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GLOBAL AND NON-GLOBAL SOLUTIONS OF A NONLINEAR PARABOLIC EQUATION

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Abstract. We study the global and non-global existence of positive solutions of a nonlinear parabolic equation. For this, we consider the forward and backward self-similar solutions of this equation. We obtain a family of radial symmetric global solutions which tend to zero as the time tends infinity. Next, we show that there are initial data for which the corresponding solutions blow up in finite time. Finally, we also construct some self-similar single-point blow-up patterns with different oscillations.

1. INTRODUCTION

In this paper, we study the following nonlinear parabolic equation

$$u_t = u^\sigma(\Delta u + u^p), x \in \mathbf{R}^n, t > 0, \tag{1.1}$$

where $\sigma \in \mathbf{R}$, $p > 1$, and $n \geq 1$. Equation (1.1) has been extensively studied; see, for example, [2, 4, 10] for $\sigma = 0$, [8] for $\sigma < 0$, [3] for $\sigma \in (0, 1)$, [5, 6] for $\sigma = 1$.

In all of these works, much attentions have been paid to the blow-up behaviours of solutions in order to understand the mechanism of thermal runaway in combustion problems. We also refer readers to the references listed in the books of Bebernes *et al.* [1] and Samarskii *et al.* [9].

We are concerned with the global and non-global existence of positive solutions of (1.1) for $\sigma > 1$. To study this problem, we study the existence of both forward and backward self-similar positive solutions of (1.1) in the forms

$$U(x, t) = (t + 1)^{-\alpha}\phi(\frac{|x|}{(t + 1)^\beta}), \tag{1.2}$$

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Dedicated to Professor Hwai-Chiuan Wang on the occasion of his retirement.

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$$V(x,t) = (T-t)^{-\alpha}\phi(\frac{|x|}{(T-t)^{\beta}}), \tag{1.3}$$

where $T > 0$ is given and the similarity exponents are necessarily given by

$$\alpha = \frac{1}{p+\sigma-1}, \quad \beta = \frac{p-1}{2(p+\sigma-1)}. \tag{1.4}$$

Set $\xi = |x|/(t+1)^{\beta}$. It follows that U satisfies (1.1) if and only if ϕ satisfies the equation

$$\phi'' + \frac{n-1}{\xi}\phi' + \phi^p + \alpha\phi^{1-\sigma} + \beta\xi\phi^{-\sigma}\phi' = 0, \ \xi > 0, \tag{1.5}$$

and $\phi'(0) = 0$.

Similarly, for $\xi = |x|/(T-t)^{\beta}$, V satisfies (1.1) if and only if ϕ satisfies the equation

$$\phi'' + \frac{n-1}{\xi}\phi' + \phi^p - \alpha\phi^{1-\sigma} - \beta\xi\phi^{-\sigma}\phi' = 0, \ \xi > 0, \tag{1.6}$$

and $\phi'(0) = 0$.

In Section 2, we shall prove that the solution of the initial value problem (1.5), with the initial conditions

$$\phi'(0) = 0, \quad \phi(0) = \eta, \tag{1.7}$$

exists globally for any $\eta > 0$. In Section 3, we study the asymptotic behaviour as $\xi \to \infty$ of the positive solution of (1.5). We show that for any positive solution $\phi(\xi)$ of (1.5) the limit

$$\lim_{\xi\to\infty}[\xi^{\alpha/\beta}\phi(\xi)] = A \tag{1.8}$$

exists and $A > 0$. Therefore, we obtain a family of global solutions of (1.1) with monotone decreasing symmetric initial data u_0 satisfying

$$u_0(0) = \eta > 0, \ u_0(x) = \phi(|x|), \ \lim_{|x|\to\infty}[|x|^{\alpha/\beta}u_0(x)] = A > 0.$$

Moreover, these global solutions tend to zero as $t \to \infty$.

In Section 4, we show that there are initial data for which the solution of (1.1) blows up in finite time. Then, following the method of [6], we derive some self-similar single-point blow-up patterns with different oscillations in Section 5. We remark that any solution of these patterns has the same number of critical points before the blow-up time and these critical points merge together to a single point ($x = 0$) at the blow-up time.

