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# BIPARTITE STEINHAUS GRAPHS*

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**Abstract.** A Steinhaus matrix is a symmetric 0-1 matrix $[a_{i,j}]_{n\times n}$ such that $a_{i,i} = 0$ for $0 \le i \le n-1$ and $a_{i,j} \equiv (a_{i-1,j-1} + a_{i-1,j}) \pmod 2$ for $1 \le i < j \le n-1$. A Steinhaus graph is a graph whose adjacency matrix is a Steinhaus matrix. In this paper, we present a new characterization of bipartite Steinhaus graphs.

## 1. Introduction

A *Steinhaus matrix* is a symmetric 0-1 matrix $[a_{i,j}]_{n\times n}$ such that $a_{i,i} = 0$ for $0 \le i \le n-1$ and $a_{i,j} \equiv (a_{i-1,j-1} + a_{i-1,j}) \pmod 2$ for $1 \le i < j \le n-1$. A *Steinhaus triangle* is the upper triangular part of a Steinhaus matrix. Note that a Steinhaus matrix and a Steinhaus triangle determine each other. A *Steinhaus graph* is a graph whose adjacency matrix is a Steinhaus matrix. Fig. 1 shows a Steinhaus matrix and its corresponding graph. Note that a binary string $a_{0,0}a_{0,1}\ldots a_{0,n-1}$ (with $a_{0,0} = 0$) completely determines a Steinhaus matrix (graph). It is often said that the binary string *generates* the Steinhaus matrix (graph).

The concept of Steinhaus triangles was first introduced by Steinhaus [16]. Harborth [12, 13], Wang [17], and Chang [5] studied the number of ones in Steinhaus triangles. Molluzzo [15] introduced the concept of Steinhaus graphs. This class of graphs was then extensively studied by Dymàček [7, 8, 9] (also see [1, 2, 3, 4]). Recently, Dymàček and Whaley [11] characterized all binary strings that generate bipartite Steinhaus graphs, and gave a recursive formula for the number $b(n)$ of bipartite Steinhaus graphs of order $n$. For a good survey, see [10].

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FIG. 1. A Steinhaus matrix and its corresponding graph.

In this paper, we give a new characterization of bipartite Steinhaus graphs, which is also proved in [6] alternatively and used to give a solution of $b(n)$ in terms of the binary representation of $n-2$ (also see [14]).

For any Steinhaus graph $G$ with adjacency matrix $[a_{i,j}]_{n\times n}$, the Steinhaus graph generated by $a_{r,r}a_{r,r+1}\ldots a_{r,s}$, where $0\le r\le s\le n-1$, is precisely the subgraph of $G$ induced by the vertex subset $\{r, r+1, \ldots, s\}$. Denote by $\mathrm{Adj}(i)$ the set of all vertices adjacent to $i$ and $\mathrm{Adj}^+(i)$ the set of all vertices $j>i$ adjacent to $i$. Note that $\mathrm{Adj}(0)$ completely determines a Steinhaus graph. For instance, the Steinhaus graph with $\mathrm{Adj}(0)=\{1\}$ (respectively, $\{n-1\}$, $\emptyset$) is a path (respectively, a star, $\overline{K_n}$). It is also the case that a Steinhaus graph is completely determined by $v\equiv\min\mathrm{Adj}(0)$ and $\mathrm{Adj}^+(v)$. Note that $v\equiv\min\mathrm{Adj}(0)$ gives that $a_{i,j}=0$ and $a_{i,v}=1$ for all $0\le i<j<v$. This together with $\mathrm{Adj}^+(v)$ determines $\mathrm{Adj}^+(v-1)$, and then $\mathrm{Adj}^+(v-2)$, ... etc.

## 2. Characterizations of Bipartite Steinhaus Graphs

This section gives a new characterization of bipartite Steinhaus graphs (see Theorem 7).

