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STOCHASTIC STRATONOVICH CALCULUS fBm FOR FRACTIONAL BROWNIAN MOTION WITH HURST PARAMETER LESS THAN 1/2

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Abstract. In this paper we introduce a Stratonovich type stochastic integral with respect to the fractional Brownian motion with Hurst parameter less than 1/2. Using the techniques of the Malliavin calculus, we provide sufficient conditions for a process to be integrable. We deduce an Itô formula and we apply these results to study stochastic differential equations driven by a fractional Brownian motion with Hurst parameter less than 1/2.

1. Introduction

The fractional Brownian motion of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with the covariance function (see [16])

$$E(B_t^H B_s^H) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1}$$

The purpose of this paper is to study stochastic integrals with respect to the process B^H in the case $H < 1/2$. In [16], the authors derive the integral representation

$$B_t^H = a_H \int_0^t (t-s)^{H-\frac{1}{2}} dW_s + Z_t, \tag{2}$$

where W is a standard Wiener process and Z is a process with absolutely continuous paths. Different approaches have been recently used to define stochastic integrals with respect to B^H in the case $H < 1/2$:

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(i) Using the representation (2), we defined in [1] a stochastic integral $\int_0^T u_s dB_s^H$ as the limit as ε tends to zero of the integrals with respect to the regularized process $a_H \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s + Z_t$. This integral requires the trace condition

$$\int_0^T \int_0^r |D_s u_r| (r-s)^{H-\frac{3}{2}} \, ds \, dr < \infty \tag{3}$$

almost surely, where D denotes the derivative in the sense of Malliavin calculus with respect to the Wiener process W. This condition is very strong and it is not satisfied in simple cases like $u_t = W_t$ or $u_t = B_t^H$. Moreover, under a suitable Hölder condition on the process u, this integral coincides with the limit of the forward Riemann sums

$$\sum_{i=1}^n u_{t_{i-1}} (B_{t_i}^H - B_{t_{i-1}}^H),$$

where $t_i = iT/n$.

(ii) Since the fractional Brownian motion is a Gaussian process, one can apply the stochastic calculus of variations (see [18]) and introduce the stochastic integral as the divergence operator with respect to B^H, that is, the adjoint of the derivative operator. This idea has been developed by Decreusefond and Üstünel [6, 7], Carmona and Coutin [3] and Alòs, Mazet and Nualart [2]. The integral constructed by this method has zero mean, and can be obtained as the limit of Riemann sums defined using Wick products. The forward integral defined in [1] can be expressed as the sum of the divergence with respect to B^H and the trace term (3).

(iii) Using the notions of fractional integral and derivative, Zähle has introduced in [23] a pathwise stochastic integral with respect to B^H, $H \in (0,1)$. If the integrator has λ -Hölder continuous paths with $\lambda > 1 - H$, then this integral can be interpreted as a Riemann-Stieltjes integral.

As we pointed out before, the forward integral $\int_0^T B_t^H dB_t^H$ does not exist. Actually, a simple argument shows that the expectation of the Riemann sums

$$\sum_{i=1}^n B_{t_{i-1}}^H (B_{t_i}^H - B_{t_{i-1}}^H)$$

diverges. In fact, if $t_i = iT/n$, then

$$\begin{aligned} E \sum_{i=1}^n B_{t_{i-1}}^H (B_{t_i}^H - B_{t_{i-1}}^H) &= \frac{1}{2} \sum_{i=1}^n \left[t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H} \right] \\ &= \frac{1}{2} T^{2H} \left(1 - n^{1-2H} \right). \end{aligned}$$

Notice, however, that the expectation of symmetric Riemann sums is constant:

$$\frac{1}{2}E\sum_{i=1}^{n}(B_{t_i}^H + B_{t_{i-1}}^H)(B_{t_i}^H - B_{t_{i-1}}^H) = \frac{1}{2}\sum_{i=1}^{n}\left[t_i^{2H} - t_{i-1}^{2H}\right] = \frac{T^{2H}}{2}.$$

Taking into account this remark, and following the approach by Russo and Vallois [20], in this paper we define a stochastic integral of Stratonovich type $\int_0^T u_s \circ dB_s^H$ as the limit in probability as ε tends to zero of

$$(2\varepsilon)^{-1}\int_0^T u_s\left(B_{(s+\varepsilon)\wedge T}^H - B_{(s-\varepsilon)\vee 0}^H\right)ds.$$

Our main result is Theorem 2 which provides sufficient conditions for the Stratonovich integral to exist, and yields a decomposition of this integral as the sum of the divergence operator and a trace term. These conditions are fulfilled, for instance, in the particular case $u_s = F(B_s^H)$, for some regular function F. Section 5 is devoted to establish an Itô's formula for the indefinite Stratonovich integral. Finally, in Section 6 we solve one-dimensional stochastic differential equations in the Stratonovich sense driven by the fractional Brownian motion with Hurst parameter less than $1/2$.

2. Preliminaries

Let $B = \{B_t, t \in [0,T]\}$ be a zero-mean Gaussian process of the form

$$B_t = \int_0^t K(t,s)dW_s,$$

where $W = \{W_t, t \in [0,T]\}$ is a Wiener process, and $K(t,s)$, $0 < s < t < T$, is a kernel satisfying $\|K\| = \sup_{t\in[0,T]}\int_0^t K(t,s)^2 ds < \infty$. The covariance $R(t,s)$ of B has the form

$$R(t,s) = \int_0^{t\wedge s} K(t,r)K(s,r)dr.$$

We will assume that the Gaussian subspaces generated by B and W coincide.

It is possible to construct a stochastic calculus of variations with respect to the Gaussian process B, which will be related to the Malliavin calculus with respect to the Wiener process W. We refer to [2] for a complete exposition of this subject. For the sake of completeness, we give the basic definitions and results of this calculus.

The Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}$ is defined as the closure of the linear span of the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R(t,s)$.

We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Consider the linear operator K^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$(K^*\varphi)(s) = \varphi(s)K(T,s) + \int_s^T [\varphi(t) - \varphi(s)]K(dt,s).$$

This operator satisfies the duality relationship (see Lemma 1 in [2])

$$\int_0^T (K^*\varphi)(t)h(t)dt = \int_0^T \varphi(t)(Kh)(dt),$$

for all $\varphi \in \mathcal{E}$ and $h \in L^2([0,T])$, where $(Kh)(t) = \int_0^t K(t,s)h(s)ds$.

As a consequence, the RKHS $\mathcal{H}$ can be represented as the closure of $\mathcal{E}$ with respect to the norm $\|\varphi\|_{\mathcal{H}} = \|K^*\varphi\|_{L^2([0,T])}$, and the operator K^* is an isometry between $\mathcal{H}$ and a closed subspace of $L^2([0,T])$, that is,

$$\mathcal{H} = (K^*)^{-1}(L^2([0,T])). \tag{4}$$

A similar relation holds for the derivative and divergence operators with respect to the processes B and W. That is,

(i) $K^*D^BF = DF$, for any $F \in \mathbb{D}^{1,2} = \mathbb{D}_B^{1,2}$, where D and D^B denote the derivative operators with respect to the processes W and B, respectively, and $\mathbb{D}^{1,2}$ and $\mathbb{D}_B^{1,2}$ are the corresponding Sobolev spaces.