2. Global Existence of Solution for (1.5)

In this section, we shall study the positive solution of the following initial value problem (P):

$$\phi'' + \frac{n-1}{\xi}\phi' + \phi^p + \alpha\phi^{1-\sigma} + \beta\xi\phi^{-\sigma}\phi' = 0, \xi > 0, \tag{2.1}$$

$$\phi'(0) = 0, \phi(0) = \eta, \tag{2.2}$$

where $\eta > 0$ is given. Note that there is no constant solution of (P). From the local existence and uniqueness theorem of ordinary differential equations it follows that there is a unique positive local solution ϕ of (P) for each given $\eta > 0$. For convenience, let $[0, R)$ be the maximum existence interval of ϕ such that $\phi > 0$, where $0 < R \leq \infty$. Define

$$\rho(y) = exp\{\beta \int_0^y \xi\phi^{-\sigma}(\xi)d\xi\}. \tag{2.3}$$

From (2.1) it follows that

$$(\xi^{n-1}\rho(\xi)\phi'(\xi))' = -\xi^{n-1}\rho(\xi)[\phi^p(\xi) + \alpha\phi^{1-\sigma}(\xi)], \tag{2.4}$$

and so

$$\phi'(\xi) = -\frac{1}{\xi^{n-1}\rho(\xi)} \int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy \ , \xi > 0. \tag{2.5}$$

Hence $\phi(\xi)$ is monotone decreasing in $[0, R)$.

Theorem 2.1. *For any $\eta > 0$, the local solution $\phi(\xi)$ of (P) can be continued globally so that $R = \infty$.*

Proof. On the contrary, suppose that $R < \infty$. Then $\phi(\xi) \to 0$ as $\xi \to R^-$. We shall divide our discussion into three cases. Each one leads to a contradiction.

Case 1.

$$\underline{\lim}_{\xi \to R^-} \phi'(\xi) = -\infty. \tag{2.6}$$

In this case, there exists a sequence $\{\xi_k\}$ in $[0, R)$ such that $\xi_k \to R^-$, $\phi(\xi_k) \to 0^+$, and $\phi'(\xi_k) \to -\infty$ as $k \to \infty$. From (**??**) it follows that

$$\lim_{k\to\infty} \int_0^{\xi_k} y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy \ = \infty.$$

For convenience, we set

$$g(\xi) := \int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy\ .$$

Since $g(\xi)$ is an increasing continuous function, we obtain that

$$\lim_{\xi\to R^-} \int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy\ = \infty.$$

We note that the limit of $\rho(\xi)$ as $\xi \to R^-$ exists and is either infinity or a finite positive number. First, we suppose that $\lim_{\xi\to R^-}\rho(\xi) = \infty$. Applying L'Hôpital's Rule, we compute from (2.5) that

$$\begin{aligned}
\lim_{\xi\to R^-} \phi'(\xi) &= -\lim_{\xi\to R^-} \frac{1}{\xi^{n-1}\rho(\xi)} \int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy\ . \\
&= -\lim_{\xi\to R^-} \frac{\xi^{n-1}\rho(\xi)[\phi^p(\xi) + \alpha\phi^{1-\sigma}(\xi)]}{(n-1)\xi^{n-2}\rho(\xi) + \xi^{n-1}\rho(\xi)\beta\xi\phi^{-\sigma}(\xi)} \\
&= -\lim_{\xi\to R^-} \frac{\xi[\phi^{p+\sigma}(\xi) + \alpha\phi(\xi)]}{(n-1)\phi^\sigma(\xi) + \beta\xi^2} \\
&= 0.
\end{aligned}$$

This contradicts (2.6).

Next, we consider the case that $\lim_{\xi\to R^-}\rho(\xi) = B$, $0 < B < \infty$. From (2.5), we have

$$\lim_{\xi\to R^-} \phi'(\xi) = -\lim_{\xi\to R^-} \frac{1}{\xi^{n-1}\rho(\xi)} \int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy\ = -\infty.$$

On the other hand, it follows from (2.1) that

$$\begin{aligned}
\lim\nolimits_{\xi\to R^-} \phi''(\xi) &= -\lim\nolimits_{\xi\to R^-}\{\phi^p(\xi) + \phi^{-\sigma}(\xi)[\alpha\phi(\xi) + (\beta\xi + \tfrac{n-1}{\xi}\phi^\sigma(\xi))\phi'(\xi)]\} \\
&= \infty,
\end{aligned}$$

a contradiction with (2.6). Hence this case is impossible.

Case 2.

$$\varliminf_{\xi\to R^-} \phi'(\xi) = -M, 0 < M < \infty. \tag{2.7}$$

In this case, there exists a sequence $\{\xi_k\}$ in $[0, R)$ such that $\xi_k \to R^-$, $\phi(\xi_k) \to 0^+$, and $\phi'(\xi_k) \to -M$ as $k \to \infty$. If $\lim_{\xi\to R^-}\rho(\xi) = \infty$, then as in **Case 1** we have $\lim_{\xi\to R^-}\phi'(\xi) = 0$. This contradicts with (2.7).