Suppose $A=[a_{i,j}]$ is an $n\times n$ Steinhaus matrix. Denote by $M_1(A)$ the $2n\times 2n$ Steinhaus matrix $[a'_{i,j}]$ generated by $a'_{0,0}a'_{0,1}\ldots a'_{0,2n-1}$, where $a'_{0,2j}=a_{0,j}$ and $a'_{0,2j+1}=0$ for $0\le j\le n-1$. For $k\ge 2$, recursively define $M_k(A)=M_1(M_{k-1}(A))$. Note that $M_k(A)$ is precisely the $2^k n\times 2^k n$ Steinhaus matrix $[a''_{i,j}]$ generated by $a''_{0,0}a''_{0,1}\ldots a''_{0,2^k n-1}$, where $a''_{0,2^k j}=a_{0,j}$ for $0\le j\le n-1$ and all other $a''_{0,j}=0$.

**Lemma 1.** *For any $n\times n$ Steinhaus matrix $A=[a_{i,j}]$ with $M_1(A)=[a'_{i,j}]$, we have $a'_{2i,2j}=a'_{2i+1,2j}=a'_{2i+1,2j+1}=a_{i,j}$ and $a'_{2i,2j+1}=0$ for $0\le i\le j\le n-1$.*

*Proof.* We shall prove the lemma by induction on $i$. Suppose $i = 0$. By the definition of $M_1(A)$, we have $a'_{2i,2j} = a'_{0,2j} = a_{0,j} = a_{i,j}$ and $a'_{2i,2j+1} = a'_{0,2j+1} = 0$. For $j = i$ $(= 0)$,

$$a'_{2i+1,2j} = a'_{2j,2i+1} = 0 = a_{i,j} \text{ and } a'_{2i+1,2j+1} = 0 = a_{i,j}.$$

For $j > i$ $(= 0)$,

$$a'_{2i+1,2j} = (a'_{2i,2j-1} + a'_{2i,2j}) \bmod 2 = (0 + a_{i,j}) \bmod 2 = a_{i,j} \text{ and}$$

$$a'_{2i+1,2j+1} = (a'_{2i,2j} + a'_{2i,2j+1}) \bmod 2 = (a_{i,j} + 0) \bmod 2 = a_{i,j}.$$

Therefore, the lemma holds for $i = 0$. Suppose the lemma is true for any $i' < i$. Consider the case with $i \geq 1$. For any $j \geq i$ $(\geq 1)$,

$$a'_{2i,2j+1} = (a'_{2(i-1)+1,2j} + a'_{2(i-1)+1,2j+1}) \bmod 2.$$

By the induction hypothesis, $a'_{2(i-1)+1,2j} = a'_{2(i-1)+1,2j+1} = a_{i-1,j}$. Therefore, $a'_{2i,2j+1} = 0$. For $j = i$ $(\geq 1)$, since $a'_{2i+1,2j} = a'_{2j,2i+1} = 0$,

$$a'_{2i,2j} = a'_{2i+1,2j} = a'_{2i+1,2j+1} = 0 = a_{i,j}.$$

For $j > i$ $(\geq 1)$, by the induction hypothesis, we also have

$$\begin{aligned} a'_{2i,2j} &= (a'_{2(i-1)+1,2(j-1)+1} + a'_{2(i-1)+1,2j}) \bmod 2 \\ &= (a_{i-1,j-1} + a_{i-1,j}) \bmod 2 = a_{i,j}, \end{aligned}$$

$$a'_{2i+1,2j} = (a'_{2i,2(j-1)+1} + a'_{2i,2j}) \bmod 2 = (0 + a_{i,j}) \bmod 2 = a_{i,j}, \quad \textit{and}$$

$$a'_{2i+1,2j+1} = (a'_{2i,2j} + a'_{2i,2j+1}) \bmod 2 = (a_{i,j} + 0) \bmod 2 = a_{i,j}. \qquad \blacksquare$$

