TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 5, pp. 1245-1259, September 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

SEMICONTINUITY OF METRIC PROJECTIONS IN c_0-DIRECT SUMS

S. Lalithambigai and Darapaneni Narayana

Abstract. Let $\{X_i : i \in \mathbb{N}\}$ be a family of Banach space and let $Y_i \subseteq X_i$ be a closed subspace in X_i for each $i \in \mathbb{N}$ such that at least two Y_i's are non-trivial. Consider $X = (\oplus_{c_0} X_i)_{i\in\mathbb{N}}$ and $Y = (\oplus_{c_0} Y_i)_{i\in\mathbb{N}}$. We show that Y is strongly proximinal in X if and only if P_Y is *upper Hausdorff semi-continuous* on X if and only if Y_i is strongly proximinal subspace in X_i for each $i \in \mathbb{N}$. This shows that in [9, Theorem 3.4], strong proximinality of Y_i's is a necessary assumption. We also show that *lower semi-continuity* of metric projections is stable in c_0-direct sums.

1. INTRODUCTION

We work only with real Banach spaces. For a Banach space X, we denote by B_X, S_X and $NA(X)$ the closed unit ball of X, unit sphere of X and the set of norm attaining functionals on X. For x in X and $r > 0$, we denote by $B_X(x,r)$ and $B_X[x,r]$ resp. the open ball of X centered at x with radius r and the closed ball of X centered at x with radius r. Let X be a Banach space and Y be a closed subspace of X. Let $x \in X$ and $\delta > 0$. Consider the maps $P_Y(\cdot) : X \rightrightarrows Y$ and $P_Y(\cdot,\delta) : X \rightrightarrows Y$ defined by

$$\begin{aligned} P_Y(x) &= \{y \in Y : \|x-y\| = d(x,Y)\} \\ P_Y(x,\delta) &= \{y \in Y : \|x-y\| < d(x,Y)+\delta\} \end{aligned}$$

The set $P_Y(x)$ is the set of all best approximations of x in Y. This defines a set-valued mapping and P_Y is called the metric projection onto Y. Y is said to be a proximinal subspace of X if for every $x \in X$, $P_Y(x)$ is nonempty. A proximinal

Received September 30, 2004; revised January 26, 2005.
Communicated by Bor-Luh Lin.
2000 *Mathematics Subject Classification*: Primary 41A65, Secondary 46B20.
Key words and phrases: Proximinal, Metric projection, Lower Semi-Continuity, Upper Hausdorff Semi-Continuity.

subspace Y of X is said to be strongly proximinal if for all $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in P_Y(x, \delta)$, $d(y, P_Y(x)) < \varepsilon$([5] and [4]).

The following is shown in [9] :

Theorem 1.1. *Let X_1 and X_2 be two Banach spaces and let Y_1 and Y_2 be closed subspaces in X_1 and X_2 resp. Then*

(i) If P_{Y_1} and P_{Y_2} are *lower Hausdorff semi-continuous* , then $P_{Y_1 \oplus_{\ell_\infty} Y_2}$ is *lower Hausdorff semi-continuous* on $X_1 \oplus_{\ell_\infty} X_2$.

(ii) If P_{Y_1} and P_{Y_2} are *upper Hausdorff semi-continuous*, and Y_1 and Y_2 are strongly proximinal in X_1 and X_2 resp. then $P_{Y_1 \oplus_{\ell_\infty} Y_2}$ is *upper Hausdorff semi-continuous* on $X_1 \oplus_{\ell_\infty} X_2$.

And also it is proved that if Y is a finite codimensional proximinal subspace of c_0, then Y is proximinal in ℓ_∞ and P_Y is Hausdorff metric continuous.

In this note we prove that strong proximinality of Y_1 and Y_2 in Theorem 1.1(ii) is necessary and sufficient for *upper Hausdorff semi-continuous* of $P_{Y_1 \oplus_{\ell_\infty} Y_2}$ and the above result holds true for infinite c_0-direct sums. Indeed we prove the following. Our techniques are different from [9].

Let $\{X_i : i \in \mathbb{N}\}$ be a family of Banach spaces and Y_i be a subspace in X_i for each $i \in \mathbb{N}$. Assume that at least two Y_is are nontrivial. Consider the following direct sums $X = (\oplus_{c_0} X_i)_{i \in \mathbb{N}}$ and $Y = (\oplus_{c_0} Y_i)_{i \in \mathbb{N}}$.

Theorem 1.2. *Then:*

(i) The following are equivalent :

- (a) Y_i is strongly proximinal in X_i for each $i \in \mathbb{N}$
- (b) Y is strongly proximinal in X.
- (c) P_Y is *upper Hausdorff semi-continuous* in X.

(ii) The following are equivalent :

- (a) P_{Y_i} is *lower semi-continuous* (resp. *lower Hausdorff semi-continuous*) in X_i for each $i \in \mathbb{N}$
- (b) P_Y is *lower semi-continuous* (resp. *lower Hausdorff semi-continuous*) in X.

We also prove that if Y is a factor reflexive proximinal subspace of a Banach space X, where X is a c_0-direct sum of a family of reflexive Banach spaces, then Y is proximinal in all duals of even order of X. If P_Y is *lower semicontinuous* (*lower Hausdorff semicontinuous* or *upper Hausdorff semicontinuous* or *Hausdorff metric continuous* resp.) on X, then P_Y also has the same continuity properties on all its duals of even order.

Now a few definitions are in order :

Let X be a Banach space and let Y be a proximinal subspace of X. Let P_Y be the metric projection of X onto Y. P_Y is said to be *lower semicontinuous (l.s.c)* at $x \in X$ if for each open set $V \subset Y$ with $P_Y(x) \cap V \neq \phi$, there exists a neighborhood U of x such that $P_Y(z) \cap V \neq \phi$ for all $z \in U$. This is equivalent to saying :

For x in X, $\varepsilon > 0$ and y in $P_Y(x)$, there exists $\delta > 0$ such that $P_Y(z) \cap B_Y(y, \varepsilon) \neq \emptyset$ for all z in $B_X(x, \delta)$. If the choice of δ is independent of the choice of $y \in P_Y(x)$, equivalently

(1) $$P_Y(z) \cap B_Y(y, \varepsilon) \neq \emptyset, \quad \forall\, y \in P_Y(x) \text{ and } \forall\, z \in B_X(x, \delta)$$

then we say P_Y is *lower Hausdorff semi continuous (l.H.s.c)* at x.

