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LOCAL Δ_2^E CONDITION IN GENERALIZED CALDERÓN-LOZANOVSKIĬ SPACES

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Abstract. First we present the local Δ_2^E condition in generalized Calderón-Lozanovskiĭ spaces E_φ and we discuss the relationships between the local and the global Δ_2^E condition in such spaces. We also give a full characterisation for a point of $B(E_\varphi)$ to have an order continuous norm. Then we apply the main result to particular spaces, i.e. Calderón-Lozanovskiĭ spaces and Orlicz-Lorentz spaces.

1. INTRODUCTION

The geometry of Banach spaces is a part of functional analysis which has been intensively developed recently [5, 9, 15]. The order continuity is one of the most important tools in this subject area. It is natural to study this property in a local point of view. This brings us to the notion of point of order continuity. On the other hand, Calderón-Lozanovskiĭ spaces, E_φ, are one of important classes of Banach lattices, especially due to their role in the interpolation theory [1, 6, 10, 16, 17]. The full criterion for a point of order continuity in Calderón-Lozanovskiĭ spaces has been established in [12]. In the paper we shall generalise this result to the case of generalised Calderón-Lozanovskiĭ spaces. It is worth mentioning that it requires to apply new ideas and new methods in proofs in comparison to the non-parameter case discussed in [12]. The main reason is that constants a_φ and b_φ from the Calderón-Lozanovskiĭ spaces become measurable functions $a_\varphi(t)$, $b_\varphi(t)$ in generalised Calderón-Lozanovskiĭ spaces.

It is worth mentioning that the local $\Delta_2^E(x)$ and the global Δ_2^E conditions are applicable in relatively close areas. Namely the local $\Delta_2^E(x)$ condition is necessary and sufficient in studying LLUM points in E_φ (lower locally uniformly monotone points), and the global Δ_2^E condition appears in criteria for ULUM points in E_φ (upper locally uniformly monotone points), see [12, 13].

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2. Preliminaries

Throughout the paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ denote the set of real, positive real and positive integer numbers, respectively. Let (T, Σ, μ) be a σ-finite and complete measure space. Denote by Ω the nonatomic part of T and by N the purely atomic part of T. Hence the measure space (T, Σ, μ) can be written as the direct sum $(\Omega, \Sigma \cap \Omega, \mu|_\Omega) \oplus (N, \Sigma \cap N, \mu|_N)$.

By $L^0 = L^0(T)$ we mean the set of all μ-equivalence classes of real valued measurable functions defined on T. A Banach space, $E = (E, \|\cdot\|_E)$ is a *Köthe space* if E is a linear subspace of L^0 and:

(i) if $x \in E$, $y \in L^0$ and $|y| \leqslant |x|$ μ-a.e., then $y \in E$ and $\|y\|_E \leqslant \|x\|_E$,

(ii) there exists a function $x \in E$ that is positive on the whole T.

Every Köthe space is a Banach lattice under the natural partial order ($x \geqslant 0$ if $x(t) \geqslant 0$ μ-a.e. in T). Considering a Köthe space over the nonatomic (purely atomic) part we say that E is a *Köthe function space (resp. Köthe sequence space)*.

A point $x \in E$ is said to have an *order continuous norm* if for any sequence $(x_n) \subset E$ that $0 \leqslant x_n \leqslant |x|$ and $x_n \to 0$ μ-a.e. we have $\|x_n\| \to 0$. It is clear that in case of a Köthe sequence space the condition $x_n \to 0$ μ-a.e. is equivalent to the condition $x_n \to 0$ pointwisely. We say a Köthe space E is *order continuous* ($E \in (OC)$ for short) if every element of E has an order continuous norm. E_a stands for the subspace of order continuous elements of E. It is well known that $x \in E_a$ if and only if $\|x\chi_{A_n}\|_E \to 0$ for any sequence (A_n) such that $A_n \to \emptyset$, i.e. $A_{n+1} \subset A_n$ for every n and $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$.

Let φ denote an *Orlicz function*, i.e. a $\varphi\colon \mathbb{R} \to [0, \infty]$, which is convex, even, vanishing and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero.

In the whole paper, if not stated otherwise, by φ we mean a *Musielak-Orlicz function*, a generalisation of Orlicz function, i.e. a function $\varphi\colon T \times \mathbb{R} \to [0, \infty]$ such that $\varphi(t, \cdot)$ is an Orlicz function for μ-a.e. $t \in T$ and $\varphi(\cdot, u)$ is a Σ-measurable function for every $u \in \mathbb{R}$. We denote $\varphi(t, x(t))$ by $(\varphi \circ x)(t)$.

By a *generalised Calderón-Lozanovskiĭ space* we mean

$$E_\varphi = \left\{x \in L^0\colon \varphi \circ (lx) \in E \text{ for some } l > 0\right\}$$

equipped with so called *Luxemburg-Nakano norm* defined by

$$\|x\|_\varphi = \inf\left\{\lambda > 0\colon I_\varphi\left(\frac{x}{\lambda}\right) \leqslant 1\right\},$$

where I_φ is a convex semimodular defined on L^0 by

$$I_\varphi(x) = \begin{cases} \|\varphi \circ x\|_E & \text{if } \varphi \circ x \in E, \\ \infty & \text{otherwise.} \end{cases}$$

Let

$$a_\varphi(t) = \sup\{u > 0\colon \varphi(t,u) = 0\}\,,\ b_\varphi(t) = \sup\{u > 0\colon \varphi(t,u) < \infty\}\,.$$

In case of generalised Calderón-Lozanovskiĭ spaces we assume without loss of generality that the purely atomic part of T is the counting measure space $(\mathbb{N}\,, 2^{\mathbb{N}}\,, m)$ (see [8] for argumentation).

We shall denote the function φ restricted to the purely atomic part $(\varphi|_{\mathbb{N}})$ by $\varphi_a = (\varphi_i)_{i\in\mathbb{N}}$. Analogously, φ restricted to the nonatomic part $(\varphi|_\Omega)$ we denote by φ_c. Hence $\varphi = \varphi_c + \varphi_a$ for any measure space.