**Corollary 2.** *Suppose $A = [a_{i,j}]$ is an $n \times n$ Steinhaus matrix and $M_k(A) = [a''_{i,j}]$. For $0 \leq i \leq j \leq n-1$, we have $a''_{i',2^kj} = a_{i,j}$ for $2^ki \leq i' < 2^k(i+1)$ and $a''_{2^ki,j'} = 0$ for $2^kj < j' < 2^k(j+1)$.*

*Proof.* The corollary follows from Lemma 1 and an induction on $k$. $\blacksquare$

**Corollary 3.** *Suppose $G$ and $H$ are Steinhaus graphs corresponding to Steinhaus matrices $A$ and $M_k(A)$, respectively. Then $G$ is isomorphic to the subgraph of $H$ induced by $\{2^ki : 0 \leq i \leq n-1\}$.*

*Proof.* The corollary follows from $a'_{2^ki,2^kj} = a_{i,j}$ for $0 \leq i \leq j \leq n-1$. $\blacksquare$

**Lemma 4.** *Suppose $G$ and $H$ are Steinhaus graphs corresponding to Steinhaus matrices $A$ and $M_1(A)$, respectively. Then $G$ is bipartite if and only if $H$ is bipartite.*

*Proof.* The necessity follows from Corollary 3. Suppose $G$ is a bipartite graph with a bipartition $(X, Y)$. Consider the partition of $V(H)$ into $(X', Y')$ where $X' = \{2i, 2i+1 : i \in X\}$ and $Y' = \{2j, 2j+1 : j \in Y\}$. $H$ has no edge of the form $\{2i, 2j+1\}$ with $i \leq j$ since $a'_{2i,2j+1} = 0$ by Lemma 1. Also, for $i < j$ in $X$ (or $Y$), $a_{i,j} = 0$ implies $a'_{2i,2j} = a'_{2i+1,2j} = a'_{2i+1,2j+1} = a_{i,j} = 0$. So $(X', Y')$ is a bipartition for $H$. ∎

**Theorem 5.** *Suppose $G$ and $H$ are Steinhaus graphs corresponding to Steinhaus matrices $A$ and $M_k(A)$, respectively. Then $G$ is bipartite if and only if $H$ is bipartite.*

Now consider the function $f$ from positive integers $\mathbb{Z}^+$ to $\mathbb{Z}^+ \cup \{\infty\}$ defined by

$$f(w) = \begin{cases} \infty & \text{if } w = 2^k \text{ for some integer } k, \\ 2^k & \text{if } w = 2^k x, \text{ where } x \text{ is an odd integer greater than 2.} \end{cases}$$

Note that $w = 2^k x$ with $x$ an odd integer greater than 2 if and only if the binary representation of $w$ has at least two 1's.

**Lemma 6.** *If $G$ is a Steinhaus graph of $n$ vertices with* $\mathrm{Adj}(0) = \{w\}$, *then the following statements are equivalent*:

(1) *$G$ is bipartite*,

(2) *$G$ has no triangles*,

(3) *$f(w) \geq n - w$.*

*Proof.* (1) $\Longrightarrow$ (2) is clear.

(2) $\Longrightarrow$ (3). Suppose $G$ has no triangles but $f(w) < n - w$. In this case, $f(w) = 2^k$ and $w = 2^k x$ for some odd integer greater than 2. Now

$$\lceil \frac{n}{2^k} \rceil = \lceil \frac{w}{2^k} + \frac{n-w}{2^k} \rceil > x + 1.$$

Consider the Steinhaus graph $H$ of order $\lceil \frac{n}{2^k} \rceil$ with $\mathrm{Adj}(0) = \{x\}$ and adjacency matrix $A = [a_{i,j}]$. Then, $a_{i,x} = 1$ for $0 \leq i < x$. Also $a_{2i,x+1} = 0$ and $a_{2i+1,x+1} = 1$ for $0 \leq 2i < 2i+1 \leq x$. In particular, $a_{1,x} = a_{1,x+1} = a_{x,x+1} = 1$. Consider the Steinhaus graph $G''$ corresponding to $M_k(A) = [a''_{i,j}]$. By Corollary 2, $\mathrm{Adj}(0) = \{2^k x\} = \{w\}$ in $G''$. So $G$ is a subgraph of $G''$ induced by