The metric projection P_Y is said to be *upper semicontinuous*(*u.s.c.*) at x if for every open set $V \subseteq Y$ such that $P_Y(x) \subseteq V$, there exists an open neighborhood U of x such that $P_Y(z) \subseteq V$ for all $z \in U$. And P_Y is said to be *upper Hausdorff semicontinuous*(*u.H.s.c.*) at x, if for every $\varepsilon > 0$, there exists an open neighborhood U of x in X such that $P_Y(z) \subseteq P_Y(x) + \varepsilon B_Y$ for all $z \in U$.

Remark 1.3. If the metric projection is single valued, then all the above semi continuities are equivalent to continuity of a single valued map. Let Y be a closed subspace of a Banach space X and $x \in X$. If Y is strongly proximinal at x it is easy to see that P_Y is *u.H.s.c* at x.

We present a few examples of metric projections which have(or do not have) the above semi continuities.

Examples :

(i) Let Y be a finite dimensional subspace of a Banach space X. Then P_Y is *u.s.c.* and *u.H.s.c.* on X. If X is polyhedral, then P_Y is *l.s.c* and *l.H.s.c.*

A. L. Brown has constructed an example of three dimensional non-polyhedral normed linear space which has one dimensional subspace such that metric projection onto this subspace is not *l.s.c.*

(ii) Let Y be a finite co-dimensional proximinal subspace of c_0. Then P_Y is *l.H.s.c.* and *u.H.s.c.* on c_0 ([9]).

(iii) Let Y be a proximinal hyperplane in a Banach space X. Then P_Y is *l.s.c.*, *l.H.s.c.* and *u.H.s.c.* resp. on X (Proposition 2.1). In particular if we take $X = c_0$ and $Y = \ker f$ where $f = (1, 0, 0, \cdots) \in \ell_1$, then Y is proximinal and P_Y is not *u.s.c.* on $X \setminus Y$ as P_Y has non-compact images for all points in $X \setminus Y$([2]).

Let Y be a subspace of a Banach space X. Let $\mathcal{C}(Y)$ denote the class of all bounded, closed and convex subsets of Y. The Hausdorff metric on $\mathcal{C}(Y)$ is given by

$$h(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for all A and B in $\mathcal{C}(Y)$.

We say that P_Y is *Hausdorff metric continuous*(*H.m.c.*) at x if P_Y is continuous from X into $\mathcal{C}(Y)$. Observe that if Y is proximinal in X, then P_Y is *H.m.c* at x in X if and only if P_Y is both *l.H.s.c* and *u.H.s.c* at x ([9, Remark 2.8]). It follows by observing that if A and B are in $\mathcal{C}(Y)$, then $h(A, B) < \varepsilon \Leftrightarrow A \subseteq B + \varepsilon B_Y$ and $A \cap B_Y(z, \varepsilon) \neq \emptyset$ for all $z \in B$. Now by Example (ii), if Y is a finite co-dimensional proximinal subspace of c_0, then P_Y is *H.m.c.* Indeed, it will be shown that (Corollary 3.6) if Y is finite co-dimensional proximinal subspace of c_0, then Y is strongly proximinal in all even duals of c_0 and P_Y is *H.m.c.* on each of them.

2. Continuity and Strong Proximinality in ℓ_∞-Direct Sum of Finite Family of Banach Spaces

It would be interesting to see whether there is any connection between continuity of metric projection and strong proximinality. If Y is strongly proximinal then by simple observation we can show that P_Y is *u.H.s.c.* In general converse is not true i.e., continuity of the metric projection does not imply strong proximinality of the subspace. In the positive note we prove that if Y is proximinal subspace of a Banach space X, then Y is strongly proximinal in X if and only if $P_{(Y \oplus_{\ell_\infty} Y)}$ is *u.H.s.c.* in $X \oplus_{\ell_\infty} X$.

It was shown in [2] that if Y is a proximinal subspace of X and $x \in X$, then

- P_Y is u.s.c. at x if and only if P_Y is *u.H.s.c.* at x and $P_Y(x)$ is compact in Y (Theorem 1).
- If P_Y is *l.H.s.c.* at x, then P_Y is *l.s.c.* at x (Lemma 2).

  If $P_Y(x)$ is compact, then
- P_Y is *l.H.s.c.* at x if and only if P_Y is *l.s.c.* at x.

We do not have an example of a metric projection which is *l.s.c.* but not l.H.s.c. In [12], it was shown that, for any proximinal hyperplane Y in X, P_Y is both *u.H.s.c.* and *l.H.s.c.* By [2, Lemma 2], P_Y is *l.s.c.* In [16, Theorem 4], it was shown that P_Y is *l.s.c* on X when Y is a proximinal hyperplane in X. It was also noted in [13, Theorem 5] that if X is reflexive and Y is hyperplane in X, P_Y is u.H.s.c. Now we give a simple proof for all the above facts :

Proposition 2.1. *Let X be a Banach space and Y be a proximinal hyperplane in X. Then*

(i) P_Y is *u.H.s.c.* on X.

(ii) P_Y is *l.s.c.* on X

(iii) P_Y is *l.H.s.c.* on X.

If P_Y has only compact images in Y, then

(iv) P_Y is *u.s.c.* on X.

Proof. Let $x \in X$ and $\varepsilon > 0$. Let $\{x_n\}$ be in X such that $x_n \to x$. Let $f \in S_{X^*}$ such that $f \in NA(X)$ and consider $Y = \ker f$. Then Y is proximinal in X.
(i) : To prove (i) we show that there exists n_0 such that $P_Y(x_n) \subseteq P_Y(x) + \varepsilon B_Y$ for all $n \geq n_0$.