We say a Musielak-Orlicz function φ_c satisfies a *global condition* Δ_2^E ($\varphi_c \in \Delta_2^E$ for short) if there exist a constant $K > 0$ and a nonnegative, Σ-measurable function f with $\varphi_c \circ (2f) \in E$ such that

$$\varphi_c(t, 2u) \leqslant K\varphi_c(t,u)$$

for μ-a.e. $t \in \Omega$ and $u \geqslant f(t)$. We say that $\varphi_c \in \Delta_2^E(\varepsilon)$ for some $\varepsilon > 0$ if $\varphi_c \in \Delta_2^E$ with $K = K_\varepsilon$, $f = f_\varepsilon$ and $\|\varphi_c \circ (2f_\varepsilon)\|_E < \varepsilon$. See [3] for equivalent formulations of Δ_2^E condition.

We say $\varphi_c \in \Delta_l^E$ for $l > 1$ if there exist a constant $K_l > 0$ and a nonnegative, Σ-measurable function f_l with $\varphi_c \circ (lf_l) \in E$ such that $\varphi_c(t, lu) \leqslant K_l\varphi_c(t,u)$ for μ-a.e. $t \in \Omega$ and $u \geqslant f_l(t)$. We define a $\Delta_l^E(\varepsilon)$ condition in analogous way as $\Delta_2^E(\varepsilon)$ condition.

We say that φ_c satisfies the Δ_2^E condition on a set $A \subset \Omega$ with $\mu(A) > 0$ if there is $K > 0$ and a nonnegative, Σ-measurable function f with $supp(f) \subset A$, $\varphi_c \circ (2f) \in E$ such that $\varphi_c(t, 2u) \leqslant K\varphi_c(t,u)$ for μ-a.e. $t \in A$ and every $u \geqslant f(t)$. We shall write shortly $\varphi_c \in \Delta_2^E\big|_A$.

Considering a Köthe sequence space E we say φ_a satisfies a *global* δ_2^E *condition* ($\varphi_a \in \delta_2^E$) if there exist constants $\alpha, K > 0$, sequences $b = (b_i)_{i=1}^\infty \geqslant 0$ with $\varphi_a \circ (2b) \in E_+$ and $(d_i)_{i=1}^\infty$ with $\|\varphi_i(d_i)e_i\|_E = \alpha$, such that

$$\varphi_i(2u) \leqslant K\varphi_i(u) \tag{1}$$

for every $i \in \mathbb{N}$ and $u \in [b_i, d_i]$. Analogously as in case of function spaces we can define a condition $\delta_2^E(\varepsilon)$. For more details see [4].

We say φ_a satisfies δ_2^E on a countable infinite set $A \subset \mathbb{N}$ ($\varphi \in \delta_2^E\big|_A$ for short) if there are α, K, sequences $b = (b_i)_{i=1}^\infty \geqslant 0$ with $\varphi_a \circ (2b) \in E_+$ and $(d_i)_{i=1}^\infty$ with $\|\varphi_i(d_i)e_i\|_E = \alpha$, such that $\varphi_i(2u) \leqslant K\varphi_i(u)$ for every $i \in A$ and $u \in [b_i, d_i]$.

We say a Musielak-Orlicz function φ satisfies Δ_2^E condition ($\varphi \in \Delta_2^E$) provided that $\varphi_c \in \Delta_2^E$ and $\varphi_a \in \delta_2^E$.

3. Results

Definition 1. Let $x \in E_\varphi$. We say φ satisfies *a local* $\Delta_2^E(x)$ *condition* with respect to x ($\varphi \in \Delta_2^E(x)$ for short) if for each $l > 1$ there holds

$$\left\|\varphi \circ (lx)\chi_{A_k^l}\right\|_E \to 0 \text{ as } k \to \infty,$$

where

(2) $$A_k^l = \{t \in supp(x) \colon l|x(t)| < b_\varphi(t) \text{ and } \varphi(t, lx(t)) > k\varphi(t, x(t))\}.$$

Now we present an example showing that function φ may not fulfil the global Δ_2^E condition but it may satisfy the local $\Delta_2^E(x)$ condition for some $x \in E_\varphi$, which is the motivation key for Definition 1. The most convenient way to see the idea is to consider the function without parameter.

Example 2. Consider the Orlicz function $\varphi(u) = 2^u - 1$, $E = L^1(0,\infty)$ and

$$x(t) = \begin{cases} 0 & \text{for } t \in (0,1), \\ \log_2\left(\frac{1}{t^2}+1\right) & \text{for } t \geqslant 1. \end{cases}$$

Clearly $\Delta_2^E = \Delta_2(\mathbb{R}_+)$, so $\varphi \notin \Delta_2^E$. Let $l \in \mathbb{N}$, $l > 1$. Then

$$I_\varphi(lx) = \int_1^\infty \left(\frac{1}{t^2}+1\right)^l - 1 \, dt = \int_1^\infty \sum_{i=1}^{l} \binom{l}{i} \frac{1}{t^{2i}} \, dt < \infty.$$

Let $l > 1$ and notice that

$$A_k^l = \{t \in supp(x) \colon \varphi(t, lx(t)) > k\varphi(t, x(t))\}.$$

Then $\mu\left(\bigcap_{k>l} A_k^l\right) = 0$ and $A_{k'}^l \subset A_k^l$ for any $k' > k$. Since

$$\varphi \circ \left(lx\chi_{A_k^l}\right) \leqslant \varphi \circ (lx) \in L^1 = \left(L^1\right)_a,$$

then

$$\left\|\varphi \circ \left(lx\chi_{A_k^l}\right)\right\|_{L^1} \to 0 \quad \text{as} \quad k \to \infty,$$

so $\varphi \in \Delta_2^E(x)$.

Denote

(3) $$\begin{aligned} &\mathcal{A} = \{t \in T \colon a_\varphi(t) = 0\}, \quad \mathcal{A}_1 = \{t \in T \colon a_\varphi(t) = 0 \text{ and } b_\varphi(t) = \infty\}, \\ &\mathcal{B} = \{t \in T \colon a_\varphi(t) > 0\}, \quad \mathcal{B}_1 = \{t \in T \colon 0 < a_\varphi(t) < b_\varphi(t)\}, \end{aligned}$$

Remark 3. ([1, Prop. 5.1]). For any Musielak-Orlicz function φ both functions $a_\varphi(t)$ and $b_\varphi(t)$ are Σ-measurable.

We will see later that the global Δ_2^E condition implies the local $\Delta_2^E(x)$ condition for any $x \in E_\varphi$ (under some restrictions concerning the function φ, see Lemma 7).