$\{0, 1, \ldots, n-1\}$. By Corollary 2, $a''_{2^k,w} = a_{1,x} = 1$ and $a''_{2^k,w+2^k} = a_{1,x+1} = 1$ and $a''_{w,w+2^k} = a_{x,x+1} = 1$. So, $\{2^k, w, w+2^k\}$ induces a triangle $T$ in $G''$. However, $2^k = f(w) \leq n - w - 1$; i.e., $w + 2^k \leq n - 1$, and so $G$ contains the triangle $T$, which is impossible.

(3) $\Longrightarrow$ (1). Suppose $f(w) \geq n - w$. There are two cases. For the first case, $f(w) = \infty$, we have $w = 2^k$ for some integer k. Consider the Steinhaus graph $P$ of order $\lceil \frac{n}{w} \rceil$ with $\mathrm{Adj}(0) = \{1\}$. $P$ is a path and so is bipartite. If $A$ is the adjacency matrix of $P$, then the graph $H$ corresponding to $M_k(A)$ is bipartite by Theorem 5. Since $G$ is the subgraph of $H$ induced by $\{0, 1, \ldots, n-1\}$, $G$ is also bipartite.

Next, consider the case with $f(w) = 2^k$ and $w = 2^k x$ where $x$ is an odd integer greater than 2. Since $2^k \geq n - w \geq 1$, we have $\lceil \frac{n}{2^k} \rceil = \lceil \frac{w}{2^k} + \frac{n-w}{2^k} \rceil = x + 1$. Consider the Steinhaus graph $S$ of order $x+1$ with $\mathrm{Adj}(0) = \{x\}$. Now $S$ is a star and so is bipartite. If $A$ is the adjacency matrix of $S$, then the Steinhaus graph $H$ corresponding to $M_k(A)$ is bipartite by Theorem 5, and $H$ is of order $w + 2^k \geq n$ with $\mathrm{Adj}(0) = \{w\}$. Since $G$ is the subgraph of $H$ induced by $\{0, 1, \ldots, n-1\}$, $G$ is also bipartite. ∎

**Theorem 7.** *If $G$ is a Steinhaus graph of order $n$ with $v = \min \mathrm{Adj}(0)$, then the following statements are equivalent*:

(1) *$G$ is bipartite*,

(2) *$G$ has no triangles*,

(3) $\mathrm{Adj}^+(v) = \emptyset$ *or* $\mathrm{Adj}^+(v) = \{v + w\}$ *with* $f(w) \geq \max\{n - v - w, v\}$.

*Proof.* (1) $\Longrightarrow$ (2) is clear.

(2) $\Longrightarrow$ (3). Let $A = [a_{i,j}]_{n \times n}$ be the adjacency matrix of $G$. Suppose $|\mathrm{Adj}^+(v)| \geq 2$. Choose the smallest vertex $x$ and the second smallest vertex $y$ of $\mathrm{Adj}^+(v)$. By the Steinhaus property, $a_{v-1,v} = 1$. For all $v < z < x$, since $a_{v,z} = 0$, we have $a_{v-1,z} = 1$. Since $a_{v-1,x-1} = a_{v,x} = 1$, we have $a_{v-1,x} = 0$. For all $x < z < y$, since $a_{v,z} = 0$, we have $a_{v-1,z} = 0$. Since $a_{v-1,y-1} = 0$ and $a_{v,y} = 1$, $a_{v-1,y} = 1$. Thus $\{v-1, v, y\}$ induces a triangle in $G$, which is impossible.

Assume $\mathrm{Adj}^+(v) = \{v + w\}$ for some positive integer $w$. Since $G$ has no triangles, the subgraph $H$ of $G$ induced by $\{v, v+1, \ldots, n-1\}$ has no triangles. Note that $H$ is isomorphic to the Steinhaus graph of order $n - v$ with $\mathrm{Adj}(0) = \{w\}$. By Lemma 6, $f(w) \geq n - v - w$.