With out loss of generality, assume that $d(x_n, Y) = d(x, Y) = 1 = f(x)$ for all $n \geq 1$. Now there exists $n_1 > 0$ such that $f(x_n) = f(x)$ for all $n \geq n_1$. Now we have for all $n \geq n_1$

$$\begin{aligned} x_n - P_Y(x_n) &= \{z \in S_X : f(z) = f(x_n)\} \\ &= \{z \in S_X : f(z) = f(x)\} \\ &= x - P_Y(x) \end{aligned}$$

Then $P_Y(x_n) = (x_n - x) + P_Y(x)$. Since $x_n \to x$, there exists $n_2 > 0$ such that $\|x_n - x\| < \varepsilon$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then $P_Y(x_n) \subseteq P_Y(x) + \varepsilon B_Y$ for all $n \geq n_0$ which proves (i). It is easy to see that remaining part of the Proposition by the above discussion. ■

By the above Proposition and [4, Proposition 2.6], *u.H.s.c* of P_Y need not imply that Y is strongly proximinal. Now we study some cases when the converse is true. Let $\{X_i : i \in I\}$ be a finite family of Banach spaces, where $I \subseteq \mathbb{N}$ and $|I| = n < \infty$. Let Y_i be a proximinal subspace in X_i for each $i \in I$ and consider $X = (\oplus_{\ell_p} X_i)_{i \in I}$ and $Y = (\oplus_{\ell_p} Y_i)_{i \in I}$ where $1 \leq p < \infty$. By simple calculations, it can be seen that P_{Y_i} is *l.s.c*(*l.H.s.c* or *u.H.s.c* or *Hausdorff metric continuous* resp.) for each $i \in I$, if and only if P_Y is *l.s.c*(*l.H.s.c* or *u.H.s.c* or *Hausdorff metric continuous* resp.) on X. Similar result holds true for strong proximinality. When $p = +\infty$, situation is different. First we start with finite family of Banach spaces and show the semi-continuity results for ℓ_∞-direct sum of these spaces. We apply these results to show that [9, Theorem 3.4] holds true for c_0-direct sum of infinite family of Banach spaces. For this we need to introduce some notations :

We consider ℓ_∞-direct sum of finite family of Banach spaces $\{X_i : i \in I\}$, i.e., $X = (\oplus_{\ell_\infty} X_i)_{i \in I}$ where I is a finite subset of $\mathbb{N}$. For $x = (x_i) \in X$, we have $\|x\|_{\ell_\infty} = \sup_{i \in I}\{\|x_i\|\}$. Let Y_i be a proximinal subspace of X_i for each $i \in I$. Now consider ℓ_∞-direct sum of Y_i i.e., $Y = (\oplus_{\ell_\infty} Y_i)_{i \in I}$. For $x = (x_i)_{i \in I}$, we set

$$d_i(x) = d(x_i, Y_i) \qquad \text{for} \quad i \in I$$

Note that $d(x) = \sup_{i\in I}\{d_i(x)\}$ and also for any $x' = (x'_i) \in X$, we have $|d_i(x) - d_i(x')| \leq \|x_i - x'_i\|$. In particular d and d_i's are 1-Lipschitz functions on X and X_i for each $i \in I$ resp.

The following Remark is easy to verify :

Remark 2.2. Let Y_i be a proximinal subspace of X_i for $i \in I$ and consider $X = (\oplus_{\ell_\infty} X_i)_{i\in I}$ and $Y = (\oplus_{\ell_\infty} Y_i)_{i\in I}$. Then Y is proximinal in X and

$$P_Y(x) = (\oplus_{\ell_\infty} B_{X_i}[x_i, d(x)] \cap Y_i)_{i\in I}$$

Observe that if $d_i(x) = d(x)$, we have $B_{X_i}[x_i, d(x)] \cap Y_i = P_{Y_i}(x_i)$.

We need the following result([9, Fact 3.2]), in the sequel :

Lemma 2.3. ([9]) *Let X be a Banach space, Y be a proximinal subspace of X and x be in $X\backslash Y$. Let $\alpha > d(x,Y) = d_x$. Then given $\epsilon > 0$, there exists $\delta > 0$ such that for any z in $B_X(x,\delta)$ and β satisfying $|\beta - \alpha| < \delta$, we have*

$$h(B_X[x,\alpha] \cap Y, B_X[z,\beta] \cap Y) \;\leq\; \epsilon.$$

Before going further, we first introduce some more notations :

As usual, let $\{X_i : i \in I\}$ be a finite family of Banach spaces, Y_i be a proximinal subspace of X_i for each $i \in I$ and consider their ℓ_∞-direct sums $X = (\oplus_{\ell_\infty} X_i)_{i\in I}$ and $Y = (\oplus_{\ell_\infty} Y_i)_{i\in I}$, where $I \subset \mathbb{N}$ with $|I| = n < \infty$. Let us define a map I_0 from X into 2^I by

$$I_0(x) = \{i \in I : d_i(x) = d(x)\}$$

Observe that I_0 is upper semi-continuous on X. Indeed for given $x_0 \in X$, there exists $\eta > 0$ such that if $\|x - x_0\| < \eta$, then $I_0(x) \subseteq I_0(x_0)$. Let us define another map Q_{Y_i} from X into 2^{Y_i} for fixed $i \in I$ by

$$Q_{Y_i}(x) = B_{X_i}[x_i, d(x)] \cap Y_i \quad \text{for } i \in I \tag{2}$$

Then we have $P_Y(x) = (\oplus_{\ell_\infty} Q_{Y_i}(x))_{i\in I}$.