Lemma 4. *If $\mu(\mathcal{B}_1) > 0$ then $\varphi \notin \Delta_2^E(x)$ for some $x \in E_\varphi$.*

Proof. Assume that $\mu(\mathcal{B}_1) > 0$. Let us first show that $\mu(A_{l_0}) > 0$ for some $l_0 > 1$, where $A_{l_0} = \{t \in \mathcal{B}_1 \colon l_0 a_\varphi(t) < b_\varphi(t)\}$. Assume for the contrary that for every $l > 1$ the measure of A_l is zero that is for every $l > 1$ there is a set $C(l)$ of measure zero such that $l a_\varphi(t) \geqslant b_\varphi(t)$ for every $t \in \mathcal{B}_1 \backslash C(l)$. Therefore, taking a sequence $(l_n)_{n=1}^{\infty}$ such that $l_n = 1 + \frac{1}{n}$ for every $n \in \mathbb{N}$ we get $\mu(C_n) = 0$ whence $\mu\left(\sum_{n=1}^{\infty} C_n\right) = 0$ and

$$\left(1 + \frac{1}{n}\right) a_\varphi(t) \geqslant b_\varphi(t) \quad \text{for every } t \in \mathcal{B}_1 \backslash C_n.$$

Denote $C = \bigcup_{n=1}^{\infty} C_n$. Then for every $n \in \mathbb{N}$ and $t \in \mathcal{B}_1 \backslash C$ we have

$$b_\varphi(t) > a_\varphi(t) \geqslant \frac{1}{1 + \frac{1}{n}} b_\varphi(t).$$

Letting $n \to \infty$ we conclude that $a_\varphi(t) = b_\varphi(t)$ for μ-a.e. $t \in \mathcal{B}_1$, a contradiction.

Take an element $x = a_\varphi \chi_{A_{l_0}}$. Notice, that for such x we have $supp(x) = A_{l_0} = A_k^{l_0}$ for every $k \in \mathbb{N}$. Indeed, $A_k^{l_0} \subset A_{l_0}$ is obvious and the inequality $\varphi \circ (l_0 a_\varphi)\chi_{A_{l_0}} > k\varphi \circ a_\varphi \chi_{A_{l_0}} = 0$ holds for every k whence $A_k^{l_0} \supset A_{l_0}$. Therefore for every $k \in \mathbb{N}$

$$\left\|\varphi \circ (l_0 x)\chi_{A_k^{l_0}}\right\|_E = \left\|\varphi \circ (l_0 a_\varphi)\chi_{A_{l_0}}\right\|_E > 0,$$

where A_k^l is from Definition 1.

The following two Lemmas apply some methods from Lemma 2 in [3] and Lemma 2.1 in [4].

Lemma 5. *Suppose $\mu(A_\Omega) > 0$ where $A_\Omega = \mathcal{A} \cap \Omega$ and $\mathcal{A}$ is defined in (3). If $\varphi_c \in \Delta_2^E\big|_{A_\Omega}$ with $\varphi_c \circ (2f) \in E_a$ then for every $l > 1$ and $\varepsilon > 0$, $\varphi_c \in \Delta_l^E(\varepsilon)\big|_{A_\Omega}$ with $supp(f_{\varepsilon,l}) \subset A_\Omega$.*

Proof. Let f, K be from the definition of $\Delta_2^E\big|_{A_\Omega}$ condition. Let $l > 1$ and $\varepsilon > 0$. Then there is $p \in \mathbb{N}$ that $l \leqslant 2^p$ and by $\varphi_c \in \Delta_2^E\big|_{A_\Omega}$ we get that

$$\varphi_c \circ (lf) \leqslant \varphi_c \circ (2^p f) \leqslant K^{p-1} \varphi_c \circ (2f) \in E_a,$$

whence

$$\varphi_c \circ (lf) \in E_a.$$

In addition

$$0 \leqslant \left\| \varphi_c \circ \left(\frac{1}{n} l f \right) \right\|_E \leqslant \left\| \frac{1}{n} \varphi_c \circ (l f) \right\|_E = \frac{1}{n} \left\| \varphi_c \circ (l f) \right\|_E \to 0 \text{ as } n \to \infty.$$

Hence there exists $n_0 \in \mathbb{N}$ for which

$$\left\| \varphi_c \circ \left(\frac{1}{n_0} l f \right) \right\|_E \leqslant \frac{\varepsilon}{2}. \tag{4}$$

Denote

$$A_1 = \{ t \in A_\Omega \colon f(t) > 0 \}, \quad A_2 = \{ t \in A_\Omega \colon f(t) = 0 \},$$

$$C_m = \left\{ t \in A_1 \colon \forall_{u \in \mathbb{R}} \ \frac{1}{n_0} f(t) \leqslant u \leqslant f(t) \Rightarrow \frac{\varphi_c(t, lu)}{\varphi_c(t, u)} \leqslant 2^m \right\} \text{ for } m \in \mathbb{N}.$$

Notice that $C_1 \subset C_2 \subset C_3 \subset \ldots$. As we will see $\mu \left(A_1 \backslash \bigcup_{m=1}^{\infty} C_m \right) = 0$. Assume for the contrary that

$$\mu \left(A_1 \backslash \bigcup_{m=1}^{\infty} C_m \right) = \mu \left(\bigcap_{m=1}^{\infty} A_1 \backslash C_m \right) > 0.$$

Denoting $C = \bigcap_{m=1}^{\infty} A_1 \backslash C_m$ we get that

$$\forall_{t \in C} \forall_{m \in \mathbb{N}} \exists_{u_m \in \mathbb{R}} \ \frac{1}{n_0} f(t) \leqslant u_m \leqslant f(t) \quad \text{and} \quad \frac{\varphi_c(t, l u_m)}{\varphi_c(t, u_m)} > 2^m.$$

For a fixed $t \in C$ we have

$$0 < \frac{1}{n_0} f(t) \leqslant u_m \leqslant f(t) < \infty \ \text{ for all } \ m \in \mathbb{N} \quad \text{and} \quad \frac{\varphi_c(t, l u_m)}{\varphi_c(t, u_m)} \to \infty \text{ as } m \to \infty.$$

On the other hand we claim, that $b_\varphi(t) = \infty$ μ-a.e. in A_Ω. Otherwise, there is a set $B \subset A_\Omega$ such that $\mu(B) > 0$ and $b_\varphi(t) < \infty$ for $t \in B$. In addition $f(t) < b_\varphi(t)$ for $t \in B$ since $\varphi \circ (2f) \in E$. Then we can find an element $f(t) < u < b_\varphi(t)$ such that $2u > b_\varphi(t)$, which contradicts the $\Delta_2^E\big|_{A_\Omega}$ condition and proves the claim.