(ii) Dom $\delta^B = (K^*)^{-1}(\mathrm{Dom}\,\delta)$, and $\delta^B(u) = \delta(K^*u)$ for any $\mathcal{H}$-valued random variable u in Dom δ^B, where δ and δ^B denote the divergence operators with respect to the processes B and W, respectively.

Moreover, we have $\mathbb{D}_B^{1,2}(\mathcal{H}) = (K^*)^{-1}(\mathbb{L}^{1,2})$, where $\mathbb{L}^{1,2} = \mathbb{D}^{1,2}(L^2([0,T]))$, and this space is included in the domain of the divergence δ^B. We will make use of the notations $\delta(v) = \int_0^T v_s dW_s$ for any $v \in \mathrm{Dom}\,\delta$, and $\delta^B(v) = \int_0^T v_s dB_s$ for any $v \in \mathrm{Dom}\,\delta^B$. Hence, if $u \in \mathrm{Dom}\,\delta^B$, then

$$\int_0^T u_s dB_s = \int_0^T (K^*u)_s dW_s. \tag{5}$$

We will denote by c a generic constant that may be different from one formula to another one. Moreover, by convention $K(t,s) = 0$ if $s > t$.

3. The Stratonovich Integral

Suppose that the Gaussian process is the fractional Brownian motion B of Hurst parameter $H \in [0,1/2)$. The covariance of this process is given by

$$R(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

This process has the integral representation $B_t = \int_0^t K(t,r)dW_r$, where (see [2, 6])

$$K(t,s) = c_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right) \tag{6}$$

and

$$F_1(z) = c_H\left(\frac{1}{2} - H\right)\int_0^{z-1} \theta^{H-\frac{3}{2}}\left(1 - (\theta+1)^{H-\frac{1}{2}}\right)d\theta.$$

The kernel $K(t,s)$ satisfies the following conditions, where $\alpha = 1/2 - H$:

(i) $|K(t,s)| \le c\left((t-s)^{-\alpha} + s^{-\alpha}\right)$,

(ii) $\left|\frac{\partial K}{\partial t}(t,s)\right| \le c(t-s)^{-1-\alpha}$.

Condition (ii) is a consequence of (see [16])

$$\frac{\partial K}{\partial t}(t,s) = c_H\left(H - \frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}. \tag{7}$$

Consider the following seminorm on the set $\mathcal{E}$ of step functions on $[0,T]$:

$$\begin{aligned}\|\varphi\|_K^2 = &\int_0^T \varphi^2(s)K(T,s)^2 ds \\ &+ \int_0^T\left(\int_s^T |\varphi(t)-\varphi(s)|(t-s)^{-1-\alpha}dt\right)^2 ds.\end{aligned}$$

We denote by $\mathcal{H}_K$ the completion of $\mathcal{E}$ with respect to this seminorm $\|\cdot\|_K$. The space $\mathcal{H}_K$ is the class of functions φ on $[0,T]$ such that $\|\varphi\|_K < \infty$, and it is continuously included in $\mathcal{H}$.

Note that if $u = \{u_t, t \in [0,T]\}$ is a process in $\mathbb{D}^{1,2}(\mathcal{H}_K)$, then there is a sequence $\{\varphi_n\}$ of bounded simple $\mathcal{H}_K$-valued processes of the form

$$\varphi_n = \sum_{j=0}^{n-1} F_j 1_{(t_j,t_{j+1}]}, \tag{8}$$

where F_j is a smooth random variable of the form

$$F_j = f_j(B_{s_1^j}, ..., B_{s_{m(j)}^j}),$$

with f_j an infinitely differentiable function with bounded derivatives, and $0 = t_0 < t_1 < ... < t_n = T$, such that

$$E\|u-\varphi_n\|_K^2 + E\int_0^T \|D_r u - D_r\varphi_n\|_K^2 dr \longrightarrow 0, \quad \text{as} \quad n \to \infty. \tag{9}$$

Moreover, if $u \in \mathbb{D}^{1,2}(\mathcal{H}_K)$, then $u \in \operatorname{Dom}\delta^B$, $K^*u \in \mathbb{L}^{1,2}$ and (5) holds.

For a process $u = \{u_t, t \in [0,T]\}$ with integrable paths and $\varepsilon > 0$, we denote by u_t^ε the integral $(2\varepsilon)^{-1}\int_{t-\varepsilon}^{t+\varepsilon} u_s ds$, where we use the convention $u_s = 0$ for $s \notin [0,T]$.

Now we introduce a stochastic integral of Stratonovich type with respect to B.

Definition 1. *We say that a process u with integrable paths belongs to* $\operatorname{Dom}\delta_S^B$ *if*

$$(2\varepsilon)^{-1}\int_0^T u_s \left(B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0}\right) ds$$

converges in probability as $\varepsilon \downarrow 0$. In this case, we denote this limit by $\delta_S^B(u)$. We also make use of the notation $\delta_S^B(u) = \int_0^T u_r \circ dB_r$.

In order to study the relationship between the integrals δ_S^B and δ^B, we introduce the following notion of trace. We say that a process $u \in \mathbb{D}^{1,2}(\mathcal{H}_K)$ belongs to the space $\mathbb{D}_C^{1,2}(\mathcal{H}_K)$ if the limit in probability

$$\mathrm{Tr}Du := \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^T \left\langle D^B u_s, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds$$

exists. We will also make use of the notation

$$\mathrm{Tr}Du = \int_0^T (\nabla u)_s ds.$$

The following is the main result of this section.

Theorem 2. *Let $u \in \mathbb{D}_C^{1,2}(\mathcal{H}_K)$ be a process such that*

$$E\int_0^T u_s^2\left(s^{-2\alpha} + (T-s)^{-2\alpha}\right) ds < \infty, \tag{10}$$

$$E\int_0^T\int_0^T (D_r u_s)^2\left(s^{-2\alpha} + (T-s)^{-2\alpha}\right) dsdr < \infty. \tag{11}$$

Then $u \in \operatorname{Dom}\delta_S^B$ and

$$\delta_S^B(u) = \delta^B(u) + \mathrm{Tr}Du.$$

In order to prove this theorem, we need the following technical result.