Suppose $f(w) < v$. Let $w = 2^k x$, where $x$ is an odd integer greater than 2. Then, $2^k < v$ and so $u \equiv \lceil \frac{v}{2^k} \rceil \geq 2$. Consider the Steinhaus graph $H$ of order $\lceil \frac{n}{2^k} \rceil$ with $u = \min \mathrm{Adj}(0)$ and $\mathrm{Adj}^+(u) = \{u + x\}$. Let $A = [a_{i,j}]$ be the adjacency matrix of $H$. Since $\mathrm{Adj}^+(u) = \{u + x\}$, $a_{u,j} = 0$ for $u < j < u + x$

and $a_{u,u+x} = 1$. These together with $a_{u-1,u} = 1$ imply that $a_{u-1,j} = 1$ for $u < j < u + x$ and $a_{u-1,u+x} = 0$. These new values together with $a_{u-2,u} = 1$ imply $a_{u-2,j} \equiv (j - u - 1) \pmod 2$ for $u < j < u + x$ and $a_{u-2,u+x} = 1$. Let $G''$ be the Steinhaus graph whose adjacency matrix is $M_k(A) = [a''_{i,j}]$. By Corollary 2, $\min \mathrm{Adj}(0) = 2^k u \geq v$ and $\mathrm{Adj}^+(2^k u) = \{2^k(u + x)\} = \{2^k u + w\}$ in $G''$. Then, the subgraph of $G''$ induced by $\{2^k u - v, 2^k u - v + 1, \ldots, 2^k u - v + n - 1\}$ is precisely the Steinhaus graph of $n$ vertices in which $\min \mathrm{Adj}(0) = v$ and $\mathrm{Adj}^+(v) = \{v + w\}$, which is just $G$. Note that $a_{u-2,u} = a_{u-2,u+x} = a_{u,u+x} = 1$. By Corollary 2, $a''_{2^k u - 2^k - 1, 2^k u} = a''_{2^k u - 2^k - 1, 2^k u + w} = a''_{2^k u, 2^k u + w} = 1$; i.e., $\{2^k u - 2^k - 1, 2^k u, 2^k u + w\}$ induces a triangle in $G''$. But, $2^k u - v \leq 2^k u - 2^k - 1 < 2^k u < 2^k u + w \leq 2^k u - v + n - 1$. So, this triangle is also a triangle in $G$, a contradiction. Thus, $f(w) \geq v$.

(3) $\Longrightarrow$ (1). For the case of $\mathrm{Adj}^+(v) = \emptyset$, $V(G)$ can be partitioned into $X = \{0, 1, \ldots, v - 1\}$ and $Y = \{v, v + 1, \ldots, n - 1\}$ such that every edge of $G$ has one vertex in $X$ and the other vertex in $Y$. So, we may assume that $\mathrm{Adj}^+(v) = \{v + w\}$ with $f(w) \geq \max\{n - v - w, v\}$. Let $w = 2^k x$, where $x$ is a positive odd integer. Let $H$ be the Steinhaus graph of order $\lceil \frac{n-v}{2^k} \rceil + 1$ with $1 = \min \mathrm{Adj}(0)$ and $\mathrm{Adj}^+(1) = \{1 + x\}$. $H - 0$ is precisely the Steinhaus graph of order $\lceil \frac{n-v}{2^k} \rceil$ with $\mathrm{Adj}(0) = \{x\}$. Also,

$$f(x) = f\left(\frac{w}{2^k}\right) = \frac{f(w)}{2^k} \geq \frac{n - v - w}{2^k} = \frac{n - v}{2^k} - x$$