Moreover $f(t) > 0$ for $t \in C$ and $a_\varphi(t) = 0$ for every $t \in A_\Omega$. Thus $\frac{\varphi_c(t,lu)}{\varphi_c(t,u)}$ is a continuous function considered over a compact set $\left[\frac{1}{n_0} f(t), f(t) \right]$, whence $\sup_{u \in \left[\frac{1}{n_0} f(t), f(t) \right]} \frac{\varphi_c(t,lu)}{\varphi_c(t,u)} < \infty$, a contradiction. Therefore $\mu \left(A_1 \backslash \bigcup_{m=1}^{\infty} C_m \right) = 0$.

Let

$$x_m = \varphi_c \circ (l f) \chi_{A_1 \backslash C_m}$$

for $m \in \mathbb{N}$. Since $\varphi_c \circ (lf) \in E_a$ we get that $\|x_m\|_E \to 0$ as $m \to \infty$. Thus there exists $m_0 \in \mathbb{N}$ such that

$$\|x_{m_0}\|_E \leqslant \frac{\varepsilon}{2}. \tag{5}$$

Define function

$$\begin{aligned} f_{\varepsilon,l}(t) &= \frac{1}{n_0} f(t)\chi_{C_{m_0}}(t) + f(t)\chi_{A_1\setminus C_{m_0}}(t) + f(t)\chi_{A_2}(t) \\ &= \frac{1}{n_0} f(t)\chi_{C_{m_0}}(t) + f(t)\chi_{A_\Omega\setminus C_{m_0}}(t). \end{aligned}$$

Then $supp(f_{\varepsilon,l}) = A_1 \subset A_\Omega$. Inequalities (4) and (5) imply

$$\|\varphi_c \circ (lf_\varepsilon)\|_E \leqslant \left\|\varphi_c \circ \left(\frac{1}{n_0} lf\right) \chi_{C_{m_0}}\right\|_E + \left\|\varphi_c \circ (lf)\chi_{A_1\setminus C_{m_0}}\right\|_E \leqslant \varepsilon.$$

For every $u \geqslant f_{\varepsilon,l}(t)$ we have

$$\begin{aligned} &\varphi_c(t, lu) \leqslant \max\{K^p, 2^{m_0}\}\ \varphi_c(t,u) && \text{if} \quad t \in C_{m_0} \\ &\varphi_c(t, lu) \leqslant K^p \varphi_c(t,u) && \text{if} \quad t \in A_\Omega \setminus C_{m_0} \end{aligned}$$

Therefore, taking $K_{\varepsilon,l} = \max\{K^p, 2^{m_0}\}$, for μ-a.e. $t \in A_\Omega$ and every $u \geqslant f_{\varepsilon,l}(t)$ we have

$$\varphi_c(t, lu) \leqslant K_{\varepsilon,l}\ \varphi_c(t,u).$$

Lemma 6. *Suppose* $E|_{\mathbb{N}} \hookrightarrow c_0\{\|e_n\|_E\}$. *Let* $A_{\mathbb{N}} = \mathcal{A}_1 \cap \mathbb{N}$ *be an infinite countable set where* $\mathcal{A}_1$ *is defined in (3). If* $\varphi_a \in \delta_2^E\big|_{A_{\mathbb{N}}}$ *with* $\varphi_a \circ (2b) \in E_a$ *then for every* $l > 1$ *and* $\varepsilon > 0$ *there exist constants* $\alpha_{\varepsilon,l}, K_{\varepsilon,l} > 0$ *and sequences* $b_{\varepsilon,l} = \left(b_i^{\varepsilon,l}\right)_{i=1}^{\infty}$ *with* $\|\varphi \circ (lb_{\varepsilon,l})\|_E \leqslant \varepsilon$ *and* $d_{\varepsilon,l} = (d_i)_{i=1}^\infty$ *with* $\varphi_i(d_i)\,\|e_i\|_E = \alpha_{\varepsilon,l}$ *such that*

$$\varphi_i(lu) \leqslant K_{\varepsilon,l}\varphi_i(u)$$

for every $i \in A_{\mathbb{N}}$ *and* $b_i^{\varepsilon,l} \leqslant u \leqslant \frac{d_i}{l}$.

Proof. Let $\alpha, K, b = (b_i)_{i=1}^\infty, (d_i)_{i=1}^\infty$ be from the definition of $\delta_2^E\big|_{A_{\mathbb{N}}}$ condition. Take $l > 1$ and $\varepsilon > 0$. We claim that for every $p \in \mathbb{N}$ there is $i_p \in A_{\mathbb{N}}$ that

$$\varphi_a \circ \left(2^p b \chi_{\{i \in A_{\mathbb{N}}\,:\, i \geqslant i_p\}}\right) \in E_a.$$

Clearly for $p = 1$ it is obvious. If is enough to show the implication

$$\exists_{i_p \in A_{\mathbb{N}}} \ \varphi_a \circ \left(2^p b \chi_{\{i \in A_{\mathbb{N}} \, : \, i \geqslant i_p\}}\right) \in E_a$$
$$\Downarrow$$
$$\exists_{i_0 \in A_{\mathbb{N}}, i_0 \geqslant i_p} \ \ \varphi_a \circ \left(2^{p+1} b \chi_{\{i \in A_{\mathbb{N}} \, : \, i \geqslant i_0\}}\right) \in E_a.$$

By the assumption $\varphi_a \circ \left(2^p b \chi_{\{i \in A_{\mathbb{N}} \, : \, i \geqslant i_p\}}\right) \in E_a \subset E \hookrightarrow c_0\{\|e_n\|_E\}$ there is $i_0 \geqslant i_p$ that for every $i \geqslant i_0$

$$\varphi_i \left(2^p b_i\right) \|e_i\| \leqslant \alpha = \varphi_i(d_i) \|e_i\| \, .$$