Lemma 2.4. *Let $\{X_i : i \in I\}$ be a finite family of Banach spaces and let Y_i be a proximinal subspace of X_i for each $i \in I$, where $I \subset \mathbb{N}$ and $|I| = n < \infty$. Consider $X = (\oplus_{\ell_\infty} X_i)_I$ and $Y = (\oplus_{\ell_\infty} Y_i)_I$. Let Q_{Y_i} be defined as above. If Q_{Y_i} is l.s.c(l.H.s.c or u.H.s.c resp.) at $x = (x_i) \in X$ for each $i \in I$ then $P_Y(x)$ is l.s.c(l.H.s.c or u.H.s.c resp.) at x.*

Proof. Suppose Q_{Y_i} is *l.s.c* at $x = (x_i) \in X$ for each $i \in I$. Let $\varepsilon > 0$ and $y = (y_i) \in P_Y(x)$. We have to show that there exists $\delta > 0$ such that

$$P_Y(z) \cap B_Y(y,\varepsilon) \neq \emptyset \forall\, z \in B_X(x,\delta) \tag{3}$$

By *l.s.c* of Q_{Y_i} at x, there exists $\delta_i > 0$ such that

(4) $$Q_{Y_i}(z) \cap B_{Y_i}(y_i, \varepsilon) \neq \emptyset \forall\, z \in B_X(x, \delta)$$

Let $\delta = \min_{i\in I}\{\delta_i\}$. Then it is easy to see that

(5) $$P_Y(z) \cap B_Y(y, \varepsilon) \neq \emptyset \forall\, z \in B_X(x, \delta)$$

This implies that P_Y is *l.s.c* at x. Similar arguments show that P_Y is *l.H.s.c* at x if Q_{Y_i} is *l.H.s.c* at x for each $i \in I$.

Now suppose Q_{Y_i} is *u.H.s.c* at $x = (x_i) \in X$ for each $i \in I$ and $\varepsilon > 0$. By *u.H.s.c* of Q_{Y_i} at x, there exists $\delta_i > 0$ such that

(6) $$Q_{Y_i}(z) \subseteq Q_{Y_i}(x) + \varepsilon B_{Y_i} \forall\, z \in B_X(x, \delta_i)$$

Let $\delta = \min_{i\in I}\{\delta_i\}$. Then it is easy to see that

(7) $$P_Y(z) \subseteq P_Y(x) + \varepsilon B_Y \forall\, z \in B_X(x, \delta)$$

Indeed, let $z \in B_X(x, \delta)$ and $y = (y_i) \in P_Y(z)$. Then there exists $y'_i \in Q_{Y_i}(x_i)$ such that $\|y_i - y'_i\| < \varepsilon$ for each $i \in I$. We have $y' = (y'_i) \in P_Y(x)$ and $\|y - y'\|_{\ell_\infty} < \varepsilon$. Then (7) holds true and this implies *u.H.s.c* of P_Y at x. ■

In the following, we show that *l.s.c.* and *l.H.s.c.* will be preserved in ℓ_∞-direct sum of finite family of Banach spaces. Observe that our techniques are different from [9].

Proposition 2.5. *Let $\{X_i : i \in I\}$ be a finite family of Banach spaces, where $I \subset \mathbb{N}$ and $|I| = n < \infty$. Let Y_i be a proximinal subspace of X_i and each P_{Y_i} is l.s.c (l.H.s.c resp.) on respective X_i for $i \in I$. Consider $X = (\oplus_{\ell_\infty} X_i)_{i\in I}$ and $Y = (\oplus_{\ell_\infty} Y_i)_{i\in I}$. Then Y is proximinal in X and P_Y is l.s.c(l.H.s.c resp.) on X.*

Proof. Suppose Y_i is proximinal in X_i and P_{Y_i} is *l.s.c* for each $i \in I$. Proximinality of Y is obvious. Fix $x = (x_i) \in X$ and $\varepsilon > 0$ be given. Consider $P_Y(x) = (\oplus_{\ell_\infty} B[x_i, d(x)] \cap Y_i)_{i\in I}$. In order to prove that P_Y is *l.s.c* at x, by Lemma 2.4, it is enough if we prove that Q_{Y_i} is *l.s.c* at x for every $i \in I$. This is done in two cases:

Case 1. Let $i \in I_0(x)$. Then we have $P_{Y_i}(x_i) = Q_{Y_i}(x)$. Select any $y_i \in P_{Y_i}(x_i)$. Using the *l.s.c* of the map P_{Y_i} at x_i, we can get $\delta > 0$ such that

(8) $$z_i \in X_i, \quad \|x_i - z_i\| < \delta \Rightarrow B_{Y_i}(y_i, \varepsilon) \cap P_{Y_i}(z_i) \neq \emptyset$$

Since $P_{Y_i}(z_i) \subseteq B[z_i, d(z)] \cap Y_i$, (8) holds for Q_{Y_i} also. This implies that Q_{Y_i} is *l.s.c* at x.

Case 2. Let $i \in I \setminus I_0(x)$. Replacing α be $d(x)$ and d_x by $d_i(x)$ in Lemma 2.3, we can get $\delta > 0$ such that if $\|x - z\| < \delta$, then

$$h(B_{X_i}[x_i, d(x)], B_{X_i}[z_i, d(z)]) < \varepsilon$$

i.e.,

$$h(Q_{Y_i}(x), Q_{Y_i}(z)) < \varepsilon$$

Choose any $z \in B_X(x, \delta)$. If $t_i \in Q_{Y_i}(x)$, using Lemma 2.3, we can select r_i in $Q_{Y_i}(z)$ satisfying $\|t_i - r_i\| < \varepsilon$ which implies *l.s.c* of Q_{Y_i} at x. Indeed, Q_{Y_i} is *l.H.s.c* at x in this case.

Let $x = (x_i) \in X$ and $\varepsilon > 0$ be given. Now suppose $i \in I_0(x)$ and P_{Y_i} is *l.H.s.c* at x_i. Then there exists $\delta > 0$ such that

$$B_{Y_i}(y_i, \varepsilon) \cap P_{Y_i}(z_i) \neq \emptyset \quad \forall z_i \in B_{X_i}(x_i, \delta) \text{ and } \forall y_i \in P_{Y_i}(x_i)$$

To show that Q_{Y_i} is *l.H.s.c* at x, let $z' = (z_i') \in B_X(x, \delta)$. Then $z_i' \in B_{X_i}(x_i, \delta)$. Since $P_{Y_i}(z_i') \subseteq B[z_i', d(z')] \cap Y_i = Q_{Y_i}(z')$, we have $B_{Y_i}(y_i, \varepsilon) \cap Q_{Y_i}(z') \neq \emptyset$ for all $y_i \in Q_{Y_i}(x_i)$. Since $z' \in B_X(x, \delta)$ is arbitrary, we have

$$B_{Y_i}(y_i, \varepsilon) \cap Q_{Y_i}(z') \neq \emptyset \quad \forall z' \in B_X(x, \delta) \text{ and } \forall y_i \in Q_{Y_i}(x_i) = P_{Y_i}(x_i)$$

This implies that Q_{Y_i} is *l.H.s.c* at x. ∎

The following theorem shows that strong proximinality of Y_i can not be weakened in the hypothesis of [9, Theorem 3.4].

