TAIWANESE JOURNAL OF MATHEMATICS
Vol. 4, No. 1, pp. 55-64, March 2000

A NOTE ON THE DISCRETE ALEKSANDROV-BAKELMAN MAXIMUM PRINCIPLE

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Dedicated to Professor Fon-Che Liu on his sixtieth birthday

Abstract. In previous works, we have established discrete versions of the Aleksandrov -Bakelman maximum principle for elliptic operators, on general meshes in Euclidean space. In this paper, we prove a variant of these estimates in terms of a discrete analogue of the determinant of the coefficient matrix in the differential operator case. Our treatment depends on an interesting connection between the determinant and volumes of cells in the underlying mesh.

In our previous papers [8, 9], we proved discrete versions of the Aleksandrov-Bakelman maximum principle, (see [1, 2]), for linear second order elliptic partial differential operators in domains Ω in Euclidean n-space R^n. For operators $\widetilde{L}$ in the simple form

$$\widetilde{L} = a^{ij} D_{ij} u \tag{1}$$

acting on functions $u \in C^2(\Omega)$ with coefficient matrix $\mathcal{A} = [\dashv\rangle|]$ measurable and positive in Ω, the Aleksandrov-Bakelman maximum principle provides an estimate,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C(n) \operatorname{diam} \Omega \left\{ \int_{\Omega} \frac{[(\widetilde{L}u)^-]^n}{\mathcal{D}} \right\}^{1/n}, \tag{2}$$

where $C(n)$ is a constant depending on n and $\mathcal{D} = \det \mathcal{A}$ is the determinant of the coefficient matrix $\mathcal{A}$. In our papers [4, 8, 9], we have treated analogous

Received November 19, 1999.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: Primary 65N12, 35J15, 39A70; Secondary 65N40, 35B05, 39A10.
Key words and phrases: Discrete Aleksandrov-Bakelman maximum principle, balanced operator, monotone operator, elliptic operator.
†Research supported by Taiwan National Science Council under Grant No. NSC37155F.
‡Research supported by Australian Research Council Grant.

results for difference operators, with the purpose of deriving local estimates and eventually stability results for nonlinear schemes as in [5]. Here, we again consider meshes $\mathbf{E}$, which are arbitrary discrete sets in R^n, and difference operators L of the form

$$Lu = \sum_y a(x,y)u(y) \tag{3}$$

acting on mesh functions $u : \mathbf{E} \to R^n$. The coefficients $a(x,y)$ are defined on $\mathbf{E} \times \mathbf{E}$ and vanish except for finitely many y, for each x value. The operator L is called *monotone* if

$$a(x,y) \geq 0, \qquad \text{for all} \quad x,\ y \in \mathbf{E}, \tag{4}$$

and *positive*, if in addition,

$$c(x) := \sum_y a(x,y) \leq 0, \qquad \text{for all} \quad x \in \mathbf{E}. \tag{5}$$

Furthermore, L is *balanced* if

$$b(x) := \sum_y a(x,y)(y-x) = 0, \qquad \text{for all} \quad x \in \mathbf{E}. \tag{6}$$

The differential operator corresponding to L is given by, (see [8]) ,

$$\widetilde{L}u = \mathcal{A}\cdot\mathcal{D}^{\in}\sqcap + \lfloor\cdot\mathcal{D}\sqcap + \rfloor\sqcap, \tag{7}$$

with principal coefficient matrix

$$\mathcal{A}(\S) = \frac{\infty}{\in}\sum_{\dagger} \dashv(\S,\dagger)(\dagger-\S)\otimes(\dagger-\S) \tag{8}$$

and coefficients b and c as in [6] and [7]. Accordingly monotone, balanced difference operators L of the form

$$Lu(x) = \sum_y a(x,y)(u(y)-u(x)) \tag{9}$$

correspond to elliptic partial differential operators of the form (1).

Our purpose in this note is to deduce the discrete maximum principle in a form corresponding to (2), where the dependence on the coefficients of L is determined by det $\mathcal{A}$ for $\mathcal{A}$ given by (8). First we prove a lemma which gives a representation for det $\mathcal{A}$ as a sum of squares of volumes spanned by n-tuples of the vectors $\sqrt{a(x,y)}(y-x)$. For vectors $y^1,\cdots,y^n \in R^n$, let

$$V(y^1, \cdots, y^n) = \det \left[y^i_j \right] \tag{10}$$

denote the volume of the parallelpiped spanned by $y^1, \cdots, y^n$.

