because of Lemma 8.1(ix). $\square$

PROPOSITION 8.7. *Let $g \in C_b^0([0, T] \times \mathbb{R}) \cap L^1([0, T] \times \mathbb{R})$, $\varphi \in C_b^0(\mathbb{R}) \cap L^1(\mathbb{R})$. Let $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ be the C_b^0-generalized solution to $\mathcal{L}u = g$, $u(T, \cdot) = \varphi$. Then*:

(a) $\int_0^T dt \int_{\mathbb{R}} u^2(t, x)\, dx < \infty$;
(b) *$x \mapsto u(t, x)$ is absolutely continuous*,

$$\int_0^T dt \left(\int_{\mathbb{R}} (\partial_x u)^2(t, x)\, dx\right)^{1/2} < \infty$$

and in particular, for a.e. $t \in [0, T]$, $\partial_x u(t, \cdot)$ is square integrable.

REMARK 8.8. Previous assumptions imply that g and φ are also square integrable.

PROOF OF PROPOSITION 8.7. We recall the expression given in Theorem 8.4,

$$u(t, x) = \int_{\mathbb{R}} \varphi(x_1) p_{T-t}(x, x_1)\, dx_1 + \int_t^T dr \int_{\mathbb{R}} g(r, x_1) p_{T-r}(x, x_1)\, dx_1.$$

Using Lemma 8.6 and classical integration theorems, we have

$$\begin{aligned}\partial_x u(t, x) &= \int_{\mathbb{R}} \varphi(x_1)\, \partial_x p_{T-t}(x, x_1)\, dx_1 \\ &\quad + \int_t^T dr \int_{\mathbb{R}} ds\, g(s, x_1)\, \partial_x p_{T-s}(x, x_1)\, dx_1.\end{aligned} \tag{8.5}$$

Using Jensen's inequality, we have

$$\begin{aligned}|u(t, x)|^2 &\le \int_{\mathbb{R}} \varphi(x_1)^2 p_{T-t}(x, x_1)\, dx_1 \\ &\quad + (T - t) \int_t^T ds \int_{\mathbb{R}} g^2(s, x_1) p_{T-s}(x, x_1)\, dx_1.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} u^2(t,x)\,dx = \int_{\mathbb{R}} dx_1\,\varphi(x_1)^2 \int_{\mathbb{R}} dx\, p_{T-t}(x,x_1) + \int_t^T ds\,(T-t)\int_{\mathbb{R}} dx_1 \int g^2(s,x_1)\int_{\mathbb{R}} dx\, p_{T-s}(x,x_1).$$

Using Aronson estimates, this quantity is bounded by

$$\mathrm{const}\biggl(\int_{\mathbb{R}} dx_1\,\varphi(x_1)^2 \int_{\mathbb{R}} dx\,\frac{1}{\sqrt{T-t}}p\biggl(\frac{x-x_1}{\sqrt{T-t}}\biggr) + \int_t^T ds \int_{\mathbb{R}} dx_1 \int g^2(s,x_1)\int_{\mathbb{R}} dx\,\frac{1}{\sqrt{T-s}}p\biggl(\frac{x-x_1}{\sqrt{T-s}}\biggr)\biggr),$$

where p is the Gaussian $N(0,1)$ density. This is clearly equal to

$$\mathrm{const}\biggl(\int_{\mathbb{R}} dx_1\,\varphi(x_1)^2 + \int_0^T ds\int_{\mathbb{R}} dx_1\, g^2(s,x_1)\biggr).$$

This establishes point (a).

Concerning point (b), in order to simplify the framework we will suppose that $g=0$. Expression (8.4) implies that

$$\partial_x u(t,x) = \int_{\mathbb{R}} \varphi(x_1)\,\partial_x p_{T-t}(x,x_1)\,dx_1.$$

Jensen's inequality implies that

$$\partial_x u(t,x)^2 \le \biggl(\int_{\mathbb{R}} dx_1\,|\varphi(x_1)||\partial_x p_{T-t}(x,x_1)|^2\biggr)\int_{\mathbb{R}} dx_1\,|\varphi(x_1)|.$$

Integrating with respect to x and taking the square root, we get

$$\begin{aligned}\sqrt{\int_{\mathbb{R}} dx\,\partial_x u(t,x)^2} &\le \sqrt{\int_{\mathbb{R}} |\varphi(x_1)|\,dx_1}\sqrt{\int_{\mathbb{R}} dx_1\,|\varphi(x_1)|\int_{\mathbb{R}} dx\,|\partial_x p_{T-t}(x,x_1)|^2}\\ &\le \int_{\mathbb{R}} dx_1\,|\varphi(x_1)|\sqrt{\sup_{x_1}\int_{\mathbb{R}} |\partial_x p_t(x,x_1)|^2\,dx}.\end{aligned}$$