Theorem 2.6. *Let $\{X_i : i \in I\}$ be a finite family of Banach spaces and Y_i be a closed subspace of X_i for each $i \in I$ resp., where $I \subset \mathbb{N}$ and $1 < |I| = n < \infty$. Assume that at least two Y_is are nontrivial. Consider its ℓ_∞-direct sum i.e., $X = (\oplus_{\ell_\infty} X_i)_{i \in I}$ and $Y = (\oplus_{\ell_\infty} Y_i)_{i \in I}$. Then TFAE :*

(i) Y_i is strongly proximinal subspace in X_i for each $i \in I$

(ii) Y is strongly proximinal in X

(iii) P_Y is u.H.s.c on X

Proof. $(i) \Rightarrow (ii)$: Proximinality of Y follows easily. Let $x \in X$ such that $d(x, Y) = 1$ and $\varepsilon > 0$ be given. We have that $P_Y(x) = (\oplus_{\ell_\infty} B[x_i, 1] \cap Y_i)$. For $i \in I_0(x)$, by strong proximinality of Y_i at x_i, there exists $\delta_i > 0$ such that

$$P_{Y_i}(x_i, \delta_i) \subseteq P_{Y_i}(x_i) + \varepsilon B_{Y_i}$$

For $i \in I \setminus I_0(x)$, replacing α by $d(x)$ and d_x by $d_i(x)$ in the Lemma 2.3, we can get $\delta_i > 0$ such that if $\|x - z\| < \delta$, then

$$h(B_{X_i}[x_i, d(x)] \cap Y_i, B_{X_i}[z_i, d(z)] \cap Y_i) < \varepsilon$$

In particular, we have

$$B_{X_i}[x_i, d(x)+\delta_i] \cap Y_i \subseteq B_{X_i}[x_i, d(x)] + \varepsilon B_{Y_i}$$

Now, let $\delta = \min_{i\in I}\{d_i\}$. Then we have that

$$\oplus_{\ell_\infty} B_{X_i}[x_i, d(x)+\delta] \cap Y \subseteq \oplus_{\ell_\infty} B_{X_i}[x_i, d(x)] \cap Y + \varepsilon B_Y$$

which implies (ii).

$(ii) \Rightarrow (iii)$: Follows easily by definition of strong proximinality.

$(iii) \Rightarrow (i)$: Fix $i_0 \in I$ and let $x_{i_0} \in X_{i_0}$. Let $x_i \in X_i$ such that $d(x_i, Y_i) = d(x_{i_0}, Y_{i_0})$ for all $i \in \mathbb{N} \setminus \{i_0\}$. Suppose Y_j is a non-trivial subspace in X_j and $j \neq i_0$. Let x_j in X_j such that $d(x_j, Y_j) = d(x_{i_0}, Y_{i_0})$ and consider $x = (x_i)$ where $x_i = \begin{cases} x_i & \text{if } i \in \mathbb{N}\setminus\{i_0, j\} \\ x_{i_0} & \text{if } i = i_0 \\ x_j & \text{if } i = j \end{cases}$ By *u.H.s.c.* of P_Y at x, there exists $\delta > 0$ such that

$$P_Y(z) \subseteq P_Y(x) + \varepsilon B_Y \qquad \text{for every } z \in B_X(x,\delta)$$

i.e.,

$$P_{Y_{i_0}}(z_{i_0}) \subseteq B_{X_{i_0}}[z_{i_0}, d(z)] \cap Y_{i_0} \subseteq P_{Y_{i_0}}(x_{i_0}) + \varepsilon B_{Y_{i_0}} \quad \text{for all} \quad z = (z_i) \in B_X(x,\delta)$$

Let $z \in B_X(x,\delta)$ such that $d_{i_0}(z) < d(z)$ and $d(x) < \min_{i\in I}\{d_i(z)\}$. So there exists $\eta > 0$ with $\delta > \eta > 0$ such that $d_{i_0}(z) + \eta < d(z)$. Let $z_{i_0} \in B_{X_{i_0}}(x_{i_0}, \delta/3)$. Denote $z' = (z'_i)$, where $z_i = \begin{cases} z_i & \text{if } i \in \mathbb{N}\setminus\{i_0\} \\ z'_{i_0} & \text{if } i = i_0 \end{cases}$

Then

$$\begin{aligned} d_{i_0}(z', Y) &= d(z'_{i_0}, Y_{i_0}) \\ &= \inf_{y_{i_0}\in Y_{i_0}} \|z'_{i_0} - y_{i_0}\| \\ &\leq \|z'_{i_0} - x_{i_0}\| + \inf_{y_{i_0}\in Y_{i_0}} \|x_{i_0} - y_{i_0}\| \\ &< \eta/2 + d(x) \\ &< d(z) \end{aligned}$$

By *u.H.s.c.* of P_Y at x, we have that

$$B_{X_{i_0}}[z'_{i_0}, d(z')] \cap Y_{i_0} \subseteq P_{Y_{i_0}}(x_{i_0}) + \varepsilon B_{Y_{i_0}}$$