Lemma 3. *Let u be a simple process of the form* (8). *Then u^ε converges to u in $\mathbb{D}^{1,2}(\mathcal{H}_K)$ as $\varepsilon \downarrow 0$.*

Proof. Let u be given by the right-hand side of (8). Then u is a bounded process. Hence, property (i) of the kernel K and the dominated convergence theorem imply

$$E\int_0^T (u_s-u_s^\varepsilon)^2 K(T,s)^2 ds \longrightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0. \tag{12}$$

Fix an index $i \in \{0,1,...,n-1\}$. Using that $u_t-u_s=0$ for $s,t \in [t_{i,}t_{i+1}]$, we obtain

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}}\left(\int_s^T |u_t^\varepsilon-u_s^\varepsilon-(u_t-u_s)|(t-s)^{-1-\alpha}dt\right)^2 ds\\
\le{}&2\int_{t_i}^{t_{i+1}}\left(\int_s^{t_{i+1}} |u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds\\
&+2\int_{t_i}^{t_{i+1}}\left(\int_{t_{i+1}}^T |u_t^\varepsilon-u_s^\varepsilon-(u_t-u_s)|(t-s)^{-1-\alpha}dt\right)^2 ds\\
={}&2A_1(i,\varepsilon)+2A_2(i,\varepsilon).
\end{aligned} \tag{13}$$

The convergence of the term $A_2(i,\varepsilon)$ to 0, as $\varepsilon \downarrow 0$, follows from the dominated convergence theorem, the fact that u is a bounded process and that for a.a. $0 \le s < t \le T$,

$$|u_t^\varepsilon-u_s^\varepsilon-(u_t-u_s)|(t-s)^{-1-\alpha} \longrightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0.$$

Suppose that $\varepsilon < (1/4)\min_{0\le i\le n-1}|t_{i+1}-t_i|$. Then $u_t^\varepsilon - u_s^\varepsilon = 0$ if s and t belong to $[t_i+2\varepsilon, t_{i+1}-2\varepsilon]$. We can make the following decomposition

$$\begin{aligned}
&E(A_1(i,\varepsilon))\\
\le{}&8\int_{t_i}^{t_i+2\varepsilon}\left(\int_s^{t_i+2\varepsilon} |u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds\\
&+8\int_{t_{i+1}-2\varepsilon}^{t_{i+1}}\left(\int_s^{t_{i+1}} |u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds\\
&+8\int_{t_i}^{t_i+2\varepsilon}\left(\int_{t_i+2\varepsilon}^{t_{i+1}} |u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds\\
&+8\int_{t_i}^{t_{i+1}-2\varepsilon}\left(\int_{t_{i+1}-2\varepsilon}^{t_{i+1}} |u_t^\varepsilon-u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2 ds.
\end{aligned}$$

The first and second integrals converge to zero, due to the estimate

$$|u_t^\varepsilon-u_s^\varepsilon| \le \frac{c}{\varepsilon}|t-s|.$$

On the other hand, the third and fourth term of the above expression converge to zero because u_t^ε is bounded. Therefore, we have proved that

$$E\|u-u^\varepsilon\|_K^2 \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.$$

Finally, it is easy to see by the same arguments that we also have

$$E\int_0^T \|D_r u - D_r u^\varepsilon\|_K^2 dr \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus the proof is complete. ■

Now we are ready to prove Theorem 2.

Proof of Theorem 2. From the properties of the divergence operator, applying Fubini's theorem we have

$$\begin{aligned}
&(2\varepsilon)^{-1}\int_0^T u_s\left(B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0}\right) ds \\
&= (2\varepsilon)^{-1}\int_0^T \delta^B\left(u_s 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}(\cdot)\right) ds \\
&\quad + (2\varepsilon)^{-1}\int_0^T \left\langle D_\cdot^B u_s, 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}(\cdot)\right\rangle_{\mathcal{H}} ds \\
&= (2\varepsilon)^{-1}\int_0^T \left(\int_{(r-\varepsilon)\vee 0}^{(r+\varepsilon)\wedge T} u_s ds\right) dB_r \\
&\quad + (2\varepsilon)^{-1}\int_0^T \left\langle D_\cdot^B u_s, 1_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}(\cdot)\right\rangle_{\mathcal{H}} ds \\
&= \int_0^T u_r^\varepsilon dB_r + B^\varepsilon.
\end{aligned}$$

Using $u \in \mathbb{D}_C^{1,2}(\mathcal{H}_K)$, we get that B^ε converges to $\mathrm{Tr}Du$ in probability as $\varepsilon \downarrow 0$.

In order to see that $\int_0^T u_r^\varepsilon dB_r$ converges to $\delta^B(u)$ in $L^2(\Omega)$ as ε tends to zero, we will show that u^ε converges to u in the norm of $\mathbb{D}^{1,2}(\mathcal{H}_K)$. Fix $\delta > 0$. We have already noted that the definition of the space $\mathbb{D}^{1,2}(\mathcal{H}_K)$ implies that there is a bounded simple $\mathcal{H}_K$-valued processes φ as in (8) such that

$$E\|u-\varphi\|_K^2 + E\int_0^T \|D_r u - D_r\varphi\|_K^2 dr \le \delta. \tag{14}$$

Therefore, Lemma 3 implies that for ε small enough,

$$
\begin{aligned}
& E\|u-u^\varepsilon\|_K^2 + E\int_0^T \|D_r(u-u^\varepsilon)\|_K^2 dr \\
\le\, & cE\|u-\varphi\|_K^2 + cE\int_0^T \|D_r(u-\varphi)\|_K^2 dr \\
& + cE\|\varphi-\varphi^\varepsilon\|_K^2 + cE\int_0^T \|D_r(\varphi-\varphi^\varepsilon)\|_K^2 dr \\
& + cE\|\varphi^\varepsilon-u^\varepsilon\|_K^2 + cE\int_0^T \|D_r(\varphi^\varepsilon-u^\varepsilon)\|_K^2 dr \\
\le\, & 2c\delta + cE\|\varphi^\varepsilon-u^\varepsilon\|_K^2 + cE\int_0^T \|D_r(\varphi^\varepsilon-u^\varepsilon)\|_K^2 dr.
\end{aligned}
\tag{15}
$$

We have

$$
\begin{aligned}
& \int_0^T E(\varphi_s^\varepsilon - u_s^\varepsilon)^2 \; K(T,s)^2 ds \\
\le & \int_0^T E\left(\frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon}(\varphi_r-u_r)dr\right)^2 \; K(T,s)^2 ds \\
\le & \int_0^T E(\varphi_r-u_r)^2 \; \left(\frac{1}{2\varepsilon}\int_{(r-\varepsilon)\vee 0}^{(r+\varepsilon)\wedge T} K(T,s)^2 ds\right) dr.
\end{aligned}
$$

From property (i) it follows that

$$
(2\varepsilon)^{-1} \int_{(r-\varepsilon)\vee 0}^{(r+\varepsilon)\wedge T} K(T,t)^2 dt \le c\left[(T-r)^{-2\alpha} + r^{-2\alpha}\right].
$$

Hence, by the dominated convergence theorem and condition (10) we obtain

$$
\begin{aligned}
& \limsup_{\varepsilon\downarrow 0} \int_0^T E(\varphi_s^\varepsilon - u_s^\varepsilon)^2 \; K(T,s)^2 ds \\
\le & \int_0^T E(\varphi_s-u_s)^2 \; K(T,s)^2 ds \; \le \delta.
\end{aligned}
\tag{16}
$$