Suppose that $\lim_{\xi\to R^-}\rho(\xi)=B, 0<B<\infty$. Then from (2.5) it follows that

$$\lim_{k\to\infty}\int_0^{\xi_k} y^{n-1}\rho(y)[\phi^p(y)+\alpha\phi^{1-\sigma}(y)]dy \ = BMR^{n-1},$$

and so

$$\lim_{\xi\to R^-}\int_0^{\xi} y^{n-1}\rho(y)[\phi^p(y)+\alpha\phi^{1-\sigma}(y)]dy \ = BMR^{n-1}.$$

Therefore, we obtain that $\lim_{\xi\to R^-}\phi'(\xi)=-M$. It follows from (2.1) that

$$\begin{aligned}
&\lim_{\xi\to R^-}[\phi(\xi)\phi''(\xi)]\\
=& -\lim_{\xi\to R^-}\{\frac{n-1}{\xi}\phi(\xi)\phi'(\xi)+\phi^{p+1}(\xi)+\phi^{1-\sigma}(\xi)[\alpha\phi(\xi)+\beta\xi\phi'(\xi)]\}\\
=& \ \infty.
\end{aligned}$$

In particular, we have

$$\lim_{\xi\to R^-}\phi''(\xi)=\infty. \tag{2.8}$$

Differentiating (2.1) we have

$$\begin{aligned}
\phi'''(\xi) \ = \ & -\{\frac{n-1}{\xi}\phi''(\xi)-\frac{n-1}{\xi^2}\phi'(\xi)+p\phi^{p-1}(\xi)\phi'(\xi)+\alpha(1-\sigma)\phi^{-\sigma}(\xi)\phi'(\xi)+\\
& \beta\phi^{-\sigma}(\xi)\phi'(\xi)+\beta\xi(-\sigma)\phi^{-\sigma-1}(\xi)[\phi'(\xi)]^2+\beta\xi\phi^{-\sigma}(\xi)\phi''(\xi)\}.
\end{aligned}$$

Then we obtain that

$$\lim_{\xi\to R^-}\phi'''(\xi)=-\infty,$$

a contradiction with (2.1).

Case 3.

$$\underline{\lim}_{\xi\to R^-}\phi'(\xi)=0.$$

Since $\phi'(\xi)<0$ in $[0,R)$, we have

$$\lim_{\xi\to R^-}\phi'(\xi)=0. \tag{2.9}$$

We claim that

$$\lim_{\xi\to R^-}\rho(\xi)=\infty \text{ and } \lim_{x\to R^-}\rho(\xi)\phi(\xi)=\infty. \tag{2.10}$$

If $\rho(\xi) \to B$ as $\xi \to R^-$ for some $B \in (0, \infty)$, then from (2.5) it follows that

$$\lim_{\xi \to R^-} \int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy = 0.$$

This is impossible, since $y^{n-1}\rho(y)[\phi^p(y)+\alpha\phi^{1-\sigma}(y)] > 0$ for all $y > 0$. Therefore,

$$\lim_{\xi \to R^-} \rho(\xi) = \infty.$$

Next, it follows from the definition of $\rho(\xi)$ that

$$\beta \int_0^\xi y\phi^{-\sigma}(y)dy \to \infty \text{ as } \xi \to R^-.$$

Then

$$\lim_{\xi \to R^-} \frac{\beta \int_0^\xi y\phi^{-\sigma}(y)dy}{-ln[\phi(\xi)]} = \lim_{\xi \to R^-} \frac{\beta\xi\phi^{-\sigma}(\xi)}{-\phi^{-1}(\xi)\phi'(\xi)} = \infty.$$

From this and (2.1), we obtain that

$$\lim_{\xi \to R^-} \rho(\xi)\phi(\xi) = \lim_{\xi \to R^-} exp\{\beta \int_0^\xi y\phi^{-\sigma}(y)dy + ln[\phi(\xi)]\} = \infty.$$

Hence (2.10) is proved.