Integrating with respect to t gives

$$\int_0^T dt\,\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \le \int_{\mathbb{R}} dx_1\,|\varphi(x_1)|\int_0^T dt\sqrt{\sup_{x_1}\int_{\mathbb{R}} \partial_x p_t(x,x_1)^2\,dx}.$$

This quantity is finite due to Lemma 8.6(iii). $\square$

9. Relation with weak solutions of stochastic partial differential equations. As in the previous section, we will adopt assumption (Aronson). At this point, we wish to investigate the link between C_b^0-generalized solutions and the notion of SPDE's weak solutions for a corresponding Cauchy problem.

We will adopt the same conventions as in Section 4. In this section, we will suppose that coefficients σ, b are realizations of stochastic processes indexed by $\mathbb{R}$. Let us consider the formal operator $\mathcal{L} = \partial_t + L$, where L acts on the second variable.

We consider the equation

$$\begin{aligned} \mathcal{L}u &= \lambda, \\ u(T, \cdot) &= u^0. \end{aligned} \tag{9.1}$$

The aim of this section is to show that a C_b^0-generalized solution to (9.1) provides, when $\sigma = 1$, a solution to the (stochastic) PDE of the type (1.1), as defined in the Definition 1.1, that is, with the help of a symmetric integral via regularization, as defined in Section 3. We denote by $\mathcal{D}(\mathbb{R})$ the linear space of C^∞ real functions with compact support.

The link between the SPDE (1.1) and (1.2) is given in the following.

PROPOSITION 9.1. *Let $u(t,x)$, $v(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}$ be two continuous random fields a.s. in $C^{0,1}(]0,T[\times\mathbb{R})$ such that $v(t,x) = u(T-t,x)$. v is a solution to the SPDE* (1.1) *if and only if v is a solution to the SPDE* (1.2).

PROOF. We observe that $\partial_x v(t,x) = -\partial_x u(T-t,x)$. The proof is elementary. The only point to check is the following:

$$\int_{\mathbb{R}} d^\circ\eta(x)\alpha(x)\biggl(\int_t^T ds\, \partial_x u(s,x)\biggr) = -\int_{\mathbb{R}} d^\circ\eta(x)\alpha(x)\biggl(\int_0^t ds\, \partial_x v(s,x)\biggr).$$

This follows by the definition of symmetric integral and the following, obvious, identity:

$$\begin{aligned} &\int_{\mathbb{R}} dx\, \frac{\eta(x+\varepsilon) - \eta(x-\varepsilon)}{2\varepsilon}\alpha(x)\biggl(\int_t^T ds\, \partial_x u(s,x)\biggr) \\ &\qquad = -\int_{\mathbb{R}} dx\, \frac{\eta(x+\varepsilon) - \eta(x-\varepsilon)}{2\varepsilon}\alpha(x)\biggl(\int_0^t ds\, \partial_x v(s,x)\biggr) \end{aligned}$$

for every $\varepsilon > 0$. □

We continue with a lemma, still supposing σ to be general.

LEMMA 9.2. *Let λ (resp., u^0) be a random field with parameter $(t,x) \in [0,T]\times\mathbb{R}$ (resp., $x \in \mathbb{R}$) whose paths are bounded and continuous. Let σ, b be continuous stochastic processes such that Σ is defined a.s. and assumption* (Aronson) *is satisfied. Let u be the random field which is a.s. the C_b^0-generalized solution to* (9.1). *The following then holds*:

$$\int_{\mathbb{R}} dx\, \alpha(x)\left(u(t,x) - u^0(x) + \int_t^T \lambda(s,x)\,ds\right) = \int_{\mathbb{R}} e^{\Sigma(x)}\left(\int_t^T ds\, \partial_x u(s,x)\right) d^\circ\left(\alpha \frac{\sigma^2}{2} e^{-\Sigma}(x)\right)$$

for every $\alpha \in \mathcal{D}(\mathbb{R})$.