On the other hand,

(17)

$$
\begin{aligned}
&E\int_0^T\left(\int_s^T|\varphi_t^\varepsilon-u_t^\varepsilon-\varphi_s^\varepsilon+u_s^\varepsilon|(t-s)^{-1-\alpha}dt\right)^2ds\\
\le&\frac{1}{4\varepsilon^2}E\int_0^T\left(\int_{-\varepsilon}^{\varepsilon}\int_s^T|(\varphi-u)_{t-\theta}-(\varphi-u)_{s-\theta}|(t-s)^{-1-\alpha}dtd\theta\right)^2ds\\
=&\frac{1}{4\varepsilon^2}E\int_0^T\left(\int_{s-\varepsilon}^{s+\varepsilon}\int_r^{T+r-s}|(\varphi-u)_t-(\varphi-u)_r|(t-r)^{-1-\alpha}dtdr\right)^2ds\\
\le&\frac{1}{2\varepsilon}E\int_0^T\int_{s-\varepsilon}^{s+\varepsilon}\left(\int_r^{T+\varepsilon}|(\varphi-u)_t-(\varphi-u)_r|(t-r)^{-1-\alpha}dt\right)^2drds\\
=&\frac{1}{2\varepsilon}E\int_{-\varepsilon}^{T+\varepsilon}\int_{(r-\varepsilon)\vee 0}^{(r+\varepsilon)\wedge T}\left(\int_r^{T+\varepsilon}|\varphi_t-u_t-\varphi_r+u_r|(t-r)^{-1-\alpha}dt\right)^2dsdr\\
\le&E\int_{-\varepsilon}^{T+\varepsilon}\left(\int_r^{T+\varepsilon}|\varphi_t-u_t-\varphi_r+u_r|(t-r)^{-1-\alpha}dt\right)^2dr.
\end{aligned}
$$

By (16) and (17), we obtain

$$\limsup_{\varepsilon\downarrow 0}E\|\varphi^\varepsilon-u^\varepsilon\|_K^2\le 2\delta.$$

By a similar argument,

$$\limsup_{\varepsilon\downarrow 0}E\int_0^T\|D_r(\varphi^\varepsilon-u^\varepsilon)\|_K^2dr\le 2\delta.$$

Since δ is arbitrary, u^ε converges to u in the norm of $\mathbb{D}^{1,2}(\mathcal{H}_K)$ as $\varepsilon\downarrow 0$, and, as a consequence, $\int_0^T u_r^\varepsilon dB_r$ converges in $L^2(\Omega)$ to $\delta^B(u)$. Thus the proof is complete. ∎

Remark 1.
The results of this section can be easily generalized to a centered Gaussian process of the form $B_t=\int_0^t K(t,s)dW_s$, where $K(t,s)$ is a continuously differentiable kernel in the region $\{0<s<t<T\}$ satisfying conditions (i) and (ii).

4. Examples

The purpose of this section is to analyze the existence of the Stratonovich integral introduced in Definition 1 in some particular cases.

We will make use of the notation

$$T_\varepsilon(u) = (2\varepsilon)^{-1} \int_0^T \left\langle D^B u_t, 1_{[(t-\varepsilon)\vee 0,(t+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} \quad dt \tag{18}$$

for a process u in $\mathbb{D}^{1,2}(\mathcal{H}_K)$.

Let F be a continuously differentiable function satisfying the growth condition

$$\max\{|F(x)|, |F'(x)|\} \leq ce^{\lambda|x|^2}, \tag{19}$$

where c and λ are positive constants such that $\lambda < T^{-2H}/4$.

From [2] we know that if $H > 1/4$, the process $u_t = F(B_t)$ belongs to the space $L^2(\Omega; \mathcal{H}_K)$. Actually, it is not difficult to show that the process u_t belongs to $\mathbb{D}^{1,2}(\mathcal{H}_K)$. Let us check that the trace TrDu exists. To do this we first compute

$$\begin{aligned}
T_\varepsilon(u) &= (2\varepsilon)^{-1} \int_0^T F'(B_t) \langle 1_{[0,t]}, 1_{[(t-\varepsilon)\vee 0,(t+\varepsilon)\wedge T]} \rangle_{\mathcal{H}} \quad dt \\
&= (2\varepsilon)^{-1} \int_0^T F'(B_t)(R(t,(t+\varepsilon)\wedge T) - R(t,(t-\varepsilon)\vee 0)) \; dt \\
&= (4\varepsilon)^{-1} \int_0^T F'(B_t)(((t+\varepsilon)\wedge T)^{2H} - ((t-\varepsilon)\vee 0)^{2H} \\
&\quad -((t+\varepsilon)\wedge T - t)^{2H} + (t-(t-\varepsilon)\vee 0)^{2H}) dt \\
&\longrightarrow H \int_0^T F'(B_t) t^{2H-1} dt \quad \text{as} \quad \varepsilon \downarrow 0.
\end{aligned}$$

As a consequence, $F(B_t)$ belongs to the space $\mathbb{D}_C^{1,2}(\mathcal{H}_K)$, and by Theorem 2, the Stratonovich integral of $F(B_t)$ with respect to B exists. Moreover

$$\int_0^T F(B_t) \circ dB_t = \int_0^T F(B_t) dB_t + H \int_0^T F'(B_t) t^{2H-1} dt.$$

Remark 1.

The forward integral of $F(B_t)$ with respect to B defined as the limit in probability, as $\varepsilon \downarrow 0$, of

$$\varepsilon^{-1} \int_0^T F(B_t) \left(B_{(t+\varepsilon)\wedge T} - B_t \right) dt,$$

does not exist in general. For instance, in the particular case $F(x) = x$, we would find a trace term of the form

$$
\begin{aligned}
&\varepsilon^{-1}\int_0^T \left\langle 1_{[0,t]}, 1_{[t,(t+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} \; dt \\
&= \varepsilon^{-1}\int_0^T \left(R(t,(t+\varepsilon)\wedge T) - R(t,t)\right) dt \\
&= \frac{1}{2\varepsilon}\int_0^T \left(((t+\varepsilon)\wedge T)^{2H} - t^{2H} - ((t+\varepsilon)\wedge T - t)^{2H}\right) \; dt \\
&= \frac{1}{2}\left(T^{2H} - T\varepsilon^{2H-1} + \frac{2H-1}{2H+1}\varepsilon^{2H}\right),
\end{aligned}
$$

which converges to $-\infty$ as ε tends to zero.

The forward integral with respect to the fractional Brownian motion of index $H < 1/2$ has been studied in [1]. Notice that the process $F(B_t)$ does not satisfy the sufficient conditions introduced in this paper, for the forward integral to exist.

Remark 2.