Thus $b_i \leqslant 2^p b_i \leqslant d_i$ for $i \geqslant i_0$ and

$$\varphi_a \circ \left(2^{p+1} b \chi_{\{i \in A_{\mathbb{N}} \, : \, i \geqslant i_0\}}\right) \leqslant K \varphi_a \circ \left(2^p b \chi_{\{i \in A_{\mathbb{N}} \, : \, i \geqslant i_0\}}\right) \in E_a$$

which proves the claim. Therefore for any $l > 1$ there are p such that $2^p \leqslant l \leqslant 2^{p+1}$ and $i_1 \in A_{\mathbb{N}}$ such that

$$\varphi_a \circ \left(l b \chi_{\{i \in A_{\mathbb{N}} \, : \, i \geqslant i_1\}}\right) \in E_a \hookrightarrow \{c_0 \|e_i\|_E\}.$$

Since $\varphi_i(d_i) \|e_i\|_E = \alpha$ we find $i_2 \geqslant i_1$ such that

$$l b_i < d_i \quad \text{for} \quad i \geqslant i_2.$$

In addition there exists $k \in A_{\mathbb{N}}$, $k > i_2$, that

(6) $$\left\| \left(\varphi_i(l b_i)\right)_{i=k+1}^{\infty} \right\|_E \leqslant \frac{\varepsilon}{2}.$$

For every $i = 1, 2, \ldots, k$ there is $b_i' > 0$ such that $b_i' < \frac{d_i}{l}$ and

$$\left\| \varphi_i(l b_i') e_i \right\|_E \leqslant \frac{\varepsilon}{2k}.$$

Denote $b' = (b_1', b_2', \ldots, b_k', 0, 0, \ldots)$. We have

(7) $$\left\| \varphi \circ (l b') \right\|_E \leqslant \sum_{i=1}^{k} \left\| \left(\varphi_i(l b_i') e_i\right) \right\|_E \leqslant \frac{\varepsilon}{2}.$$

Let

$$A_1 = \{1, 2, \ldots, k\}, \quad A_2 = A_{\mathbb{N}} \setminus A_1.$$

and define

$$K' = \max_{i \in A_1} \sup_{b_i' \leqslant u \leqslant \frac{d_i}{l}} \frac{\varphi_i(l u)}{\varphi_i(u)}.$$

Notice that for every $i \in A_1$ the function $\frac{\varphi_i(lu)}{\varphi_i(u)}$ is continuous over the compact set $\left[b_i', \frac{d_i}{l}\right]$, since by the assumption $a_\varphi(i) = 0$ and $b_\varphi(i) = \infty$ for each $i \in A_{\mathbb{N}} = \mathcal{A}_1 \cap \mathbb{N}$. Thus $\sup\limits_{b_i' \leqslant u \leqslant \frac{d_i}{l}} \frac{\varphi_i(lu)}{\varphi_i(u)}$ is finite for every $i \in A_1$. But A_1 is a finite set, so $K' < \infty$.

Define $b_{\varepsilon,l} = \left(b_i^{\varepsilon,l}\right)_{i=1}^{\infty}$ as follows

$$b_i^{\varepsilon,l} = \begin{cases} b_i' & \text{for } i \in A_1, \\ b_i & \text{for } i \in A_2. \end{cases}$$

By (6) and (7) we get that $\|\varphi_a \circ (lb_{\varepsilon,l})\|_E \leqslant \varepsilon$. Notice that if $i \in A_2$ and $b_i^{\varepsilon,l} \leqslant u \leqslant \frac{d_i}{l}$ then $\varphi_i(lu) \leqslant K^p \varphi_i(u)$ since $2^p \leqslant l \leqslant 2^{p+1}$.

Taking $\alpha_{\varepsilon,l} = \alpha, K_{\varepsilon,l} = \max\{K^p, K'\}$ and $d_{\varepsilon,l} = (d_i)_{i=1}^{\infty}$ we obtain

$$\varphi_i(lu) \leqslant K_{\varepsilon,l}\varphi_i(u)$$

for every $i \in A_{\mathbb{N}}$ and $u \in \left[b_i^{\varepsilon,l}, \frac{d_i}{l}\right]$ what finishes the proof.

Lemma 7. *(i) Let $A_\Omega = \mathcal{A} \cap \Omega$ be a set of positive measure. If $\varphi_c \in \Delta_2^E\big|_{A_\Omega}$ with $\varphi_c \circ (2f) \in E_a$ then $\varphi_c \in \Delta_2^E(x\chi_{A_\Omega})$ for every $x \in E_\varphi$.*

(ii) Let $E|_{\mathbb{N}} \hookrightarrow c_0\{\|e_n\|_E\}$ and $A_{\mathbb{N}} = \mathcal{A}_1 \cap \mathbb{N}$ be an infinite countable set. If $\varphi_a \in \Delta_2^E\big|_{A_{\mathbb{N}}}$ with $\varphi_a \circ (2b) \in E_a$ then $\varphi_a \in \Delta_2^E(x\chi_{A_{\mathbb{N}}})$ for every $x \in E_\varphi$.

Proof. Let $x \in E_\varphi$.

(i) Take $l > 1$ and $\varepsilon > 0$. Applying Lemma 5 with $K_{\varepsilon,l}$ and $f_{\varepsilon,l}$ we get

$$\{t \in supp(x) \cap A_\Omega \colon\ |x(t)| \geqslant f_{\varepsilon,l}(t)\} \subset A_\Omega \backslash A_k^l,$$

for every $k \geqslant K_{\varepsilon,l}$ and consequently

$$A_\Omega \cap A_k^l \subset \{t \in supp(x) \cap A_\Omega \colon\ |x(t)| < f_{\varepsilon,l}(t)\}.$$

Therefore

$$\left\|\varphi_c \circ (lx)\chi_{A_\Omega}\chi_{A_k^l}\right\|_E \leqslant \left\|\varphi_c \circ (lf_{\varepsilon,l})\chi_{A_\Omega \cap A_k^l}\right\|_E \leqslant \|\varphi_c \circ (lf_{\varepsilon,l})\|_E \leqslant \varepsilon. \tag{8}$$

(ii) Take $l > 1$ and $\varepsilon > 0$. Apply Lemma 6 with $K_{\varepsilon,l}$, $\alpha_{\varepsilon,l}$, $b_{\varepsilon,l} = (b_i)_{i=1}^{\infty}$, $d_{\varepsilon,l} = (d_i)_{i=1}^{\infty}$ and notice that for every $x \in E_\varphi$ and $l > 0$ there is i_0 such that

$$\varphi_a \circ (lx)\chi_{\{i \in A_{\mathbb{N}} \,:\, i \geqslant i_0\}} \in E. \tag{9}$$