Let $y_{i_0} \in P_{Y_{i_0}}(x_{i_0}, \eta/2)$. Then

$$\begin{aligned}\|z'_{i_0} - y_{i_0}\| &\leq \|z'_{i_0} - x_{i_0}\| + \|x_{i_0} - y_{i_0}\| \\ &< \eta/2 + d(x) + \eta/2 \\ &< d(z') = d(z)\end{aligned}$$

i.e.,

$$P_{Y_{i_0}}(x_{i_0}, \eta/2) \subseteq P_{Y_{i_0}}(x_{i_0}) + \varepsilon B_{Y_{i_0}}$$

which implies that Y_{i_0} is strongly proximinal in X_{i_0}. ■

Immediate Corollary of Proposition 2.5 and Theorem 2.6 is

Corollary 2.7. *Let $\{X_i : i \in I\}$ be a finite family of Banach spaces and Y_i be a proximinal subspace in X_i, for each $i \in I$ where $I \subset \mathbb{N}$. Consider their ℓ_∞-direct sums i.e., $X = (\oplus_{\ell_\infty} X_i)_{i \in I}$ and $Y = (\oplus_{\ell_\infty} Y_i)_{i \in I}$. Then Y_i is strongly proximinal and P_{Y_i} is l.H.s.c. in X_i for each $i \in I$ if and only if P_Y is H.m.c.*

Now by Theorem 2.6, we can deduce the following:

Corollary 2.8. *Let Y be a proximinal subspace in a Banach space X. Then Y is strongly proximinal in X if and only if $P_{(Y \oplus_{\ell_\infty} Y)}$ is u.H.s.c. on $X \oplus_{\ell_\infty} X$.*

3. Semicontinuity and Strong Proximinality in c_0-Direct Sum Spaces

In the following theorem we show that *l.s.c.* is stable under general c_0-direct sums since same arguments will work for *l.H.s.c.* also, we avoid presenting later one.

Theorem 3.1. *Let $\{X_i : i \in \mathbb{N}\}$ be a family of Banach spaces and Y_i be a proximinal subspace in X_i for each $i \in \mathbb{N}$. Consider the following direct sums $X = (\oplus_{c_0} X_i)_{i \in \mathbb{N}}$ and $Y = (\oplus_{c_0} Y_i)_{i \in \mathbb{N}}$. Then P_{Y_i} is l.s.c.(resp. l.H.s.c.) in X_i for each $i \in \mathbb{N}$ if and only if P_Y is l.s.c.(resp. l.H.s.c.) in X.*

Proof. It is easy to see that Y is proximinal in X. Indeed for $x = (x_i) \in X$, we have

$$P_Y(x) = (\oplus_{c_0}(B_{X_i}[x_i, d(x)] \cap Y_i))_{i \in \mathbb{N}}$$

Suppose that P_{Y_i} is *l.s.c.* in X_i for each $i \in \mathbb{N}$. We claim that P_Y is *l.s.c.* in X. Let $x = (x_i) \in X$ and $\varepsilon > 0$. Since $x \in X$, we have $d(x_i, Y_i) = d_i(x) \to 0$ as $i \to \infty$. So there exists $n_0 \in \mathbb{N}$ such that $d_i(x) < d(x)$ for all $i > n_0$. Let

$\mathbb{N}_1 = \{i : 1 \le i \le n_0\}$ and $\mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1$. Now consider $X' = (\oplus_{\ell_\infty} X_i)_{i\in\mathbb{N}_1}$, $X'' = (\oplus_{c_0} X_i)_{i\in\mathbb{N}_2}$, $Y' = (\oplus_{\ell_\infty} Y_i)_{i\in\mathbb{N}_1}$ and $Y'' = (\oplus_{c_0} Y_i)_{i\in\mathbb{N}_2}$. Since $\mathbb{N}_1$ is finite, $P_{Y'}$ is *l.s.c.* at $x' = (x_1, \cdots, x_{n_0})$ i.e., for each $y' \in P_{Y'}(x')$, there exists $\delta_1 > 0$ such that $P_{Y'}(z') \cap B_{Y'}(y', \varepsilon) \neq \emptyset$ for all $z' \in B_{X'}(x', \delta_1)$.

Let $x'' = (x_{n_0+1}, x_{n_0+2}, \cdots)$ and $\eta_x = sup_{i\in\mathbb{N}_2}\{d_i(x)\}$. We have $d(x'', Y'') = \eta_x < d(x)$. Let $2\gamma = d(x) - \eta_x$. Replacing x by x'' and α by $d(x)$ in [9, Fact 3.2], we can get $\delta_2 > 0$ such that if $\|x - z\| < \delta_2$, then $d(z) - \eta_z > \gamma$ and $h(B_{X''}[x'', d(x)] \cap Y'', B_{X''}[z'', d(z)] \cap Y'') < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. We have $P_Y(w) = (B_{X'}[w', d(w)] \cap Y') \oplus_{\ell_\infty} (B_{X''}[w'', d(w)] \cap Y'')$ for all $w \in B_X(x, \delta)$. Let $z \in B_X(x, \delta)$. If $y' \in B_{X'}[x', d(x)] \cap Y'$ and $y'' \in B_{X''}[x'', d(x)] \cap Y''$, by using *l.s.c.* of $P_{Y'}$ at x' and [9, Fact 3.2]., we select $r' \in B_{X'}[z', d(z)] \cap Y'$ and $r'' \in B_{X''}[z'', d(z)] \cap Y''$ such that $\|y' - r'\| < \varepsilon$ and $\|y'' - r''\| < \varepsilon$. We clearly have that $r = (r', r'') \in P_Y(z)$ and this implies *l.s.c.* of P_Y at x.

Converse follows easily. ∎

Theorem 3.2. *Let $\{X_i : i \in \mathbb{N}\}$ be a family of Banach spaces and Y_i be a subspace in X_i for each $i \in \mathbb{N}$. Consider the following direct sums $X = (\oplus_{c_0} X_i)_{i\in\mathbb{N}}$ and $Y = (\oplus_{c_0} Y_i)_{i\in\mathbb{N}}$. Then TFAE:*

(i) Y_i is strongly proximinal in X_i for each $i \in \mathbb{N}$

(ii) Y is strongly proximinal in X.

(iii) P_Y is *u.H.s.c.* in X.

