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ON PATHWISE UNIQUENESS FOR STOCHASTIC HEAT EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

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We consider the existence and pathwise uniqueness of the stochastic heat equation with a multiplicative colored noise term on $\mathbb{R}^d$ for $d \geq 1$. We focus on the case of non-Lipschitz noise coefficients and singular spatial noise correlations. In the course of the proof a new result on Hölder continuity of the solutions near zero is established.

1. Introduction. This work is motivated by the following question: Does pathwise uniqueness hold in the parabolic stochastic p.d.e.

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x)\,dt + \sqrt{u(t,x)}\dot{W}(x,t)? \tag{1}$$

Here Δ denotes the Laplacian and $\dot{W}$ is space–time white noise on $\mathbb{R}_+ \times \mathbb{R}$. It is known that uniqueness in law holds for solutions to (1) in the appropriate space of continuous functions and such solutions are the density for one-dimensional super-Brownian motion (see, e.g., Section III.4 of [4]). One motivation for studying pathwise uniqueness is the hope that such an approach would be more robust and establish uniqueness for closely related equations in which $\sqrt{u(t,x)}$ could be replaced by $\sqrt{\gamma(u(t,x))u(t,x)}$. Such models arise as scaling limits of critical branching particle systems in which the branching rate at (t,x) is given by $\gamma(u(t,x))$. The method used to establish uniqueness in law for solutions of (1) is duality. This approach has the advantage of giving a rich toolkit for the study of solutions to (1), but the disadvantage of being highly nonrobust, although one of us was able to extend this method to powers of $u(t,x)$ between $1/2$ and 1 (see [3]).

The difficulty in proving pathwise uniqueness in (1) arises from the fact that $\sqrt{u}$ is non-Lipschitz. The above equation does have the advantage of having a diagonal form—that is, when viewed as a continuum-dimensional stochastic differential equation, there are no off-diagonal terms in the noise part of the equation

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and the diffusion coefficient for the x coordinate is a function of that coordinate alone. For finite-dimensional SDEs, this was the setting for Yamada and Watanabe's extension [14] of Itô's pathwise uniqueness results to Hölder continuous coefficients, and so an optimist may hope this approach can carry over to our infinite-dimensional setting. As we will be using their conditions later, let us recall the Yamada–Watanabe result. Let ρ be a strictly increasing function on $\mathbb{R}_+$ such that

$$\int_{0+} \rho^{-2}(x)\,dx = \infty. \tag{2}$$

Now assume that $\sigma : \mathbb{R} \to \mathbb{R}$ is such that, for all $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)| \le \rho(|x - y|). \tag{3}$$

Then pathwise uniqueness holds for solutions of the one-dimensional SDE

$$X(t) = X(0) + \int_0^t \sigma(X(t))\,dB(t), \tag{4}$$

where B is a standard Brownian motion. The square root function clearly satisfies the above hypotheses, but the infinite-dimensional setting has stymied attempts to carry the methodology over. Yamada and Watanabe's proof has been simplified (see, e.g., Theorem IX.3.5 of [6]) by the notion of the local time of a semimartingale and the fact that $u(t, x)$ will not be a semimartingale in t for x fixed (it will only be Hölder continuous of index $1/4$) would seem to be a serious obstacle in directly applying these methods.

We will not resolve the uniqueness question posed above, but will succeed in extending the above ideas to stochastic heat equations of the form

$$\frac{\partial}{\partial t} u(t, x) = \frac{1}{2}\Delta u(t, x)\,dt + \sigma\bigl(u(t, x)\bigr)\dot{W}(x, t) \tag{5}$$

for colored noises other than white, and appropriate Hölder continuous, but not necessarily Lipschitz continuous, σ. Here, u is a random function on $\mathbb{R}_+ \times \mathbb{R}^d$ and we sometimes write u_t for $u(t, \cdot)$. The coefficient σ is a real-valued continuous function on $\mathbb{R}$. It is assumed throughout this work to satisfy the following global growth condition: For all $u \in \mathbb{R}$, there exists a constant c_6 such that

$$|\sigma(u)| \le c_6(1 + |u|). \tag{6}$$

Here and elsewhere c_i and $c_{i.j}$ will denote fixed positive constants, while C will denote a positive constant which may change from line to line. The noises W considered here are Gaussian martingale measures on $\mathbb{R}_+ \times \mathbb{R}^d$ in the sense of Walsh [13]. W is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and $W_t(\phi) = \int_0^t \int_{\mathbb{R}^d} \phi(s, x) W(dx\,ds)$ is an $\mathcal{F}_t$-martingale for $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, the space of compactly supported, infinitely differentiable functions on $\mathbb{R}_+ \times \mathbb{R}^d$. If $W(\phi) = W_\infty(\phi)$, W can be characterized by its covariance functional

$$
\begin{aligned}
J_k(\phi, \psi) &:= \mathbb{E}[W(\phi)W(\psi)] \\
&= \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(s,x)k(x,y)\psi(s,y)\,dx\,dy\,ds,
\end{aligned}
\tag{7}
$$

for $\phi, \psi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$. We call the function $k: \mathbb{R}^{2d} \to \mathbb{R}$ the correlation kernel of W. Some sufficient conditions for the existence of a martingale measure W corresponding to k are that J_k is symmetric, positive definite and continuous. Thus, necessarily, $k(x,y) = k(y,x)$ for all $x, y \in \mathbb{R}^d$. Continuity on C_c^∞ is implied, for example, if k is integrable on compact sets. We also note that a general class of martingale measures, spatially homogeneous noises, can be described by (7), where $k(x,y) = \tilde{k}(x-y)$.

If $\sigma(u) = u$, then equation (5) arises as the diffusion limit of super-Brownian motion in $\mathbb{R}^d$ where the offspring law depends on a random environment, whose spatial correlation is described by k. For k bounded, this was proven in [11]. More general coefficients σ may be thought of as reflecting an additional dependence of the offspring law on the local particle density.

If k is bounded, Viot [12] proved pathwise uniqueness for solutions to (5) on bounded domains of $\mathbb{R}^d$ for $\sigma(u) = \sqrt{u(1-u)_+}$, where the subscript indicates that the positive part of the function is taken. We will extend this result to our setting for solutions of (5) on $\mathbb{R}^d$ with bounded k in Theorem 1.6 below. Note that white noise will correspond to the case where we set $\tilde{k}$ equal to the generalized function δ_0 in the above. Our main result (Theorem 1.4 below) will interpolate between these settings and establish pathwise uniqueness for colored noises for which the correlation is bounded by a Riesz kernel,

$$
|k(x,y)| \le c_8[|x-y|^{-\alpha} + 1] \qquad \text{for all } x, y \in \mathbb{R}^d \text{ and appropriate } \alpha > 0. \tag{8}
$$

In order to formulate a condition on the singularity of k and relate our conditions to those in the literature, we define the spectral measure, μ, of a spatially homogeneous covariance kernel $\tilde{k}$:

$$
\int_{\mathbb{R}^d} \tilde{k}(x)\phi(x)\,dx = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\mu(d\xi) \tag{9}
$$

for any rapidly decreasing test function ϕ where $\mathcal{F}\phi(\xi) = \int_{\mathbb{R}^d} \exp(-2i\pi\xi \cdot x)\phi(x)\,dx$ is the Fourier transform. Later on we will assume μ to be a tempered measure fulfilling, for some $\eta \in [0,1]$,

$$
\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty. \tag{10}
$$

To relate (8) with condition (10) used in the literature, we introduce the following:

$(A)_\eta$: ($\eta > 0$) W is a Gaussian noise with correlation kernel $|k(x, y)| \le c_{10}\tilde{k}(x - y)$, $x, y \in \mathbb{R}^d$ for some symmetric, locally bounded and positive definite kernel $\tilde{k}$ whose spectral measure satisfies (10).

$(A)_0$: W is a Gaussian noise and its correlation kernel k is bounded.

REMARK 1.1. Note that (8) implies $(A)_\eta$ for $\alpha \in (0, 2\eta \wedge d)$: Here, $\tilde{k}(x) = |x|^{-\alpha} + 1$ and the spectral measure is of the form $\mu(d\xi) = c_{1.1}[|\xi|^{\alpha-d} d\xi + \delta_0(d\xi)]$. Hence, condition (10) is satisfied if and only if $\alpha \in (0, 2\eta \wedge d)$ (see Chapter V, Lemma 2(a) of [9]). Note also that the positive definite spatially homogeneous kernels $k_\alpha(x, y) = |x - y|^{-\alpha}$ give a natural family of kernels for which our results will hold.

In order to make sense of the formal equation (5), we use the variation of constants form of solutions: Denote by p the d-dimensional heat kernel

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right). \tag{11}$$

A stochastic process $u : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, which is jointly measurable and $\mathcal{F}_t$-adapted, is said to be a solution to the stochastic heat equation (5) in the variation of constants sense with respect to the martingale measure W, defined on Ω, and initial condition u_0, if for each $t \ge 0$, a.s. for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned} u(t, x) = &\int_{\mathbb{R}^d} p_t(x - y) u_0(y)\, dy \\ &+ \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y)\sigma\big(u(s, y)\big) W(dy\, ds). \end{aligned} \tag{12}$$

Solutions to (12) have been well studied in the case where σ is Lipschitz continuous in u. A sufficient condition for strong existence and uniqueness of solutions is given by $(A)_\eta$ for $\eta \le 1$ see [1] (see also Theorem A.1 in the Appendix) and [5]. Hölder continuity of the sample paths was established by Sanz-Solé and Sarrà [8] if $\eta < 1$ (cf. Lemma A.4 in the Appendix).

To state the main results, we introduce some notation, which will be used throughout this work: We write $C(\mathbb{R}^d)$ for the space of continuous functions on $\mathbb{R}^d$. A superscript k, respectively ∞, indicates that functions are in addition k times, respectively infinitely often, continuously differentiable. A subscript b, respectively c, indicates that they are also bounded, respectively have compact support. We also define

$$\|f\|_{\lambda,\infty} := \sup_{x \in \mathbb{R}^d} |f(x)| e^{-\lambda|x|},$$

set $C_{\text{tem}} := \{f \in C(\mathbb{R}^d), \|f\|_{\lambda,\infty} < \infty \text{ for any } \lambda > 0\}$ and endow it with the topology induced by the norms $\|\cdot\|_{\lambda,\infty}$ for $\lambda > 0$. That is, $f_n \to f$ in C_{tem} iff

$\lim_{n\to\infty} \|f - f_n\|_{\lambda,\infty} = 0$ for all $\lambda > 0$. For $I \subset \mathbb{R}_+$, let $C(I, E)$ be the space of all continuous functions on I taking values in a topological space E, endowed with the topology of uniform convergence on compact subsets of I. A stochastically weak solution to (12) is a solution on some filtered space with respect to some noise W, that is, the noise and space is not specified in advance.

With this notation we can state the following standard existence result whose proof is outlined in the Appendix:

THEOREM 1.2. *Let $u_0 \in C_{\mathrm{tem}}$, and let σ be a continuous function satisfying the growth bound* (6). *Assume that* (8) *holds for some $\alpha \in (0, 2 \wedge d)$. Then there exists a stochastically weak solution to* (12) *with sample paths a.s. in $C(\mathbb{R}_+, C_{\mathrm{tem}})$.*

REMARK 1.3. (a) The proof in fact only requires that $(A)_\eta$ hold for some $\eta \in [0, 1)$, a condition which follows from the above bound on k by Remark 1.1.

(b) In the case where the correlation kernel is bounded, existence has been shown for more general initial conditions and solution spaces in [11]: Define $L^p_\lambda(\mathbb{R}^d) := L^p(\mathbb{R}^d, e^{-\lambda|x|}\,dx)$ and denote the associated norm by $\|\cdot\|_{\lambda,p}$. Then if $\mathbb{E}(\|u_0\|^p_{\lambda,p}) < \infty$, for some $p > 2$ and $\lambda > 0$, there exists a stochastically weak solution $u \in C(\mathbb{R}_+, L^p_\lambda(\mathbb{R}^d))$ to (12) which satisfies

$$\mathbb{E}\biggl(\sup_{0\le t\le T} \|u(t,\cdot)\|^p_{\lambda,p}\biggr) < \infty \qquad \text{for any } T > 0. \tag{13}$$

We say pathwise uniqueness holds for solutions of (12) in $C(\mathbb{R}_+, C_{\mathrm{tem}})$ if, for every $u_0 \in C_{\mathrm{tem}}$, any two solutions to (12) with sample paths a.s. in $C(\mathbb{R}_+, C_{\mathrm{tem}})$ must be equal with probability 1. For Lipschitz continuous σ, it is easy to modify Theorem 13 of [1] and Theorem 2.1 of [8] to get pathwise uniqueness and Hölder continuity of solutions for $\alpha < 2 \wedge d$. Also, Theorem 11 and Remark 12 of [1] show that function-valued solutions will not exist for $\alpha > 2 \wedge d$. Here then is our main result—it holds in any spatial dimension d:

THEOREM 1.4. *Assume that, for some $\alpha \in (0, 1)$, $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies* (6), *is Hölder continuous of index γ for some $\gamma \in (\frac{1+\alpha}{2}, 1]$, and*

$$|k(x, y)| \le c_{1.4}[|x - y|^{-\alpha} + 1] \qquad \textit{for all } x, y \in \mathbb{R}^d.$$

Then pathwise uniqueness holds for solutions of (12) *in $C(\mathbb{R}_+, C_{\mathrm{tem}})$.*

REMARK 1.5. The Hölder condition on σ may be weakened to the local Hölder condition: For any $K > 0$, there exists $L = L(K)$ such that

$$|\sigma(u) - \sigma(v)| \le L(|u - v|^\gamma + |u - v|) \qquad \forall u, v : |u|, |v| \le K,$$

where γ is as in Theorem 1.4. The required modifications in the proof are elementary.

It also looks possible to weaken the pointwise bound on k to the following condition:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)h(x)h(y)\,dx\,dy \le c\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [|x-y|^{-\alpha}+1]|h(x)\|h(y)|\,dx\,dy,$$

for all h in an appropriate class of functions decaying to 0 at infinity. We have used the stronger pointwise bound as it is more convenient and explicit.

In the above result there is a trade-off between the Hölder continuity of σ and the singularity of the covariance kernel of the noise. For $d=1$, letting $\alpha \to 1-$ and renormalizing will give white noise. More specifically, if $\tilde{k}_\alpha(x-y) = \frac{1-\alpha}{2}|x-y|^{-\alpha}$, then for $\phi, \psi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$, $\lim_{\alpha\to 1-} J_{\tilde{k}_\alpha}(\phi,\psi) = \int_0^\infty \int \phi(s,x)\psi(s,x)\,dx\,ds$. The Hölder condition in Theorem 1.4 approaches Lipschitz continuity. (As $\tilde{k}$ should be locally integrable, we cannot expect to take $\alpha = 1$.) Hence, although the result does not say anything about white noise itself, it at least coincides with the known Lipschitz conditions which imply pathwise uniqueness in the limit as α approaches 1. The same cannot be said for higher dimensions. Here, the aforementioned results of Dalang, and Sanz-Solé and Sarrá show that, for $\alpha < 2$, we will have pathwise unique continuous solutions when the coefficients are Lipschitz continuous. Unfortunately, our hypotheses become vacuous in the above uniqueness theorem when α exceeds 1 and so we believe our condition on the Hölder index in Theorem 1.4 is nonoptimal in dimensions greater than 1. At the other end of the scale, we see that as α approaches 0, the required Hölder exponent approaches 1/2, the critical power in the one-dimensional results of Yamada and Watanabe. In fact, if the covariance kernel is bounded, we can weaken the Hölder condition on σ to precisely the Yamada–Watanabe condition (2), (3) introduced above. Again, the result holds in any spatial dimension.

THEOREM 1.6. *Assume that $(A)_0$ holds and that $\sigma:\mathbb{R}\to\mathbb{R}$ satisfies* (6) *and* (3). *Then pathwise uniqueness holds for solutions of* (12) *in* $C(\mathbb{R}_+, C_{\mathrm{tem}})$.

REMARK 1.7. (a) The conclusions of Theorems 1.2, 1.4 and 1.6 remain valid if we allow for an additional drift term in the heat equation. More precisely, we can add a term of the form $\int_0^t \int p_{t-s}(x-y) f(u(s,y))\,dy\,ds$ to the right-hand side of (12), where f satisfies the growth bound (6), is continuous in the existence theorem, Theorem 1.2, and is Lipschitz continuous for the uniqueness results, Theorems 1.4 and 1.6. The additional arguments are standard.

(b) The pathwise uniqueness conclusions of Theorems 1.4 and 1.6, and weak existence given by Theorem 1.2 imply the existence of a strong solution to (12), that is, a solution which is adapted with respect to the canonical filtration of the noise W. The proof follows just as in the classical SDE argument of Yamada and Watanabe (see, e.g., Theorem IX.1.7 of [6]).

(c) Theorem 1.6 holds true if we consider solutions with paths in $C(\mathbb{R}_+, L^p_\lambda(\mathbb{R}^d))$ as was done in Viot's work [12]. In fact, the arguments given in Sections 2 and 3 remain the same in this case. The only difference is that a bit more care has to be taken to justify some of the convergences as the solutions are not necessarily continuous. But this can be done in a straightforward way.

The proof of our pathwise uniqueness theorems will require some moment bounds for *arbitrary* continuous C_{tem}-valued solutions to the equation (12). Let $S_t\phi(x) = \int p_t(y-x)\phi(y)\,dy$. The following result will be proved in the Appendix.

PROPOSITION 1.8. *Let $u_0 \in C_{\mathrm{tem}}$, and let σ be a continuous function satisfying the growth bound* (6). *Assume that* (8) *holds for some $\alpha \in (0, 2\wedge d)$. Then any solution $u \in C(\mathbb{R}_+, C_{\mathrm{tem}})$ to* (12) *has the following properties*:

(a) *For any $T, \lambda > 0$ and $p \in (0,\infty)$*,

$$\mathbb{E}\Big(\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d} |u(t,x)|^p e^{-\lambda|x|}\Big) < \infty. \tag{14}$$

(b) *For any $\xi \in (0, 1-\alpha/2)$, the process $u(\cdot,\cdot)$ is a.s. uniformly Hölder continuous on compacts in $(0,\infty)\times\mathbb{R}^d$, and the process $Z(t,x) \equiv u(t,x) - S_t u_0(x)$ is uniformly Hölder continuous on compacts in $[0,\infty)\times\mathbb{R}^d$, both with Hölder coefficients $\frac{\xi}{2}$ in time and ξ in space.*

Moreover, for any $T, R > 0$, and $0 \le t, t' \le T, x, x' \in \mathbb{R}^d$ such that $|x-x'| < R$, as well as $p \in [2,\infty)$ and $\xi \in (0, 1-\alpha/2)$, there exists a constant $c_{15} = c_{15}(T, p, \lambda, R, \xi)$ such that

$$\mathbb{E}\big(|Z(t,x) - Z(t',x')|^p e^{-\lambda|x|}\big) \le c_{15}\big(|t-t'|^{(\xi/2)p} + |x-x'|^{\xi p}\big). \tag{15}$$

REMARK 1.9. The proof of the above will only require $(A)_\eta$ for some $\eta \in [0,1)$, a condition which is implied by the hypotheses above (see Remark 1.1). In this case we should take $\xi \in (0, 1-\eta)$ in (b) as is done in the proof in the Appendix.

It is straightforward to show that, under the hypotheses of Theorem 1.2, solutions to (12) with continuous C_{tem}-valued paths are also solutions to the heat equation in its distributional form for suitable test functions Φ. More specifically, for $\Phi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} u(t,x)\Phi(x)\,dx \\
&\quad = \int_{\mathbb{R}^d} u_0(x)\Phi(x)\,dx + \int_0^t \int_{\mathbb{R}^d} u(s,x)\tfrac{1}{2}\Delta\Phi(x)\,dx\,ds \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} \sigma\big(u(s,x)\big)\Phi(x)W(dx\,ds) \qquad \forall t \ge 0 \text{ a.s.}
\end{aligned} \tag{16}$$

In fact, given an appropriate class of test functions, the two notions of solutions (12) and (16) are equivalent. In our case, $\{\Phi \in C^\infty(\mathbb{R}^d) : \Phi(x) \le Ce^{-\lambda|x|}$ for some $C > 0$ and all $x \in \mathbb{R}^d\}$ is a suitable class of test functions. For the details of the proof, we refer to [10], Proposition 3.2.3. There, the setting is a bit different as it works in the setting of Remark 1.3 with bounded k. However, the arguments do not change for the case of k unbounded as long as the stochastic integral in (16) is well defined, which can easily be checked.