Lemma 1. *For $y^1, \cdots, y^N \in R^n$, $N \geq n$, we have*

$$\det \left\{ \sum_{i=1}^{N} y^i \otimes y^i \right\} = \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq N} V^2(y^{i_1}, \cdots, y^{i_n}). \tag{11}$$

Proof. We proceed by induction on N. Accordingly suppose (11) is true for $N \geq n$, for each n, and consider vectors $y^1, \cdots, y^{N+1} \in R^n$. We may choose coordinates so that

$$y^{N+1} = \alpha\, e_1,$$

where e_1 is the unit vector directed along the x_1 coordinate axis. Then

$$\sum_{i=1}^{N+1} y^i \otimes y^i = \alpha^2\, e_1 \otimes e_1 + \sum_{i=1}^{N} y^i \otimes y^i$$

and hence

$$\det \sum_{i=1}^{N+1} y^i \otimes y^i = \det \sum_{i=1}^{N} y^i \otimes y^i + \alpha^2 \det \sum_{i=1}^{N} \bar{y}^i \otimes \bar{y}^i,$$

where $\bar{y}^i = (y^i_2, \cdots, y^i_n) \in R^{n-1}$. By our induction hypothesis, we then obtain

$$\begin{aligned} &\det \sum_{i=1}^{N+1} y^i \otimes y^i \\ &= \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq N} V^2(y^{i_1}, \cdots, y^{i_n}) + \sum_{1 \leq i_1 < i_2 \cdots < i_{n-1} \leq N} V^2(y^{i_1}, \cdots, y^{i_{n-1}}, y^{N+1}) \\ &= \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq N+1} V^2(y^{i_1}, \cdots, y^{i_n}). \end{aligned}$$

It therefore remains to show (11) when $N = n$. Again we may proceed by induction. Taking $y^n = \alpha e_1$, we obtain, as before,

$$\sum_{i=1}^{n} y^i \otimes y^i = \alpha^2 e_1 \otimes e_1 + \sum_{i=1}^{n-1} y^i \otimes y^i$$

and hence the validity of (11) for $N = n-1$ in R^{n-1} implies that for $N = n$ in R^n. Obviously (11) is true for $N = n = 1$ and we are done. ∎

For the difference operator L, given by (3), and the mesh $\mathbf{E}$, let us introduce a volume element $V(x)$ at a point $x \in \mathbf{E}$ by setting

$$
\begin{aligned}
Y_x &= \{y \in \mathbf{E} \,|\, a(x,y) > 0\}\,, \\
V(x) &= \max_{y^1,\cdots,y^n \in Y_x} V(y^1 - x, \cdots, y^n - x).
\end{aligned} \tag{12}
$$

The set Y_x consists of those points y which are directly connected to x through L. For future use, we will let $N = N(x)$ denote the number of points in Y_x. Recall from [8, 9] that for a bounded subset D of $\mathbf{E}$, the interior D^o and boundary D^b of D, with respect to L, are defined by

$$
\begin{aligned}
D^o &= \{x \in D \,|\, a(x,y) = 0 \quad \forall \;\; y \neq D\}, \\
D^b &= D - D^o,
\end{aligned} \tag{13}
$$

respectively. We can now state the following discrete maximum principle.

Theorem 2. *Let u be a mesh function satisfying the difference inequality,*

$$
Lu + f \geq 0 \tag{14}
$$

in the interior D^o of a bounded set D in $\mathbf{E}$, with L positive and balanced in D^o and $u \leq 0$ on the boundary D^b. Then we have the estimate

$$
\max_D u \leq C \cdot \operatorname{diam} D \left\{ \sum_{x \in D^o} \frac{|f(x)|^n V(x)}{\det \mathcal{A}(\S)} \right\}^{1/n}, \tag{15}
$$

where C is a constant depending on n and $N_o = \max_{D^o} N$.

Remarks:

(i) The condition $\det \mathcal{A} > \prime$, together with the balance condition (6), imply $N \geq n+1$. It would be interesting to remove the dependence on N from Theorem 2, although normally one would expect $N \leq O(n)$.