The process $u = W$ does not belong to the space $\mathbb{D}_C^{1,2}(\mathcal{H}_K)$, and we cannot apply Theorem 2 to deduce the existence of the Stratonovich integral $\int_0^T W_t \circ dB_t$. In fact, as a consequence of (7),

$$
\begin{aligned}
&\frac{1}{2\varepsilon}\int_0^T \left\langle K((t+\varepsilon)\wedge T,\cdot) - K((t-\varepsilon)\vee 0,\cdot), \mathbf{1}_{[0,t]}\right\rangle_{L^2([0,T])} dt \\
&= c_H\left(H-\frac{1}{2}\right)\frac{1}{2\varepsilon}\int_0^T\int_0^t\int_{(t-\varepsilon)\vee 0}^{(t+\varepsilon)\wedge T}\left(\frac{r}{s}\right)^{\frac{1}{2}-H}(s-r)_+^{H-\frac{3}{2}} ds dr dt \\
&= c_H\left(H-\frac{1}{2}\right)\int_0^T\int_0^s \frac{(s+\varepsilon)\wedge T - (s-\varepsilon)\vee r}{2\varepsilon}\left(\frac{r}{s}\right)^{\frac{1}{2}-H}(s-r)_+^{H-\frac{3}{2}} dr ds,
\end{aligned}
$$

which by Fatou's lemma, tends to $-\infty$ as ε tends to zero.

Remark 3.

The fact that $F(B_t)$ is Stratonovich integrable with respect to B_t is still true for kernels satisfying conditions (i) and (ii) other than the fractional Brownian motion case. For instance, consider the Gaussian process $B_t = \int_0^t (t-s)^{-\alpha} dW_t$, with $\alpha \in [0, 1/2)$. That is, $K(t,s) = (t-s)^{-\alpha}$. The covariance function of this process is given by

$$
\begin{aligned}
R(t,s) &= \int_0^s (t-r)^{-\alpha}(s-r)^{-\alpha} dr = \int_0^s (t-s+r)^{-\alpha} r^{-\alpha} dr \\
&= s^{-2\alpha}\int_0^s \left(\frac{t-s+r}{s}\right)^{-\alpha}\left(\frac{r}{s}\right)^{-\alpha} dr = s^{1-2\alpha} G\left(\frac{t-s}{s}\right),
\end{aligned}
$$

with

$$G(t) = \int_0^1 (t+r)^{-\alpha} r^{-\alpha} dr.$$

As in the case of the fractional Brownian motion, the process $u_t = F(B_t)$ belongs to the space $\mathbb{D}^{1,2}(\mathcal{H}_K)$ if F is a continuously differentiable function satisfying condition (19) and $\alpha < 1/4$. Let us show that the process $u_t = F(B_t)$ belongs to the space $\mathbb{D}_C^{1,2}(\mathcal{H}_K)$. We have

$$\begin{aligned}
T_\varepsilon(u) &= (2\varepsilon)^{-1} \int_0^T F'(B_t)(R((t+\varepsilon)\wedge T, t) - R(t, (t-\varepsilon)\vee 0))dt \\
&= (2\varepsilon)^{-1} \int_\varepsilon^{T-\varepsilon} F'(B_t)\left(t^{1-2\alpha} - (t-\varepsilon)^{1-2\alpha}\right) G\left(\frac{\varepsilon}{t}\right) dt \\
&\quad + (2\varepsilon)^{-1} \int_\varepsilon^{T-\varepsilon} F'(B_t)(t-\varepsilon)^{1-2\alpha} \left(G\left(\frac{\varepsilon}{t}\right) - G\left(\frac{\varepsilon}{t-\varepsilon}\right)\right) dt \\
&\quad + (2\varepsilon)^{-1} \left(\int_0^\varepsilon F'(B_t) R(t+\varepsilon, t) dt \right.\\
&\quad \left. + \int_{T-\varepsilon}^T F'(B_t)(R(T,t) - R(t, t-\varepsilon))dt\right) \\
&= I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon}.
\end{aligned}$$

The term $I_{3,\varepsilon}$ tends to zero as ε goes to zero. By the dominated convergence theorem, the term $I_{1,\varepsilon}$ converges to

$$\left(\frac{1}{2} - \alpha\right) G(0) \int_0^T F'(B_t) t^{-2\alpha}\, dt.$$

On the other hand, for $s, r > 0$, we have

$$\frac{d}{dr}\left(s^{-\alpha}(s+r)^{-\alpha}\right) = -\alpha s^{-\alpha}(s+r)^{-1-\alpha}.$$

Thus, for $\delta > 0$ such that $2\alpha + \delta < 1$, we obtain

$$\left|\frac{d}{dr}\left(s^{-\alpha}(s+r)^{-\alpha}\right)\right| \le \alpha s^{-1+\delta} r^{-2\alpha-\delta}.$$

Therefore,

$$G'(r) \le \alpha r^{-2\alpha-\delta} \int_0^1 s^{-1+\delta} ds = c_\delta r^{-2\alpha-\delta}.$$

Hence we have that for $t \in [\varepsilon, T-\varepsilon]$, there is $\theta_{t,\varepsilon} \in (\varepsilon/t,\ \varepsilon/(t-\varepsilon))$ such that

$$\begin{aligned}
&(2\varepsilon)^{-1}(t-\varepsilon)^{1-2\alpha}\left|G\left(\frac{\varepsilon}{t}\right) - G\left(\frac{\varepsilon}{t-\varepsilon}\right)\right| \\
&\le c_\delta \varepsilon t^{-1}(t-\varepsilon)^{-2\alpha}(\theta_{t,\varepsilon})^{-2\alpha-\delta}
\end{aligned}$$

$$\leq c_\delta (t-\varepsilon)^{-2\alpha} \left(\frac{\varepsilon}{t}\right)^{1-2\alpha-\delta} \tag{21}$$

$$\leq c_\delta (t-\varepsilon)^{-2\alpha}. \tag{22}$$

Note that (21) implies

$$\begin{aligned} &(2\varepsilon)^{-1}(t-\varepsilon)^{1-2\alpha} \left| G\left(\frac{\varepsilon}{t}\right) - G\left(\frac{\varepsilon}{t-\varepsilon}\right) \right| 1_{[\varepsilon, T-\varepsilon]}(t) \\ &\longrightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0 \end{aligned}$$

and (22) gives

$$I_{2,\varepsilon} \longrightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0.$$

Observe that as in the case of the fractional Brownian motion, the process $u = W$ does not belong to the space $\mathbb{D}_C^{1,2}(\mathcal{H}_K)$. In fact, (18) implies

$$\begin{aligned} T_\varepsilon(u) =& \frac{1}{2\varepsilon}\left(\int_0^T \int_0^t ((t+\varepsilon)\wedge T - s)^{-\alpha} ds dt \right. \\ & \left. - \int_0^T \int_0^{(t-\varepsilon)\vee 0} ((t-\varepsilon)_+ - s)^{-\alpha} ds dt \right) \\ =& \frac{1}{2\varepsilon}(1-\alpha)^{-1}(2-\alpha)^{-1}\left(T^{2-\alpha} - 2\varepsilon^{2-\alpha} - (T-\varepsilon)^{2-\alpha}\right) \\ & - \frac{1}{2\varepsilon}(1-\alpha)^{-1}\varepsilon^{1-\alpha}(T-\varepsilon) + \frac{1}{2}(1-\alpha)^{-1}T^{1-\alpha}, \end{aligned}$$

which does not converge as $\varepsilon \downarrow 0$.