We now briefly outline the proof of our main result (Theorem 1.4) and the contents of the paper. To emulate Yamada and Watanabe, consider a pair of solutions, u^1 and u^2, to (12), set $\tilde{u} = u^1 - u^2$, and use (16) and Itô's lemma to derive a semimartingale decomposition for $\int_0^t \int |\tilde{u}(s,x)|\Psi_s(x)\,dx\,ds$, where $\Psi_s(x) \ge 0$ is a smooth test function. This involves approximating $|\tilde{u}(s,x)|$ by $\psi_n(\langle \tilde{u}_s, \Phi_m(\cdot - x)\rangle)$ as $m, n \to \infty$, where $\{\psi_n\}$ are smooth functions approximating the absolute value function as in [14], and $\{\Phi_m\}$ is a smooth approximate identity. In Section 2 the martingale and standard drift terms which arise are handled in a relatively straightforward manner in a general setting, including that of both Theorems 1.6 and 1.4 (see Lemma 2.2). Here we may let $m, n \to \infty$ in any manner. The problematic term, called $I_3^{m,n}$ below, is the one arising from the $\psi_n''/2$ term in using Itô's lemma and so will involve the quadratic variation of the martingale term. In the context of the Yamada–Watanabe proof, it is the one which leads to the local time at 0 of the difference of two solutions to the SDE, $L_t^0(X^1 - X^2)$. There, this term is shown to be 0 using the modulus of continuity of σ and the regularity of the sample paths of the solutions (the latter implicitly, as one needs the stochastic calculus associated with continuous semimartingales).

In Section 3 $I_3^{m,n}$ is shown to approach 0 if we first let $m \to \infty$ and then $n \to \infty$ in the simpler context of Theorem 1.6. This leads to

$$\int \mathbb{E}(|\tilde{u}(t,x)|)\Psi_t(x)\,dx \le \int_0^t \int \mathbb{E}(|\tilde{u}(s,x)|)\left|\tfrac{1}{2}\Delta\Psi_s(x) + \dot{\Psi}_s(x)\right|dx\,ds, \tag{17}$$

from which $\tilde{u} = 0$ follows easily by taking $\Psi_s(x) = \int p_{t-s}(y-x)\phi(x)\,dx$. We feel the ease of this argument is partly related to the greater path regularity $\tilde{u}$ in this context—it is Hölder continuous in space with index $1 - \varepsilon$ and in time with index $\frac{1}{2} - \varepsilon$ by results of Sanz-Solé and Sarra (see [8] and Lemma A.4 below).

In Section 4 we complete the proof of Theorem 1.4 by showing $\lim_{n\to\infty} I_3^{m_n,n} = 0$ for a judicious choice of m_n, which again leads to (17). In this setting $\tilde{u}(t,x)$ is only Hölder continuous of index $\frac{1-\alpha/2}{2} - \varepsilon$ in time and $1 - \frac{\alpha}{2} - \varepsilon$ in space (see Lemma A.4 or [8]) and this additional irregularity makes the argument more involved. In the Yamada–Watanabe context, the key fact that $L_t^0(X^1 - X^2) = 0$ reflects the fact that the solutions must separate "slowly" if they do so at all. In our setting we will argue along similar lines by showing that $\tilde{u}(t,x)$ is more regular in (t,x) at small values of $\tilde{u}(t,x)$, that is, when the solutions are close (see Theorem 4.1). For example, they will be Hölder of index $\frac{1-\alpha/2}{1-\gamma} \wedge 1 - \varepsilon$ in space near space–time points where $\tilde{u}$ is sufficiently small (see Corollary 4.2). Theorem 4.1 is proved in Section 5 and is the key to the proof of Theorem 1.4 which is completed in Section 4. This improved modulus of continuity result may be of independent interest. In fact, a similar result to Theorem 4.1 was derived independently by Mueller and Tribe in the context of white noise, in their ongoing work on the zero set of solutions to (1). The continuity results of Sanz-Solé and Sarra [8] and the factorization method they use (see [2]) play a critical role in the proof of Theorem 4.1 in our colored noise setting. The Appendix includes the proofs of the weak existence theorem (Theorem 1.2) and the required moment estimates (Proposition 1.8).

2. Some auxiliary results. Let ρ be as in (2). An elementary argument shows that $\int_{0+}(\rho(x)+\sqrt{x})^{-2}\,dx=+\infty$ [e.g., consider $\liminf_{x\downarrow 0}\rho^{-2}(x)x\ge 1$ and $\liminf_{x\downarrow 0}\rho^{-2}(x)x<1$ separately]. As we will be using ρ as a modulus of continuity [see (3)], we may replace ρ with $\rho(x)+\sqrt{x}$ and so assume

$$\rho(x)\ge\sqrt{x}. \tag{18}$$

As in the proof of Yamada and Watanabe [14], we may define a sequence of functions ϕ_n in the following way. First, let $a_n\downarrow 0$ be a strictly decreasing sequence such that $a_0=1$, and

$$\int_{a_n}^{a_{n-1}}\rho^{-2}(x)\,dx=n. \tag{19}$$

Second, we define functions $\psi_n\in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}(\psi_n)\subset(a_n,a_{n-1})$, and that

$$0\le\psi_n(x)\le\frac{2\rho^{-2}(x)}{n}\le\frac{2}{nx}\qquad\text{for all } x\in\mathbb{R}\text{ as well as }\int_{a_n}^{a_{n-1}}\psi_n(x)\,dx=1. \tag{20}$$

Finally, set

$$\phi_n(x)=\int_0^{|x|}\int_0^y\psi_n(z)\,dz\,dy. \tag{21}$$

From this, it is easy to see that $\phi_n(x)\uparrow|x|$ uniformly in $x\ge 0$. Note that each ψ_n and, thus, also each ϕ_n, is identically zero in a neighborhood of zero. This implies that $\phi_n\in C^\infty(\mathbb{R})$ despite the absolute value in its definition. We have

$$\phi_n'(x)=\operatorname{sgn}(x)\int_0^{|x|}\psi_n(y)\,dy, \tag{22}$$

$$\phi_n''(x)=\psi_n(|x|). \tag{23}$$

Thus, $|\phi_n'(x)|\le 1$, and $\int\phi_n''(x)h(x)\,dx\to h(0)$ for any function h which is continuous at zero.

Now let u^1 and u^2 be two solutions of (12) with sample paths in $C(\mathbb{R}_+, C_{\mathrm{tem}})$ a.s., with the same initial condition, $u^1(0) = u^2(0) = u_0 \in C_{\mathrm{tem}}$, and the same noise W in either the setting of Theorem 1.6 or Theorem 1.4. We proceed assuming Proposition 1.8 which will be derived in the Appendix. Define $\tilde{u} \equiv u^1 - u^2$. Let $\Phi \in C_c^\infty(\mathbb{R}^d)$ be a positive function with $\mathrm{supp}(\Phi) \subset B(0,1)$ (the open ball centered at 0 with radius 1) such that $\int_{\mathbb{R}^d} \Phi(x)\,dx = 1$ and set $\Phi_x^m(y) = m^d \Phi(m(x-y))$. Let $\langle \cdot, \cdot \rangle$ denote the scalar product on $L^2(\mathbb{R}^d)$. By applying Itô's formula to the semimartingale $\langle \tilde{u}_t, \Phi_x^m \rangle$ of (16), it follows that

$$
\begin{aligned}
&\phi_n(\langle \tilde{u}_t, \Phi_x^m \rangle) \\
&\quad = \int_0^t \int_{\mathbb{R}^d} \phi_n'(\langle \tilde{u}_s, \Phi_x^m \rangle)\bigl(\sigma\bigl(u^1(s,y)\bigr) - \sigma\bigl(u^2(s,y)\bigr)\bigr)\Phi_x^m(y) W(dy\,ds) \\
&\qquad + \int_0^t \phi_n'(\langle \tilde{u}_s, \Phi_x^m \rangle)\langle \tilde{u}_s, \tfrac{1}{2}\Delta \Phi_x^m \rangle\,ds \\
&\qquad + \tfrac{1}{2}\int_0^t \int_{\mathbb{R}^{2d}} \psi_n(|\langle \tilde{u}_s, \Phi_x^m \rangle|) \\
&\qquad\qquad \times \bigl(\sigma\bigl(u^1(s,y)\bigr) - \sigma\bigl(u^2(s,y)\bigr)\bigr)\bigl(\sigma\bigl(u^1(s,z)\bigr) - \sigma\bigl(u^2(s,z)\bigr)\bigr) \\
&\qquad\qquad \times \Phi_x^m(y)\Phi_x^m(z) k(y,z)\,dy\,dz\,ds.
\end{aligned}
$$

We integrate this function of x against another nonnegative test function $\Psi \in C_c^\infty([0,t] \times \mathbb{R}^d)$. Assume

$$
\Gamma \equiv \{x : \Psi_s(x) > 0\ \exists s \le t\} \subset B(0,K) \qquad \text{for some } K > 0. \tag{24}
$$

We then obtain by the classical and stochastic version of Fubini's theorem, and arguing as in the proof of Proposition II.5.7 of [4] to handle the time dependence in ψ, that, for any $t \ge 0$,

$$
\begin{aligned}
&\langle \phi_n(\langle \tilde{u}_t, \Phi_\cdot^m \rangle), \Psi_t \rangle \\
&\quad = \int_0^t \int_{\mathbb{R}^d} \langle \phi_n'(\langle \tilde{u}_s, \Phi_\cdot^m \rangle)\Phi_\cdot^m(y), \Psi_s \rangle \bigl(\sigma\bigl(u^1(s,y)\bigr) - \sigma\bigl(u^2(s,y)\bigr)\bigr) W(dy\,ds) \\
&\qquad + \int_0^t \langle \phi_n'(\langle \tilde{u}_s, \Phi_\cdot^m \rangle)\langle \tilde{u}_s, \tfrac{1}{2}\Delta \Phi_\cdot^m \rangle, \Psi_s \rangle\,ds \\
&\qquad + \tfrac{1}{2}\int_0^t \int_{\mathbb{R}^{3d}} \psi_n(|\langle \tilde{u}_s, \Phi_x^m \rangle|) \\
&\qquad\qquad \times \bigl(\sigma\bigl(u^1(s,y)\bigr) - \sigma\bigl(u^2(s,y)\bigr)\bigr)\bigl(\sigma\bigl(u^1(s,z)\bigr) - \sigma\bigl(u^2(s,z)\bigr)\bigr) \\
&\qquad\qquad \times \Phi_x^m(y)\Phi_x^m(z) k(y,z)\,dy\,dz\,\Psi_s(x)\,dx\,ds \\
&\qquad + \int_0^t \langle \phi_n(\langle \tilde{u}_s, \Phi_\cdot^m \rangle), \dot{\Psi}_s \rangle\,ds \\
&\quad \equiv I_1^{m,n}(t) + I_2^{m,n}(t) + I_3^{m,n}(t) + I_4^{m,n}(t).
\end{aligned} \tag{25}
$$

We need a calculus lemma. For $f \in C^2(\mathbb{R}^d)$, let $\|D^2 f\|_\infty = \max_i \|\frac{\partial^2 f}{\partial x_i^2}\|_\infty$.

LEMMA 2.1. *Let $f \in C_c^2(\mathbb{R}^d)$ be nonnegative and not identically zero. Then*

$$\sup\left\{\left(\frac{\partial f}{\partial x_i}(x)\right)^2 f(x)^{-1} : f(x) > 0\right\} \le 2\|D^2 f\|_\infty.$$

PROOF. Assume first $d = 1$. Choose x so that $f(x)|f'(x)| > 0$. Without loss of generality, assume $f'(x) > 0$. Let

$$x_1 = \sup\{x' < x : f'(x') = 0\} \in (-\infty, x).$$

By the Cauchy (or generalized mean value) theorem, there is an $x_2 \in (x_1, x)$ so that

$$\big(f'(x)^2 - f'(x_1)^2\big)f'(x_2) = \big(f(x) - f(x_1)\big)\frac{d((f')^2)}{dx}(x_2)$$

and, as $f'(x_2) > 0$, we get

$$f'(x)^2 = \big(f(x) - f(x_1)\big)2f''(x_2).$$

Since f is strictly increasing on (x_1, x), and $f(x_1) \ge 0$,

$$\frac{f'(x)^2}{f(x)} \le \frac{f'(x)^2}{f(x) - f(x_1)} \le 2\|f''\|_\infty.$$

For the d-dimensional case, assume x satisfies $f(x) > 0$ and let e_i be the ith unit basis vector. Now apply the one-dimensional result to $g(t) = f(x + te_i)$, $t \in \mathbb{R}$, at $t = 0$. $\square$

We now consider the expectation of expression (25) stopped at a stopping time T, which we will specify later on. For all the terms except $I_3^{m,n}$, we can give a unified treatment for the settings of both Theorems 1.4 and 1.6.

LEMMA 2.2. *For any stopping time T and constant $t \ge 0$, we have the following*:

(a)

$$\mathbb{E}\big(I_1^{m,n}(t \wedge T)\big) = 0 \qquad \textit{for all } m, n. \tag{26}$$

(b)

$$\limsup_{m,n\to\infty} \mathbb{E}\big(I_2^{m,n}(t \wedge T)\big) \le \mathbb{E}\left(\int_0^{t\wedge T} \int_{\mathbb{R}} |\tilde{u}(s,x)| \tfrac{1}{2}\Delta\Psi_s(x)\, dx\, ds\right). \tag{27}$$

(c)

$$\lim_{m,n\to\infty}\mathbb{E}\big(I_4^{m,n}(t\wedge T)\big)=\mathbb{E}\bigg(\int_0^{t\wedge T}|\tilde{u}(s,x)|\dot{\Psi}_s(x)\,ds\bigg). \tag{28}$$

PROOF. (a) Let $g_{m,n}(s,y)=\langle\phi_n'(\langle\tilde{u}_s,\Phi_{\cdot}^m\rangle)\Phi_{\cdot}^m(y),\Psi_s\rangle$. Note first that $I_1^{m,n}(t\wedge T)$ is a continuous local martingale with square function

$$\begin{aligned}\langle I_1^{m,n}\rangle_{t\wedge T}&=\int_0^{t\wedge T}\int\int g_{m,n}(s,y)g_{m,n}(s,z)\big(\sigma\big(u^1(s,y)\big)-\sigma\big(u^2(s,y)\big)\big)\\&\qquad\times\big(\sigma\big(u^1(s,z)\big)-\sigma\big(u^2(s,z)\big)\big)k(y,z)\,dy\,dz\,ds\\&\le C\int_0^{t\wedge T}\int\int|g_{m,n}(s,y)||g_{m,n}(s,z)|\big(|u^1(s,y)|+|u^2(s,y)|+1\big)\\&\qquad\times\big(|u^1(s,z)|+|u^2(s,z)|+1\big)(|z-y|^{-\alpha}+1)\,dy\,dz\,ds.\end{aligned}$$

An easy calculation shows that $|g_{m,n}(s,y)|\le\|\Psi\|_\infty\mathbb{1}(|y|\le K+1)$, where K is defined in (24). Now use Hölder's inequality and (14) to conclude that

$$\begin{aligned}&\mathbb{E}(\langle I_1^{m,n}\rangle_{t\wedge T})\\&\quad\le C\int_0^t\int\int\mathbb{1}(|y|\le K+1)\mathbb{1}(|z|\le K+1)(|y-z|^{-\alpha}+1)\,dy\,dz\,ds<\infty\\&\qquad\forall t>0.\end{aligned}$$

This shows $I_1^{m,n}(t\wedge T)$ is a square integrable martingale and so has mean 0, as required.

(b) In order to rewrite $I_2^{m,n}$, we note that both $\phi_n'(\langle\tilde{u}_s,\Phi_{\cdot}^m\rangle)$, as well as $\langle\tilde{u}_s,\frac{1}{2}\Delta\Phi_{\cdot}^m\rangle$, are in $C^\infty(\mathbb{R}^d)$ a.s. This follows from the infinite differentiability of the test functions ϕ_n and Φ and from (14). Denote by Δ_x the Laplacian acting with respect to x. Since $\tilde{u}_s$ is locally integrable and Φ smooth, we have, for $|x|\le K$,

$$\begin{aligned}\int_{\mathbb{R}^d}\tilde{u}(s,y)\tfrac{1}{2}\Delta_y\Phi^m(x-y)\,dy&=\int_{\mathbb{R}^d}\tilde{u}(s,y)\tfrac{1}{2}\Delta_x\Phi^m(x-y)\,dy\\&=\tfrac{1}{2}\Delta_x\int_{\mathbb{R}^d}\tilde{u}(s,y)\Phi^m(x-y)\,dy,\end{aligned}\tag{29}$$

for all m. This implies, for any $t\ge 0$,

$$\begin{aligned}I_2^{m,n}(t)&=\int_0^t\int_{\mathbb{R}^d}\phi_n'(\langle\tilde{u}_s,\Phi_x^m\rangle)\frac{1}{2}\Delta_x(\langle\tilde{u}_s,\Phi_x^m\rangle)\Psi_s(x)\,dx\,ds\\&=-\sum_{i=1}^d\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\frac{\partial}{\partial x_i}\big(\phi_n'(\langle\tilde{u}_s,\Phi_x^m\rangle)\big)\frac{\partial}{\partial x_i}(\langle\tilde{u}_s,\Phi_x^m\rangle)\Psi_s(x)\,dx\,ds\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{d}\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\phi_n'(\langle\tilde{u}_s,\Phi_x^m\rangle)\frac{\partial}{\partial x_i}(\langle\tilde{u}_s,\Phi_x^m\rangle)\frac{\partial}{\partial x_i}\Psi_s(x)\,dx\,ds\\
&=-\sum_{i=1}^{d}\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\psi_n(|\langle\tilde{u}_s,\Phi_x^m\rangle|)\left(\frac{\partial}{\partial x_i}\langle\tilde{u}_s,\Phi_x^m\rangle\right)^2\Psi_s(x)\,dx\,ds\\
&\quad-\sum_{i=1}^{d}\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\phi_n'(\langle\tilde{u}_s,\Phi_x^m\rangle)\frac{\partial}{\partial x_i}(\langle\tilde{u}_s,\Phi_x^m\rangle)\frac{\partial}{\partial x_i}\Psi_s(x)\,dx\,ds\\
&=-\sum_{i=1}^{d}\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\psi_n(|\langle\tilde{u}_s,\Phi_x^m\rangle|)\left(\frac{\partial}{\partial x_i}\langle\tilde{u}_s,\Phi_x^m\rangle\right)^2\Psi_s(x)\,dx\,ds\\
&\quad+\sum_{i=1}^{d}\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\psi_n(\langle\tilde{u}_s,\Phi_x^m\rangle)\frac{\partial}{\partial x_i}(\langle\tilde{u}_s,\Phi_x^m\rangle)\langle\tilde{u}_s,\Phi_x^m\rangle\frac{\partial}{\partial x_i}\Psi_s(x)\,dx\,ds\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\phi_n'(\langle\tilde{u}_s,\Phi_x^m\rangle)\langle\tilde{u}_s,\Phi_x^m\rangle\frac{1}{2}\Delta\Psi_s(x)\,dx\,ds\\
&=\int_0^t I_{2,1}^{m,n}(s)+I_{2,2}^{m,n}(s)+I_{2,3}^{m,n}(s)\,ds.
\end{aligned}
$$

Above, we have used that $\phi_n''=\psi_n$ and we have repeatedly used integration by parts, the product rule as well as the chain rule on $\phi_n'(\langle\tilde{u}_s,\Phi_x^m\rangle)$. In order to deal with the various parts of $I_2^{m,n}$, we will first jointly consider $I_{2,1}^{m,n}$ and $I_{2,2}^{m,n}$. For fixed s and $i=1,\dots,d$, we define, a.s.,

$$
\begin{aligned}
A_i^s&=\left\{x:\left(\frac{\partial}{\partial x_i}\langle\tilde{u}_s,\Phi_x^m\rangle\right)^2\Psi_s(x)\le\langle\tilde{u}_s,\Phi_x^m\rangle\frac{\partial}{\partial x_i}\langle\tilde{u}_s,\Phi_x^m\rangle\frac{\partial}{\partial x_i}\Psi_s(x)\right\}\\
&\qquad\cap\{x:\Psi_s(x)>0\}\\
&=A_i^{+,s}\cup A_i^{-,s}\cup A_i^{0,s},
\end{aligned}
$$

where

$$
\begin{aligned}
A_i^{+,s}&=A_i^s\cap\left\{\frac{\partial}{\partial x_i}\langle\tilde{u}_s,\Phi_x^m\rangle>0\right\},\\
A_i^{-,s}&=A_i^s\cap\left\{\frac{\partial}{\partial x_i}\langle\tilde{u}_s,\Phi_x^m\rangle<0\right\},\\
A_i^{0,s}&=A_i^s\cap\left\{\frac{\partial}{\partial x_i}\langle\tilde{u}_s,\Phi_x^m\rangle=0\right\}.
\end{aligned}
$$