PROOF. We fix a realization ω. Theorem 8.4 says that the unique solution to equation (9.1) is given by

$$u(s,x) = \int_{\mathbb{R}} u^0(y) p_{T-s}(x,y)\,dy + \int_s^T dr \int_{\mathbb{R}} \lambda(r,y) p_{T-r}(x,y)\,dy, \tag{9.2}$$

where $(p_t(x,y))$ is the density law of the solution to the martingale problem related to L at point x at time s.

Proposition 8.7(b) implies that $\partial_x u$ exists and is integrable on $]0,T[\times\mathbb{R}$.

According to Proposition 7.8, we know that

$$\bar{u}(t,z) = u(t, k^{-1}(z))$$

is a C_b^0-generalized solution to

$$\begin{aligned} \mathcal{L}^1 \bar{u} &= \bar{\lambda}, \\ \bar{u}(T,\cdot) &= \bar{u}^0, \end{aligned} \tag{9.3}$$

where

$$\bar{\lambda}(t,z) = \lambda(t, k^{-1}(z)), \qquad \bar{u}^0(z) = u^0(k^{-1}(z)).$$

On the other hand, $\bar{u}$ can be represented via (8.2) in Theorem 8.2 through fundamental solutions $(\nu_t) = (r_t(x,y))$ of

$$\partial_t \nu_t(\cdot, y) = L^1 \nu_t(\cdot, y), \qquad \nu_0(\cdot, y) = \delta_y.$$

Since the previous equation holds in the Schwarz distribution sense, by inspection, it is not difficult to show that $\bar{u}$ is a solution (in the sense of distributions) to (9.3), which means that we have the following:

$$\begin{aligned} &\int_{\mathbb{R}} \alpha(z)\left(\bar{u}^0(z) - \bar{u}(t,z)\right) dz - \tfrac{1}{2}\int_t^T ds \int_{\mathbb{R}} \alpha'(z)\, \partial_z \bar{u}(s,z) \sigma_k^2(z) \\ &\quad = \int_t^T ds \int_{\mathbb{R}} \alpha(z) \bar{\lambda}(s,z) \end{aligned} \tag{9.4}$$

for every test function $\alpha \in \mathcal{D}(\mathbb{R})$, $t \in [0,T]$. We recall, in particular, that $\partial_z \bar{u}$ is in $L^1(]0,T[\times\mathbb{R})$.

We set

$$D(t,z)=\int_t^T \partial_z\bar{u}(s,z)\,ds, \qquad \mathcal{D}(t,z)=D(t,z)\frac{\sigma_k^2(z)}{2}.$$

Expression (9.4) shows that

$$\partial_z\mathcal{D}(t,\cdot)=-\bar{u}^0+\bar{u}(t,\cdot)+\int_t^T\bar{\lambda}(s,\cdot)\,ds, \tag{9.5}$$

in the sense of distributions. So for each $t\in[0,T]$, $\mathcal{D}$ is of class C^1.

For $t\in[0,T]$ and $x\in\mathbb{R}$, we set $A(t,x)=\int_t^T\partial_x u(s,x)\,ds$, $\mathcal{A}(t,x)=A(t,x)e^{\Sigma(x)}$. We recall that

$$u(s,x)=\bar{u}(s,k(x)), \qquad \partial_x u(s,x)=\partial_x\bar{u}(s,k(x))k'(x).$$

Therefore,

$$A(t,x)=D(t,k(x))k'(x)$$

so that

$$\begin{aligned}A(t,x)&=2\mathcal{D}(t,k(x))\frac{k'(x)}{\sigma_k^2(k(x))}=\mathcal{D}(t,k(x))\frac{2}{\sigma^2(x)k'(x)}\\&=2\mathcal{D}(t,k(x))e^{-\Sigma(x)}.\end{aligned}$$

Therefore, $\mathcal{A}(t,x)=2\mathcal{D}(t,k(x))$ and so $\mathcal{A}$ is of class C^1.

Since $\partial_x\mathcal{A}(t,x)=2\partial_z\mathcal{D}(t,k(x))k'(x)$, (9.5) gives

$$\partial_x\mathcal{A}(t,x)=\left(-u^0(x)+u(t,x)+\int_t^T\lambda(s,x)\,ds\right)2\frac{e^{\Sigma}}{\sigma^2}(x). \tag{9.6}$$

Consequently,

$$u(t,x)-u^0(x)+\int_t^T\lambda(s,x)\,ds=\partial_x\left(e^{\Sigma(x)}\int_t^T ds\,\partial_x u(s,x)\right)e^{-\Sigma(x)}\frac{\sigma^2(x)}{2}.$$