5. Itô's Formula for Fractional Brownian Motion Integrals

Our purpose in this section is to prove a change-of-variable formula for the Stratonovich integral defined in Section 3.

We will assume the following condition on the integrand process u.

(C) u and $D_r u$ are λ-Hölder continuous in the norm of the space $\mathbb{D}^{1,4}$ for some $\lambda > \alpha$, and the function

$$\gamma_r = \sup_{0 \leq s \leq T} \|D_r u_s\|_{1,4} + \sup_{0 \leq s \leq T} \frac{\|D_r u_t - D_r u_s\|_{1,4}}{|t-s|^\lambda}$$

satisfies $\int_0^T \gamma_r^p dr < \infty$ for some $p > 2/(1-4\alpha)$.

Then we can prove the following result.

Theorem 4. *Suppose $\alpha < 1/4$. Let u be an adapted process in $\mathbb{D}^{2,2}(\mathcal{H}_K)$ satisfying* (10), (11) *and condition* (C) *and such that the following limit exists in probability,*

$$\int_0^T \left| (\nabla u)_s - \frac{1}{2\varepsilon} \left\langle D^B u_s, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} \right| ds \to 0,$$

for some process $(\nabla u)_s$ in $\mathbb{L}^{1,2}$. Define $X_t = \int_0^t u_s \circ dB_s$. Then, for all $F \in \mathcal{C}_b^2(\mathbb{R})$ the process $F^{'}(X_s)u_s$ is Stratonovich integrable with respect to B and

$$F(X_t) = F(0) + \int_0^t F^{'}(X_s)u_s \circ dB_s.$$

Proof. We can write, by Theorem 2,

$$X_t = \int_0^t u_s dB_s + \int_0^t (\nabla u)_s ds.$$

Then, by a straightforward extension of Theorem 3 in [2], we obtain that $F'(X_s)u_s$ is Skorohod integrable with respect to B, and

$$\begin{aligned} F(X_t) = & F(0) + \int_0^t F'(X_s)u_s dB_s \\ & + \int_0^t F^{''}(X_s)u_s \left(\int_0^s \frac{\partial K}{\partial s}(s,r) \left(\int_0^s D_r(K_s^* u)_\theta dW_\theta \right) dr \right) ds \\ & + \frac{1}{2} \int_0^t F^{''}(X_s) \frac{\partial}{\partial s} \left(\int_0^s (K_s^* u)_r^2 dr \right) ds \\ & + \int_0^t F^{'}(X_s)(\nabla u)_s ds \\ & + \int_0^t F^{''}(X_s)u_s \int_0^s \left(\int_r^s D_r(\nabla u)_\theta d\theta \right) \frac{\partial K}{\partial s}(s,r) dr ds. \end{aligned}$$

Then we only need to check that the following limit in probability exists:

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \left\langle D^B \left(F^{'}(X_s)u_s \right), \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds,$$

and that it is equal to

$$\int_0^t F''(X_s)u_s\left(\int_0^s \frac{\partial K}{\partial s}(s,r)\left(\int_0^s D_r(K_s^*u)_\theta dW_\theta\right)dr\right)ds$$
$$+\frac{1}{2}\int_0^t F''(X_s)\frac{\partial}{\partial s}\left(\int_0^s (K_s^*u)_r^2 dr\right)ds$$
$$+\int_0^t F'(X_s)(\nabla u)_s ds$$
$$+\int_0^t F''(X_s)u_s\int_0^s\left(\int_0^s D_r(\nabla u)_\theta d\theta\right)\frac{\partial K}{\partial s}(s,r)drds.$$

We can write

$$\begin{aligned}
&\frac{1}{2\varepsilon}\int_0^t \left\langle D^B(F'(X_s)u_s), \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\\
=&\frac{1}{2\varepsilon}\int_0^t F'(X_s)\left\langle D^B u_s, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\\
&+\frac{1}{2\varepsilon}\int_0^t F''(X_s)u_s\left\langle D^B X_s, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\\
=&\frac{1}{2\varepsilon}\int_0^t F'(X_s)\left\langle D^B u_s, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\\
&+\frac{1}{2\varepsilon}\int_0^t F''(X_s)u_s\left\langle D^B\left(\int_0^s u_r dB_r\right), \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\\
&+\frac{1}{2\varepsilon}\int_0^t F''(X_s)u_s\left\langle D^B\left(\int_0^s (\nabla u)_r dr\right), \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\\
=&T_1^\varepsilon+T_2^\varepsilon+T_3^\varepsilon.
\end{aligned}$$

Easily, the first term converges to $\int_0^t F'(X_s)(\nabla u)_s ds$ in probability.

On the other hand, by the relationship between the derivative operators with respect to B and with respect to W, it follows that

$$\begin{aligned}
T_2^\varepsilon =& \frac{1}{2\varepsilon}\int_0^t F''(X_s)u_s\left[\int_0^{s+\varepsilon} D_\theta\left(\int_0^s (K_s^*u)_r dW_r\right)K(s+\varepsilon,\theta)d\theta\right.\\
&\left.-\int_0^{s-\varepsilon} D_\theta\left(\int_0^s (K_s^*u)_r dW_r\right)K(s-\varepsilon,\theta)d\theta\right]ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\varepsilon}\int_0^t F''(X_s)u_s\left[\int_0^s (K_s^*u)_\theta K(s+\varepsilon,\theta)d\theta\right.\\
&\quad\left.-\int_0^{s-\varepsilon}(K_s^*u)_\theta K(s-\varepsilon,\theta)d\theta\right]ds\\
&\quad+\frac{1}{2\varepsilon}\int_0^t F''(X_s)u_s\left[\int_0^{s+\varepsilon}\left(\int_0^s D_\theta(K_s^*u)_r dW_r\right)K(s+\varepsilon,\theta)d\theta\right.\\
&\quad\left.-\int_0^{s-\varepsilon}\left(\int_0^s D_\theta(K_s^*u)_r dW_r\right)K(s-\varepsilon,\theta)d\theta\right]ds\\
&= T_{2,1}^\varepsilon + T_{2,2}^\varepsilon.
\end{aligned}
$$