Now, we suppose first that

$$\lim_{\xi \to R^-} \int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy = \infty.$$

Using (2.5) and L'Hôpital's Rule, we obtain that

$$\begin{aligned}
\lim_{\xi \to R^-} \frac{\phi'(\xi)}{\phi(\xi)} &= -\lim_{\xi \to R^-} \{\frac{1}{\xi^{n-1}\rho(\xi)\phi(\xi)} \int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy\} \\
&= -\lim_{\xi \to R^-} \{\frac{\xi^{n-1}\rho(\xi)[\phi^p(\xi) + \alpha\phi^{1-\sigma}(\xi)]}{(n-1)\xi^{n-2}\rho(\xi)\phi(\xi)+\xi^{n-1}\rho(\xi)\beta\xi\phi^{-\sigma}(\xi)\phi(\xi) + \xi^{n-1}\rho(\xi)\phi'(\xi)}\} \\
&= -\lim_{\xi \to R^-} \{\frac{[\phi^{p+\sigma-1}(\xi) + \alpha]}{\frac{(n-1)}{\xi}\phi^\sigma(\xi) + \xi\beta + \phi^{\sigma-1}(\xi)\phi'(\xi)}\} \\
&= -\frac{\alpha}{R\beta}.
\end{aligned}$$

This implies that there exists $\xi_0 > 0$ such that

(2.11) $$\frac{\phi'(\xi)}{\phi(\xi)} \geq -\frac{3\alpha}{2R\beta} \text{ for all } \xi \in (\xi_0, R).$$

Integrating (2.11) from ξ_0 to $\xi > \xi_0$, we have

$$\ln\phi(\xi) - \ln\phi(\xi_0) \geq -\frac{3\alpha}{2R\beta}\xi + \frac{3\alpha}{2R\beta}\xi_0,$$

i.e.,

$$\phi(\xi) \geq Ce^{-3\alpha\xi/(2R\beta)}$$

for all $\xi \in (\xi_0, R)$, where $C = \phi(\xi_0)e^{\frac{3\alpha}{2R\beta}\xi_0} > 0$. This contradicts with $\phi(\xi) \to 0$ as $\xi \to R^-$.

Next, we suppose that

$$\lim_{\xi\to R^-}\int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy \ = B$$

for some $B \in (0,\infty)$. Then

$$\lim_{\xi\to R^-}\frac{\phi'(\xi)}{\phi(\xi)} = -\lim_{\xi\to R^-}\{\frac{1}{\xi^{n-1}\rho(\xi)\phi(\xi)}\int_0^\xi y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy\ \} = 0.$$

This implies that there exists $\xi_0 > 0$ such that

$$\frac{\phi'(\xi)}{\phi(\xi)} \geq -1 \text{ for all } \xi \in (\xi_0, R). \tag{2.12}$$

Similarly, integrating (2.12) from ξ_0 to $\xi > \xi_0$, we obtain that

$$\phi(\xi) \geq Ce^{-\xi}$$

for all $\xi \in (\xi_0, R)$, for some positive constant C. Again, this is a contradiction.

This complete the proof of the theorem. ■

3. Asymptotic Behavior of Solution for (1.5)

In this section, we shall study the asymptotic behavior of positive solution $\phi(\xi)$ of (P) as $\xi \to \infty$. We first study the limits of $\phi(\xi)$ and $\phi'(\xi)$ as $\xi \to \infty$.

Lemma 3.1. *There holds $\phi(\xi) \to 0$ as $\xi \to \infty$.*

Proof. Since $\phi(\xi)$ is monotone decreasing, the limit

$$l = \lim_{\xi\to\infty}\phi(\xi)$$

exists and $l \geq 0$. Suppose $l > 0$, we define

$$H(\xi) = \frac{1}{2}[\phi'(\xi)]^2 + G(\phi(\xi)), \quad \xi > 0, \tag{3.1}$$

where

$$G(\phi) = \int_\eta^\phi (s^p + \alpha s^{1-\sigma})ds, \quad l \leq \phi < \eta. \tag{3.2}$$

Since

$$H'(\xi) = -(\frac{n-1}{\xi} + \beta\xi\phi^{-\sigma})\phi'(\xi) \geq 0$$

and

$$G(\phi) \geq -(\eta^p + \alpha l^{1-\sigma})(\eta - l),$$

the limit

$$L = \lim_{\xi\to\infty} H(\xi)$$

exists and $L > -\infty$. By the definition of $H(\xi)$, the limit

$$-K = \lim_{\xi\to\infty} \phi'(\xi)$$

exists and $K \geq 0$. Since

$$\int_0^\infty \phi'(\xi)d\xi = l - \eta,$$

there exists a sequence $\{\xi_k\}$ such that $\xi_k \to \infty$ and $\phi'(\xi_k) \to 0$ as $k \to \infty$. Hence $K = 0$. Therefore, there exists $M > 0$ such that $\mid \phi(\xi) \mid \leq M$ and $\mid \phi'(\xi) \mid \leq M$, $\forall \xi \geq 0$. Dividing (2.1) by ξ, we have

$$\frac{\phi''(\xi)}{\xi} + \frac{n-1}{\xi^2}\phi'(\xi) = -\frac{\phi^p(\xi)}{\xi} - \alpha\frac{\phi^{1-\sigma}(\xi)}{\xi} - \beta\phi^{-\sigma}(\xi)\phi'(\xi).$$