(ii) As in previous works, the summation over D^o in the estimate (15) can be replaced by summation over the upper contact set $\Gamma^+ = \Gamma^+_u$ defined by

$$
\begin{aligned}
\Gamma^+ = \{\, x \in D^o | \exists p \in R^n & \text{ satisfying} \\
& u(y) \leq u(x) + p \cdot (y - x) \quad \forall \; y \;\in\; D\}.
\end{aligned} \tag{16}
$$

Proof. The estimate (15) can be extracted from the proof of [9, Theorem 1]. For completeness, we provide the details here. First, we recall for a mesh function u, its normal mapping $\chi = \chi_u$ over the domain D is defined by

$$\chi(x) = \{p \in R^n | u(y) \le u(x) + p \cdot (y - x) \quad \forall\, y \in D\} \tag{17}$$

for $x \in D$, so that from (16) we see that

$$\Gamma^+ = \{x \in D^o \mid \chi_u(x) \ne \emptyset\}.$$

Note that $\chi_u(x)$ being nonempty at x means that u is concave at the point x. To prove Theorem 2, we need to estimate $|\chi_u(x)|$ at points $x \in \Gamma^+$. Let us fix a point $x \in \Gamma^+$ and a vector $p \in \chi_u(x)$. Without loss of generality, we can assume $u(x) > 0$. Writing

$$v(z) = u(z) - p \cdot (z - x), \tag{18}$$

we then have $v(y) \le v(x)$ for all $y \in D$. Using the difference inequality (14) and the fact that L is positive and balanced, we then have

$$\begin{aligned} &\sum a(x,y)(v(x) - v(y)) \\ &\quad = \sum a(x,y)(u(x) - u(y)) \\ &\quad = -Lu(x) + c(x)u(x) \\ &\quad \le f(x). \end{aligned} \tag{19}$$

Now let $Z = Z_x$ be given by

$$Z_x = \{x + a(x,y)(y - x) \mid y \in E\}. \tag{20}$$

The condition that L is balanced means that Z_x is centred at x. Let us suppose for the time being that Z_x consists only of extreme points. Defining a new function w by

$$\begin{aligned} w(x) &= v(x), \\ w(y) &= v(x) + a(x,y)(v(y) - v(x)) \end{aligned} \tag{21}$$

for $y \in Z_x$, we have

$$\chi_w(x) = \chi_v(x) \tag{22}$$

and by (19),

$$\sum_{y \in Z} (w(x) - w(y)) \le f(x). \tag{23}$$

Hence

$$w(x) - w(y) \le f(x) \tag{24}$$

for all $y \in Y$. Now let $k = k_x$ be the function given by

$$\begin{aligned} &k(x) = 1, \\ &k(y) = 0 \quad \text{for} \quad y \in Z. \end{aligned} \tag{25}$$

Then by (24), we have

$$\chi_w(x) \subset \chi_{|f(x)|k}(x)$$

and hence, by (22),

$$|\chi_v(x)| \le |f(x)|^n \, |\chi_k(x)|. \tag{26}$$

Noting that

$$\begin{aligned} Z_x^* &= \chi_k(x) \\ &= \{p \in R^n | p \cdot (y - x) \le 1 \quad \text{for all} \ \ y \in Z\}, \end{aligned} \tag{27}$$

is the *polar* of the convex hull $\widehat{Z}$ with respect to x, we have $|\chi_k(x)| = |Z_x^*|$ and hence by (26),

$$|\chi_v(x)| \le |f(x)|^n |Z_x^*|. \tag{28}$$

To estimate the polar volume $|Z^*|$, we use the following geometric inequality.

Lemma 3. *Let K be a convex body in R^n and K^* its polar with respect to its centre x. Then*

$$|K| \, |K^*| \le C \tag{29}$$

for some constant C depending only on n.

Proof. To show (29), we observe first that for an ellipsoid E, we have the equality

$$|E| \, |E^*| = \omega_n^2, \tag{30}$$

where ω_n denotes the volume of the unit ball in R^n. The inequality (29) then follows from the existence of a minimal ellipsoid E, with centre x, satisfying

$$n^{-\frac{3}{2}} E \subset K \subset E, \tag{31}$$

where for $\gamma > 0$, γE denotes the γ dilation of E with respect to x. For a proof of (31), see, for example, [1, Lemma 25.6]. ■