Using the definition of K_s^*u, we can write

$$
\begin{aligned}
&\frac{1}{2\varepsilon}\left[\int_0^s (K_s^*u)_\theta\ K(s+\varepsilon,\theta)d\theta-\int_0^{s-\varepsilon}(K_s^*u)_\theta K(s-\varepsilon,\theta)d\theta\right]\\
&= \frac{1}{2\varepsilon}\left[\int_0^s K(s,\theta)u_\theta K(s+\varepsilon,\theta)d\theta-\int_0^{s-\varepsilon}K(s,\theta)u_\theta K(s-\varepsilon,\theta)d\theta\right]\\
&\quad+\frac{1}{2\varepsilon}\left[\int_0^s\left(\int_\theta^s \frac{\partial K}{\partial r}(r,\theta)(u_r-u_\theta)dr\right)K(s+\varepsilon,\theta)d\theta\right.\\
&\quad\left.-\int_0^{s-\varepsilon}\left(\int_\theta^s \frac{\partial K}{\partial r}(r,\theta)(u_r-u_\theta)dr\right)K(s-\varepsilon,\theta)d\theta\right].
\end{aligned}
$$

We add and substract u_s in the first two integrals of the above expression and obtain

$$
\begin{aligned}
&\frac{u_s}{2\varepsilon}[R(s,s+\varepsilon)-R(s,s-\varepsilon)]\\
&+\frac{1}{2\varepsilon}\left[\int_0^T K(s,\theta)(u_\theta-u_s)[K(s+\varepsilon,\theta)-K(s-\varepsilon,\theta)]d\theta\right]\\
&+\frac{1}{2\varepsilon}\int_0^T\left(\int_\theta^s \frac{\partial K}{\partial r}(r,\theta)(u_r-u_\theta)dr\right)[K(s+\varepsilon,\theta)-K(s-\varepsilon,\theta)]d\theta.
\end{aligned}
$$

Substituting the above expression into $T_{2,1}^\varepsilon$, it is easy to see that this term converges in $L^1(\Omega)$ to

$$
\begin{aligned}
&H\int_0^t F''(X_s)u_s^2\ s^{2H-1}ds\\
&+\frac{1}{2}\int_0^t F''(X_s)u_s\ \left(\int_0^s (u_\theta-u_s)\frac{\partial K^2}{\partial s}(s,\theta)d\theta\right)ds\\
&+\int_0^t F''(X_s)u_s\ \int_0^s\left(\int_\theta^s \frac{\partial K}{\partial r}(r,\theta)(u_r-u_\theta)dr\right)\frac{\partial K}{\partial s}(s,\theta)d\theta ds\\
&= \frac{1}{2}\int_0^t F''(X_s)\frac{\partial}{\partial s}\int_0^s (K_s^*u)_\theta^2 d\theta.
\end{aligned}
$$

The term $T_{2,2}^{\varepsilon}$ converges in $L^1(\Omega)$ to

$$\int_0^t F''(X_s)u_s \left(\int_0^s \frac{\partial K}{\partial s}(s,\theta) \left(\int_0^s D_\theta (K_s^* u)_r dW_r \right) d\theta \right) ds.$$

It remains now to prove the convergence of the term T_3^{ε}. Using again the relationship between the derivative operators with respect to B and with respect to W, we can write

$$\begin{aligned} T_3^{\varepsilon} = & \frac{1}{2\varepsilon} \int_0^t F''(X_s)u_s \left[\int_0^{s+\varepsilon} \left(\int_0^s D_\theta (\nabla u)_r dr \right) K(s+\varepsilon,\theta) d\theta \right. \\ & \left. - \int_0^{s-\varepsilon} \left(\int_0^s D_\theta (\nabla u)_r dr \right) K(s-\varepsilon,\theta) d\theta \right] ds, \end{aligned}$$

from which we deduce that T_3^{ε} converges in $L^1(\Omega)$ to

$$\int_0^t F''(X_s)u_s \int_0^s \left(\int_0^s D_r (\nabla u)_\theta d\theta \right) \frac{\partial K}{\partial s}(s,r) dr ds.$$

Now the proof is complete. ∎

6. Application to Stochastic Differential Equations

Let $B = \{B_t, t \in [0,T]\}$ the fractional Brownian motion with parameter $H \in (1/4, 1/2)$. Consider the equation

$$X_t = x + \int_0^t a(X_s) \circ dB_s + \int_0^t b(X_s) ds, \tag{23}$$

where $x \in \mathbb{R}$ and a, b are measurable functions.

Definition 5. *We will say that a process $X = \{X_t, t \in [0,T]\}$ is a solution to* (23) *if the integrals of the right-hand side of this equation are well defined and* (23) *holds.*

Then, using the pathwise representation result for one-dimensional stochastic differential equations due to Doss [8], we have the following result:

Proposition 6. *Assume that $a \in \mathcal{C}_b^2(\mathbb{R})$ and $b \in \mathcal{C}_b^1(\mathbb{R})$. Then the unique solution of* (23) *is given by*

$$X_t = \alpha(B_t, Y_t),$$

where Y_t is the solution of

$$Y_t = x + \int_0^t \left(\frac{\partial \alpha}{\partial y}(B_s, Y_s) \right)^{-1} b(\alpha(B_s, Y_s)) ds$$

and $\alpha(x, y)$ is the solution of

$$\begin{cases} \frac{\partial \alpha}{\partial x}(x, y) = a(\alpha(x, y)) \\ \alpha(0, y) = y. \end{cases}$$

Proof. For any $\varepsilon > 0$, set

$$B_t^\varepsilon = \frac{1}{2\varepsilon} \int_0^t (B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0}) ds$$

and

$$X_t^\varepsilon = \alpha(B_t^\varepsilon, Y_t).$$

Using the usual rules of the deterministic integral calculus, it follows that

$$\begin{aligned} X_t^\varepsilon &= \alpha(B_t^\varepsilon, Y_t) \\ &= x + \frac{1}{2\varepsilon} \int_0^t a(\alpha(B_s^\varepsilon, Y_s))(B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0}) ds \\ &\quad + \int_0^t \left(\frac{\partial \alpha}{\partial y}(B_s^\varepsilon, Y_s) \right) \left(\frac{\partial \alpha}{\partial y}(B_s, Y_s) \right)^{-1} b(\alpha(B_s, Y_s)) ds \\ &= x + \frac{1}{2\varepsilon} \int_0^t a(\alpha(B_s, Y_s))(B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0}) ds \\ &\quad + \frac{1}{2\varepsilon} \int_0^t [a(\alpha(B_s^\varepsilon, Y_s)) - a(\alpha(B_s, Y_s))](B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0}) ds \\ &\quad + \int_0^t \left(\frac{\partial \alpha}{\partial y}(B_s^\varepsilon, Y_s) \right) \left(\frac{\partial \alpha}{\partial y}(B_s, Y_s) \right)^{-1} b(\alpha(B_s, Y_s)) ds. \end{aligned} \tag{24}$$

Now the proof will be decomposed into several steps.

Step 1. Using $a \in \mathcal{C}_b^2(\mathbb{R})$ and the fact that b is bounded, it is easy to check that the last term in (24) converges a.s. to

$$\int_0^t b(\alpha(B_s, Y_s)) ds$$

and that the left-hand side of this equality converges a.s. to X_t.