On $A_i^{+,s}$ we have

$$
0<\left(\frac{\partial}{\partial x_i}\langle\tilde{u}_s,\Phi_x^m\rangle\right)\Psi_s(x)\le\langle\tilde{u}_s,\Phi_x^m\rangle\frac{\partial}{\partial x_i}\Psi_s(x),
$$

and therefore, for any $t \geq 0$,

$$\begin{aligned}
&\int_0^t \int_{A_i^{+,s}} \psi_n(|\langle \tilde{u}_s, \Phi_x^m \rangle|) \langle \tilde{u}_s, \Phi_x^m \rangle \frac{\partial}{\partial x_i} \Psi_s(x) \frac{\partial}{\partial x_i} \langle \tilde{u}_s, \Phi_x^m \rangle \, dx \, ds \\
&\quad \leq \int_0^t \int_{A_i^{+,s}} \psi_n(|\langle \tilde{u}_s, \Phi_x^m \rangle|) \langle \tilde{u}_s, \Phi_x^m \rangle^2 \frac{(\partial/\partial x_i \Psi_s(x))^2}{\Psi_s(x)} \, dx \, ds \\
&\quad \leq \int_0^t \int_{A_i^{+,s}} \frac{2}{n} \mathbb{1}_{\{a_{n-1} \leq |\langle \tilde{u}_s, \Phi_x^m \rangle| \leq a_n\}} |\langle \tilde{u}_s, \Phi_x^m \rangle| \frac{(\partial/\partial x_i \Psi_s(x))^2}{\Psi_s(x)} \, dx \, ds \qquad \text{by (20)} \\
&\quad \leq \frac{2a_n}{n} \int_0^t \int_{\mathbb{R}^d} \mathbb{1}\bigl(\Psi_s(x) > 0\bigr) \frac{(\frac{\partial}{\partial x_i} \Psi(x))^2}{\Psi_s(x)} \, dx \, ds \\
&\quad \leq \frac{2a_n}{n} \int_0^t 2\|D^2 \Psi_s\|_\infty \operatorname{Area}(\Gamma) \, ds \equiv \frac{2a_n}{n} C(\Psi),
\end{aligned}$$

where Lemma 2.1 is used in the last line, and Γ is defined in (24). Similarly, on the set $A_i^{-,s}$,

$$0 > \frac{\partial}{\partial x_i} \langle \tilde{u}_s, \Phi_x^m \rangle \Psi_s(x) \geq \langle \tilde{u}_s, \Phi_x^m \rangle \frac{\partial}{\partial x_i} \Psi_s(x).$$

Hence, with the same calculation,

$$\begin{aligned}
&\int_0^t \int_{A_i^{-,s}} \psi_n(|\langle \tilde{u}_s, \Phi_x^m \rangle|) \langle \tilde{u}_s, \Phi_x^m \rangle \frac{\partial}{\partial x_i} \Psi_s(x) \frac{\partial}{\partial x_i} \langle \tilde{u}_s, \Phi_x^m \rangle \, dx \, ds \\
&\quad \leq \frac{2a_n}{n} \int_0^t \int_{\mathbb{R}^d} \mathbb{1}\bigl(\Psi_s(x) > 0\bigr) \frac{(\partial/\partial x_i \Psi_s(x))^2}{\Psi_s(x)} \, dx \, ds \\
&\quad \leq \frac{2a_n}{n} C(\Psi).
\end{aligned}$$

Finally, for any $t \geq 0$,

$$\int_0^t \int_{A_i^{0,s}} \psi_n(|\langle \tilde{u}_s, \Phi_x^m \rangle|) \langle \tilde{u}_s, \Phi_x^m \rangle \frac{\partial}{\partial x_i} \Psi_s(x) \frac{\partial}{\partial x_i} \langle \tilde{u}_s, \Phi_x^m \rangle \, dx \, ds = 0,$$

and we conclude that

$$\mathbb{E}\bigl(I_{2,1}^{m,n}(t \wedge T) + I_{2,2}^{m,n}(t \wedge T)\bigr) \leq 4C(\Psi) \frac{a_n}{n},$$

which tends to zero as $n \to \infty$. For $I_{2,3}^{m,n}$, recall that $\phi_n'(u)u \uparrow |u|$ uniformly in u as $n \to \infty$, and that $\langle \tilde{u}_s, \Phi_x^m \rangle$ tends to $\tilde{u}(s, x)$ as $m \to \infty$ for all s, x a.s. by the a.s. continuity of $\tilde{u}$. This implies that $\phi_n'(\langle \tilde{u}_s, \Phi_x^m \rangle) \langle \tilde{u}_s, \Phi_x^m \rangle \to |\tilde{u}(s, x)|$ pointwise a.s. as $m, n \to \infty$, where it is unimportant how we take the limit. We also have the bound

$$|\phi_n'(\langle \tilde{u}_s, \Phi_x^m \rangle) \langle \tilde{u}_s, \Phi_x^m \rangle| \leq |\langle \tilde{u}_s, \Phi_x^m \rangle| \leq \langle |\tilde{u}_s|, \Phi_x^m \rangle. \tag{30}$$

The a.s. continuity of $\tilde{u}$ implies a.s. convergence for all s, x of $\langle|\tilde{u}_s|, \Phi_x^m\rangle$ to $|\tilde{u}(s,x)|$ as $m \to \infty$. A simple application of Jensen's inequality and (14) shows that $|\langle|\tilde{u}_s|, \Phi_x^m\rangle|$ is L^p bounded on $([0,t] \times B(0,K) \times \Omega, ds \times dx \times \mathbb{P})$ uniformly in m. This implies

$$\{\langle|\tilde{u}_s|, \Phi_x^m\rangle : m\} \text{ is uniformly integrable on } \big([0,t] \times B(0,K) \times \Omega\big) \tag{31}$$

and so gives uniform integrability of $\{|\phi_n'(\langle\tilde{u}_s, \Phi_x^m\rangle)\langle\tilde{u}_s, \Phi_x^m\rangle| : m, n\}$ by our earlier bound (30). This implies

$$\lim_{m,n\to\infty} \mathbb{E}\big(I_{2,3}^{m,n}(t \wedge T)\big) = \mathbb{E}\bigg(\int_0^{t\wedge T} \int |\tilde{u}(s,x)| \tfrac{1}{2}\Delta\Psi_s(x)\,dx\,ds\bigg).$$

Collecting the pieces, we have shown that (27) holds.

(c) As in the above argument, we have

$$\phi_n(\langle\tilde{u}_s, \Phi_x^m\rangle) \to |\tilde{u}(s,x)| \qquad \text{as } m, n \to \infty \text{ a.s. for all } x \text{ and all } s \le t. \tag{32}$$

The uniform integrability in (31) and the bound $\phi_n(\langle\tilde{u}_s, \Phi_x^m\rangle) \le \langle|\tilde{u}_s|, \Phi_x^m\rangle$ imply

$$\{\phi_n(\langle\tilde{u}_s, \Phi_x^m\rangle : n, m\} \text{ is uniformly integrable on } [0,t] \times B(0,K) \times \Omega.$$

Therefore, the result now follows from the above convergence and the bound

$$|\dot{\Psi}_s(x)| \le C1(|x| \le K). \qquad \square$$

3. Proof of Theorem 1.6. Here, we let $T = t$ be deterministic. Given the results from Section 2, it now remains to estimate $\mathbb{E}(I_3^{m,n}(t))$. We will then let $m \to \infty$ before letting $n \to \infty$. By the boundedness of the correlation kernel k and Jensen's inequality, $I_3^{m,n}(t)$ is bounded by

$$\begin{aligned}
&\tfrac{1}{2}\|k\|_\infty \int_0^t \int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \big|\sigma\big(u^1(s,y)\big) - \sigma\big(u^2(s,y)\big)\big| \Phi_x^m(y)\,dy\bigg)^2 \\
&\qquad \times \psi_n(|\langle\tilde{u}_s, \Phi_x^m\rangle|)\Psi_s(x)\,dx\,ds \\
&\quad \le \tfrac{1}{2}\|k\|_\infty \int_0^t \int_{\mathbb{R}^d} \big(\sigma\big(u^1(s,y)\big) - \sigma\big(u^2(s,y)\big)\big)^2 \\
&\qquad \times \bigg(\int_{\mathbb{R}^d} \psi_n(|\langle\tilde{u}_s, \Phi_x^m\rangle|)\Phi_x^m(y)\Psi_s(x)\,dx\bigg)\,dy\,ds.
\end{aligned}$$

The integral in parentheses is bounded by a constant, independent of m, is zero for all m if $|y| > K+1$, and as $m \to 0$, converges to $\psi_n(\tilde{u}(s,y))\Psi_s(y)$ for all (s,y) by the continuity of $\tilde{u}$. Our growth condition on σ and (14) imply the integrability of

$$\int_0^t \int \big(\sigma\big(u^1(s,y)\big) - \sigma\big(u^2(s,y)\big)\big)^2 \mathbb{1}_{\{|y|\le K+1\}}\,dy\,ds.$$

Therefore, the dominated convergence theorem implies that

$$\begin{aligned}\limsup_{m\to\infty}\mathbb{E}(I_3^{m,n}(t)) &\le \frac{1}{2}\|k\|_\infty\mathbb{E}\biggl(\int_0^t\langle\psi_n(\tilde{u}_s)\bigl(\sigma(u_s^1)-\sigma(u_s^2)\bigr)^2,\Psi_s\rangle\,ds\biggr)\\ &\le C(\Psi)\|k\|_\infty\frac{t}{n},\end{aligned}\tag{33}$$

where the last line follows by (3) and (20).

Return to equation (25) and let first $m\to\infty$ and then $n\to\infty$. Use the above and Lemma 2.2 on the right-hand side, and (32) and Fatou's lemma on the left-hand side, to conclude that

$$\int_{\mathbb{R}^d}\mathbb{E}\bigl(|\tilde{u}(t,x)|\bigr)\Psi_t(x)\,dx \le \int_0^t\int_{\mathbb{R}^d}\mathbb{E}(|\tilde{u}_s(x)|)\bigl|\tfrac{1}{2}\Delta\Psi_s(x)+\dot{\Psi}_s(x)\bigr|\,dx\,ds.\tag{34}$$

Let $\{g_N\}$ be a sequence of functions in $C_c^\infty(\mathbb{R}^d)$ such that $g_N:\mathbb{R}^d\to[0,1]$,

$$B(0,N)\subset\{x:g_N(x)=1\},\qquad B(0,N+1)^c\subset\{x:g_N(x)=0\}$$

and

$$\sup_N[\|\nabla g_N\|_\infty+\|D^2g_N\|_\infty]\equiv C(g)<\infty,$$

where ∇g_N denotes the gradient with respect to the spatial variables. Now let $\phi\in C_c^\infty(\mathbb{R}^d)$, and for $(s,x)\in[0,t]\times\mathbb{R}^d$, set $\Psi_N(s,x)=(S_{t-s}\phi(x))g_N(x)$. It is then easy to check that $\Psi_N\in C_c^\infty([0,t]\times\mathbb{R}^d)$ and for $\lambda>0$, there is a $C=C(\lambda,\phi)$ such that, for all N,

$$\begin{aligned}\left|\frac{\Delta}{2}\Psi_N(s,x)+\dot{\Psi}_N(s,x)\right| &= \left|\sum_{i=1}^d\frac{\partial}{\partial x_i}S_{t-s}\phi(x_i)\frac{\partial}{\partial x_i}g_N(x_i)+S_{t-s}\phi(x)\frac{\Delta}{2}g_N(x)\right|\\ &\le Ce^{-\lambda|x|}\mathbb{1}_{\{|x|>N\}}.\end{aligned}$$

Use this in (34) to conclude that

$$\int_{\mathbb{R}^d}\mathbb{E}\bigl(|\tilde{u}(t,x)|\bigr)\phi(x)\,dx\le C\int_0^t\int_{\mathbb{R}^d}\mathbb{E}\bigl(|\tilde{u}(s,x)|\bigr)e^{-\lambda|x|}\mathbb{1}_{\{|x|>N\}}\,dx\,ds.$$

By Proposition 1.8, the right-hand side of the above approaches zero as $N\to\infty$ and we see that

$$\mathbb{E}\biggl(\int_{\mathbb{R}^d}|\tilde{u}(t,x)|\,dx\biggr)=0.$$

Therefore, $u^1(t)=u^2(t)$ for all $t\ge 0$ a.s. by a.s. continuity.

4. Proof of Theorem 1.4. We continue to use the notation of Section 2 and also assume the hypotheses of Theorem 1.4. In particular, u^1 and u^2 are solutions of (12), $\tilde{u} = u^1 - u^2$, σ is Hölder continuous with exponent γ,

$$|\sigma(u) - \sigma(v)| \leq L|u - v|^\gamma \qquad \text{for } u, v \in \mathbb{R},$$

and $|k(x, y)| \leq c_{1.4}[|x - y|^{-\alpha} + 1]$ for some $\alpha \in (0, 1)$. We choose $\rho(x) = \sqrt{x}$ for our smooth approximation of the absolute value function throughout noting that (3) is not necessarily satisfied for large values. Nevertheless, we will use the test function ϕ_n and its derivatives as defined in (21) to (23) corresponding to this ρ.

Fix some $\lambda > 0$ and let $T_K = \inf\{t \geq 0 : \sup_{x\in\mathbb{R}^d}(|u^1(t, x)| + |u^2(t, x)|)e^{-\lambda|x|} > K\} \wedge K$. Note that

$$T_K \to \infty, \qquad P\text{-a.s.}, \tag{35}$$

since $u^i \in C(\mathbb{R}_+, C_{\text{tem}})$.

Also define a metric d by

$$d\big((t, x), (t', x')\big) = \sqrt{|t - t'|} + |x - x'|, \qquad t, t' \in \mathbb{R}_+, x, x' \in \mathbb{R}^d,$$

and set

$$Z_{K,N,\xi} = \big\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d : t \leq T_K, |x| \leq K, d\big((t, x), (\hat{t}, \hat{x})\big) < 2^{-N} \text{ for some } (\hat{t}, \hat{x}) \in [0, T_K] \times \mathbb{R}^d \text{ satisfying } |\tilde{u}(\hat{t}, \hat{x})| \leq 2^{-N\xi}\big\}.$$

We will now use the following key result on the improved Hölder continuity of $\tilde{u}$ when $\tilde{u}$ is small. It will be proved in Section 5.

THEOREM 4.1. *Assume the hypotheses of Theorem 1.4, except now allow $\gamma \in (0, 1]$. Let $u_0 \in C_{\text{tem}}$ and $\tilde{u} = u^1 - u^2$, where u^i is a solution of (12) with sample paths in $C(\mathbb{R}_+, C_{\text{tem}})$ a.s. for $i = 1, 2$. Let $\xi \in (0, 1)$ satisfy*

$$\begin{aligned} &\exists N_\xi = N_\xi(K, \omega) \in \mathbb{N} \text{ a.s. such that } \forall N \geq N_\xi, (t, x) \in Z_{K,N,\xi} \\ &d\big((t', y), (t, x)\big) \leq 2^{-N}, \qquad t, t' \leq T_K \Rightarrow |\tilde{u}(t, x) - \tilde{u}(t', y)| \leq 2^{-N\xi}. \end{aligned} \tag{36}$$

Let $0 < \xi_1 < [\xi\gamma + 1 - \frac{\alpha}{2}] \wedge 1$. Then there is an $N_{\xi_1} = N_{\xi_1}(K, \omega) \in \mathbb{N}$ a.s. such that, for any $N \geq N_{\xi_1}$ in $\mathbb{N}$ and any $(t, x) \in Z_{K,N,\xi}$,

$$d\big((t', y), (t, x)\big) \leq 2^{-N}, \qquad t, t' \leq T_K \Rightarrow |\tilde{u}(t, x) - \tilde{u}(t', y)| \leq 2^{-N\xi_1}. \tag{37}$$

Moreover, there are strictly positive constants $R, \delta, c_{38.1}, c_{38.2}$ depending only on (ξ, ξ_1) and $N(K) \in \mathbb{N}$, which also depends on K, such that

$$\mathbb{P}\big(N_{\xi_1} \geq N\big) \leq c_{38.1}\big(\mathbb{P}(N_\xi \geq N/R) + K^{d+1}\exp(-c_{38.2}2^{N\delta})\big), \tag{38}$$

provided that $N \geq N(K)$.

REMARK. Results similar to the above for white noise were independently found by Carl Mueller and Roger Tribe in their parallel work on level sets of solutions of SPDEs.

Recall $\lambda > 0$ is a fixed parameter used in the definition of T_K.

COROLLARY 4.2. *Assume the hypotheses of Theorem 1.4, except now allow* $\gamma \in (0, 1]$. *Let* u_0 *and* $\tilde{u}$ *be as in Theorem 4.1, and* $1 - \frac{\alpha}{2} < \xi < \frac{1-\alpha/2}{1-\gamma} \wedge 1$. *There is an a.s. finite positive random variable* $C_{\xi,K}(\omega)$ *such that, for any* $\varepsilon \in (0, 1]$, $t \in [0, T_K]$ *and* $|x| \leq K$, *if* $|\tilde{u}(t, \hat{x})| \leq \varepsilon^{\xi}$ *for some* $|\hat{x} - x| \leq \varepsilon$, *then* $|\tilde{u}(t, y)| \leq C_{\xi,K}\varepsilon^{\xi}$ *whenever* $|x - y| \leq \varepsilon$. *Moreover, there are strictly positive constants* $\delta, c_{39.1}, c_{39.2}$, *depending on* ξ, *and an* $r_0(K)$, *which also depends on* K, *such that*

$$\begin{aligned}\mathbb{P}(C_{\xi,K} \geq r) \leq c_{39.1}\Bigg[\bigg(\frac{r-6}{(K+1)e^{\lambda(K+1)}}\bigg)^{-\delta} \\ + K^{d+1}\exp\bigg(-c_{39.2}\bigg(\frac{r-6}{(K+1)e^{\lambda(K+1)}}\bigg)^{\delta}\bigg)\Bigg]\end{aligned} \tag{39}$$

for all $r \geq r_0(K) > 6 + (K+1)e^{\lambda(K+1)}$.

PROOF. By Proposition 1.8(b) and the equality $\tilde{u} = Z^1 - Z^2$, where $Z^i(t, x) = u^i(t, x) - S_t u_0(x)$, we have (36) with $\xi = \xi_0 = \frac{1}{2}(1 - \frac{\alpha}{2})$. Indeed, $\tilde{u}$ is uniformly Hölder continuous on compacts in $[0, \infty) \times \mathbb{R}^d$ with coefficient ξ in space and $\frac{\xi}{2}$ in time provided that $\xi < 1 - \frac{\alpha}{2}$.

Inductively, define $\xi_{n+1} = [(\xi_n\gamma + 1 - \frac{\alpha}{2}) \wedge 1](1 - \frac{1}{n+3})$ so that $\xi_n \uparrow \frac{1-\alpha/2}{1-\gamma} \wedge 1$. Fix n_0 so that $\xi_{n_0} \geq \xi > \xi_{n_0-1}$. Apply Theorem 4.1 inductively n_0 times to get (36) for ξ_{n_0-1} and, hence, (37) with $\xi_1 = \xi_{n_0}$.

First consider $\varepsilon \leq 2^{-N_{\xi_{n_0}}}$. Choose $N \in \mathbb{N}$ so that $2^{-N-1} < \varepsilon \leq 2^{-N}$ ($N \geq N_{\xi_{n_0}}$), and assume $t \leq T_K$, $|x| \leq K$ and $|\tilde{u}(t, \hat{x})| \leq \varepsilon^{\xi} \leq 2^{-N\xi} \leq 2^{-N\xi_{n_0-1}}$ for some $|\hat{x} - x| \leq \varepsilon \leq 2^{-N}$. Then $(t, x) \in Z_{K,N,\xi_{n_0-1}}$. Therefore, (37) with $\xi_1 = \xi_{n_0}$ implies that, if $|y - x| \leq \varepsilon \leq 2^{-N}$, then

$$\begin{aligned}|\tilde{u}(t, y)| &\leq |\tilde{u}(t, \hat{x})| + |\tilde{u}(t, \hat{x}) - \tilde{u}(t, x)| + |\tilde{u}(t, x) - \tilde{u}(t, y)| \\ &\leq 2^{-N\xi} + 2 \cdot 2^{-N\xi_{n_0}} \leq 3 \cdot 2^{-N\xi} \leq 3(2\varepsilon)^{\xi} \leq 6\varepsilon^{\xi}.\end{aligned}$$

For $\varepsilon > 2^{-N_{\xi_{n_0}}}$, we have for (t, x) and (t, y), as in the corollary,

$$|\tilde{u}(t, y)| \leq (K+1)e^{\lambda(K+1)} \leq (K+1)e^{\lambda(K+1)}2^{N_{\xi_{n_0}}}\varepsilon^{\xi}.$$

This gives the conclusion with $C_{\xi,K} = (K+1)e^{\lambda(K+1)}2^{N_{\xi_{n_0}}} + 6$. A short calculation and (38) now imply that there are strictly positive constants $\tilde{R}, \tilde{\delta}, c_{40.1}, c_{40.2}$, depending on ξ and K, such that

$$\begin{aligned}\mathbb{P}(C_{\xi,K} \geq r) \leq c_{40.1}\biggl[\mathbb{P}\biggl(N_{1/2(1-\alpha/2)} \geq \frac{1}{\tilde{R}}\log_2\biggl(\frac{r-6}{(K+1)e^{\lambda(K+1)}}\biggr)\biggr)\\ + K^{d+1}\exp\biggl(-c_{40.2}\biggl(\frac{r-6}{(K+1)e^{\lambda(K+1)}}\biggr)^{\tilde{\delta}}\biggr)\biggr]\end{aligned} \tag{40}$$

for all $r \geq r_0(K)$. The usual Kolmogorov continuity proof applied to (15) with $\tilde{u} = Z^1 - Z^2$ in place of Z [and $\xi = \frac{1}{2}(1 - \frac{\alpha}{2})$] shows there are $\tilde{\varepsilon}, \tilde{c}_3 > 0$ such that

$$\mathbb{P}(N_{1/2(1-\alpha/2)} \geq M) \leq \tilde{c}_3 2^{-M\tilde{\varepsilon}}$$

for all $M \in \mathbb{R}$. Thus, (39) follows from (40). $\square$

Now fix α, γ satisfying the conditions of Theorem 1.4, so $\alpha < (2\gamma - 1)$ and notice that since $1 \geq \gamma > \frac{1}{2}$, this implies that $\frac{1-\alpha/2}{1-\gamma} > 1$. Hence, we can choose $\xi \in (0, 1)$ such that

$$\alpha < \xi(2\gamma - 1) \tag{41}$$

and $1 - \frac{\alpha}{2} < \xi < \frac{1-\alpha/2}{1-\gamma} \wedge 1$. This means that ξ satisfies the conditions of Corollary 4.2.