Integrating it from 1 to ξ_k, we obtain

$$\Big|\int_1^{\xi_k} \frac{\phi''(\xi)}{\xi} + \frac{n-1}{\xi^2}\phi'(\xi)\Big| = \Big|\frac{\phi'(\xi_k)}{\xi_k} - \phi'(1) + \int_1^{\xi_k} \frac{n}{\xi^2}\phi'(\xi)\Big| \leq M + \Big|\phi'(1)\Big| + nM,$$

$$\Big|\int_1^{\xi_k} \beta\phi^{-\sigma}(\xi)\phi'(\xi)\Big| = \Big|\frac{\beta}{1-\sigma}(\phi^{1-\sigma}(\xi_k) - \phi^{1-\sigma}(1))\Big| \leq \frac{\beta}{\sigma-1}(l^{1-\sigma} + \phi^{1-\sigma}(1))$$

But,

$$\int_1^{\xi_k} \frac{\phi^p(\xi) + \alpha\phi^{1-\sigma}(\xi)}{\xi} \geq \alpha\, l^{1-\sigma} \int_1^{\xi_k} \frac{1}{\xi} \to \infty \text{ as } k \to \infty,$$

a contradiction. Hence, $l = 0$. ∎

Lemma 3.2. *It holds $\phi'(\xi) \to 0$ as $\xi \to \infty$.*

Proof. Using (2.5) and applying L'Hôpital's Rule, we have

$$\begin{aligned}
\lim_{\xi\to\infty} \phi'(\xi) &= -\lim_{\xi\to\infty} \frac{1}{\xi^{n-1}\rho(\xi)} \int_0^{\xi} y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy \,. \\
&= -\lim_{\xi\to\infty} \frac{\xi^{n-1}\rho(\xi)[\phi^p(\xi) + \alpha\phi^{1-\sigma}(\xi)]}{(n-1)\xi^{n-2}\rho(\xi) + \xi^{n-1}\rho(\xi)\beta\xi\phi^{-\sigma}(\xi)} \\
&= -\lim_{\xi\to\infty} \frac{[\phi^{p+\sigma}(\xi) + \alpha\phi(\xi)]}{\frac{(n-1)\phi^{\sigma}(\xi)}{\xi} + \xi\beta} \\
&= 0.
\end{aligned}$$

The lemma follows. ■

The following lemma will be useful to obtain the asymptotic behavior of $\phi(\xi)$.

Lemma 3.3. *There holds $\phi(\xi)\rho(\xi) \to \infty$ as $\xi \to \infty$.*

Proof. Since

$$(\phi\rho)'(\xi) = \rho(\xi)[\phi'(\xi) + \phi^{1-\sigma}(\xi)\beta\xi],$$

we obtain that $\lim_{\xi\to\infty}(\phi\rho)'(\xi) = \infty$. Therefore, $\phi(\xi)\rho(\xi) \to \infty$ as $\xi \to \infty$. ■

Using (2.5) and L'Hôpital's Rule, we obtain

$$\begin{aligned}
\lim_{\xi\to\infty} \frac{\phi'(\xi)}{\phi(\xi)} &= -\lim_{\xi\to\infty} \{\frac{1}{\xi^{n-1}\rho(\xi)\phi(\xi)} \int_0^{\xi} y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy \,\} \\
&= -\lim_{\xi\to\infty} \{\frac{\xi^{n-1}\rho(\xi)[\phi^p(\xi)+\alpha\phi^{1-\sigma}(\xi)]}{(n-1)\xi^{n-2}\rho(\xi)\phi(\xi)+\xi^{n-1}\rho(\xi)\beta\xi\phi^{-\sigma}(\xi)\phi(\xi)+\xi^{n-1}\rho(\xi)\phi'(\xi)}\} \\
&= -\lim_{\xi\to\infty} \{\frac{\phi^{p+\sigma-1}(\xi) + \alpha}{\frac{(n-1)}{\xi}\phi^{\sigma}(\xi) + \xi\beta + \phi^{\sigma-1}(\xi)\phi'(\xi)}\} \\
&= 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\lim_{\xi\to\infty} \frac{\xi\phi'(\xi)}{\phi(\xi)} &= -\lim_{\xi\to\infty} \{\frac{1}{\xi^{n-2}\rho(\xi)\phi(\xi)} \int_0^{\xi} y^{n-1}\rho(y)[\phi^p(y) + \alpha\phi^{1-\sigma}(y)]dy \,\} \\
&= -\lim_{\xi\to\infty} \{\frac{\xi^{n-1}\rho(\xi)[\phi^p(\xi)+\alpha\phi^{1-\sigma}(\xi)]}{(n-2)\xi^{n-3}\rho(\xi)\phi(\xi)+\xi^{n-2}\rho(\xi)\beta\xi\phi^{-\sigma}(\xi)\phi(\xi)+\xi^{n-2}\rho(\xi)\phi'(\xi)}\} \\
&= -\lim_{\xi\to\infty} \{\frac{\phi^{p+\sigma-1}(\xi) + \alpha}{\frac{(n-2)}{\xi^2}\phi^{\sigma}(\xi) + \beta + \frac{\phi^{\sigma-1}(\xi)\phi'(\xi)}{\xi}}\} \\
&= -\frac{\alpha}{\beta}.
\end{aligned}$$