Proof. $(i) \Rightarrow (ii)$, : Suppose each P_{Y_i} is strongly proximinal in X_i for each $i \in \mathbb{N}$. It is easy to see that Y is proximinal in X. Indeed for $x = (x_i) \in X$, we have

$$P_Y(x) = (\oplus_{c_0}(B_{X_i}[x_i, d(x)] \cap Y_i))_{i\in\mathbb{N}}$$

Let $x = (x_i) \in X$ and $\varepsilon > 0$. Since $x \in X$, we have $d(x_i, Y_i) = d_i(x) \to 0$ as $i \to \infty$. So there exists $n_0 \in \mathbb{N}$ such that $d_i(x) < d(x)$ for all $i > n_0$. Let $\mathbb{N}_1 = \{i : 1 \le i \le n_0\}$ and $\mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1$. Now consider $X' = (\oplus_{\ell_\infty} X_i)_{i\in\mathbb{N}_1}$, $X'' = (\oplus_{c_0} X_i)_{i\in\mathbb{N}_2}$, $Y' = (\oplus_{\ell_\infty} Y_i)_{i\in\mathbb{N}_1}$ and $Y'' = (\oplus_{c_0} Y_i)_{i\in\mathbb{N}_2}$. Since $\mathbb{N}_1$ is finite, Y' is strongly proximinal at $x' = (x_1, \cdots, x_{n_0})$ i.e., there exists $\delta_1 > 0$ such that $P_{Y'}(x', \delta_1) \subseteq P_{Y'}(x') + \varepsilon B_{Y'}$.

Let $x'' = (x_{n_0+1}, x_{n_0+2}, \cdots)$ and $\eta_x = sup_{i\in\mathbb{N}_2}\{d_i(x)\}$. We have $d(x'', Y'') = \eta_x < d(x)$. Let $2\gamma = d(x) - \eta_x$. Replacing x by x'' and α by $d(x)$ in [9, Fact 3.2], we can get $\delta_2 > 0$ such that if $\|x - z\| < \delta_2$, then $d(z) - \eta_z > \gamma$ and $h(B_{X''}[x'', d(x)] \cap Y'', B_{X''}[z'', d(z)] \cap Y'') < \varepsilon$. In particular, we have that

$$B_{X''}[x'', d(x) + \delta_2] \cap Y'' \subseteq B_{X''}[x'', d(x)] + \varepsilon B_{Y''}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. With this δ, now the result follows.

$(ii) \Rightarrow (iii)$: Follows by definition of strong proximinality.
$(iii) \Rightarrow (i)$: Follows by Theorem 2.6. ■

In the rest of this section, we consider mainly c_0-direct sum of reflexive spaces. Let us recall that a closed subspace Y of a Banach space X is called factor reflexive if X/Y is reflexive. We show that factor reflexive proximinal subspaces in c_0-direct sum of reflexive spaces are proximinal in all duals of even order of c_0-direct sum. First let us recall some facts about M-embedded spaces from [6] and [15]:

Let X be a Banach space and P be a linear projection from X into X.

(i) P is said to be an M-projection if

$$\|x\| = \max\{\|Px\|, \|x - Px\|\} \text{ for all } x \in X$$

(ii) P is said to be an L-projection if

$$\|x\| = \|Px\| + \|x - Px\| \text{ for all } x \in X$$

A closed subspace Y of a Banach space X is called an L-summand if it is the rang of an L-projection. Similarly M-summand will be defined. A closed subspace Y of a Banach space X is called an M-ideal if $Y^{\perp}$ is an L-summand in X^*, where $Y^{\perp} = \{f \in X^* : f(y) = 0 \quad \forall \quad y \in Y\}$. A Banach space X is called an M-embedded if X is an M-ideal in X^{**} and X is called an L-embedded if X is an L-summand in X^{**}. Standard examples of M-embedded spaces are c_0, and $\mathcal{K}(\ell_2)$, compact operators on ℓ_2. By Theorem III.1.6 of [6], c_0 direct sum of M-embedded spaces is M-embedded. In particular c_0-direct sum of reflexive spaces is M-embedded. If X is an M-embedded space, then X^* is an L-embedded.

We consider a Banach space as canonically embedded in its bidual. We denote by i_X, the canonical injection. For a Banach space X and for $n \geq 0$, we denote by $X^{(n)}$, the n^{th} dual of X where $X^{(0)} = X$. In [15], it is proved that if X is M-embedded, then X is an M-ideal in all duals of even order of X. It is also proved in [15] that if X is M-embedded, then X^* is L-embedded in all duals of odd order of X. By Proposition II.1.1 and Proposition II.1.8 from [6], if Y is an M-ideal in a Banach space X, then Y is strongly proximinal in X and P_Y is Hausdorff metric continuous (see also Proposition 2.1, [9]).

Let Y be a closed subspace of an M-embedded space X. By Theorem 5 from [11] and Theorem 2 from [15], if Y has n-ball property in X, where $n \geq 1\frac{1}{2}$, then Y has n-ball property in all duals of even order of X(since we are not using ball properties here, we have not given definitions of these). In the following, we prove that if Y is a factor reflexive proximinal subspace of $X = (\oplus_{c_0} X_i)_{i \in \mathbb{N}}$, where X_i is reflexive for each $i \in \mathbb{N}$, then Y is proximinal in all duals of even order of X.

Remark 3.3. Let $\{X_i : i \in \mathbb{N}\}$ be a family of reflexive spaces and consider its c_0-direct sum i.e., $X = (\oplus_{c_0} X_i)_{\mathbb{N}}$. We observe that

$NA(X) = \{f = (f_i) \in X^* = (\oplus_{\ell_1} X_i^*)_{i\in\mathbb{N}}$: only finitely many f_i's are non-zero $\}$

Let M be a norm closed subspace of $NA(X)$. It is proved in [10, Lemma 2.3] that, there exists n_0 such that if $f = (f_i) \in M$, then $f_i = 0$ for all $i > n_0$. In otherwords, number of the non-zero coordinates of elements of M are uniformly bounded. Indeed, M is norm closed subspace in $NA(X)$ if and only if M is reflexive in $NA(X)$.