We return to the setting and notation in Section 2. In particular, $\Psi \in C_c^\infty([0, t] \times \mathbb{R}^d)$ with $\Gamma = \{x : \Psi_s(x) > 0\ \exists s \leq t\} \subset B(0, K)$. Recall Lemma 2.2 is valid in the setting of Theorem 1.4.

Let $m^{(n)} := a_{n-1}^{-1/\xi}$. Note that $m^{(n)} \geq 1$ for all n. We set $c_0(K) := r_0(K) \vee K^2 e^{\lambda K}$ [where $r_0(K)$ is chosen as in Corollary 4.2] and define the stopping time

$$\begin{aligned}T_{\xi,K} = \inf\{t \geq 0 : t > T_K \text{ or } t \leq T_K \text{ and there exist } \varepsilon \in (0, 1],\\ \hat{x}, x, y \in \mathbb{R} \text{ with } |x| \leq K, |\tilde{u}(t, \hat{x})| \leq \varepsilon^\xi, |x - \hat{x}| \leq \varepsilon,\\ |x - y| \leq \varepsilon \text{ such that} |\tilde{u}(t, y)| > c_0(K)\varepsilon^\xi\}.\end{aligned}$$

Assuming our filtration is completed as usual, $T_{\xi,K}$ is a stopping time by the standard projection argument. Note that, for any $t \geq 0$, by Corollary 4.2,

$$\begin{aligned}\mathbb{P}(T_{\xi,K} \leq t) &\leq \mathbb{P}(T_K \leq t) + \mathbb{P}(C_{\xi,K} > c_0(K))\\ &\leq \mathbb{P}(T_K \leq t)\\ &\quad + c_{39.1}\biggl[\biggl(\frac{K^2 e^{\lambda K} - 6}{(K+1)e^{\lambda(K+1)}}\biggr)^{-\delta} \qquad (42)\\ &\quad + K^{d+1}\exp\biggl(-c_{39.2}\biggl(\frac{K^2 e^{\lambda K} - 6}{(K+1)e^{\lambda(K+1)}}\biggr)^{\delta}\biggr)\biggr], \qquad (43)\end{aligned}$$

which tends to zero as $K \to \infty$ due to (35).

With this set-up we can show the following lemma:

LEMMA 4.3. *For all $x \in \Gamma$ and $s \in [0, T_{\xi,K}]$, if $|\langle \tilde{u}_s, \Phi_x^{m^{(n)}} \rangle| \le a_{n-1}$, then*

$$\sup_{y \in B(x, 1/m^{(n)})} |\tilde{u}(s, y)| \le c_0(K) a_{n-1}.$$

PROOF. Since $|\langle \tilde{u}_s, \Phi_x^{m^{(n)}} \rangle| \le a_{n-1}$ and $\tilde{u}_s(\cdot)$ is continuous, there exists a $\hat{x} \in B(x, \frac{1}{m^{(n)}})$ such that $|\tilde{u}(s, \hat{x})| \le a_{n-1}$. Apply the definition of the stopping time with $\varepsilon = 1/m^{(n)} \in (0, 1]$ and so $\varepsilon^{\xi} = a_{n-1}$ to obtain the required bound. □

Next, we bound $|I_3^{m^{(n)},n}|$ using the Hölder continuity of σ, as well as the definition of ψ_n. If $|\sigma(x) - \sigma(y)| \le L|x - y|^{\gamma}$, then

$$\begin{aligned} &\left|I_3^{m^{(n)},n}(t \wedge T_{\xi,K})\right| \\ &\quad \le \frac{c_8 L^2}{n} \int_0^{t \wedge T_{\xi,K}} \int_{\mathbb{R}^{3d}} \mathbb{1}_{\{a_n \le |\langle \tilde{u}_s, \Phi_x^{m^{(n)}} \rangle| \le a_{n-1}\}} a_n^{-1} |\tilde{u}_s(y)|^{\gamma} |\tilde{u}_s(z)|^{\gamma} \\ &\qquad \times \Phi_x^{m^{(n)}}(y) \Phi_x^{m^{(n)}}(z) [|y - z|^{-\alpha} + 1] \, dy \, dz \, \Psi_s(x) \, dx \, ds. \end{aligned}$$

Now set $\Gamma^1 = \{x \in \mathbb{R}^d, d(x, \Gamma) < 1\}$. Since $\Phi(x) \le C \mathbb{1}_{B(0,1)}(x)$ and

$$\begin{aligned} &\mathbb{1}_{B(0,1)}\big(m^{(n)}(x - y)\big) \cdot \mathbb{1}_{B(0,1)}\big(m^{(n)}(x - z)\big) \\ &\quad \le \mathbb{1}_{B(0,1)}\big(m^{(n)}(x - y)\big) \cdot \mathbb{1}_{B(0,1)}\big(\tfrac{1}{2} m^{(n)}(y - z)\big), \end{aligned}$$

we obtain from Lemma 4.3

$$\begin{aligned} &\left|I_3^{m^{(n)},n}(t \wedge T_{\xi,K})\right| \\ &\quad \le c_8 L^2 c_0(K)^{2\gamma} \frac{a_{n-1}^{2\gamma}}{n a_n} \\ &\qquad \times \int_0^{t \wedge T_{\xi,K}} \int_{\mathbb{R}^{3d}} \mathbb{1}_{\{a_n \le |\langle \tilde{u}_s, \Phi_x^{m^{(n)}} \rangle| \le a_{n-1}\}} \\ &\qquad\qquad \times \Phi_x^{m^{(n)}}(y) \Phi_x^{m^{(n)}}(z) [|y - z|^{-\alpha} + 1] \, dy \, dz \, \Psi_s(x) \, dx \, ds \\ &\quad \le \frac{c_8 L^2 \|\Psi\|_{\infty} c_0(K)^{2\gamma}}{n} \frac{a_{n-1}^{2\gamma}}{a_n} \\ &\qquad \times \int_0^{t \wedge T_{\xi,K}} \int_{\Gamma^1 \times \Gamma^1} \left(\int_{\Gamma} \Phi_x^{m^{(n)}}(y) \Phi_x^{m^{(n)}}(z) \, dx \right) [|y - z|^{-\alpha} + 1] \, dy \, dz \, ds \\ &\quad \le \frac{c_8 L^2 \|\Psi\|_{\infty} c_0(K)^{2\gamma} t}{n} \frac{a_{n-1}^{2\gamma}}{a_n} \\ &\qquad \times \int_{\Gamma^1 \times \Gamma^1} \big(m^{(n)}\big)^d \mathbb{1}_{B(0,1)}\big(\tfrac{1}{2} m^{(n)}(y - z)\big) [|y - z|^{-\alpha} + 1] \, dy \, dz \end{aligned}$$

$$\begin{aligned}&\leq \frac{C(c_8, L, \Psi, \Phi)c_0(K)^{2\gamma} t}{n} \frac{a_{n-1}^{2\gamma}}{a_n}\big[\big(m^{(n)}\big)^{\alpha} + 1\big]\\&= \frac{C(c_8, L, \Psi, \Phi)c_0(K)^{2\gamma} t}{n} \frac{a_{n-1}^{(2\gamma-\alpha/\xi)}}{a_n}.\end{aligned}$$

Observe now that $\int_{a_n}^{a_{n-1}} x^{-1}\,dx \sim n$ so that $\frac{a_{n-1}}{a_n} \sim e^n$ or (using that $a_0 = 1$) $a_n \sim e^{-(n(n+1))/2}$. Thus,

$$\lim_{n\to\infty} \mathbb{E}\big(\big|I_3^{m^{(n)},n}(t \wedge T_{\xi,K})\big|\big) = 0 \tag{44}$$

if $n(n+1) - (2\gamma - \frac{\alpha}{\xi})(n-1)n < 0$ for n large. This is equivalent to

$$1 - \left(2\gamma - \frac{\alpha}{\xi}\right) < 0 \quad \Longleftrightarrow \quad \alpha < \xi(2\gamma - 1),$$

which holds by (41).

Use (32) and Fatou's lemma on the left-hand side of (25), and Lemma 2.2 and (44) on the right-hand side, to take limits in this equation and so conclude

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathbb{E}\big(|\tilde{u}(t \wedge T_{\xi,K}, x)|\big)\Psi_t(x)\,dx\\&\qquad \leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} \mathbb{E}\big(\phi_n\big(\langle \tilde{u}_{t\wedge T_{\xi,K}}, \Phi_x^{m^{(n)}}\rangle\big)\big)\Psi_t(x)\,dx\\&\qquad \leq \mathbb{E}\left(\int_0^{t\wedge T_{\xi,K}} \int_{\mathbb{R}^d} |\tilde{u}(s,x)|\tfrac{1}{2}\big(\Delta\Psi_s(x) + \dot{\Psi}_s(x)\big)\,dx\,ds\right)\\&\qquad \leq \int_0^t \int_{\mathbb{R}^d} \mathbb{E}\big(|\tilde{u}(s,x)|\big)\big|\tfrac{1}{2}\Delta\Psi_s(x) + \dot{\Psi}_s(x)\big|\,dx\,ds.\end{aligned}$$

Since $T_{\xi,K}$ tends in probability to infinity as $K \to \infty$ according to (42), we know that $\tilde{u}(t \wedge T_{\xi,K}, x) \to \tilde{u}(t,x)$ and so we finally conclude with another application of Fatou's lemma that

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathbb{E}\big(|\tilde{u}(t,x)|\big)\Psi_t(x)\,dx\\&\qquad \leq \int_0^t \int_{\mathbb{R}^d} \mathbb{E}\big(|\tilde{u}(s,x)|\big)\big|\tfrac{1}{2}\Delta\Psi_s(x) + \dot{\Psi}_s(x)\big|\,dx\,ds.\end{aligned}$$

This is (34) of Section 3 and the conclusion now follows as in the proof of Theorem 1.6 given there.

5. Proof of Theorem 4.1. In this section we will first prove three technical lemmas needed for the proof of Theorem 4.1.

LEMMA 5.1. *Let B be a standard d-dimensional Brownian motion. For $\alpha < d$, there exists a constant $c_{5.1} = c_{5.1}(\alpha, d)$ such that, for all $x, y \in \mathbb{R}^d$ and $t, t' > 0$,*

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_t(x-w)p_{t'}(y-z)|w-z|^{-\alpha}\,dw\,dz &= \mathbb{E}_{x-y}(|B_{t+t'}|^{-\alpha}) \\ &\le \mathbb{E}_0(|B_{t+t'}|^{-\alpha}) \\ &\le c_{5.1}(t+t')^{-\alpha/2}.\end{aligned}\tag{45}$$

In addition, for any $\lambda' > 0$, $c \ge 0$ and $0 < t \le t'$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\lambda'(|w|+|z|)}p_t(x-w)p_{t'}(y-z)[|w-z|^{-\alpha}+c]\,dw\,dz \\ &\quad\le c_{5.1}e^{2(\lambda')^2t'}e^{\lambda'(|x|+|y|)}[(t+t')^{-\alpha/2}+c].\end{aligned}\tag{46}$$

PROOF. The first equality of (45) is immediate from change of variables. The second inequality then follows from a simple coupling argument: Let $|B_t^i|$ for $i = 1, 2$ be the radial part of a d-dimensional Brownian motion started at 0 and $|x-y|$, respectively. Define the stopping time $T := \inf\{t \ge 0 : |B_t^1| > |B_t^2|\}$. Then

$$|B_t^3| = \begin{cases} |B_t^2|, & \text{for } t \le T, \\ |B_t^1|, & \text{for } t > T, \end{cases}$$

has the same law as $|B^2|$ and the property that $|B_t^3| \ge |B_t^1|$ for all $t \ge 0$ a.s., which implies the inequality of the expectations in (45). We finally compute by setting $r = \frac{|w|^2}{2t}$,

$$\begin{aligned}\mathbb{E}_0(|B_t|^{-\alpha}) &= \int_{\mathbb{R}^d} |w|^{-\alpha}(2\pi t)^{d/2}\exp\left(-\frac{|w|^2}{2t}\right)dw \\ &= c_d\int_0^\infty r^{(d-\alpha)/2-1}\exp(-r)\,dr\cdot t^{-\alpha/2} \\ &= c(\alpha, d)t^{-\alpha/2},\end{aligned}$$

provided that $\alpha < d$. This shows (45). For proving (46), we note that, for $0 < t \le t'$,

$$e^{\lambda'|w|}p_t(w) \le 2^{d/2}\exp\left(\lambda'|w| - \frac{1}{4t}|w|^2\right)p_{2t}(w) \le c_d e^{(\lambda')^2t'}p_{2t}(w)\tag{47}$$

since $\lambda'|w| - \frac{1}{4t}|w|^2 \le (\lambda')^2 t$. Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\lambda'(|w|+|z|)}p_t(x-w)p_{t'}(y-z)[|w-z|^{-\alpha}+c]\,dw\,dz \\ &\quad\le e^{\lambda'(|x|+|y|)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\lambda'(|w|+|z|)}p_t(w)p_{t'}(z)[|w-z+x-y|^{-\alpha}+c]\,dw\,dz\end{aligned}$$

$$\le c_d^2 e^{2(\lambda')^2 t'} e^{\lambda'(|x|+|y|)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{2t}(w) p_{2t'}(z)[|w-z+x-y|^{-\alpha}+c]\,dw\,dz$$
$$\le c(\alpha,d) e^{2\lambda^2 t'} e^{\lambda(|x|+|y|)}[(t+t')^{-\alpha/2}+c].$$

Here, we have used a shift of variables in the first and (47) in the second inequality, as well as (45) in the third. This shows (46). □

The next lemma provides some estimates of the temporal and spatial differences of the heat kernels:

LEMMA 5.2. *There are constants* $c_{48}(d)$ *and* $c_{49}(\alpha, d)$ *such that if* $0 < \beta \le 1$ *and* $\lambda' \ge 0$, *then for any* $x, y \in \mathbb{R}^d$, $0 < t \le t'$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |p_t(x-w) - p_{t'}(y-w)| e^{\lambda'|w|}\,dw \\
&\quad \le c_{48} e^{2(\lambda')^2 t'} \bigl(e^{\lambda'|x|} + e^{\lambda'|y|}\bigr) e^{2\beta\lambda'(|x-y|)} \\
&\qquad \times (t^{-\beta/2}|x-y|^\beta + t^{-\beta}|t'-t|^\beta),
\end{aligned} \tag{48}$$

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |p_t(x-w) - p_t(y-w)||p_t(x-z) - p_t(y-z)| \\
&\qquad \times [|w-z|^{-\alpha} + 1]\,dw\,z \\
&\quad \le c_{49}[t^{-1-\alpha/2} + t^{-1}]|x-y|^2
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |p_t(x-w) - p_{t'}(x-w)||p_t(x-z) - p_{t'}(x-z)| \\
&\qquad \times [|w-z|^{-\alpha} + 1]\,dw\,dz \\
&\quad \le c_{49}[t^{-2-\alpha/2} + t^{-2}]|t'-t|^2.
\end{aligned} \tag{50}$$

PROOF. We consider the space and time differences separately. For the former, define $v = x - y$, and set $\hat{v}_0 = 0$, $\hat{v}_d = v$ and $\hat{v}_i - \hat{v}_{i-1} = v_i e_i$, where v_i is the ith component of v and e_i is the ith unit vector in $\mathbb{R}^d$. Therefore,

$$\begin{aligned}
&\left|\exp\left(-\frac{|w+v|^2}{2t}\right) - \exp\left(-\frac{|w|^2}{2t}\right)\right| \\
&\quad \le \sum_{i=1}^d \left|\exp\left(-\frac{|w+\hat{v}_i|^2}{2t}\right) - \exp\left(-\frac{|w+\hat{v}_{i-1}|^2}{2t}\right)\right| \\
&\quad = \sum_{i=1}^d \left|\int_0^{v_i} \frac{w_i + r_i}{t} \exp\left(-\frac{|w+\hat{v}_{i-1}+r_i e_i|^2}{2t}\right) dr_i\right|.
\end{aligned}$$

Hence, by a change of variables, (47) and using $|w| \le |\hat{w}_i| + |w_i|$ ($\hat{w}_i = w - w_i e_i$), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |p_t(x-w) - p_t(y-w)| e^{\lambda'|w|}\, dw \\
&\quad \le e^{\lambda'|x|}(2\pi t)^{d/2} \\
&\qquad \times \sum_{i=1}^d \int_0^{|v_i|} \int_{\mathbb{R}^d} \frac{|w_i + r_i|}{t} \exp\biggl(-\frac{|w + \hat{v}_{i-1} + r_i e_i|^2}{2t}\biggr) e^{\lambda'|w|}\, dw\, dr_i \\
&\quad \le c_d e^{(\lambda')^2 t'} e^{\lambda'(|x|+|v|)} t^{-1/2} \\
&\qquad \times \sum_{i=1}^d \int_0^{|v_i|} \int_{-\infty}^{\infty} \frac{|w_i + r_i|}{t} \exp\biggl(-\frac{(w_i + r_i)^2}{2t}\biggr) e^{\lambda'|w_i|}\, dw_i\, dr_i \\
&\quad \le c_d e^{(\lambda')^2 t'} e^{\lambda'(|x|+|v|)} t^{-1/2} \Biggl(\sum_{i=1}^d e^{\lambda'|v_i|} |v_i|\Biggr) \int_0^\infty \frac{r}{t} \exp\biggl(-\frac{r^2}{4t}\biggr)\, dr \\
&\quad \le c_d e^{(\lambda')^2 t'} e^{\lambda'|x| + 2\lambda'|x-y|} t^{-1/2} |x-y|.
\end{aligned}
\tag{51}
$$

Similarly, using that $a \le c \exp(a^2/4)$ for all $a \in \mathbb{R}_+$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |p_t(x-w) - p_t(y-w)| \cdot |p_t(x-z) - p_t(y-z)| [|w-z|^{-\alpha} + 1]\, dw\, dz \\
&\quad \le C t^{-d} \\
&\qquad \times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{i,j=1}^d \int_0^{|v_i|} \int_0^{|v_j|} \frac{|w_i + r_i|}{t} \\
&\qquad\qquad \times \exp\biggl(-\frac{|w + \hat{v}_{i-1} + r_i e_i|^2}{2t}\biggr) \\
&\qquad\qquad \times \frac{|z_i + \tilde{r}_i|}{t} \exp\biggl(-\frac{|z + \hat{v}_{i-1} + \tilde{r}_i e_i|^2}{2t}\biggr)\, dr_i\, d\tilde{r}_i \\
&\qquad\qquad \times [|w-z|^{-\alpha} + 1]\, dw\, dz \\
&\quad \le C t^{-d-1} \\
&\qquad \times \sum_{i,j=1}^d \int_0^{|v_i|} \int_0^{|v_j|} \biggl(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\biggl(-\frac{|w + \hat{v}_{i-1} + r_i e_i|^2}{2t} \\
&\qquad\qquad + \frac{|w_i + r_i|^2}{4t}\biggr)
\end{aligned}
$$

$$\times \exp\biggl(-\frac{|z+\hat{v}_{i-1}+\tilde{r}_i e_i|^2}{2t} + \frac{|z_i+\tilde{r}_i|^2}{4t}\biggr) \times [|w-z|^{-\alpha}+1]\,dw\,dz\biggr)\,dr_i\,d\tilde{r}_i$$

$$\le C(\alpha,d)t^{-1}\sum_{i,j=1}^{d}\int_0^{|v_i|}\int_0^{|v_j|}[t^{-\alpha/2}+1]\,dr_i\,d\tilde{r}_i$$

$$\le C(\alpha,d)[t^{-1-\alpha/2}+t^{-1}]|x-y|^2,$$

where we have used an appropriate shift of variables and Lemma 5.1 in the previous to last line. This shows (49).