Indeed, first take an element $x \in B(E_\varphi)$. Then $\varphi_a \circ x \in E \hookrightarrow c_0\{\|e_i\|_E\}$, whence

$$\varphi_i(x(i))\,\|e_i\|_E \to 0 \quad \text{as} \quad i \to \infty.$$

Thus there is $i_0 \in A_{\mathbb{N}}$ that for every $i \geqslant i_0$ we have

$$\varphi_i(x(i)) \|e_i\| \leqslant \alpha_{\varepsilon,l} = \varphi_i(d_i) \|e_i\| ,$$

whence

$$x(i) \leqslant d_i \quad \text{for every} \quad i \geqslant i_0.$$

Denote

$$N_1 = \{i \geqslant i_0 \colon x(i) \in [b_i, d_i]\} \quad \text{and} \quad N_2 = \{i \geqslant i_0 \colon x(i) < b_i\} .$$

By the $\delta_l^E(\varepsilon)$ condition we obtain that we have

$$\varphi_a \circ (lx)\chi_{N_1} \leqslant K_{\varepsilon,l}\ \varphi_a \circ x\chi_{N_1}.$$

In addition

$$\varphi_a \circ (lx)\chi_{N_2} \leqslant K_{\varepsilon,l}\ \varphi_a \circ b\chi_{N_2}.$$

Therefore $\varphi \circ (lx\chi_{\{i\in A_{\mathbb{N}}\, :\, i \geqslant i_0\}}) \in E$ if $x \in B(E_\varphi)$.

Consider $x \in E_\varphi$ with $\|x\|_\varphi > 1$, set $u = \frac{x}{\|x\|_\varphi}$. Take $l > 0$. Then $lx = l\,\|x\|_\varphi\, u$ and denoting $l_0 = l\,\|x\|_\varphi$ we get by previous reasoning that there is i_0 that $\varphi_a \circ (l_0 u)\chi_{\{i\in A_{\mathbb{N}}\, :\, i \geqslant i_0\}} \in E$. Hence

$$\varphi_a \circ (lx)\chi_{\{i\in A_{\mathbb{N}}\, :\, i \geqslant i_0\}} = \varphi_a \circ (l_0 u)\chi_{\{i\in A_{\mathbb{N}}\, :\, i \geqslant i_0\}} \in E$$

what finishes the proof of statement (9).

Since $E \hookrightarrow c_0\{\|e_i\|_E\}$ there is $i_1 > i_0$ that

$$\varphi_i(lx(i)) \|e_i\| \leqslant \alpha_{\varepsilon,l} = \varphi_i(d_i) \|e_i\| \quad \text{for every} \quad i \geqslant i_1,$$

whence

$$lx(i) \leqslant d_i \quad \text{for} \quad i \geqslant i_1.$$

Therefore for every $x \in E$ there is an index i_0 that the above inequality holds for all $i \geqslant i_0$.

Let us divide the set $A_k^l \cap A_{\mathbb{N}}$ into three subsets:

$$B_1 = A_k^l \cap A_{\mathbb{N}} \cap \{i \in \mathbb{N} : i \leqslant i_0\},$$
$$B_2 = A_k^l \cap A_{\mathbb{N}} \cap \{i \in \mathbb{N} : i \geqslant i_0 \text{ and } x(i) \geqslant b_i\},$$
$$B_3 = A_k^l \cap A_{\mathbb{N}} \cap \{i \in \mathbb{N} : i \geqslant i_0 \text{ and } x(i) < b_i\}.$$

Since B_1 is a finite set and $a_\varphi(i) = 0$ and $b_\varphi(i) = \infty$ for $i \in A_{\mathbb{N}}$ then

$$K_0 = \max_{i\in B_1} \frac{\varphi_i(lx(i))}{\varphi_i(x(i))} < \infty,$$

whence for every $k \geqslant K_0$ we get that $B_1 = \emptyset$.

Notice, that for every $i \in B_2$ we get by Lemma 6 that

$$\varphi_i(lx(i)) < K_{\varepsilon,l}\ \varphi_i(x(i)),$$

whence for every $k > K_{\varepsilon,l}$ we get that $B_2 = \emptyset$.

In addition Lemma 6 implies that

$$\|\varphi_a \circ (lx)\chi_{B_3}\| < \|\varphi_a \circ (lb)\chi_{B_3}\| \leqslant \varepsilon$$

for every $k \in \mathbb{N}$. Summarising, for every $k > \max\{K_0, K_{\varepsilon,l}\}$ we obtain

$$\left\|\varphi_a \circ (lx)\chi_{A_k^l \cap A_{\mathbb{N}}}\right\| = \|\varphi_a \circ (lx)\chi_{B_3}\| < \varepsilon. \tag{10}$$

Lemma 8. *(i) Assume $supp\,(E|_\Omega)_a = \Omega$ and $\mu(\mathcal{A}_1 \cap \Omega) > 0$. Then $\varphi_c \in \Delta_2^E\big|_{\mathcal{A}_1\cap\Omega}$ if and only if $\varphi \in \Delta_2^E\,(x\chi_{\mathcal{A}_1\cap\Omega})$ for every $x \in E_\varphi$.*

(ii) Suppose $E|_{\mathbb{N}} \hookrightarrow c_0\{\|e_n\|_E\}$ and $m(\mathcal{A}_1 \cap \mathbb{N}) = \infty$. Then $\varphi_a \in \delta_2^E\big|_{\mathcal{A}_1\cap\mathbb{N}}$ if and only if $\varphi \in \delta_2^E\,(x\chi_{\mathcal{A}_1\cap\mathbb{N}})$ for every $x \in E_\varphi$.

Proof. *Necessity* of (*i*) and (*ii*) follows from Lemma 7.

(*i*) Sufficiency. Conversely suppose that $\varphi_c \notin \Delta_2^E\big|_{\mathcal{A}_1\cap\Omega}$. We apply the element $x = \sum\limits_{n=1}^{\infty} g_n\chi_{B_n}$ constructed in Lemma 4 in [3]. We will show $\varphi_c \notin \Delta_2^E\,(x\chi_{\mathcal{A}_1\cap\Omega})$.