Let Y be a factor reflexive proximinal subspace of $X = (\oplus_{c_0} X_i)_{\mathbb{N}}$. It is easy to see that $Y^\perp \subseteq NA(X)$ (Proposition 2.14, [10]). Let n_0 be in $\mathbb{N}$ such that if $f = (f_i) \in Y^\perp$, then $f_i = 0$ for all $i > n_0$. Let $I = \{i : 1 \le i \le n_0\}$. Now consider $Z_1 = (\oplus_{\ell_\infty} X_i)_I$ and $Z_2 = (\oplus_{c_0} X_i)_{\mathbb{N}\setminus I}$. Let us take $Y_1 = \{x = (x_i) \in Z_1 : \sum_{i=1}^{n_0} f_i(x_i) = 0 : \forall f = (f_i) \in Y^\perp\}$ and $Y_2 = Z_2$. Observe that $Y = Y_1 \oplus_{\ell_\infty} Y_2$.

For the following Proposition, we adopt notations from Remark 3.3.

Lemma 3.4. *Let $\{X_i : i \in \mathbb{N}\}$ be a family of reflexive spaces and consider its c_0-direct sum i.e., $X = (\oplus_{c_0} X_i)_{\mathbb{N}}$. Let Y be factor reflexive proximinal subspace in X. Then TFAE :*

(a) *Y_1 is strongly proximinal in Z_1.*

(b) *P_Y is u.H.s.c. on $X^{(2n)}$ for some $n > 0$.*

(c) *P_Y is u.H.s.c. on $X^{(2n)}$ for all $n \ge 0$.*

Proof. Follows by Theorem 2.6 and observing that Z_2 is an M-summand, so it is strongly proximinal in its all duals even order. ∎

The following Lemma follows from Proposition 2.5 and Lemma 3.4.

Lemma 3.5. *Let $\{X_i : i \in \mathbb{N}\}$ be a family of reflexive spaces and consider its c_0-direct sum i.e., $X = (\oplus_{c_0} X_i)_{\mathbb{N}}$. Let Y be factor reflexive proximinal subspace in X. Then TFAE :*

(a) *P_Y is l.s.c.(l.H.s.c. or H.m.c. resp.) on X.*

(b) *P_Y is l.s.c.(l.H.s.c. or H.m.c. resp.) on $X^{(2n)}$ for some $n \ge 0$.*

(c) *P_Y is l.s.c.(l.H.s.c. or H.m.c. resp.) on $X^{(2n)}$ for all $n \ge 0$.*

If each $X_i = \mathbb{R}$, we have more to say i.e., when $X = c_0$. Let X be a finite dimensional polyhedral space and Y be a subspace of X. Then it was proved in [7] that the metric projection P_Y from X onto Y is *l.s.c.* on X. By compactness arguments we can in fact show that P_Y is *l.H.s.c.*

Now by Lemma 3.4 and Lemma 3,.5 we have :

Corollary 3.6. *Let Y be a proximinal subspace of finite codimension in c_0. Then Y is strongly proximinal in $c_0^{(2n)}$ and the metric projection from $c_0^{(2n)}$ onto Y is Hausdorff metric continuous for every $n \geq 0$.*

Acknowledgment

We thank our teacher Prof. V. Indumathi for the discussions we had while working on this paper. The first named author's research was supported by CSIR Research Fellowship and she would like to thank CSIR for their financial support.

References

1. G. Debs, G. Godefroy and J. Saint-Raymond, Topological properties of the set of norm-attaining linear functionals, *Canad. J. Math.*, **47** (1995) 318-329.
2. F. Deutsch, W. Pollul and I. Singer, On set-valued metric projections, Hahn-Banach extension maps, and spherical image maps, *Duke Math. J.*, **40** (1973), 355-370.
3. G. Godefroy and N. J. Kalton, Lipschitz-free Banach spaces , *Studia Mathematica*, **159(1)** (2003) 121-141.
4. G. Godefroy and V. Indumathi, Strong proximinality and polyhedral spaces, *Rev. Mat. Complut.*, **14** (2001), 105-125.
5. G. Godefroy, V. Indumathi and F. Lust-Piquard, Strong subdifferentiability of convex functionals and proximinality, *J. Approx. Theory*, **116** (2002), 397-415.
6. H. Harmand, D. Werner and W. Werner, *M-Ideals in Banach spaces and Banach algebras* , Lecture Notes in Mathematics 1547, Springer-Verlag, Berlin, 1993.
7. R. B. Holmes, *On the continuity of best approximation operators*, Sympos. on infinite dimensional topology, Princeton, Princeton University Press, 1972.
8. V. Indumathi, Proximinal subspaces of finite codimension in direct sum spaces, *Proc. Indian Acad. Sci. (Math. Sci.), Vol. 111, No. 2,* May 2001, pp. 229-239.
9. V. Indumathi, Semi-continuity of metric projection in ℓ_∞-direct sums, To appear in *Proc. A.M.S.*
10. D. Narayana and T. S. S. R. K. Rao, *Transitivity of proximinality and norm attaining functionals,* Preprint, 2003.
11. R. Pay$\acute{a}$ and D. Yost, The two-ball property: transitivity and examples, *Mathematika*, **35(2)** (1988), 190-197.
12. W. Pollul, *Topologien auf Mengen von Teilmengen und Stetigkeit von mengenwertigen metrischen Projektionen*, Diplomarbeit, Bonn, 1967.

13. E. V. Osman, On the continuity of the metric projection in Banach space, *Math. USSR Sb.*, **9** (1969), 171-182.

14. E. V. Osman, On continuity of metric projection onto some classes of subspaces in a Banach space, *Soviet Math. Dokl.*, **11** (1970), 1521-1523.

15. T. S. S. R. K. Rao, On the geometry of higher duals of a Banach space, *Illinois J. Math.*, **45(4)** (2001), 1389-1392.

16. I. Singer, *On set-valued metric projections*, Linear operators and approximation, Proc. Conf., Math. Res. Inst., Oberwolfach, 1971, pp. 217-233. Internat. Ser. Numer. Math., Vol. 20, Birkhuser, Basel, 1972.

S. Lalithambigai
Department of Mathematics,
Pondicherry University,
Kalapet, Pondicherry 605 014,
India
E-mail: s_lalithambigai@yahoo.co.in

Darapaneni Narayana
Department of Mathematics,
Indian Institute of Science,
Bangalore 560012,
India
E-mail: narayana@math.iisc.ernet.in