For the time differences observe that for some $C = C(\alpha, d)$,

$$\begin{aligned}
&|p_t(w)-p_{t'}(w)| \\
&\quad\le C|t^{d/2}-t'^{d/2}|\exp\biggl(-\frac{|w|^2}{2t}\biggr) \\
&\qquad + Ct'^{d/2}\biggl|\exp\biggl(-\frac{|w|^2}{2t}\biggr)-\exp\biggl(-\frac{|w|^2}{2t'}\biggr)\biggr| \\
&\quad\le C|t'-t|t^{d/2-1}\exp\biggl(-\frac{|w|^2}{2t}\biggr)+Ct'^{d/2}\int_t^{t'}\exp\biggl(-\frac{|w|^2}{2s}\biggr)\frac{|w|^2}{2s^2}\,ds \\
&\quad\le Ct^{-1}|t'-t|\bigl(p_t(w)+p_{2t'}(w)\bigr),
\end{aligned}\tag{52}$$

since $\frac{|w|^2}{4s} \le \exp(\frac{|w|^2}{4s})$. Therefore, another application of (47) yields

$$\int_{\mathbb{R}^d}|p_t(w)-p_{t'}(w)|e^{\lambda'|w|}\,dw \le C(d)e^{(\lambda')^2t'}t^{-1}|t'-t|.$$

Taking this estimate with a change of variables, together with (51), we obtain

$$\begin{aligned}
&\int |p_t(x-w)-p_{t'}(y-w)|e^{\lambda'|w|}\,dw \\
&\qquad \le C(d)e^{2\lambda^2t'}e^{\lambda'|x|}\biggl[e^{2\lambda'|x-y|}\frac{|x-y|}{\sqrt{t}}+\frac{|t'-t|}{t}\biggr].
\end{aligned}\tag{53}$$

An application of (47) and a change of variables also show that

$$\int |p_t(x-w)-p_{t'}(y-w)|e^{\lambda'|w|}\,dw \le C(d)e^{(\lambda')^2t'}\bigl(e^{\lambda'|x|}+e^{\lambda'|y|}\bigr).\tag{54}$$

If $\beta \in (0,1]$, the inequality $z \wedge 1 \le z^\beta$ for $z \ge 0$, and the previous two bounds now show that

$$\begin{aligned}\int |p_t(x-w) - p_{t'}(y-w)| e^{\lambda'|w|}\, dw \\ \le C(d) e^{2(\lambda')^2 t'} \big(e^{\lambda'|x|} + e^{\lambda'|y|}\big) \\ \times \big[e^{2\beta\lambda'|x-y|}|x-y|^\beta t^{-\beta/2} + |t'-t|^\beta t^{-\beta}\big],\end{aligned}$$

which implies (48). Similarly, using (52) and (45),

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |p_t(x-w) - p_{t'}(x-w)| \\ &\qquad \times |p_t(x-z) - p_{t'}(x-z)|[|w-z|^{-\alpha}+1]\, dw\, dz \\ &\quad \le C t^{-2}|t'-t|^2 \\ &\qquad \times \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(p_t(w) + p_{2t'}(w)\big)\big(p_t(z) + p_{2t'}(z)\big)[|w-z|^{-\alpha}+1]\, dw\, dz \\ &\quad \le C(\alpha, d)(t^{-2-\alpha/2} + t^{-2})|t'-t|^2,\end{aligned}$$

which proves (50). $\square$

We will also need the following rather technical lemma:

LEMMA 5.3. *For $b, c \ge 0$ with $c < \frac{1}{2}(b + 1 - \frac{\alpha}{2})$, and $a \in (c, 1 - \alpha/2)$, there is a finite constant $c_{5.3} = c_{5.3}(a, b, c, \alpha)$ such that $t \ge 0$,*

$$\begin{aligned}&Q(t, a, b, c, \alpha) \\ &\quad := \int_0^t \int_0^t (t-r)^{a-1-c}(t-r')^{a-1-c} \\ &\qquad\qquad \times \int_0^{r\wedge r'} (t-s)^b (r-s)^{-a}(r'-s)^{-a} \\ &\qquad\qquad \times \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{r-s}(w) p_{r'-s}(z)[|w-z|^{-\alpha}+1]\, dw\, dz\, ds\, dr\, dr' \\ &\quad \le c_{5.3}[t^{b+1-\alpha/2-2c} + t^{b+1-2c}].\end{aligned}$$

PROOF. By Lemma 5.1, it suffices to estimate

$$\begin{aligned}&\int_0^t \int_0^t (t-r)^{a-1-c}(t-r')^{a-1-c} \\ &\qquad \times \int_0^{r\wedge r'} (t-s)^b (r-s)^{-a}(r'-s)^{-a}[(r-s+r'-s)^{-\alpha/2}+1]\, ds\, dr\, dr'\end{aligned}$$

$$= 2\int_0^t \int_s^t (t-r)^{a-1-c}(t-s)^b(r-s)^{-a} \\ \times \left(\int_r^t (t-r')^{a-1-c}[(r'-s)^{-a-\alpha/2} + (r'-s)^{-a}]\,dr'\right) dr\,ds,$$

where we have used the symmetry in r and r' and concentrated on the case $r \le r'$. Substituting $v = \frac{r'-r}{t-r}$ and using that $c < a < 1 - \frac{\alpha}{2}$, we calculate, for $t \ge r \ge s$,

$$\int_r^t (t-r')^{a-1-c}[(r'-s)^{-a-\alpha/2} + (r'-s)^{-a}]\,dr'$$

$$= (t-r)^{-\alpha/2-c} \int_0^1 (1-v)^{a-1-c}\left[\left(v + \frac{r-s}{t-r}\right)^{-a-\alpha/2} + (t-r)^{\alpha/2}\left(v + \frac{r-s}{t-r}\right)^{-a}\right] dv$$

$$\le C(a,c,\alpha)(t-r)^{a-c} \\ \times [(t-r)^{-a-\alpha/2} \wedge (r-s)^{-a-\alpha/2} + (t-r)^{-a} \wedge (r-s)^{-a}].$$

Hence, the required Q is at most $C(a,c,\alpha)$ times the sum of the following integral, $I(\beta)$, for $\beta = a$ and $\beta = a + \alpha/2$:

$$I(\beta) = \int_0^t (t-s)^b \int_s^t (t-r)^{2a-1-2c}(r-s)^{-a}\big((t-r)^{-\beta} \wedge (r-s)^{-\beta}\big)\,dr\,ds.$$

For these values of β, $I(\beta)$ is at most

$$\int_0^t (t-s)^b \left(\int_s^{(t+s)/2} (t-r)^{2a-1-2c-\beta}(r-s)^{-a}\,dr + \int_{(t+s)/2}^t (t-r)^{2a-1-2c}(r-s)^{-a-\beta}\,dr\right) ds$$

$$\le C(a,\alpha)\int_0^t \left((t-s)^{b+2a-1-2c-\beta}\int_s^{(t+s)/2} (r-s)^{-a}\,dr + (t-s)^{b-a-\beta}\int_{(t+s)/2}^t (t-r)^{2a-1-2c}\,dr\right) ds$$

$$\le C(a,c,\alpha)\int_0^t (t-s)^{b+a-2c-\beta}\,ds \le C(a,b,c,\alpha)t^{b+a-2c-\beta+1}.$$

Here, we have used that $t - r \ge \frac{t-s}{2}$ for $r \in [s, \frac{t+s}{2}]$ and, analogously, $r - s \ge \frac{t-s}{2}$ for $r \in [\frac{t+s}{2}, t]$, as well as our assumption of $a > c$ and $c < \frac{1}{2}(b + 1 - \frac{\alpha}{2})$. The result follows upon summing over the two values of β. □

PROOF OF THEOREM 4.1. Fix arbitrary (deterministic) (t,x), (t',y) such that $d((t,x),(t',y)) \le \varepsilon \equiv 2^{-N}$ ($N \in \mathbb{N}$) and $t \le t'$ (the case $t' \le t$ works analogously).

As $\xi_1 < (\xi\gamma + 1 - \alpha/2) \wedge 1$, we may choose $\delta \in (0, 1-\alpha/2)$ so that

$$1 > \xi\gamma + 1 - \alpha/2 - \delta > \xi_1. \tag{55}$$

Note that $\xi\gamma < 1$ shows we may choose δ in the required range. Next choose $\delta' \in (0, \delta)$ and $p \in (0, \xi\gamma)$ so that

$$1 > p + 1 - \alpha/2 - \delta > \xi_1 \tag{56}$$

and

$$1 > \xi\gamma + 1 - \alpha/2 - \delta' > \xi_1. \tag{57}$$

Now consider, for some random $N_1 = N_1(\omega, \xi, \xi_1)$ to be chosen below,

$$\begin{aligned} &\mathbb{P}\big(|\tilde{u}(t,x) - \tilde{u}(t,y)| \geq |x-y|^{1-\alpha/2-\delta}\varepsilon^p, (t,x) \in Z_{K,N,\xi}, N \geq N_1\big) \\ &\quad + \mathbb{P}\big(|\tilde{u}(t',x) - \tilde{u}(t,x)| \geq |t'-t|^{1/2(1-\alpha/2-\delta)}\varepsilon^p, \\ &\qquad (t,x) \in Z_{K,N,\xi}, t' \leq T_K, N \geq N_1\big). \end{aligned} \tag{58}$$

In order to simplify notation, we define

$$\begin{aligned} D^{x,y,t,t'}(w,z,s) &= |p_{t-s}(x-w) - p_{t'-s}(y-w)||p_{t-s}(x-z) - p_{t'-s}(y-z)| \\ &\quad \times |\tilde{u}(s,w)|^\gamma |\tilde{u}(s,z)|^\gamma [|w-z|^{-\alpha} + 1], \\ D^{x,t'}(w,z,s) &= p_{t'-s}(x-w)p_{t'-s}(x-z)|\tilde{u}(s,w)|^\gamma |\tilde{u}(s,z)|^\gamma [|w-z|^{-\alpha} + 1]. \end{aligned}$$

With this notation expression (58) is bounded by

$$\begin{aligned} &\mathbb{P}\bigg(|\tilde{u}(t,x) - \tilde{u}(t,y)| \geq |x-y|^{1-\alpha/2-\delta}\varepsilon^p, (t,x) \in Z_{K,N,\xi}, N \geq N_1, \\ &\qquad \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^{x,y,t,t}(w,z,s)\,dw\,dz\,ds \leq |x-y|^{2-\alpha-2\delta'}\varepsilon^{2p}\bigg) \\ &\quad + \mathbb{P}\bigg(|\tilde{u}(t',x) - \tilde{u}(t,x)| \geq |t'-t|^{1/2(1-\alpha/2-\delta)}\varepsilon^p, \\ &\qquad (t,x) \in Z_{K,N,\xi}, t' \leq T_K, N \geq N_1, \\ &\qquad \int_t^{t'} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^{x,t'}(w,z,s)\,dw\,dz\,ds \\ &\qquad + \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^{x,x,t,t'}(w,z,s)\,dw\,dz\,ds \leq (t'-t)^{1-\alpha/2-\delta'}\varepsilon^{2p}\bigg) \\ &\quad + \mathbb{P}\bigg(\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^{x,y,t,t}(w,z,s)\,dw\,dz\,ds > |x-y|^{2-\alpha-2\delta'}\varepsilon^{2p}, \\ &\qquad (t,x) \in Z_{K,N,\xi}, N \geq N_1\bigg) \end{aligned} \tag{59}$$

$$+\mathbb{P}\biggl(\int_t^{t'}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^{x,t'}(w,z,s)\,dw\,dz\,ds$$
$$+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^{x,x,t,t'}(w,z,s)\,dw\,dz\,ds$$
$$> (t'-t)^{1-\alpha/2-\delta'}\varepsilon^{2p}, (t,x)\in Z_{K,N,\xi}, t'\le T_K, N\ge N_1\biggr)$$
$$=: P_1+P_2+P_3+P_4.$$

Notice that the processes

$$\tilde{t}\mapsto \int_0^{\tilde{t}}\int_{\mathbb{R}^d} p_{t-s}(x-w)\bigl(\sigma\bigl(u^1(s,w)\bigr)-\sigma\bigl(u^2(s,w)\bigr)\bigr)W(dw\,ds)$$

are continuous local martingales for any fixed x,t on $0\le\tilde{t}\le t$. We bound the appropriate differences of these integrals by considering the respective quadratic variations of $\tilde{u}(t,x)-\tilde{u}(t,y)$ and $\tilde{u}(t',x)-\tilde{u}(t,x)$ [see (12)]. If $|\sigma(u)-\sigma(v)|\le L|u-v|^\gamma$ and recalling that $|k(x,y)|\le c_{1.4}[|x-y|^{-\alpha}+1]$, we see that the time integrals in the above probabilities differ from the appropriate square functions by a multiplicative factor of $L^2c_{1.4}$.

If $\delta''=\delta-\delta'>0$, B is a standard one-dimensional Brownian motion with $B(0)=0$, and $B^*(t):=\sup_{0\le s\le t}|B(s)|$, then the first two probabilities of (59) can be bounded using the Dubins–Schwarz theorem

$$\begin{aligned}P_1&\le\mathbb{P}\bigl(B^*(c_{1.4}L^2|x-y|^{2-\alpha-2\delta'}\varepsilon^{2p})\ge|x-y|^{1-\alpha/2-\delta}\varepsilon^p\bigr)\\&=\mathbb{P}\bigl(B^*(1)\sqrt{c_{1.4}}L|x-y|^{1-\alpha/2-\delta'}\varepsilon^p\ge|x-y|^{1-\alpha/2-\delta}\varepsilon^p\bigr)\\&=\mathbb{P}\bigl(B^*(1)\ge\bigl(\sqrt{c_{1.4}}L\bigr)^{-1}|x-y|^{-\delta''}\bigr)\\&\le c_{60}\exp(-c'_{60}|x-y|^{-\delta''}),\end{aligned}\tag{60}$$

where we have used the reflection principle in the last line. Likewise,

$$\begin{aligned}P_2&\le\mathbb{P}\bigl(B^*(c_{1.4}L^2|t'-t|^{1-\alpha/2-\delta'}\varepsilon^{2p})\ge|t'-t|^{1/2(1-\alpha/2-\delta)}\varepsilon^p\bigr)\\&=\mathbb{P}\bigl(B^*(1)\ge\bigl(\sqrt{c_{1.4}}L\bigr)^{-1}|t'-t|^{-\delta''/2}\bigr)\\&\le c_{60}\exp(-c'_{60}|t'-t|^{-\delta''/2}).\end{aligned}\tag{61}$$

Here the constants c_{60} and c'_{60} depend on d, L and $c_{1.4}$.

In order to bound P_3 and P_4, we estimate the respective integral expressions by splitting them up in several parts: Let $\delta_1\in(0,\frac{1}{2}(1-\frac{\alpha}{2}))$ and $t_0=0, t_1=t-\varepsilon^2, t_2=t$ and $t_3=t'$. We also define

$$\begin{aligned}&A_1^{1,s}(x)=\bigl\{w\in\mathbb{R}^d:|x-w|\le2\sqrt{t-s}\,\varepsilon^{-\delta_1}\bigr\}\quad\text{and}\\&A_2^{1,s}(x)=\mathbb{R}^d\setminus A_1^{1,s}(x),\end{aligned}\tag{62}$$

$$A_1^2(x)=\{w\in\mathbb{R}^d:|x-w|\le2\varepsilon^{1-\delta_1}\}\quad\text{and}\quad A_2^2(x)=\mathbb{R}^d\setminus A_1^2(x).\tag{63}$$

For notational convenience, we will sometimes omit the index s for $A_i^1(x)$. We continue to write

$$Q^{x,y,t,t'} := \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D^{x,y,t,t'}(w,z,s)\,dw\,dz\,ds = \sum_{i,j,k=1,2} Q_{i,j,k}^{x,y,t,t'},$$

where

$$Q_{i,j,k}^{x,y,t,t'} := \int_{t_{i-1}}^{t_i} \int_{A_j^i(x)} \int_{A_k^i(x)} D^{x,y,t,t'}(w,z,s)\,dw\,dz\,ds.$$

And likewise,

$$Q^{x,t,t'} := \int_t^{t'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D^{x,t'}(w,z,s)\,dw\,dz\,ds = \sum_{j,k=1,2} Q_{j,k}^{x,t,t'},$$

where

$$Q_{j,k}^{x,t,t'} := \int_t^{t'} \int_{A_j^2(x)} \int_{A_k^2(x)} D^{x,t'}(w,z,s)\,dw\,dz\,ds.$$

Before we proceed, let us note that $\tilde{u}$ can be bounded on the sets A_1^i as follows: Set

$$N_1(\omega) = \left[\frac{5N_\xi(\omega)}{\delta_1}\right] \geq \left[\frac{N_\xi(\omega)+4}{1-\delta_1}\right] \in \mathbb{N}, \tag{64}$$

where $[\cdot]$ is the greatest integer function and assume $N \geq N_1$ in the following.

Recall $\lambda > 0$ is a fixed constant used in the definition of T_K and, hence, $Z_{K,N,\xi}$. As it is fixed, we often suppress dependence on λ in our notation. $\square$

LEMMA 5.4. *Let $N \geq N_1$. Then on $\{\omega : (t,x) \in Z_{K,N,\xi}\}$,*

$$|\tilde{u}(s,w)| \leq 10\varepsilon^{(1-\delta_1)\xi} \qquad \text{for } s \in [t-\varepsilon^2, t'], w \in A_1^2(x), \tag{65}$$

$$|\tilde{u}(s,w)| \leq (8 + 3K2^{N_\xi \xi})e^{\lambda|w|}(t-s)^{\xi/2}\varepsilon^{-\delta_1\xi} \qquad \text{for } s \in [0, t-\varepsilon^2], w \in A_1^{1,s}(x). \tag{66}$$

PROOF. We choose $N' \in \mathbb{N}$ so that $2^{-N'-1} \leq 3\varepsilon^{1-\delta_1} \leq 2^{-N'}$. Then $2^{-N'-3} < 2^{-N(1-\delta_1)} \leq 2^{-N'-1}$, and so by (64),

$$N' > N(1-\delta_1) - 3 \geq N_1(1-\delta_1) - 3 \geq N_\xi. \tag{67}$$

Assume $(t,x) \in Z_{K,N,\xi}$, $0 \leq t' \leq T_K$, and choose $(\hat{t},\hat{x})$ such that

$$\hat{t} \leq T_K, d\big((t,x),(\hat{t},\hat{x})\big) < \varepsilon = 2^{-N} \quad \text{and} \quad |\tilde{u}(\hat{t},\hat{x})| \leq 2^{-N\xi} = \varepsilon^\xi. \tag{68}$$

We first observe that for $s \in [t-\varepsilon^2, t']$ and $w \in A_1^2(x)$ so that $|w-x| \leq 2\varepsilon^{1-\delta_1}$, we have

$$d\big((s,w),(t,x)\big) \leq \varepsilon + 2\varepsilon^{1-\delta_1} \leq 3\varepsilon^{1-\delta_1} \leq 2^{-N'}. \tag{69}$$

Therefore, by (36) and (67), for $s \in [t-\varepsilon^2, t']$ and $w \in A_1^2(x)$,

$$
\begin{aligned}
|\tilde{u}(s,w)| &\le |\tilde{u}(\hat{t},\hat{x})| + |\tilde{u}(\hat{t},\hat{x}) - \tilde{u}(t,x)| + |\tilde{u}(t,x) - \tilde{u}(s,w)| \\
&\le 2 \cdot 2^{-N\xi} + 2^{-N'\xi} \\
&\le 2\varepsilon^{\xi} + (8\varepsilon^{1-\delta_1})^{\xi} \\
&\le 10\varepsilon^{(1-\delta_1)\xi},
\end{aligned}
\tag{70}
$$

which proves (65). Similarly, if $s \in [0, t-\varepsilon^2]$ and $w \in A_1^{1,s}(x)$, meaning that $|w-x| \le 2\sqrt{t-s}\,\varepsilon^{-\delta_1}$, we have

$$
d\big((s,w),(t,x)\big) \le \sqrt{t-s} + 2\sqrt{t-s}\,\varepsilon^{-\delta_1} \le 3\sqrt{t-s}\,\varepsilon^{-\delta_1}. \tag{71}
$$

Notice that if $3\sqrt{t-s}\,\varepsilon^{-\delta_1} \le 2^{-N_\xi}$, then there exists an $N' \ge N_\xi$ such that $2^{-(N'+1)} \le 3\sqrt{t-s}\,\varepsilon^{-\delta_1} \le 2^{-N'}$ so that we can as in (65) bound

$$
\begin{aligned}
|\tilde{u}(s,w)| &\le |\tilde{u}(\hat{t},\hat{x})| + |\tilde{u}(\hat{t},\hat{x}) - \tilde{u}(t,x)| + |\tilde{u}(t,x) - \tilde{u}(s,w)| \\
&\le 2^{-N\xi} + 2^{-N\xi} + 2^{-N'\xi} \\
&\le 2 \cdot 2^{-N\xi} + 2 \cdot 2^{-(N'+1)\xi} \\
&\le 2(t-s)^{\xi/2} + 2 \cdot 3^{\xi}(t-s)^{\xi/2}\varepsilon^{-\delta_1\xi} \\
&\le 8(t-s)^{\xi/2}\varepsilon^{-\delta_1\xi},
\end{aligned}
\tag{72}
$$

since $\varepsilon = 2^{-N} \le \sqrt{t-s}$. If, on the other hand, $3\sqrt{t-s}\,\varepsilon^{-\delta_1} > 2^{-N_\xi}$, then we bound

$$
\begin{aligned}
|\tilde{u}(s,w)| &\le K e^{\lambda|w|} \\
&= \big(K(t-s)^{-\xi/2}\big)e^{\lambda|w|}(t-s)^{\xi/2} \\
&\le (K\varepsilon^{-\delta_1\xi}3^{\xi}2^{N_\xi\xi})e^{\lambda|w|}(t-s)^{\xi/2}.
\end{aligned}
\tag{73}
$$