From Lemma 3, we have

$$|Z_x^*| \leq \frac{C}{|\widehat{Z}_x|}\,, \tag{32}$$

where $C = n^{3n/2}\omega_n^2$ is the constant in (29). To proceed further, we write

$$\begin{aligned} Y(x) &= \{y^1, \cdots, y^N\}, \\ z^i &= x + a(x, y^i)(y^i - x), \\ a(x, z^i) &= a^i, \qquad i = 1, \cdots, N, \end{aligned} \tag{33}$$

and apply Lemma 1 to estimate

$$\begin{aligned} \det \mathcal{A} &= \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq N} V^2(\sqrt{a^{i_1}}(y^{i_1} - x), \cdots, \sqrt{a^{i_n}}(y^{i_n} - x)) \\ &= \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq N} V(z^{i_1} - x, \cdots, z^{i_n} - x)\, V(y^{i_1} - x, \cdots, y^{i_n} - x) \\ &\leq V(x) \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq N} V(z^{i_1} - x, \cdots, z^{i_n} - x) \\ &\leq C(N) V(x)\, |\widehat{Z}_x|\,. \end{aligned} \tag{34}$$

Consequently, we obtain from (32), (28), (18),

$$\begin{aligned} |\chi_u(x)| &= |\chi_v(x)| \\ &\leq C(N) \frac{V(x)}{\det \mathcal{A}(\S)}\, |f(x)|^n \end{aligned} \tag{35}$$

and hence

$$|\chi_u(\Gamma^+)| \leq C(N) \sum_{x \in \Gamma^+} \frac{V(x)}{\det \mathcal{A}(\S)}\, |f(x)|^n. \tag{36}$$

The estimate (15) then follows from [9, Lemma 2.2].

Returning to the general case, we write each point $z \in Z_x$ as a convex combination,

$$z = \sum_{i=1}^{\ell} \alpha_i(z) z^i\,, \tag{37}$$

of the extreme points $z^1, \cdots z^\ell$, where $0 \leq \alpha_i(z) \leq 1$, $\sum \alpha_i = 1$ and $\ell = \ell(x) < N(x)$ is the number of extreme points of Z_x. We then define a new set $\widetilde{Z}$ by

$$\widetilde{Z} = \left\{ (\widetilde{z})^1, \cdots, (\widetilde{z})^\ell \right\}, \tag{38}$$

where

$$(\widetilde{z})^i = \left(\sum_{z\in Z} \alpha_i(z)\right)(z^i - x) + x, \qquad i = 1, \cdots, \ell, \tag{39}$$

and extend the function v by defining

$$\begin{aligned} w(x) &= v(x), \\ w((\widetilde{z})^i) &= \left(\sum_{z\in Z} \alpha_i(z)\right)(v(z^i) - v(x)) + v(x), \qquad i = 1, \cdots, \ell. \end{aligned} \tag{40}$$

Continuing the process, we end up with a set $\widetilde{Z} = \widetilde{Z}_x$ consisting of only extreme points, centred at x by the balance condition of L, and a function $\widetilde{w}$ on $\widetilde{Z} \cup \{x\}$ for which

$$\chi_v(x) \subset \chi_{\underset{w}{\sim}}(x)\,, \tag{41}$$

but which satisfies

$$\sum (\widetilde{w}(z) - \widetilde{w}(x)) \leq C(N)|f(x)| \tag{42}$$

instead of (23). We obtain thus estimate (36) and as before conclude (15). ■

Remarks :

(i) If K is a convex body in R^n and K^* its polar with respect to some interior point x, we have a complementary lower bound,

$$|K|\,|K^*| \geq C, \tag{43}$$

where C is a positive constant depending on n, to the upper bound (29) (see, for example, [1]). Consequently, from (36), we infer a sharper form of the estimate (15) with diam D replaced by $|\widehat{D}|^{1/n}$.

(ii) When the convex body K in Lemma 3 is centrally symmetric, inequality (29) with the sharp constant $C = \omega_n^2$ is a consequence of the Blashcke-Santalo inequality (see, for example, [11]).

(iii) From (24), applied to the extreme points of Z_x, we deduce the estimate

$$|\chi_u(x)| \leq |f(x)|^n |\widehat{Z}_x^*|, \qquad x \in \Gamma^+, \tag{44}$$

which is more general than (28), leading to Theorem 1 in [9]. Under the further assumption of nondegeneracy

$$B_\rho(x) \subset \widehat{Z}_x, \tag{45}$$

where $B_\rho(x)$ denotes the ball of radius $\rho = \rho(x)$ and centre x in R^n, we obtain, in place of (15),

$$\max_\Omega \leq C(n)\,\mathrm{diam}\, D \left\{ \sum_{x \in D^o} \left(\frac{|f(x)|}{\rho} \right)^n \right\}^{1/n}, \tag{46}$$

which is the basis for our treatment of local estimates (Harnack inequality, Hölder estimate, Liouville theorem) in [8, 9].

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