This implies that $\forall\ \epsilon > 0$ there exists $K, R > 0$ depending only on ϵ such that

$$\phi(\xi) \le K\xi^{-\frac{\alpha}{\beta}+\epsilon}, \forall \xi \ge R. \tag{3.3}$$

Let $\psi(\xi) = \phi'(\xi)/\phi(\xi)$. Note that

$$\lim_{\xi\to\infty} \psi(\xi) = 0, \quad \lim_{\xi\to\infty} \xi\psi(\xi) = -\frac{\alpha}{\beta}.$$

By (2.1), $\psi(\xi)$ satisfies the following equation

$$\psi'(\xi) + \frac{n-1}{\xi}\psi(\xi) + \beta\xi\phi^{-\sigma}(\xi)\psi(\xi) = -[\phi^{p-1}(\xi) + \alpha\phi^{-\sigma}(\xi) + \psi^2(\xi)],$$

i.e.,

$$(\xi^{n-1}\rho(\xi)\psi(\xi))' = -\xi^{n-1}\rho(\xi)[\phi^{p-1}(\xi) + \alpha\phi^{-\sigma}(\xi) + \psi^2(\xi)].$$

Hence

$$\psi(\xi) = -\frac{1}{\xi^{n-1}\rho(\xi)}\int_0^\xi y^{n-1}\rho(y)[\phi^{p-1}(y) + \alpha\phi^{-\sigma}(y) + \psi^2(y)]dy. \tag{3.4}$$

Now we are ready to prove the main theorem of this section as follows.

Theorem 3.4. *The limit $lim_{\xi\to\infty}\{\xi^{\alpha/\beta}\phi(\xi)\}$ exists and is positive.*

Proof. From (3.4) and applying L'Hôpital's Rule, it follows that

$$\begin{aligned}&[\xi\psi(\xi) + \frac{\alpha}{\beta}]\xi^\lambda \\ &= \frac{-\int_0^\xi y^{n-1}\rho(y)[\phi^{p-1}(y) + \alpha\phi^{-\sigma}(y) + \psi^2(y)]dy + \frac{\alpha}{\beta}\xi^{n-2}\rho(\xi)}{\xi^{n-2-\lambda}\rho(\xi)},\end{aligned} \tag{3.5}$$

where $\lambda \in (0, 2)$. Then we obtain

$$\begin{aligned}&\lim_{\xi\to\infty}[\xi\psi(\xi) + \frac{\alpha}{\beta}]\xi^\lambda \\ &= \lim_{\xi\to\infty}\frac{-\xi^{n-1}\rho(\xi)[\phi^{p-1}(\xi)+\alpha\phi^{-\sigma}(\xi)+\psi^2(\xi)]+\frac{\alpha}{\beta}(n-2)\xi^{n-3}\rho(\xi)+\alpha\xi^{n-1}\phi^{-\sigma}(\xi)\rho(\xi)}{(n-2-\lambda)\xi^{n-3-\lambda}\rho(\xi)+\beta\xi^{n-1-\lambda}\phi^{-\sigma}(\xi)\rho(\xi)} \\ &= \lim_{\xi\to\infty}\frac{-[\xi^\lambda\phi^{p-1}(\xi) + \xi^\lambda(\psi)^2(\xi)] + \frac{\alpha}{\beta}(n-2)\xi^{\lambda-2}}{(n-2-\lambda)\xi^{-2} + \beta\phi^{-\sigma}(\xi)} \\ &= 0.\end{aligned}$$

From this and by integration, we obtain

$$\phi(\xi) = A\xi^{-\frac{\alpha}{\beta}}[1 + o(\xi^{-\lambda})] \text{ as } \xi \to \infty \tag{3.6}$$

for some positive constant A. Hence the theorem follows. ∎

4. Blow-up for Large Initial Data

In this section we prove that finite-time blow-up actually occurs for the Cauchy problem. We consider the case of positive solutions for the general n-dimensional case. One can see that the classical analysis of Kaplan [7], based on an ordinary differential inequality for the first Fourier coefficient, can be applied to (1.1).