implies $f(x) \geq \lceil \frac{n-v}{2^k} \rceil - x$. By Lemma 6, $H - 0$ is bipartite. Note that in $H$, $\mathrm{Adj}(0) = \{1, 2, \ldots, x\}$ and $x + 1$ is adjacent to $1, 2, \ldots, x$. Then, $H$ is also bipartite. Let $A$ be the adjacency matrix of $H$, and $G''$ the Steinhaus graph whose adjacency matrix is $M_k(A)$. By Corollary 2, in $G''$ we have $2^k = \min \mathrm{Adj}(0)$ and $\mathrm{Adj}^+(2^k) = \{2^k + 2^k x\} = \{2^k + w\}$. Then the subgraph of $G''$ induced by $\{2^k - v, 2^k - v + 1, \ldots, 2^k - v + n - 1\}$ is precisely the Steinhaus graph of $n$ vertices in which $\min \mathrm{Adj}(0) = v$ and $\mathrm{Adj}^+(v) = \{v + w\}$, which is $G$. By Theorem 5, $G''$ is bipartite and so is $G$. ∎

We close this paper by noting that the equivalence of (1) and (2) in Theorem 7 was also proved in [9]; and (3) is also proved in [6] in an alternative way and is used to obtain a formula for the number of bipartite Steinhaus graphs of order $n$ in terms of $n - 2$ (also see [14]).

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## References


1. N. Brand, Almost all Steinhaus graphs have diameter 2, *J. Graph Theory* **16** (1992), 213-219.
2. N. Brand, S. Curran, S. Das and T. Jacob, Probability of diameter two for Steinhaus graphs, *Discrete Appl. Math.* **41** (1993), 165-171.
3. R. C. Brigham, J. R. Carrington and R. D. Dutton, Embedding in Steinhaus graphs, *J. Combin. Inform. System Sci.* **17** (1992), 257-270.
4. R. C. Brigham and R. D. Dutton, Distances and diameters in Steinhaus graphs, *Congr. Numer.* **76** (1990), 7-14.
5. G. J. Chang, Binary triangles, *Bull. Inst. Math. Acad. Sinica* **11** (1983), 209-225.
6. G. J. Chang, B. DasGupta, W. M. Dymàček, M. Fürer, M. Koerlin, Y.-S. Lee and T. Whaley, Characterizations of bipartite Steinhaus graphs, submitted.
7. W. M. Dymàček, Steinhaus graphs, *Congr. Numer.* **23-24** (1979), 399-412.
8. W. M. Dymàček, Complements of Steinhaus graphs, *Discrete Math.* **37** (1981), 167-180.
9. W. M. Dymàček, Bipartite Steinhaus graphs, *Discrete Math.* **59** (1986), 9-20.
10. W. M. Dymàček, M. Koerlin and T. Whaley, A survey of Steinhaus graphs, to appear in *the Proceedings of the Eighth International Conference on Graph Theory, Combinatorics, Algorithms and Applications* (Kalamazoo, Michigan, 1996).
11. W. M. Dymàček and T. Whaley, Generating strings for bipartite Steinhaus graphs, *Discrete Math.* **141** (1995), 95-107.
12. H. Harborth, Solution of Steinhaus's problem with plus and minus signs, *J. Combin. Theory, Ser. A* **12** (1972), 253-259.
13. H. Harborth, Aufgabe 785, *Elem. Math.* **33** (1978), 49-50; solution by O. P. Lossers.
14. Y.-S. Lee, *Counting Bipartite Steinhaus Graphs*, Master Thesis, Department of Applied Math., National Chiao Tung University, Hsinchu, Taiwan, 1994.
15. J. C. Molluzzo, Steinhaus graphs, in: *Theory and Applications of Graphs*, Y. Alavi and D. R. Lick, eds. (Kalamazoo, Michigan, 1976), Lecture Notes in Math., Vol. 642, Springer, Berlin, 1978, pp.394-402.
16. H. Steinhaus, *One Hundred Problems in Elementary Mathematics*, Dover, New York, 1979. This is a republication of the English translation first published in 1964 by Basic Books, Inc., 10 E. 53rd St., New York, NY 10022.
17. E. T. H. Wang, Problem E 2541, *Amer. Math. Monthly* **82** (1975), 659-660; solution by M. Joseph in same journal **83** (1976), 660-661.

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