Take $\lambda > 1$. There exists $m_0 \in \mathbb{N}$ such that $\lambda \geqslant 1 + \frac{1}{m_0}$. Notice that for every $t \in B_m$, where $m \geqslant m_0$ we have

$$\varphi_c(t, \lambda x(t)) \geqslant \varphi_c\left(t, \left(1 + \frac{1}{m}\right)x(t)\right) \geqslant 2^{m+1}\varphi_c(t, x(t)).$$

Since the latest inequality holds for every $t \in B_m$ we conclude that $B_m \subset A_{2^{m+1}}^{\lambda}$ for every $m \geqslant m_0$. Therefore

$$\left\|\varphi_c \circ (\lambda x)\chi_{A_{2^{m+1}}^{\lambda}}\right\|_E \geqslant \|\varphi_c \circ (\lambda x)\chi_{B_m}\|_E \geqslant 1 \quad \text{for every} \quad m \geqslant m_0.$$

(ii) Sufficiency. The proof goes analogously as in (*i*). Suppose $\varphi_a \notin \delta_2^E\big|_{\mathcal{A}_1\cap\mathbb{N}}$. Construct an element $x = (x(n))_{n=1}^{\infty}$ such that $x = \sum_{m=1}^{\infty}\sum_{n\in N_m} u_n^m e_n$, analogously like in Lemma 2.4 in [4]. Let $\lambda > 1$ and take m_0 such that $\lambda \geqslant 1 + \frac{1}{m_0}$. For every $m \geqslant m_0$ and $n \in N_m$ we have

$$\varphi_n(\lambda x(n)) \geqslant \varphi_n\left(\left(1 + \frac{1}{m}\right)x(n)\right) = \varphi_n\left(\left(1 + \frac{1}{m}\right)u_n^m\right) \geqslant 2^{m+2}\varphi_n\left(u_n^m\right),$$

hence $N_m \subset A^{\lambda}_{2^{m+2}}$ for every $m \geqslant m_0$. Finally

$$\left\|\Big(\varphi_n(\lambda x(n))\Big)_{n\in A^{\lambda}_{2^{m+2}}}\right\|_e \geqslant \left\|\Big(\varphi_n(\lambda x(n))\Big)_{n\in N_m}\right\|_e \geqslant 1 \quad \text{for every} \quad m \geqslant m_0$$

what finishes the proof.

Recall that in [12] the local $\Delta_2^E(x)$ condition has been formulated in a little different way, namely the set A_k^l has been defined as follows

(11) $$A_k^l = \left\{t \in supp(x) \colon l^2|x(t)| < b_\varphi \text{ and } \varphi(lx(t)) > k\varphi(x(t))\right\}.$$

Denote by $\widetilde{\Delta_2^E(x)}$ the definition with the above formula of A_k^l. It is easy to see that in the context of the criterion for a point of order continuity of E_φ (see Theorem 11) both of these formulations are equivalent. The following example shows the differences between the two formulations as well as the fact that it should be assumed in Lemma 7 in [12] that $b_\varphi = \infty$ instead of $\varphi(b_\varphi) = \infty$.

Example 9. Let E be a Köthe space such that $L^\infty \hookrightarrow E$ or ($L^\infty \not\hookrightarrow E$ and $E \not\hookrightarrow L^\infty$). Let φ be an Orlicz function defined by $\varphi(u) = \frac{1}{1-u} - 1$. For such function $b_\varphi = 1$. Obviously $\varphi \notin \Delta_2^E$ since $\Delta_2^E = \Delta_2(\infty)$ or $\Delta_2^E = \Delta_2(\mathbb{R}_+)$. We are about to show that $\varphi \in \widetilde{\Delta_2^E(x)}$ for every $x \in E_\varphi$.

Take $x \in E_\varphi$ and $l > 1$. We apply the formula (11) from A_k^l in [12]. If $t \in A_k^l$ then $|x(t)| < \frac{1}{l^2}$ and

$$\frac{1}{1 - lx(t)} - 1 > k\left(\frac{1}{1-x(t)} - 1\right)$$

whence

$$\frac{lx(t)}{1 - lx(t)} > k\frac{x(t)}{1-x(t)}.$$

Consequently

(12) $$\frac{l}{k} \cdot \frac{1-x(t)}{1-lx(t)} > 1.$$

Since $lx(t) < \frac{1}{l}$ we get that

$$\frac{1-x(t)}{1-lx(t)} < \frac{1}{1-\frac{1}{l}} = \frac{l}{l-1}.$$

Hence instead of (12) we can write

(13) $$\frac{l}{k} \cdot \frac{l}{l-1} > 1$$

But for $k > \frac{l^2}{l-1}$ we have

$$\frac{l}{k} \cdot \frac{l}{l-1} < \frac{l}{\frac{l^2}{l-1}} \cdot \frac{l}{l-1} = 1,$$

a contradiction. Therefore, $\mu(A_k^l) = 0$ for sufficiently large k. Thus $\varphi \in \widetilde{\Delta_2^E(x)}$.

On the other hand consider such function φ and a particular Köthe function space $E = L^1[0,1]$. We will show that using a formula of the set A_k^l from Definition 1 there exists $x \in E_\varphi$ for which $\varphi \notin \Delta_2^E(x)$.

Take a sequence $u_n \to \frac{1}{2}$, $0 < u_n < \frac{1}{2}$ such that $\varphi(2u_n) > 2^n$ for every $n \in \mathbb{N}$. Let $(A_n) \subset [0,1]$, $A_n \cap A_m = \emptyset$ for every $n \neq m$ and $\mu(A_n) = \frac{1}{2^n}$. Take

$$x_0 = \sum_{n=1}^{\infty} u_n \chi_{A_n}.$$

Then

$$I_\varphi(x_0) = \sum_n \varphi(u_n)\mu(A_n) \leqslant \varphi\left(\frac{1}{2}\right) \sum_n \frac{1}{2^n} = \varphi\left(\frac{1}{2}\right),$$

whence $x_0 \in E_\varphi$. We shall show $\left\|\varphi \circ (2x_0)\chi_{A_k^2}\right\|_{L^1} \not\to 0$ as $k \to \infty$.