Indeed, for any $\epsilon > 0$, let $\phi_\epsilon(x) = A(\epsilon)\exp(-\epsilon|x|^2)$, where $A(\epsilon)$ is defined so that $\int_{\mathbf{R}^n} \phi_\epsilon(x)dx = 1$. Suppose that u is a bounded positive solution for $t \in [0, T)$ for some $T > 0$ with the initial condition $u(x, 0) = u_0(x)$. Set

$$g(t) := \int_{\mathbf{R}^n} u(x,t)\phi_\epsilon(x)dx, \quad G(t) := -\frac{1}{\sigma-1}\int_{\mathbf{R}^n} u^{1-\sigma}(x,t)\phi_\epsilon(x)dx.$$

Note that by Jensen's Inequality we have

$$\int_{\mathbf{R}^n} u^p(x,t)\phi_\epsilon(x)dx \geq g^p(t).$$

Then we have the inequality

$$G'(t) \geq g^p(t) - 2\epsilon n g(t),\ t \in [0, T). \tag{4.1}$$

Again, by Jensen's Inequality we have

$$\int_{\mathbf{R}^n} u^{1-\sigma}(x,t)\phi_\epsilon(x)dx \geq \left(\int_{\mathbf{R}^n} u(x,t)\phi_\epsilon(x)dx\right)^{1-\sigma},\ t \in [0, T).$$

This implies that

$$G(t) \leq -\frac{1}{\sigma-1} g^{1-\sigma}(t),\ t \in [0, T). \tag{4.2}$$

It follows by an integration of (4.1) that

$$G(t) \geq G(0) + \int_0^t [g^p(s) - 2\epsilon n g(s)]ds,\ t \in [0, T).$$

Hence it follows from (4.2) that

$$\int_0^t [g^p(s) - 2\epsilon n g(s)]ds + \frac{1}{\sigma-1} g^{1-\sigma}(t) \leq -G(0),\ t \in [0, T). \tag{4.3}$$

Now, we choose u_0 such that

$$-G(0) < \frac{1}{\sigma-1}(\kappa+\eta)^{1-\sigma} \tag{4.4}$$

for some $\eta > 0$, where κ is the unique positive zero of $g^p - 2\epsilon n g$ (i.e., $\kappa^{p-1} = 2\epsilon n$). Note that $g(0) > \kappa + \eta$. Then by (4.3) we have

$$g(t) \geq \kappa + \eta,\ t \in [0, T). \tag{4.5}$$

Indeed, if (4.5) were not true, then there exists the smallest $t_0 > 0$ such that $g(t_0) = \kappa + \eta$. Then by (4.3) and (4.4) we have

$$\frac{1}{\sigma - 1} g^{1-\sigma}(t_0) \leq -G(0) < \frac{1}{\sigma - 1}(\kappa + \eta)^{1-\sigma}.$$

This is a contradiction and the estimate (4.5) follows.

Suppose that u exists globally, i.e., $T = \infty$, for some u_0 satisfying (4.4) for some $\eta > 0$. Then by (4.5) there is a positive constant δ such that

$$g^p(t) - 2\epsilon n g(t) \geq \delta \ \text{ for all } \ t \geq 0.$$

We reach a contradiction by letting $t \to \infty$ in (4.3). Therefore, we conclude that u blows up in finite time if u_0 satisfies (4.4) for some $\eta > 0$.

5. Blow-up Patterns

We denote the initial value problem (1.6) with $\phi'(0) = 0$ and $\phi(0) = \eta > 0$ by (Q). Let $\kappa = \alpha^\alpha$. Then $\phi \equiv \kappa$ is the only trivial constant solution of (Q). It is obvious that there is a unique local solution ϕ of (Q) for each given $\eta > 0$.