Taking (72) and (73) together, we obtain (66). $\square$

In the rest of this section $C(K)$ denotes a constant depending on K (and possibly λ) which may change from line to line. We will first consider the terms for which $j = k = 1$ so that we can use the bounds (65) and (66) of Lemma 5.4:

LEMMA 5.5. *If* $0 < \beta < 1 - \frac{\alpha}{2}$, $\beta' < \xi\gamma + 1 - \frac{\alpha}{2}$, *and* $\beta' \le 1$, *then on* $\{\omega : (t,x) \in Z_{K,N,\xi}\}$,

$$
Q_{2,1,1}^{x,y,t,t} \le c_{74}(\alpha,d,\beta,K)\varepsilon^{2(1-\delta_1)\xi\gamma}|x-y|^{2\beta}, \tag{74}
$$

$$
Q_{2,1,1}^{x,x,t,t'} \le c_{74}(\alpha,d,\beta,K)\varepsilon^{2(1-\delta_1)\xi\gamma}|t'-t|^{\beta}, \tag{75}
$$

$$
Q_{1,1,1}^{x,y,t,t} \le c_{76}(\alpha,d,\beta',\xi\gamma,K)(8 + 3K2^{N_\xi\xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma}|x-y|^{2\beta'}, \tag{76}
$$

$$Q_{1,1,1}^{x,x,t,t'} \le c_{76}(\alpha, d, \beta', \xi\gamma, K)(8+3K2^{N_\xi \xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma}|t'-t|^{\beta'}, \tag{77}$$

$$Q_{1,1}^{x,t,t'} \le c_{78}(\alpha, d)\varepsilon^{2\gamma\xi(1-\delta_1)}|t'-t|^{1-\alpha/2}. \tag{78}$$

PROOF. Using the bounds (65) and (66) of Lemma 5.4, we obtain

$$\begin{aligned} Q_{2,1,1}^{x,y,t,t'} \le{}& 100^\gamma \varepsilon^{2(1-\delta_1)\xi\gamma} \\ &\times \int_{t-\varepsilon^2}^{t}\int_{A_1^2(x)}\int_{A_1^2(x)} |p_{t-s}(x-w)-p_{t'-s}(y-w)| \\ &\qquad\times |p_{t-s}(x-z)-p_{t'-s}(y-z)| \\ &\qquad\times [|w-z|^{-\alpha}+1]\,dw\,dz\,ds, \end{aligned} \tag{79}$$

$$\begin{aligned} Q_{1,1,1}^{x,y,t,t'} \le{}& (8+3K2^{N_\xi\xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma} \\ &\times \int_0^{t-\varepsilon^2}(t-s)^{\xi\gamma}\int_{A_1^1(x)}\int_{A_1^1(x)} e^{\lambda\gamma|w|}e^{\lambda\gamma|z|} \\ &\qquad\times |p_{t-s}(x-w)-p_{t'-s}(y-w)| \\ &\qquad\times |p_{t-s}(x-z)-p_{t'-s}(y-z)| \\ &\qquad\times [|w-z|^{-\alpha}+1]\,dw\,dz\,ds. \end{aligned} \tag{80}$$

Note that the above integrals only become larger if we integrate over the domain $[0,t]\times\mathbb{R}^{2d}$, which we will do in the following. We will use a version of the factorization method first introduced in [2] to estimate them. Noting that, for $s \le t$ and $0 < a < 1$,

$$\int_s^t (t-r)^{a-1}(r-s)^{-a}\,dr = \frac{\pi}{\sin(\pi a)}, \tag{81}$$

and that for $s \le r \le t$,

$$\begin{aligned} &|p_{t-s}(x-w)-p_{t'-s}(y-w)| \\ &\quad\le \int_{\mathbb{R}^d} p_{r-s}(w'-w)\cdot|p_{t-r}(x-w')-p_{t'-r}(y-w')|\,dw', \end{aligned} \tag{82}$$

we obtain, with (79),

$$\begin{aligned} Q_{2,1,1}^{x,y,t,t'} \le{}& C(a)\varepsilon^{2(1-\delta_1)\xi\gamma} \\ &\times \int_0^t\int_0^t (t-r)^{a-1}(t-r')^{a-1} \\ &\qquad\times \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J_{r,r'}^2(w',z') \\ &\qquad\qquad\times |p_{t-r}(x-w')-p_{t'-r}(y-w')| \\ &\qquad\qquad\times |p_{t-r'}(x-z')-p_{t'-r'}(y-z')|\,dw'\,dz'\,dr\,dr', \end{aligned} \tag{83}$$

where

$$J^2_{r,r'}(w',z') := \int_0^{r\wedge r'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (r-s)^{-a}(r'-s)^{-a} p_{r-s}(w'-w) p_{r'-s}(z'-z) \times [|w-z|^{-\alpha}+1]\, dw\, dz\, ds, \tag{84}$$

where $J^2_{r,r'}(w',z') \le J^2_{r,r'}(0,0)$ according to (45) of Lemma 5.1. So we get

$$\begin{aligned} Q^{x,y,t,t'}_{2,1,1} &\le C(a)\varepsilon^{2(1-\delta_1)\xi\gamma} \\ &\quad \times \int_0^t \int_0^t (t-r)^{a-1}(t-r')^{a-1} J^2_{r,r'}(0,0) \\ &\quad \times \left(\int_{\mathbb{R}^d} |p_{t-r}(x-w') - p_{t'-r}(y-w')|\, dw' \right) \\ &\quad \times \left(\int_{\mathbb{R}^d} |p_{t-r'}(x-z') - p_{t'-r'}(y-z')|\, dz' \right) dr\, dr'. \end{aligned} \tag{85}$$

The integrals in brackets can now be estimated with the help of (48) in Lemma 5.2. Recall that $(t,x) \in Z_{K,N,\xi}$ and $|x-y| \le 2^{-N}$, so that $|x| \le K$, $|y| \le K+1$, and $t \le K$, and so (48) implies

$$\begin{aligned} Q^{x,y,t,t}_{2,1,1} &\le C(a,\alpha,d)\varepsilon^{2(1-\delta_1)\xi\gamma}|x-y|^{2\beta} Q(t,a,0,\beta/2,\alpha) \\ &\le C(\alpha,d,\beta,K)\varepsilon^{2(1-\delta_1)\xi\gamma}|x-y|^{2\beta} t^{1-\alpha/2-\beta}[1+t^{\alpha/2}]. \end{aligned} \tag{86}$$

Here we use $\beta < 1-\frac{\alpha}{2}$ and choose $a \in (\beta/2, 1-(\alpha/2))$ so that Lemma 5.3 may be applied in the last line. As $t \le K$, (74) follows. Likewise, we get for the time differences, $\beta < 1-\frac{\alpha}{2}$ and $\frac{\beta}{2} < a < 1-\alpha/2$ (use Lemma 5.2 with $\beta/2$ in place of β),

$$\begin{aligned} Q^{x,x,t,t'}_{2,1,1} &\le C(a,d)\varepsilon^{2(1-\delta_1)\xi\gamma}|t'-t|^{\beta} Q(t,a,0,\beta/2,\alpha) \\ &\le C(\beta,\alpha,d,K)\varepsilon^{2(1-\delta_1)\xi\gamma}|t'-t|^{\beta}, \end{aligned} \tag{87}$$

which is (75).

With an analogous calculation as in (81) to (85) except now using (80) instead of (79), we obtain that

$$\begin{aligned} Q^{x,y,t,t'}_{1,1,1} &\le C(a)(8+3K2^{N_\xi \xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma} \\ &\quad \times \int_0^t \int_0^t (t-r)^{a-1}(t-r')^{a-1} \\ &\quad \times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J^1_{r,r'}(w',z') \end{aligned} \tag{88}$$

$$\times |p_{t-r}(x-w') - p_{t'-r}(y-w')|$$

$$\times |p_{t-r'}(x-z') - p_{t'-r'}(y-z')|\, dw'\, dz'\, dr\, dr',$$

where

$$\begin{aligned}
J^1_{r,r'}(w',z') &= \int_0^{r\wedge r'} (t-s)^{\xi\gamma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (r-s)^{-a}(r'-s)^{-a} \\
&\qquad\qquad \times p_{r-s}(w'-w)p_{r'-s}(z'-z) \\
&\qquad\qquad \times e^{\lambda\gamma(|w|+|z|)}[|w-z|^{-\alpha}+1]\, dw\, dz\, ds \\
&\le c_d^2 e^{\lambda^2(r+r')} e^{\lambda\gamma(|w'|+|z'|)} \\
&\quad \times \int_0^{r\wedge r'} (t-s)^{\xi\gamma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (r-s)^{-a}(r'-s)^{-a} \\
&\qquad\qquad \times p_{2(r-s)}(w'-w)p_{2(r'-s)}(z'-z) \\
&\qquad\qquad \times [|w-z|^{-\alpha}+1]\, dw\, dz\, ds \\
&=: c_d^2 e^{\lambda^2(r+r')} e^{\lambda\gamma(|w'|+|z'|)} \tilde{J}^1_{r,r'}(w',z'),
\end{aligned} \tag{89}$$

where in the second inequality we have bounded $|w| \le |w'| + |w'-w|$ (and likewise for z) and then used (47). Again, $\tilde{J}^1_{r,r'}(w',z') \le \tilde{J}^1_{r,r'}(0,0)$ independent of w' and z' due to (45) of Lemma 5.1. Hence, we get

$$\begin{aligned}
Q^{x,y,t,t'}_{1,1,1} &\le C(d,a)e^{2\lambda^2 K}(8+3K2^{N_\xi \xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma} \\
&\quad \times \int_0^t\int_0^t (t-r)^{a-1}(t-r')^{a-1}\tilde{J}^1_{r,r'}(0,0) \\
&\qquad \times \left(\int_{\mathbb{R}^d} |p_{t-r}(x-w') - p_{t'-r}(y-w')| e^{\lambda\gamma|w'|}\, dw'\right) \\
&\qquad \times \left(\int_{\mathbb{R}^d} |p_{t-r'}(x-z') - p_{t'-r'}(y-z')| e^{\lambda\gamma|z'|}\, dz'\right) dr\, dr'.
\end{aligned} \tag{90}$$

And so, after a change of variables, the spatial differences are bounded by

$$\begin{aligned}
Q^{x,y,t,t}_{1,1,1} &\le C(d,a)e^{2\lambda^2 K}(8+3K2^{N_\xi \xi})^{2\gamma} e^{4\lambda^2 K + 2\lambda(K+1)+4\lambda} \\
&\quad \times \varepsilon^{-2\delta_1\xi\gamma}|x-y|^{2\beta'} Q\left(2t, a, \xi\gamma, \frac{\beta'}{2}, \alpha\right) \\
&\le C(d,\xi\gamma,\beta',\alpha,K)(8+3K2^{N_\xi\xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma}|x-y|^{2\beta'}
\end{aligned} \tag{91}$$

if $\beta' < \xi\gamma + 1 - \frac{\alpha}{2}$, $\beta' \le 1$, and a is chosen in $(\beta'/2, 1-\alpha/2) \ne \varnothing$ (recall $\alpha < 1$), according to (48) of Lemma 5.2 and Lemma 5.3 combined with (47). This proves (76). Similarly, for the time differences, we obtain

$$
\begin{aligned}
Q_{1,1,1}^{x,x,t,t'} &\le C(d,a)e^{2\lambda^2 K}(8+3K2^{N_\xi \xi})^{2\gamma} e^{4\lambda^2(K+1)+2\lambda(K+1)+4\lambda} \\
&\qquad \times \varepsilon^{-2\delta_1\xi\gamma}|t'-t|^{\beta'} Q\Big(2t, a, \xi\gamma, \frac{\beta'}{2}, \alpha\Big) \qquad (92)\\
&\le C(d,\xi\gamma,\beta',\alpha,K)(8+3K2^{N_\xi\xi})^{2\gamma}\varepsilon^{-2\delta_1\xi\gamma}|t'-t|^{\beta'}
\end{aligned}
$$

if again $\beta' < \xi\gamma + 1 - \frac{\alpha}{2}$, $\beta' \le 1$ and a is chosen as above. This shows (77).

Finally, we address the remaining case using (65) of Lemma 5.4 to bound $\tilde{u}$ and Lemma 5.1:

$$
\begin{aligned}
Q_{1,1}^{x,t,t'} &\le C\varepsilon^{2\gamma\xi(1-\delta_1)} \\
&\qquad \times \int_t^{t'} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t'-s}(x-w)p_{t'-s}(x-z)[|w-z|^{-\alpha}+1]\,dw\,dz\,ds \\
&\le C(\alpha,d)\varepsilon^{2\gamma\xi(1-\delta_1)}\int_t^{t'}[(t'-s)^{-\alpha/2}+1]\,ds \qquad (93)\\
&\le C(\alpha,d)\varepsilon^{2\gamma\xi(1-\delta_1)}[|t'-t|^{1-\alpha/2}+|t'-t|],
\end{aligned}
$$

and (78) follows as $|t'-t| \le 1$. $\square$

Next, we consider all the terms for which $j = k = 2$. Here, we will use that, for $t \le T_K$, we can bound $|\tilde{u}(t,x)| \le Ke^{\lambda|x|}$.

LEMMA 5.6. *For $0 < \beta < 1 - \frac{\alpha}{2}$, we obtain, for $i = 1, 2$, and on $\{\omega : (t,x) \in Z_{K,N,\xi}\}$,*

$$Q_{i,2,2}^{x,y,t,t} \le c_{94}(d,\alpha,K)\exp\big(-\tfrac{1}{4}\varepsilon^{-2\delta_1}(1-\beta)\big)|x-y|^{2\beta}, \qquad (94)$$

$$Q_{i,2,2}^{x,x,t,t'} \le c_{95}(d,\alpha,K)\exp\Big(-\frac{1}{4}\varepsilon^{-2\delta_1}\Big(1-\frac{\beta}{2}\Big)\Big)|t'-t|^{\beta}, \qquad (95)$$

$$Q_{2,2}^{x,t,t'} \le c_{96}(d,\alpha,\beta,K)\exp\big(-\tfrac{1}{2}\varepsilon^{-2\delta_1}(1-\beta)\big)|t'-t|^{1-\alpha/2}. \qquad (96)$$

PROOF. Recall $d((t,x),(t',y)) \le \varepsilon$. For $i = 1$, we are interested in the case $s \in [0, t-\varepsilon^2]$ and $|x-w| > 2\sqrt{t-s}\,\varepsilon^{-\delta_1}$. Since $|x-y| < \varepsilon$, this implies that $|y-w| \ge ||x-w|-|x-y|| > 2\sqrt{t-s}\,\varepsilon^{-\delta_1} - \varepsilon > \sqrt{t-s}\,\varepsilon^{-\delta_1}$. Furthermore, $t'-s = t'-t+t-s \le \varepsilon^2 + t - s \le 2(t-s)$. This implies

$$
\begin{aligned}
&\exp\Big(-\frac{|x-w|^2}{4(t'-s)}\Big) \vee \exp\Big(-\frac{|y-w|^2}{4(t'-s)}\Big) \\
&\qquad \le \exp\Big(-\frac{|x-w|^2}{8(t-s)}\Big) \vee \exp\Big(-\frac{|y-w|^2}{8(t-s)}\Big) \qquad (97)\\
&\qquad \le \exp\Big(-\frac{1}{8}\varepsilon^{-2\delta_1}\Big).
\end{aligned}
$$

Therefore, for $v = x$ or $v = y$ and $r = t$ or $r = t'$,

$$p_{r-s}(v-w) \le 2^{d/2} \exp\bigl(-\tfrac{1}{8}\varepsilon^{-2\delta_1}\bigr) p_{2(r-s)}(v-w). \tag{98}$$

Using this, we obtain for any $\beta \in (0,1)$ by applying Hölder's inequality that $Q_{1,2,2}^{x,y,t,t'}$ is bounded by

$$\begin{aligned}
&\int_0^{t-\varepsilon^2} \Bigl(\int_{A_2^{1,s}(x)} \int_{A_2^{1,s}(x)} \bigl(p_{t-s}(x-w) + p_{t'-s}(y-w)\bigr) \\
&\qquad\qquad \times \bigl(p_{t-s}(x-z) + p_{t'-s}(y-z)\bigr) \\
&\qquad\qquad \times [|w-z|^{-\alpha} + 1] \\
&\qquad\qquad \times |\tilde{u}(s,w)|^{\gamma/(1-\beta)} |\tilde{u}(s,z)|^{\gamma/(1-\beta)}\, dw\, dz \Bigr)^{1-\beta} \\
&\quad \times \Bigl(\int_{A_2^{1,s}(x)} \int_{A_2^{1,s}(x)} |p_{t-s}(x-w) - p_{t'-s}(y-w)| \\
&\qquad\qquad \times |p_{t-s}(x-z) - p_{t'-s}(y-z)| \\
&\qquad\qquad \times [|w-z|^{-\alpha} + 1]\, dw\, dz \Bigr)^{\beta} ds \\
&\le C(d,\alpha) e^{2\lambda^2 K} K^{2\gamma} e^{2\lambda\gamma(K+1)} \exp\bigl(-\tfrac{1}{4}\varepsilon^{-2\delta_1}(1-\beta)\bigr) \\
&\quad \times \int_0^{t-\varepsilon^2} \bigl[(t-s)^{-\alpha/2(1-\beta)} + 1\bigr] \\
&\qquad \times \Bigl(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |p_{t-s}(x-w) - p_{t'-s}(y-w)| \\
&\qquad\qquad \times |p_{t-s}(x-z) - p_{t'-s}(y-z)| \\
&\qquad\qquad \times [|w-z|^{-\alpha} + 1]\, dw\, dz \Bigr)^{\beta} ds.
\end{aligned} \tag{99}$$

Here, we have used that $\tilde{u}(t,x) \le K e^{\lambda|x|}$ and (98), as well as (46) of Lemma 5.1. We have also used the fact that $e^{\lambda\gamma|x|}$ and $e^{\lambda\gamma|y|}$ are both bounded by $e^{\lambda\gamma(K+1)}$ since $|x| < K$ and $|x-y| < \varepsilon < 1$. Using (49) of Lemma 5.2 to estimate the integral in parentheses when $t = t'$, we obtain

$$\begin{aligned}
Q_{1,2,2}^{x,y,t,t} &\le C(d,\alpha) e^{2\lambda^2 K} \bigl(K e^{\lambda(K+1)}\bigr)^2 \exp\bigl(-\tfrac{1}{4}\varepsilon^{-2\delta_1}(1-\beta)\bigr) |x-y|^{2\beta} \\
&\quad \times \int_0^{t-\varepsilon^2} (t-s)^{-\alpha/2-\beta} + (t-s)^{-\beta}\, ds \\
&\le C(d,\alpha) K^3 e^{2(\lambda^2+\lambda)(K+1)} \exp\bigl(-\tfrac{1}{4}\varepsilon^{-2\delta_1}(1-\beta)\bigr) |x-y|^{2\beta},
\end{aligned} \tag{100}$$

provided that $\beta < 1 - \frac{\alpha}{2}$, showing (94) for $i = 1$. Likewise, using (50) of Lemma 5.2 for the time differences when $x = y$ implies (95) for $i = 1$ if again $0 < \beta < 1 - \frac{\alpha}{2}$—here we replace β with $\beta/2$ in the above.

For $i = 2$ and $Q_{2,2}^{x,t,t'}$, we will proceed analogously. We merely have to establish (97) in the case: $s \in [t - \varepsilon^2, t']$ and $|x - w| > 2\varepsilon^{1-\delta_1}$. Since $|x - y| < \varepsilon$, this implies now $|y - w| \geq ||x - w| - |x - y|| > 2\varepsilon^{1-\delta_1} - \varepsilon > \varepsilon^{1-\delta_1}$. Furthermore, $t' - s = t' - t + t - s \leq \varepsilon^2 + t - s \leq 2\varepsilon^2$. From this, the bound (97) follows and we obtain immediately (94) and (95) for $i = 2$, provided that $\beta < 1 - \frac{\alpha}{2}$.