We integrate the previous expression against a test function $\alpha\in\mathcal{D}(\mathbb{R})$ to obtain

$$\begin{aligned}&\int_{\mathbb{R}}dx\,\alpha(x)\left(u(t,x)-u^0(x)+\int_t^T\lambda(s,x)\,ds\right)\\&\quad=\int_{\mathbb{R}}dx\,\alpha(x)\left\{\partial_x\left(e^{\Sigma(x)}\int_t^T ds\,\partial_x u(s,x)\right)e^{-\Sigma(x)}\frac{\sigma^2(x)}{2}\right\}.\end{aligned}$$

Remark 3.1 and integration by parts for the symmetric integral provided by Remark 3.2(c) allow us to conclude the proof of the lemma. $\square$

Finally, we are able to state the theorem concerning the existence of weak solutions for the SPDE.

THEOREM 9.3. *Let λ (resp., u^0) be a random field with parameter in $(t, x) \in [0, T] \times \mathbb{R}$ (resp., $x \in \mathbb{R}$) whose paths are bounded and continuous.*

We suppose that $\sigma = 1$ and that η is a (two-sided) zero strong cubic variation process such that there are two finite and strictly positive random variables Z_1, Z_2 with $Z_1 \le e^{\eta(x)} \le Z_2$ a.s.

Let u be the random field which is ω a.s. a C_b^0-generalized solution to (9.1) for $b = \eta(\omega)$. We set $v(t, x) = u(T - t, x)$. Then v is a (weak) solution of the SPDE (1.1).

PROOF. Proposition 9.1 says that it will be enough to verify that

$$\begin{aligned} & -\int_{\mathbb{R}} \alpha(x) u(t, x)\, dx + \int_{\mathbb{R}} \alpha(x) u^0(x)\, dx \\ & \quad - \frac{1}{2} \int_{\mathbb{R}} \alpha'(x) \left(\int_t^T ds\, \partial_x u(s, x) \right) dx + \int_{\mathbb{R}} \alpha(x) \left(\int_t^T ds\, \partial_x u(s, x) \right) d^\circ \eta(x) \\ & \quad = \int_t^T ds \int_{\mathbb{R}} dx\, \alpha(x) \lambda(s, x) \end{aligned}$$

for every test function α and every $t \in [0, T]$.

After making the identification $b = \eta(\omega)$, the previous Lemma 9.2 says that

$$\begin{aligned} & \int_{\mathbb{R}} dx\, \alpha(x) \left(u(t, x) - u^0(x) + \int_t^T \lambda(s, x)\, ds \right) \\ & \quad = \int_{\mathbb{R}} e^{2\eta(x)} \left(\int_t^T ds\, \partial_x u(s, x) \right) d^\circ \left(\frac{\alpha e^{-2\eta}}{2}(x) \right). \end{aligned}$$

Since η is a zero strong cubic variation process, Proposition 3.8 implies that e^η is also a zero strong cubic variation process. Then the Itô chain rule from Proposition 3.9, applied with $F(x, \eta(x)) = \alpha(x) e^{\eta(x)}$, and Remark 3.1 say that the right-hand side of previous expression gives

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} \left(\int_t^T ds\, \partial_x u(s, x) \right) d^0 \left(\alpha e^{-2\eta(x)} \right) \\ & \quad = \frac{1}{2} \int_{\mathbb{R}} \left(\int_t^T ds\, \partial_x u(s, x) \right) e^{2\eta(x)} \left(\alpha'(x) e^{-2\eta}(x)\, dx + \alpha(x)\, d^\circ e^{-2\eta(x)} \right) \\ & \quad = \frac{1}{2} \int_{\mathbb{R}} \left(\int_t^T ds\, \partial_x u(s, x) \right) \alpha'(x)\, dx \\ & \qquad - \int_{\mathbb{R}} \left(\int_t^T ds\, \partial_x u(s, x) \right) \alpha(x)\, d^\circ \eta(x). \end{aligned}$$

This concludes the proof. $\square$

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INSTITUT GALILÉE, MATHÉMATIQUES
UNIVERSITÉ PARIS 13
99, AVENUE J.B. CLÉMENT
F-93430 VILLETANEUSE
FRANCE
E-MAIL: russo@math.univ-paris13.fr

UNIVERSITÄT BIELEFELD
FAKULTÄT FÜR MATHEMATIK
POSTFACH 10 01 31
D-33501 BIELEFELD
GERMANY
E-MAIL: trutnau@math.uni-bielefeld.de