Let $[0, R)$ be the maximal existence interval of ϕ, where $0 < R \leq \infty$. As before, we define

$$H(\xi) = \frac{1}{2}[\phi'(\xi)]^2 + G(\phi(\xi)), \xi \in [0, R),$$

where

$$G(\phi) = \int_\kappa^\phi (s^p - \alpha s^{1-\sigma}) ds, \phi \geq 0.$$

In this section, we always assume that $\sigma \in (1, 2)$. Note that $G(0) \in (0, \infty)$, $G(\infty) = \infty$, $G(\phi) > 0$ for all $\phi \geq 0$ except $\phi = \kappa$, and there is a unique $\tilde{\kappa} > \kappa$ such that $G(\tilde{\kappa}) = G(0)$.

For a given solution ϕ of (Q), we define

$$\rho(y) = \exp[-\beta \int_0^y \xi \phi^{-\sigma}(\xi) d\xi], \quad 0 \leq y < R. \tag{5.1}$$

Then we have

$$\phi'(\xi) = -\frac{1}{\xi^{n-1}\rho(\xi)} \int_0^\xi y^{n-1}\rho(y)[\phi^p(y) - \alpha\phi^{1-\sigma}(y)] dy, \quad 0 \leq \xi < R. \tag{5.2}$$

Next, we study the solution for η near κ. Set $\eta = \kappa + \epsilon$. If we write $\phi(\xi) = \kappa + \epsilon v(\xi)$, then, as $\epsilon \to 0$, v satisfies the limiting problem

$$\begin{aligned}&v'' + \frac{n-1}{\xi}v' - \frac{\beta}{\kappa^\sigma}\xi v' + \frac{1}{\kappa^\sigma}v = 0,\ \xi > 0,\\ &v'(0) = 0, \quad v(0) = 1.\end{aligned}$$

It is well-known that v has exactly N zeros in $(0, \infty)$, where $-N$ is the largest integer which is less or equal to $-1/(2\beta) = -(p + \sigma - 1)/(p - 1)$. Notice that $N \geq 2$, since $p > 1$. By the standard theory on continuous dependence, it is easy to show that for η sufficiently close to κ, the solution ϕ of (Q) intersects κ at least N times.

Now, for $n = 1$, with some necessary modifications of the proofs of lemmas in section 2 of [6], we can easily obtain the following theorem.

Theorem 5.1. *Suppose that $n = 1$. Then the problem (Q) has at least $N - 1$ distinct positive global solutions such that $\phi(\xi) \to 0$ as $\xi \to \infty$, where $-N$ is the largest integer which is less than or equal to $-(p + \sigma - 1)/(p - 1)$.*

Set $p_c(n) = (n+2)/(n-2)$ for $n \geq 3$ and $p_c(2) = \infty$. Similar to section 3 of [6], we can also derive the following theorem for the multi-dimensional case.

Theorem 5.2. *Suppose that $n \geq 2$ and $1 < p < p_c(n)$. Then the problem (Q) has at least N_1 distinct positive global solutions such that $\phi(0) > \kappa$ and $\phi(\xi) \to 0$ as $\xi \to \infty$, where N_1 is the largest integer which is less than or equal to $N/2$.*

For the asymptotic behavior of any bounded global solution of (1.6), we have

Theorem 5.3. *The limit $B := \lim_{\xi\to\infty}[\xi^{\alpha/\beta}\phi(\xi)]$ exists and is positive.*

Note that $B = B(\eta)$. It follows that

$$\lim_{t\to T^-} V(x,t) = B|x|^{-\alpha/\beta}, \quad x \neq 0, \tag{5.3}$$

for any nonconstant bounded global solution ϕ of (1.6).

From these results there always exists a symmetric positive monotone self-similar solution V of (1.1) in the form (1.3) such that V blows up only at the single point $x = 0$ at T for a given finite time T. Note that $N = 2$ for $p \geq \sigma + 1$ and $N \geq 3$ for $p \in (1, \sigma + 1)$. Also, there are some other self-similar single-point blow-up patterns with different oscillations, if $p \in (1, \sigma + 1)$. Notice that any self-similar solution we constructed above has the same number of critical points at any time before the blow-up time, yet the critical points of the self-similar solution merge together to a single point at the blow-up time.

It is interesting to remark that for $u_0(x) = \phi(|x|)$ (with $T = 1$) we have

$$\lim_{|x|\to\infty} [|x|^{\alpha/\beta} u_0(x)] = B > 0$$

and the corresponding solution u blows up in finite time $T = 1$. In particular, for the monotone decreasing backward self-similar solution ϕ, this spatial asymptotic behavior is the same as the one for the forward self-similar solution (except the constants A and B). However, one is a global solution and the other is non-global.

It will be very interesting to determine the global and/or non-global existence for different constants A or B. This is left as an open problem.

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