Last, we obtain with the help of (97) (verified above) and (46) of Lemma 5.1,

$$
\begin{aligned}
Q_{2,2}^{x,t,t'} &\leq C(d,\beta)K^{2\gamma} \exp\left(-\frac{1}{2}\varepsilon^{-2\delta_1}(1-\beta)\right) \\
&\qquad \times \int_t^{t'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\lambda\gamma(|w|+|z|)} p_{(t'-s)/\beta}(x-w) p_{(t'-s)/\beta}(x-z) \\
&\qquad\qquad \times [|w-z|^{-\alpha} + 1]\, dw\, dz\, ds \\
&\leq C(d,\beta,\alpha)K^{2\gamma} e^{2\lambda^2(K+1)/\beta} e^{2\lambda(K+1)} \\
&\qquad \times \exp\left(-\frac{1}{2}\varepsilon^{-2\delta_1}(1-\beta)\right) \int_t^{t'} \left[\left(\frac{2}{\beta}(t'-s)\right)^{-\alpha/2} + 1\right] ds \\
&\leq C(d,\alpha,\beta,K) \exp\left(-\frac{1}{2}\varepsilon^{-2\delta_1}(1-\beta)\right) |t'-t|^{1-\alpha/2},
\end{aligned} \tag{101}
$$

which is (96) and, hence, completes the proof. $\square$

It remains to consider the "mixed terms" for which $j = 2$ and $k = 1$ or vice versa. Say $j = 2$. In this case (97) holds for the exponential in the w integral, and we can bound the exponential in the z integral by one. Otherwise, we follow the same steps as in Lemma 5.6, treating the case $j = k = 2$. In this manner, we obtain the same bounds as in (94) to (96) with the only difference that $\exp(-\frac{1}{4}\varepsilon^{-2\delta_1}(1-\beta))$ is replaced by $\exp(-\frac{1}{8}\varepsilon^{-2\delta_1}(1-\beta))$ and $\exp(-\frac{1}{2}\varepsilon^{-2\delta_1}(1-\beta))$ by $\exp(-\frac{1}{4}\varepsilon^{-2\delta_1}(1-\beta))$.

We are now taking the estimates (74), (76) and (94) together with those for the mixed terms and choosing $\beta = 1 - \frac{\alpha}{2} - \delta'$, respectively, $\beta' = 1 - \frac{\alpha}{2} - \delta' + \xi\gamma < 1$ [by (57)], in those estimates. This shows that, for $(t,x) \in Z_{K,N,\xi}$, $|x - y| < \varepsilon = 2^{-N}$ and $N \geq N_1$,

$$
\begin{aligned}
Q^{x,y,t,t} &\leq C(K)|x-y|^{2(1-\alpha/2-\delta')} \\
&\qquad \times \Big[\varepsilon^{2(1-\delta_1)\xi\gamma} + (8 + 3K2^{N_\xi\xi})^{2\gamma} \varepsilon^{-2\delta_1\xi\gamma}|x-y|^{2\xi\gamma} \\
&\qquad\qquad + \exp\left(-\frac{1}{4}\varepsilon^{-2\delta_1}\left(\frac{\alpha}{2} + \delta'\right)\right) + \exp\left(-\frac{1}{8}\varepsilon^{-2\delta_1}\left(\frac{\alpha}{2} + \delta'\right)\right)\Big]
\end{aligned} \tag{102}
$$

$$\leq C(K)|x-y|^{2(1-\alpha/2-\delta')}\bigg[\varepsilon^{2(1-\delta_1)\xi\gamma}2^{2N_\xi\xi\gamma}+\exp\bigg(-\frac{\alpha}{16}\varepsilon^{-2\delta_1}\bigg)\bigg].$$

We have the analogous bounds for $Q^{x,x,t,t'}+Q^{x,t,t'}$ with the help of (75), (77), (78), (95) and (96); just replace $|x-y|^2$ with $|t'-t|$ and use $|t'-t|<\varepsilon^2$. We deduce that, for $N\geq N_1$ and $(t,x)\in Z_{K,N,\xi}$,

$$\begin{aligned}&Q^{x,x,t,t'}+Q^{x,t,t'}\\ &\quad\leq C(K)|t'-t|^{1-\alpha/2-\delta'}\bigg[\varepsilon^{2(1-\delta_1)\xi\gamma}2^{2N_\xi\xi\gamma}+\exp\bigg(-\frac{\alpha}{16}\varepsilon^{-2\delta_1}\bigg)\bigg].\end{aligned}\tag{103}$$

We can finally conclude that, in (59), $P_3=P_4=0$ if

$$C(K)\bigg[\varepsilon^{2(1-\delta_1)\xi\gamma}2^{2N_\xi\xi\gamma}+\exp\bigg(-\frac{\alpha}{16}\varepsilon^{-2\delta_1}\bigg)\bigg]<\varepsilon^{2p}.\tag{104}$$

For this, it is sufficient that

$$C(K)\varepsilon^{2(1-\delta_1)\xi\gamma}2^{2N_\xi\xi\gamma}<\tfrac{1}{2}\varepsilon^{2p},\tag{105}$$

$$C(K)\exp\bigg(-\frac{\alpha}{16}\varepsilon^{-2\delta_1}\bigg)<\frac{1}{2}\varepsilon^{2p}.\tag{106}$$

Since (105) is equivalent to $2C(K)<2^{2N[(1-\delta_1)\xi\gamma-p]-2N_\xi\xi\gamma}$, it suffices to choose $\delta_1>0$ small enough so that $(1-\delta_1)\xi\gamma-p>0$ (which is possible since $\xi\gamma>p$) and then to assume $N\geq[C_0(\xi,\delta_1)N_\xi]\in\mathbb{N}$, as well as $N\geq N_0(K,\xi,\delta_1,p)\in\mathbb{N}$ deterministic, so that both (105) and (106) hold. Note that the constants depend ultimately on ξ,ξ_1 and K. Hence, (59), (60) and (61) imply that if $N_2(\omega,\xi,\xi_1,K)=[\frac{5N_\xi}{\delta_1}r]\vee[C_0(\xi,\delta_1)N_\xi]\vee N_0(K,\xi,\delta_1,p)$, then for $d((t,x),(t',y))\leq 2^{-N}$, $t\leq t'$,

$$\begin{aligned}&\mathbb{P}\big(|\tilde u(t,x)-\tilde u(t,y)|\geq|x-y|^{1-\alpha/2-\delta}2^{-Np},(t,x)\in Z_{K,N,\xi},N\geq N_2\big)\\ &\quad+\mathbb{P}\big(|\tilde u(t',x)-\tilde u(t,x)|\geq|t'-t|^{1/2(1-\alpha/2-\delta)}2^{-Np},\\ &\qquad\qquad(t,x)\in Z_{K,N,\xi},t'\leq T_K,N\geq N_2\big)\\ &\quad\leq c_{60}\big(\exp(-c'_{60}|x-y|^{-\delta''}r)+\exp(-c'_{60}|t'-t|^{-\delta''/2}r)\big).\end{aligned}\tag{107}$$

Now let e_l be the lth unit vector in $\mathbb{R}^d$ and set

$$\begin{aligned}M_{n,N,K}=\max\Bigg\{&\sum_{l=1}^{d}\big|\tilde u\big(j2^{-2n},(z+e_l)2^{-n}\big)-\tilde u(j2^{-2n},z2^{-n})\big|\\ &+\big|\tilde u\big((j+e)2^{-2n},z2^{-n}\big)-\tilde u(j2^{-2n},z2^{-n})\big|:\\ &|z|\leq K2^n,(j+e)2^{-2n}\leq T_K,j\in\mathbb{Z}_+,z\in\mathbb{Z}^d,\\ &e\in\{1,2,3\},(j2^{-2n},z2^{-n})\in Z_{K,N,\xi}\Bigg\}.\end{aligned}$$

(107) implies that if

$$A_N = \{\omega\colon \text{ for some } n \geq N, M_{n,N,K} \geq (d+1) \cdot 2^{-n(1-\alpha/2-\delta)} 2^{-Np}, N \geq N_2\},$$

then for some fixed constants $C(d), c_1, c_2 > 0$,

$$\begin{aligned}\mathbb{P}\biggl(\bigcup_{N' \geq N} A_{N'}\biggr) &\leq C(d) \sum_{N'=N}^{\infty} \sum_{n=N'}^{\infty} K^{d+1} 2^{(d+2)n} e^{-c_1 2^{n\delta''}} \\ &\leq C(d) K^{d+1} \eta_N,\end{aligned}$$

where $\eta_N = e^{-c_2 2^{N\delta''}}$. Therefore, $N_3(\omega) = \min\{N \in \mathbb{N} : \omega \in A^c_{N'} \text{for all } N' \geq N\} < \infty$ a.s. and, in fact,

$$\mathbb{P}(N_3 > N) = \mathbb{P}\biggl(\bigcup_{N' \geq N} A_{N'}\biggr) \leq C(d) K^{d+1} \eta_N. \tag{108}$$

Choose $m \in \mathbb{N}$ with $m > \log_2(3 + \sqrt{d})$ and assume $N \geq (N_3 + m) \vee N_2$. Let $(t, x) \in Z_{K,N,\xi}$, $d((t', y), (t, x)) \leq 2^{-N}$, and $t' \leq T_K$. For $n \geq N$, let $t_n \in 4^{-n}\mathbb{Z}_+$ and $x_{n,i} \in 2^{-n}\mathbb{Z}$ $(i = 1, \dots, d)$ be the unique points so that $t_n \leq t < t_n + 4^{-n}$, $x_{n,i} \leq x_i < x_{n,i} + 2^{-n}$ for $x_i \geq 0$ and $x_{n,i} - 2^{-n} < x_i \leq x_{n,i}$ if $x_i < 0$. Similarly, define t'_n and y_n with (t', y) in place of (t, x). Choose $(\hat{t}, \hat{x})$ as in the definition of $Z_{K,N,\xi}$ [recall $(t, x) \in Z_{K,N,\xi}$]. If $n \geq N$, then

$$\begin{aligned}d\bigl((t'_n, y_n), (\hat{t}, \hat{x})\bigr) &\leq d\bigl((t'_n, y_n), (t', y)\bigr) + d\bigl((t', y), (t, x)\bigr) + d\bigl((t, x), (\hat{t}, \hat{x})\bigr) \\ &\leq \sqrt{|t'_n - t'|} + |y - y_n| + 2^{-N} + 2^{-N} \\ &< \bigl(3 + \sqrt{d}\bigr) 2^{-N} < 2^{m-N}.\end{aligned}$$

Therefore, $(t'_n, y_n) \in Z_{K,N-m,\xi}$, and similarly (and slightly more simply), $(t_n, x_n) \in Z_{K,N-m,\xi}$. Our definitions imply that t_N and t'_N are equal or adjacent in $4^{-N}\mathbb{Z}_+$ and similarly for the components of x_N and y_N in $2^{-N}\mathbb{Z}_+$. This, together with the continuity of $\tilde{u}$, the triangle inequality and our lower bound on N (which shows $N - m \geq N_3$), implies

$$\begin{aligned}&|\tilde{u}(t, x) - \tilde{u}(t', y)| \\ &\quad \leq |\tilde{u}(t_N, x_N) - \tilde{u}(t'_N, y_N)| \\ &\qquad + \sum_{n=N}^{\infty} |\tilde{u}(t_{n+1}, x_{n+1}) - \tilde{u}(t_n, x_n)| + |\tilde{u}(t'_{n+1}, y_{n+1}) - \tilde{u}(t'_n, y_n)| \\ &\quad \leq M_{N,N-m,K} + \sum_{n=N}^{\infty} 2M_{n+1,N-m,K}\end{aligned}$$

$$
\begin{aligned}
&\le 4 \sum_{n=N}^{\infty} (d+1) \cdot 2^{-n(1-\alpha/2-\delta)} 2^{-(N-m)p} \\
&\le c_0(d,p) 2^{-N(1-\alpha/2-\delta+p)} \\
&\le 2^{-N\xi_1}.
\end{aligned}
$$

The last line is valid for $N \ge N_4$ because $1 - \frac{\alpha}{2} - \delta + p > \xi_1$ by (56). Here N_4 is deterministic and may depend on p, ξ_1, δ, c_0 and, hence, ultimately on ξ, ξ_1. This proves the required result with

$$
N_{\xi_1}(\omega) = \max\left(N_3(\omega) + m, \left[\frac{5N_\xi(\omega)}{\delta_1}\right], [C_0(\xi,\delta_1)N_\xi], N_0 \vee N_4\right).
$$

Therefore, if $R = 5/\delta_1 \vee C_0(\xi,\delta_1)$ and $N \ge N(K) := N_0 \vee N_4$ (deterministic), (108) implies that

$$
\begin{aligned}
\mathbb{P}(N_{\xi_1} \ge N) &\le \mathbb{P}(N_3 \ge N - m) + 2\mathbb{P}(N_\xi \ge N/R) \\
&\le c(d) K^{d+1} \eta_{N-m} + 2\mathbb{P}(N_\xi \ge N/R),
\end{aligned}
$$

which gives the required probability bound (38).

APPENDIX: PROOFS OF THEOREM 1.2 AND PROPOSITION 1.8

In this appendix we briefly describe the construction of solutions to (12) with colored noise and non-Lipschitz coefficients. We start by citing the following result which states necessary conditions for the existence of solutions to (12) with Lipschitz coefficients and bounded initial conditions (see [1]):

THEOREM A.1. *Let u_0 be measurable and bounded and let σ be a Lipschitz continuous function. Assume that $(A)_\eta$ holds for $\eta = 1$. Then there exists a pathwise unique solution u to (12) which is also a strong solution. The process u satisfies a uniform moment bound: For any $T > 0$, and $p \in [1,\infty)$,*

$$
\sup_{0 \le t \le T} \sup_{x \in \mathbb{R}^d} \mathbb{E}(|u(t,x)|^p) < \infty. \tag{109}
$$

We would like to remark that the original theorem of Dalang [1] stipulates that the noise be spatially homogeneous. However, it is not hard to see that all that is needed is that it be bounded by an appropriate spatially homogeneous term in the sense of condition $(A)_\eta$.

Denote $L^\infty_{\text{tem}} = \{u : \operatorname{ess\,sup}_{x\in\mathbb{R}^d} |u(x)| e^{-\lambda|x|} < \infty \text{ for all } \lambda > 0\}$. Here the ess sup is of course with respect to the Lebesgue measure.

We introduce some frequently used notation. For any function $v : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and stopping time τ, we set

$$
J^{a-1}v(t,x) = \frac{\sin(\pi a)}{\pi} \int_0^t \int_{\mathbb{R}^d} (t-s)^{a-1} p_{t-s}(x-y) v(s,y)\, dy\, ds, \tag{110}
$$

as well as

$$J_a^\tau v(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{1}(s \le \tau)(t-s)^{-a} p_{t-s}(x-y)\sigma\big(v(s,y)\big) W(dy\,ds). \tag{111}$$

The stochastic Fubini theorem implies

$$J^{a-1} J_a^\tau v(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{1}(s \le \tau) p_{t-s}(x-y)\sigma\big(v(s,y)\big) W(dy\,ds). \tag{112}$$

We will use the notation $J_a v(t,x) = J_a^t v(t,x)$, when $\tau = t$ in the above. Also, set

$$G_{\lambda,p}^\tau v(t,x) := \mathbb{E}\big(|v(t,x)|^p \mathbb{1}(t \le \tau) e^{-\lambda|x|}\big),$$

and again $G_{\lambda,p} v(t,x) \equiv G_{\lambda,p}^t v(t,x)$ whenever $\tau = t$.

LEMMA A.2. *Let σ be a continuous function satisfying the growth condition*

$$|\sigma(u)| \le c_{113}(1+|u|). \tag{113}$$

Assume that $(A)_\eta$ holds for some $\eta \in [0,1)$ and let $a < (1-\eta)/2$. Let $v : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{P}(\mathcal{F}_\cdot) \times \mathcal{B}(\mathbb{R}^d)$-measurable [$\mathcal{P}(\mathcal{F}_\cdot)$ is the $(\mathcal{F}_t)$-predictable σ-field]. Then for any $T, \lambda > 0$, $p \ge 2$, and stopping time τ,

$$\begin{aligned} &\sup_{0\le s\le t} \sup_{x\in\mathbb{R}^d} \mathbb{E}\big(|J_a^\tau v(t,x)|^p e^{-\lambda|x|}\big) \\ &\qquad \le C(T,\lambda,p) c_{113}^p \sup_{0\le s\le t} \sup_{x\in\mathbb{R}^d} \big(1 + G_{\lambda,p}^\tau v(s,x)\big) \qquad \forall t \le T. \end{aligned} \tag{114}$$

PROOF. First fix arbitrary $p \ge 2$ and $x \in \mathbb{R}^d$. Then, using the growth condition on σ, as well as Burkholder's inequality and $|k(x,y)| \le c_{10}\tilde{k}(x-y)$, we get

$$\begin{aligned} &\mathbb{E}\big(|J_a^\tau v(t,x)|^p\big) \\ &\quad \le C c_{113}^p \mathbb{E}\bigg(\bigg(\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (t-s)^{-2a} p_{t-s}(x-y) p_{t-s}(x-z) \tilde{k}(y-z) \\ &\qquad\qquad \times \big(1 + \mathbb{1}(s\le\tau)|v(s,y)|\big) \\ &\qquad\qquad\qquad \times \big(1 + \mathbb{1}(s\le\tau)|v(s,z)|\big)\, dy\,dz\,ds\bigg)^{p/2}\bigg) \\ &\quad \le C c_{113}^p \bigg(\int_0^t (t-s)^{-2a} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t-s}(y) p_{t-s}(z) \tilde{k}(y-z)\, dy\,dz\,ds\bigg)^{p/2-1} \\ &\qquad \times \bigg(\int_0^t (t-s)^{-2a} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t-s}(y) p_{t-s}(z) \tilde{k}(y-z) \\ &\qquad\qquad \times \mathbb{E}\Big(\big(1 + \mathbb{1}(s\le\tau)|v(s,y-x)|\big)^{p/2} \\ &\qquad\qquad \times \big(1 + \mathbb{1}(s\le\tau)|v(s,z-x)|\big)^{p/2}\Big)\, dy\,dz\,ds\bigg). \end{aligned}$$

Apply Hölder's inequality to the expected value in this expression and shift variables to bound it by

$$\begin{aligned}&\mathbb{E}\big(\big(1+\mathbb{1}(s\le\tau)|v(s,y-x)|\big)^p\big)^{1/2}\mathbb{E}\big(\big(1+\mathbb{1}(s\le\tau)|v(s,z-x)|\big)^p\big)^{1/2}\\&\qquad\le C(\lambda,p)e^{\lambda/2(|y|+|z|)+\lambda|x|}\Big(1+\sup_{\tilde z\in\mathbb{R}^d}G^\tau_{\lambda,p}v(s,\tilde z)\Big).\end{aligned}\tag{115}$$

Hence, we arrive at

$$\begin{aligned}&\mathbb{E}\big(|J^\tau_a v(t,x)|^p e^{-\lambda|x|}\big)\\&\quad\le C(\lambda,p)c_{113}^p\\&\qquad\times\bigg(\int_0^t(t-s)^{-2a}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_{t-s}(y)p_{t-s}(z)\tilde k(y-z)\,dy\,dz\,ds\bigg)^{p/2-1}\\&\qquad\times\bigg(\int_0^t(t-s)^{-2a}\\&\qquad\qquad\times\bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{\lambda/2(|y|+|z|)}p_{t-s}(y)p_{t-s}(z)\tilde k(y-z)\,dy\,dz\bigg)\\&\qquad\qquad\qquad\times\Big(1+\sup_{\tilde z\in\mathbb{R}^d}G^\tau_{\lambda,p}v(s,\tilde z)\Big)\,ds\bigg)\\&\quad\le C(\lambda,p)c_{113}^p\bigg(\int_0^t f\Big(\frac{s}{2}\Big)\,ds\bigg)^{p/2-1}\\&\qquad\times\bigg(\int_0^t f(t-s)\Big(1+\sup_{\tilde z\in\mathbb{R}^d}G^\tau_{\lambda,p}v(s,\tilde z)\Big)\,ds\bigg)\\&\quad\le C(T,\lambda,p)c_{113}^p\sup_{0\le s\le t}\sup_{\tilde z\in\mathbb{R}^d}\big(1+G^\tau_{\lambda,p}v(s,\tilde z)\big)\qquad\forall t\le T,x\in\mathbb{R}^d,\end{aligned}\tag{116}$$

where

$$f(r)=r^{-2a}\bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_{2r}(y)p_{2r}(z)\tilde k(y-z)\,dy\,dz\bigg).$$

Here, we have used that $e^{\lambda/2|y|}p_t(y)\le C(T,\lambda)p_{2t}(y)$ for $t\le T$ see (47). We have also used the fact that f is integrable on $[0,T]$ for $a<\frac{1-\eta}{2}$ (cf. proof of Lemma 2.2 of [8]). This proves (114) for all $p\ge 2$. □

LEMMA A.3. *Let $u_0\in L^\infty_{\mathrm{tem}}$ and let σ be a continuous function satisfying the growth condition* (113). *Assume that $(A)_\eta$ holds for some $\eta\in[0,1)$. If u is any solution to* (12) *such that*

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\mathbb{E}\big(|u(t,x)|^p e^{-\lambda|x|}\big)<\infty\qquad\forall T>0,p>0,\lambda>0,\tag{117}$$

then for any $T, p, \lambda > 0$, *there exists* $\tilde{p} \geq p$ *such that*

$$\mathbb{E}\Big(\sup_{0\le t\le T}\sup_{x\in\mathbb{Q}^d}|u(t,x)|^p e^{-\lambda|x|}\Big) \le C_{T,\lambda,p}(c_{113}, \|u_0\|_{\lambda/p,\infty}) \times \Big(1+\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d} G_{\lambda/2,\tilde{p}}u(t,x)\Big), \tag{118}$$

where $C_{T,\lambda,p}(\cdot,\cdot)$ *is bounded on the compacts of* $\mathbb{R}_+ \times \mathbb{R}_+$.