Step 2. The process $a(\alpha(B_s, Y_s))$ is Stratonovich integrable. Observe that

$$
\begin{aligned}
Y_s &= x + \int_0^s \left(\frac{\partial \alpha}{\partial y}(B_u, Y_u)\right)^{-1} b(\alpha(B_u, Y_u))du \\
&= x + \int_0^s \exp\left(-\int_0^{B_u} a^{'}(\alpha(z, Y_u))dz\right) b(\alpha(B_u, Y_u))du \qquad (25)\\
&= x + \int_0^s F(B_u, Y_u)du,
\end{aligned}
$$

where $F(x, y) = \exp(-\int_0^x a^{'}(\alpha(z, y))dz)b(\alpha(x, y))$. Fix an integer N. Let φ_N be an infinitely differentiable function with compact support such that $\varphi_N(x) = x$ if $|x| \leq N$. Set $F_N(x, y) = \varphi_N(x)F(x, y)$, and let Y^N be the solution to Eq. (25) with F replaced by F_N. Notice that the processes Y and Y^N coincide on the set

$$\Omega_N = \left\{\omega \in \Omega : \sup_{t \leq T} |B_t| < N\right\}.$$

Taking into account that $\Omega = \cup \Omega_N$, it suffices to show that $a(\alpha(B_s, Y_s^N))$ is Stratonovich integrable for each N. It is clear that Y^N belongs to $\mathbb{D}^{1,2}(\mathcal{H})$ and we have

$$D_r Y_s^N = \int_r^s \frac{\partial F_N}{\partial x}(B_u, Y_u^N)K(u, r)du + \int_r^s \frac{\partial F_N}{\partial y}(B_u, Y_u^N)(D_r Y_u^N)du.$$

From here it follows that

$$|D_r Y_s^N| \leq C_N \int_r^s K(u, r)du \leq C_N (s - r)^{1-\alpha} r^{-\alpha}.$$

Hence we obtain that $a(\alpha(B, Y^N)) \in \mathbb{D}^{1,2}(\mathcal{H})$ and

$$
\begin{aligned}
D_r[a(\alpha(B_s, Y_s^N))] &= a^{'}(\alpha(B_s, Y_s^N))a(\alpha(B_s, Y_s^N))K(s, r) \\
&\quad + a^{'}(\alpha(B_s, Y_s^N))\frac{\partial \alpha}{\partial y}(B_s, Y_s^N)D_r Y_s^N.
\end{aligned}
$$

Let us study now the trace term. Using the notation $A(x, y) = a(\alpha(x, y))$ we can write

$$
\begin{aligned}
&\frac{1}{2\varepsilon}\int_0^T \left\langle D^B a(\alpha(B_s, Y_s^N)), \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds \\
&= \frac{1}{2\varepsilon}\int_0^T \left\langle \frac{\partial A}{\partial x}(B_s, Y_s^N)\mathbf{1}_{[0,s]}, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds \\
&+\frac{1}{2\varepsilon}\int_0^T \left\langle \frac{\partial A}{\partial y}(B_s, Y_s^N)D^B Y_s^N, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds \qquad (26)\\
&= \frac{1}{2\varepsilon}\int_0^T \frac{\partial A}{\partial x}(B_s, Y_s^N)[R((s+\varepsilon)\wedge T, s) - R((s-\varepsilon)\vee 0, s)]ds \\
&+\frac{1}{2\varepsilon}\int_0^T \frac{\partial A}{\partial y}(B_s, Y_s^N)\left\langle D^B Y_s^N, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds.
\end{aligned}
$$

Easily, the first term of (26) converges to

$$
H\int_0^T \frac{\partial A}{\partial x}(B_s, Y_s^N)s^{2H-1}ds.
$$

On the other hand, by the relationship between the derivatives with respect to B and the derivative with respect to W, it follows that

$$
\begin{aligned}
&\frac{1}{2\varepsilon}\int_0^T \frac{\partial A}{\partial y}(B_s, Y_s^N)\left\langle D^B Y_s^N, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds \\
&= \frac{1}{2\varepsilon}\int_0^T \frac{\partial A}{\partial y}(B_s, Y_s^N)\left[\int_0^{(s+\varepsilon)\wedge T} D_\theta Y_s^N K((s+\varepsilon)\wedge T, \theta)d\theta \right.\\
&\left. -\int_0^{(s-\varepsilon)\vee 0} D_\theta Y_s^N K((s-\varepsilon)\vee 0, \theta)d\theta\right] ds.
\end{aligned}
$$

Using the estimate $|D_\theta Y_s^N| \le C_N(s-\theta)^{1-\alpha}\theta^{-\alpha}$, the above term converges a.s. to

$$
\int_0^T \frac{\partial A}{\partial y}(B_s, Y_s^N)\left(\int_0^s D_\theta Y_s^N \frac{\partial K}{\partial s}(s,\theta)d\theta\right) ds.
$$

Step 3. By the previous steps and taking the limit as ε tends to zero in (24), we know that

$$
\frac{1}{2\varepsilon}\int_0^t [a(\alpha(B_s^\varepsilon, Y_s^N)) - a(\alpha(B_s, Y_s^N))](B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0})ds \qquad (27)
$$

converges in probability to

$$
X_t - x - \int_0^t a(X_s)\circ dB_s - \int_0^t b(X_s)ds.
$$

Therefore, it suffices to check that the limit in probability of (27) is zero. Let G be a smooth and cylindrical random variable. Then we can write

$$
\begin{aligned}
&\frac{1}{2\varepsilon}E\left[G\int_0^t [A(B_s^\varepsilon, Y_s^N) - A(B_s, Y_s^N)](B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0})ds\right] \\
&= \frac{1}{2\varepsilon}E\left[\int_0^t \left\langle D^B[G(A(B_s^\varepsilon, Y_s^N) - A(B_s, Y_s^N))], \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\right] \\
&= \frac{1}{2\varepsilon}E\left[\int_0^t [A(B_s^\varepsilon, Y_s^N) - A(B_s, Y_s^N)] \left\langle D^B G, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\right] \\
&\quad + \frac{1}{2\varepsilon}E\left[G\int_0^t \frac{\partial A}{\partial x}(B_s^\varepsilon, Y_s^N) \left\langle D^B B_s^\varepsilon - \mathbf{1}_{[0,s]}, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\right] \\
&\quad + \frac{1}{2\varepsilon}E\left[G\int_0^t \left(\frac{\partial A}{\partial x}(B_s^\varepsilon, Y_s^N) - \frac{\partial A}{\partial x}(B_s, Y_s^N)\right)\right. \\
&\quad \left.\times \left\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\right] \\
&\quad + \frac{1}{2\varepsilon}E\left[G\int_0^t \left(\frac{\partial A}{\partial y}(B_s^\varepsilon, Y_s^N) - \frac{\partial A}{\partial y}(B_s, Y_s^N)\right)\right. \\
&\quad \left.\times \left\langle D^B Y_s^N, \mathbf{1}_{[(s-\varepsilon)\vee 0,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\right].
\end{aligned}
$$

By the dominated convergence theorem it is not difficult to check that each term in the above expression converges to zero as ε tends to zero.

The proof is now complete. ∎

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