PROOF.

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}\sup_{x\in\mathbb{Q}^d}|u(t,x)|^p e^{-\lambda|x|}\Big)\\
&\quad\le C\mathbb{E}\Big(\sup_{0\le t\le T}\sup_{x\in\mathbb{Q}^d}\Big|\int_{\mathbb{R}^d} p_t(x-y)u_0(y)\,dy\Big|^p e^{-\lambda|x|}\Big)\\
&\qquad + C\mathbb{E}\Big(\sup_{0\le t\le T}\sup_{x\in\mathbb{Q}^d}\Big|\int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)\\
&\qquad\qquad \times \sigma\big(u(s,y)\big)W(dy\,ds)\Big|^p e^{-\lambda|x|}\Big).
\end{aligned}\tag{119}$$

The first term on the right-hand side of (119) is bounded by

$$\begin{aligned}
&\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} p_t(x-y)|u_0(y)|^p\,dy\,e^{-\lambda|x|}\Big|\\
&\quad\le C(T,\lambda)\|u_0\|^p_{\lambda/p,\infty}\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} p_t(x-y)e^{\lambda|y|}\,dy\,e^{-\lambda|x|}\Big|\\
&\quad\le C(T,\lambda,p)\|u_0\|^p_{\lambda/p,\infty}.
\end{aligned}\tag{120}$$

In this calculation we have used Jensen's inequality, as well as the fact that

$$\int_{\mathbb{R}^d} p_t(x-y)e^{\lambda|y|}\,dy \le C(T,\lambda)e^{\lambda|x|} \tag{121}$$

for $t \le T$ and $\lambda \in \mathbb{R}$ (see Lemma 6.2 of [7]).

We bound the second term on the right-hand side of (119) with the help of the factorization method of [2] [cf. (81) and (82)]. Let $0 < a < (1-\eta)/2$ and choose arbitrary $p^* > \frac{1+d/2}{a} > 2$. Assume that $p \ge p^*$. Recall $\|v\|_{\lambda,p} = [\int |v(x)|^p e^{-\lambda|x|}\,dx]^{1/p}$. Use (112) and apply Hölder's inequality to get

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\le T}\sup_{x\in\mathbb{Q}^d}\Big|\int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma\big(u(s,y)\big)W(dy\,ds)\Big|^p e^{-\lambda|x|}\Big)\\
&\quad= \mathbb{E}\Big(\sup_{t\le T}\sup_{x\in\mathbb{Q}^d}|J^{a-1}J_a u(t,x)|^p e^{-\lambda|x|}\Big)
\end{aligned}\tag{122}$$

$$
\begin{aligned}
&\le C\mathbb{E}\biggl(\sup_{t\le T}\sup_{x\in\mathbb{Q}^d}\biggl|\int_0^t (t-s)^{a-1} \\
&\qquad\qquad \times \biggl(\int_{\mathbb{R}^d} p_{t-s}(x-y)e^{\lambda/2|y|} \\
&\qquad\qquad\qquad \times |J_a u(s,y)|^{p/2}e^{-\lambda/2|y|}\,dy\biggr)^{2/p}\,ds\biggr|^p e^{-\lambda|x|}\biggr) \\
&\le C\mathbb{E}\biggl(\sup_{t\le T}\sup_{x\in\mathbb{Q}^d}\biggl|\int_0^t (t-s)^{a-1}\biggl(\int_{\mathbb{R}^d} p_{t-s}(x-y)^2 e^{\lambda|y|}\,dy\biggr)^{1/p} \\
&\qquad\qquad \times \|J_a u_s\|_{\lambda,p}\,ds,\ |^p e^{-\lambda|x|}\biggr) \\
&\le C(T,\lambda)\mathbb{E}\biggl(\sup_{t\le T}\biggl(\int_0^t (t-s)^{a-1-d/(2p)}\cdot\|J_a u_s\|_{\lambda,p}\,ds\biggr)^p\biggr) \\
&\le C(T,\lambda)\biggl(\int_0^T s^{(a-1-d/(2p))p/(p-1)}ds\biggr)^{p-1}\cdot\int_0^T \mathbb{E}(\|J_a u_s\|_{\lambda,p}^p)\,ds.
\end{aligned}
$$

Here, we have also used (121) and $p_t(x)\le Ct^{d/2}$. Lemma A.2 implies

$$
\begin{aligned}
&\mathbb{E}\biggl(\int_{\mathbb{R}^d}|J_a u(t,x)|^p e^{-\lambda|x|}\,dx\biggr) \\
&\qquad \le C(T,\lambda,p)c_{113}^p\sup_{0\le s\le t}\sup_{x\in\mathbb{R}^d}\bigl(1+G_{\lambda/2,p}u(s,x)\bigr).
\end{aligned}
$$

Recall that $a<\frac{1-\eta}{2}$ and $p\ge p^*>\frac{1+d/2}{a}$. A bit of algebra shows that the whole expression in (122) is finite and bounded by

$$
C(T,\lambda,p)c_{113}^p\sup_{0\le s\le T}\sup_{x\in\mathbb{R}^d}\bigl(1+G_{\lambda/2,p}u(s,x)\bigr).
$$

This together with (119) and (120) proves (118) for all $p\ge p^*$ with $\tilde p=p$. Note, however, that if $p<p^*$, then (118) also holds with $\tilde p=p^*$ due to the fact that $u^p\le 1+u^{p^*}$ for any $u\ge 0$, $p<p^*$. Hence, we are done. $\square$

The next result gives bounds on spatial and temporal differences of stochastic convolution integrals which, in particular, will imply they are Hölder continuous. The result is an adaptation of Theorem 2.1 of [8] to our situation.

LEMMA A.4. *Let u be a solution to* (12) *satisfying the assumptions of Lemma* A.3. *Define*

$$
Z(t,x)=\int_0^t\int p_{t-s}(x-y)\sigma\bigl(u(s,y)\bigr)W(ds\,dy),\qquad t\ge 0, x\in\mathbb{R}^d. \tag{123}
$$

Then, for $T, R > 0$, and $0 \le t, t' \le T$, $x, x' \in \mathbb{R}^d$ such that $|x - x'| < R$, as well as $p \in [2, \infty)$ and $\xi \in (0, 1 - \eta)$,

$$\begin{aligned}
&\mathbb{E}\big(|Z(t,x) - Z(t',x')|^p e^{-\lambda|x|}\big)\\
&\quad \le C(T,\lambda,p)c_{113}^p\Big(1 + \sup_{0\le s\le T}\sup_{z\in\mathbb{R}^d} G_{\lambda/(p+1),p}u(s,z)\Big) \qquad (124)\\
&\qquad \times (|t-t'|^{\xi/2p} + |x-x'|^{\xi p}).
\end{aligned}$$

In particular, if $G_{\lambda/(p+1),p}u(\cdot,\cdot)$ is bounded on $[0,T]\times\mathbb{R}^d$, then there is a version of Z which is uniformly Hölder continuous on compact subsets of $[0,T]\times\mathbb{R}^d$ with coefficients $\frac{\xi}{2}$ in time and ξ in space.

PROOF. The proof follows the proof of Theorem 2.1 in [8]. We use the same notation as in the proof of Lemma A.3, so $Z(t,x) = J^{a-1}J_a u(t,x)$ by (112). Now assume that $t' \ge t$. By Lemma A.2 and Hölder's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\big(|Z(t',x') - Z(t,x)|^p\big)e^{-\lambda|x|}\\
&\quad \le C(p)\mathbb{E}\bigg(\bigg|\int_0^t\int_{\mathbb{R}^d}\big(p_{t'-s}(x'-y)(t'-s)^{a-1}\\
&\qquad\qquad - p_{t-s}(x-y)(t-s)^{a-1}\big)\\
&\qquad\qquad \times J_a u(s,y)\,dy\,ds\bigg|^p\bigg)e^{-\lambda|x|}\\
&\qquad + C(p)\mathbb{E}\bigg(\bigg|\int_t^{t'}\int_{\mathbb{R}^d} p_{t'-s}(x'-y)(t'-s)^{a-1}J_a u(s,y)\,dy\,ds\bigg|^p\bigg)e^{-\lambda|x|}\\
&\quad \le C(T,\lambda,p)c_{113}^p\Big(1 + \sup_{0\le s\le T}\sup_{x\in\mathbb{R}^d} G_{\lambda/(p+1),p}u(s,x)\Big)e^{-\lambda|x|}\\
&\qquad \times\bigg\{\bigg(\int_0^t\int_{\mathbb{R}^d}|p_{t'-s}(x'-y)(t'-s)^{a-1}\\
&\qquad\qquad - p_{t-s}(x-y)(t-s)^{a-1}|e^{\lambda/(p+1)|y|}\,dy\,ds\bigg)^p\\
&\qquad\qquad + \bigg(\int_t^{t'}\int_{\mathbb{R}^d} p_{t'-s}(x'-y)(t'-s)^{a-1}e^{\lambda/(p+1)|y|}\,dy\,ds\bigg)^p\bigg\}.
\end{aligned}$$

Here, we have in the second inequality also inserted additional factors of $e^{-\lambda/(p+1)|y|}e^{\lambda/(p+1)|y|}$ so that we could apply Lemma A.2 to bound the expectation of $J_a u$ by using Jensen's inequality. From this point, we proceed as in [8] (proof of Theorem 2.1), the only difference being that we have to take care of the additional nuisance factors $e^{\lambda/(p+1)|y|}$. This can be done with the help of (121) and (48) of Lemma 5.2 using the remaining factor $e^{-\lambda|x|}$. □

The next lemma assures that, for any $u \in C(\mathbb{R}_+, C_{\mathrm{tem}})$ which solves (12), $G_{\lambda,p}u(t,x)$ is bounded.

LEMMA A.5. *Let $u_0 \in C_{\mathrm{tem}}$ and let σ be a continuous function satisfying the growth condition (113). Assume that $(A)_\eta$ holds for some $\eta \in [0,1)$. If $u \in C(\mathbb{R}_+, C_{\mathrm{tem}})$ a.s. is a solution to (12), then it satisfies the following moment bound. For any $T > 0$ and $p \geq 1$,*

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\mathbb{E}\big(|u(t,x)|^p e^{-\lambda|x|}\big) \le C_{T,\lambda,p}(c_{113}, \|u_0\|_{\lambda/p,\infty}), \tag{125}$$

where $C_{T,\lambda,p}(\cdot,\cdot)$ is bounded on the compacts of $\mathbb{R}_+ \times \mathbb{R}_+$.

PROOF. Define

$$\tau_n = \inf\{t : \|u_t\|_{\lambda/p,\infty} \geq n\}.$$

We set

$$G^{\tau_n}_{\lambda,p}u(t,x) := \mathbb{E}\big(|u(t,x)|^p \mathbb{1}(t \le \tau_n) e^{-\lambda|x|}\big).$$

Note that, by definition,

$$\sup_{s\le t}\sup_{x\in\mathbb{R}^d} G^{\tau_n}_{\lambda,p}u(t,x) \le n^p \qquad \forall t \geq 0, n \geq 1. \tag{126}$$

From (12), we get

$$\begin{aligned}
&\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d} G^{\tau_n}_{\lambda,p}u(t,x)\\
&\quad\le C \sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\mathbb{E}\left(\left|\int_{\mathbb{R}^d} p_t(x-y)u_0(y)\,dy\right|^p e^{-\lambda|x|}\right) \qquad (127)\\
&\qquad + C \sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\mathbb{E}\left(\left|\int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)\mathbb{1}(s\le\tau_n)\right.\right. \qquad (128)\\
&\qquad\qquad \left.\left.\times\,\sigma\big(u(s,y)\big)W(dy\,ds)\right|^p e^{-\lambda|x|}\right).
\end{aligned}$$

By (120), the term on line (127) is bounded by

$$C(T,\lambda,p)\|u_0\|^p_{\lambda/p,\infty}. \tag{129}$$

Again, as in Lemma A.3, we use the factorization method to bound the term in (128). First, we assume that $p > \frac{2}{1-\eta} > 2$ so that we can choose a constant a with $0 < \frac{1}{p} < a < \frac{1-\eta}{2} < 1$. Recall (112), and by several applications of Hölder's inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\big(|J^{a-1}J_a^{\tau_n}u(t,x)|^p e^{-\lambda|x|}\big)\\
&\quad = C\mathbb{E}\bigg(\bigg|\int_0^t\int_{\mathbb{R}^d}(t-s)^{a-1}p_{t-s}(x-y)J_a^{\tau_n}u(s,y)\,dy\,ds\bigg|^p e^{-\lambda|x|}\bigg)\\
&\quad \le C\mathbb{E}\bigg(\bigg(\int_0^t (t-s)^{a-1}\\
&\qquad\qquad \times\bigg(\bigg|\int_{\mathbb{R}^d}p_{t-s}(x-y)J_a^{\tau_n}u(s,y)\,dy\bigg|^p e^{-\lambda|x|}\bigg)^{1/p}ds\bigg)^p\bigg)\\
&\quad \le C(T,\lambda)\mathbb{E}\bigg(\bigg(\int_0^t (t-s)^{a-1}\\
&\qquad\qquad \times\bigg(\int_{\mathbb{R}^d}|J_a^{\tau_n}u(s,y)|^p p_{t-s}(x-y)e^{-\lambda|x|}\,dy\bigg)^{1/p}ds\bigg)^p\bigg)\\
&\quad \le C(T,\lambda)\bigg(\int_0^T s^{p/(p-1)(a-1)}\,ds\bigg)^{p-1}\\
&\qquad \times\bigg(\int_0^t\int_{\mathbb{R}^d}\mathbb{E}\big(|J_a^{\tau_n}u(s,y)|^p\big)e^{-\lambda|x|}p_{t-s}(x-y)\,dy\,ds\bigg)\\
&\quad \le C(T,\lambda,p)c_{113}^p\bigg(1+\int_0^t\sup_{0\le r\le s}\sup_{z\in\mathbb{R}^d}G_{\lambda,p}^{\tau_n}u(r,z)\,ds\bigg)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t\le T, x\in\mathbb{R}^d,
\end{aligned}\tag{130}
$$

where we have also used Lemma A.2 and (121) in the last inequality, as well as $a>\frac{1}{p}$. Taking (129) together with (130), we obtain that there is a constant $C=C(T,\lambda,p)$ independent of n such that, for all $t\le T$,

$$
\begin{aligned}
&\sup_{0\le s\le t}\sup_{x\in\mathbb{R}^d}G_{\lambda,p}^{\tau_n}u(t,x)\\
&\qquad \le C\big(c_{113}^p+\|u_0\|_{\lambda/p,\infty}^p\big)\\
&\qquad\quad \times\bigg(1+\int_0^t\sup_{0\le r\le s}\sup_{x\in\mathbb{R}^d}G_{\lambda,p}^{\tau_n}u(r,x)\,ds\bigg)\qquad\forall n\ge 1.
\end{aligned}\tag{131}
$$

But the left-hand side is bounded [due to (126)]. Thus, by Gronwall's lemma,

$$
\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}G_{\lambda,p}^{\tau_n}u(t,x)\le C_{T,\lambda,p}(c_{113},\|u_0\|_{\lambda/p,\infty})\qquad\forall n\ge 1,\tag{132}
$$

where $C_{T,\lambda,p}(\cdot,\cdot)$ is bounded on the compacts of $\mathbb{R}_+\times\mathbb{R}_+$. (We have obtained this result with the restriction $p>\frac{2}{1-\eta}$, which then immediately implies that it is true for all $p>0$ since we are considering L^p norms with respect to a finite measure.)

Now, recall that $u \in C(\mathbb{R}_+, C_{\mathrm{tem}})$ a.s. so that $\tau_n \uparrow \infty$ a.s., and, hence,

$$\begin{aligned}\mathbb{E}\big(|u(t,x)|^p e^{-\lambda|x|}\big) &= \mathbb{E}\Big(\lim_{n\to\infty} |u(t,x)|^p \mathbb{1}(t \le \tau_n) e^{-\lambda|x|}\Big)\\ &\le \liminf_{n\to\infty} G^{\tau_n}_{\lambda,p} u(t,x),\end{aligned}$$

where the second inequality follows by Fatou's lemma. Use this and the fact that the right-hand side of (132) does not depend on n to obtain

$$\sup_{0\le t\le T} \sup_{x\in\mathbb{R}^d} \mathbb{E}\big(|u(t,x)|^p e^{-\lambda|x|}\big) \le C_{T,\lambda,p}(c_{113}, \|u_0\|_{\lambda/p,\infty}) \qquad \forall n \ge 1, \tag{133}$$

where $C_{T,\lambda,p}(\cdot,\cdot)$ is bounded on the compacts of $\mathbb{R}_+ \times \mathbb{R}_+$. $\square$

PROOF OF THEOREM 1.2. Recall our hypotheses imply $(A)_\eta$ holds for some $\eta \in [0, 1)$ (see Remark 1.1). We can choose a sequence of Lipschitz continuous functions σ_n on $\mathbb{R}^d$ such that the growth bound (6) holds uniformly [$\sigma_n(u) \le c_6(1 + |u|)$ for all $u \in \mathbb{R}, n \in \mathbb{N}$], and such that the σ_n converge uniformly to σ as $n \to \infty$. We also set

$$u_0^m(x) = \begin{cases} u_0(x), & \text{if } |u_0(x)| < m,\\ m, & \text{if } u_0(x) \ge m,\\ -m, & \text{if } u_0(x) \le -m,\end{cases}$$

which implies that $u_0^m \in C_b(\mathbb{R}^d)$ and

$$\sup_{m\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} |u_0^m(x)| e^{-\lambda|x|} < \infty. \tag{134}$$

Hence, by Theorem A.1, for each m, n, there exists a unique solution to

$$\begin{aligned}u^{m,n}(t,x) &= \int_{\mathbb{R}^d} p_t(x-y) u_0^m(y)\,dy\\ &\quad + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma_n\big(u^{m,n}(s,y)\big) W(dy\,ds).\end{aligned} \tag{135}$$

It is easy to check that the first term on the right-hand side of (135) is jointly continuous on $[0,\infty) \times \mathbb{R}^d$. Moreover, by Theorem A.1, $\sup_{0\le t\le T} \sup_{x\in\mathbb{R}^d} \mathbb{E}(|u^{m,n}(t, x)|^p) < \infty$, and, hence, by Lemmas A.3 and A.4 we obtain that

$$u^{m,n} \in C(\mathbb{R}_+, C_{\mathrm{tem}}). \tag{136}$$

Now let us go to the limit as $m, n \to \infty$. Let $Z^{m,n}$ denote that stochastic integral on the right-hand side of (135). Since $u^{m,n} \in C(\mathbb{R}_+, C_{\mathrm{tem}})$, we may apply Lemmas A.4 and A.5 to get

$$\mathbb{E}\big(|Z^{m,n}(t,x) - Z^{m,n}(t',x')|^p e^{-\lambda|x|}\big) \le C_{T,\lambda,p}(c_6, \|u_0^m\|_{\lambda/p,\infty}), \tag{137}$$

where $C_{T,\lambda,p}(\cdot,\cdot)$ is bounded on the compacts of $\mathbb{R}_+ \times \mathbb{R}_+$. Inequality (137) combined with a Kolmogorov type tightness criterion (see Lemma 6.3 of [7]) now implies that the stochastic integrals $Z^{m,n}$ are tight in $C(\mathbb{R}_+, C_{\mathrm{tem}})$. It is also not hard to show that $(t,x) \mapsto \int_{\mathbb{R}^d} p_t(x-y)u_0^m\,dy$ are tight in $C(\mathbb{R}_+, C_{\mathrm{tem}})$ by using the Arzela–Ascoli theorem and the uniformity in m as in (134).

Therefore, $u^{m,n}$ are tight in $C(\mathbb{R}_+, C_{\mathrm{tem}})$ and we can choose an appropriate probability space and define $u^{m,n}$ on it identical in distribution to a subsequence of the original sequence of solutions which converge a.s. in $C(\mathbb{R}_+, C_{\mathrm{tem}})$ to some process u. It is routine to establish from this that all the terms in (12) converge a.s. to the appropriate limits so that the limit $u \in C(\mathbb{R}_+, C_{\mathrm{tem}})$ is indeed a solution to (12) with the desired σ and $u_0 \in C_{\mathrm{tem}}$. $\square$

PROOF OF PROPOSITION 1.8. Recall from Remark 1.1 that our hypotheses imply $(A)_\eta$ for any $\eta \in (\alpha/2, 1)$. Lemmas A.3 and A.5 now imply (a). Now use Lemma A.4 (and Lemma A.5) to derive part (b) for Z. For $u_0 \in C_{\mathrm{tem}}$, $S_t u_0(x) \equiv \int p_t(y-x)u_0(y)\,dy$ is smooth on $(0,\infty) \times \mathbb{R}^d$, and so is uniformly Lipschitz on compact subsets of $(0,\infty) \times \mathbb{R}^d$. This gives the required Hölder continuity for u. $\square$

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