Fix $k \in \mathbb{N}$ and take n_0 such that $2^{n_0} > k\varphi\left(\frac{1}{2}\right)$. Then $2^{n_0} > k\varphi(u_n)$ for every $n \in \mathbb{N}$ and

$$\bigcup_{n=n_0}^{\infty} A_n \subset A_k^2 = \{t \in supp(x_0)\colon 2|x_0(t)| < 1 \text{ and } \varphi(2x_0(t)) > k\varphi(x_0(t)\}.$$

Thus

$$\left\|\varphi \circ (2x_0)\chi_{A_k^2}\right\|_{L^1} = I_\varphi\left((2x_0)\chi_{A_k^2}\right) \geqslant \sum_{n=n_0}^{\infty} \varphi(2u_n)\mu(A_n) > \sum_{n=n_0}^{\infty} 1 = \infty.$$

Note also that for such function φ we have $\varphi \in \Delta_2^E(0)$, so in the function case it suggests that it may happen $\varphi \in \Delta_2^E$ and $\varphi \notin \Delta_2^E(x)$ for some x. But $\Delta_2^E = \Delta_2^E(0)$ concerns the case $E \hookrightarrow L^\infty$ and consequently $E_a = \{0\}$. Hence this is a trivial case in the context of a point of order continuity in E_φ. On the other hand $L^1[0,1] \not\hookrightarrow L^\infty$.

Lemma 10. *Suppose* $x \in E_\varphi$ *and* $\varphi \in \Delta_2^E(x)$. *If* $\varphi \circ x \in E_a$, *then* $\varphi \circ (lx)\chi_{B_l} \in E_a$ *for every* $l > 1$, *where* $B_l = \{t \in supp(x)\colon l\,|x(t)| < b_\varphi(t)\}$.

The above Lemma is a generalisation of Lemma 9 in [12] and the proof goes the same way.

Theorem 11. *Let E be a Köthe space and $x \in B(E_\varphi)$. Then $x \in (E_\varphi)_a$ if and only if:*

(i) $\varphi \circ x \in E_a$;

(ii) $\varphi \in \Delta_2^E(x\chi_C)$, *where* $C = \{t \in supp(x) \colon a_\varphi(t) < |x(t)|\}$;

(iii) *For each* $m \in \mathbb{N}$ *set*

$$C_m = \left\{ t \in supp(x) \colon \frac{1}{m} a_\varphi(t) \leqslant |x(t)| \leqslant a_\varphi(t) \right\};$$

$$C_m^\Omega = C_m \cap \Omega, \quad C_m^{\mathbb{N}} = C_m \cap \mathbb{N}.$$

Then for every $m \in \mathbb{N}$

$$\varphi \circ (ma_\varphi)\chi_{C_m^\Omega} \in E_a$$

and for every $m \in \mathbb{N}$, *if* $card\left(C_m^{\mathbb{N}}\right) = \aleph_0$ *then there is* $i_0 = i_0(m) \in \mathbb{N}$ *that*

$$ma_\varphi \chi_{C_m^{\mathbb{N}} \cap \{i \geqslant i_0\}} < b_\varphi \chi_{C_m^{\mathbb{N}} \cap \{i \geqslant i_0\}} \quad \textit{and} \quad \varphi \circ (ma_\varphi)\chi_{C_m^{\mathbb{N}} \cap \{i \geqslant i_0\}} \in E_a;$$

(iv) $\mu(supp(x) \cap D) = 0$, *where* $D = \{t \in \Omega \colon b_\varphi(t) < \infty\}$;

(v) $\limsup\limits_{i \in \mathbb{N}} \frac{|x(i)|}{b_\varphi(i)} = 0$.

Proof. Necessity. (i) The proof goes analogous to the proof of Lemma 7 in [7]. (ii) The part of the proof that considers $\varphi \in \Delta_2^E(x\chi_C)$ goes analogically as in the proof of Theorem 11 in [12].
(iv) Assume $\mu\{supp(x) \cap D\} > 0$. Let $\Omega_m = \left\{t \in supp(x) \cap D \colon |x(t)| \geqslant \frac{1}{m}\right\}$. There is $m_0 \in \mathbb{N}$ such that $\mu(\Omega_{m_0}) > 0$. Consider sequence $(\Omega_k) \subset \Omega_{m_0}$ such that $\Omega_k = \{t \in \Omega_{m_0} \colon b_\varphi(t) < k\}$. There is k_0 for which $\mu(\Omega_{k_0}) > 0$. Indeed, in opposite case for every $k \in \mathbb{N}$, $\mu(\Omega_k) = 0$, whence $\mu(\Omega_{m_0} \backslash \Omega_k) = \mu(\Omega_{m_0}) > 0$. Thus for every $t \in \Omega_{m_0} \backslash \Omega_k$ and every $k \in \mathbb{N}$ we get $b_\varphi(t) \geqslant k$, a contradiction.

Take $(C_n)_{n=1}^\infty \subset \Omega_{k_0}$ with $0 < \mu(C_n) \to 0$ as $n \to \infty$. Define $x_n = \frac{1}{m_0}\chi_{C_n}$. Then $0 \leqslant x_n \leqslant |x|$ and $x_n \to 0$ μ-a.e. in Ω_{k_0}. Take $\varepsilon > 0$ and λ such that $\frac{1}{\lambda} = (1+\varepsilon)k_0 m_0$. Hence

$$I_\varphi\left(\frac{1}{\lambda}x_n\right) = I_\varphi\left((1+\varepsilon)k_0 m_0 \frac{1}{m_0}\chi_{C_n}\right) > I_\varphi\left((1+\varepsilon)b_\varphi \chi_{C_n}\right) = \infty$$

and $\|x_n\|_\varphi \geqslant \lambda$. Thus $x \notin (E_\varphi)_a$.
(v) Let us denote

$$N_1 = \{i \in \mathbb{N} : b_\varphi(i) < \infty\}. \tag{14}$$

Notice, that it is enough to prove that $\limsup\limits_{i \in N_1} \frac{|x(i)|}{b_\varphi(i)} = 0$ when $card(N_1) = \aleph_0$. Suppose that $a = \limsup\limits_{i \in N_1} \frac{|x(i)|}{b_\varphi(i)} > 0$ and $card(N_1) = \aleph_0$. There is a sequence

$(i_k) \subset N_1$ with $\frac{|x(i_k)|}{b_\varphi(i_k)} \geqslant \frac{a}{2}$. Let $y_k = |x(i_k)|\, e_{i_k}$. Then $y_k \leqslant |x|$, $y_k \to 0$ pointwisely. On the other hand, taking $\lambda < \frac{a}{2}$ we obtain $\frac{y_k}{\lambda} = \frac{|x(i_k)|}{\lambda} \geqslant \frac{a}{2\lambda} b_\varphi(i_k)$ so $I_\varphi(\frac{y_k}{\lambda}) = \infty$. Therefore $\|y_n\|_\varphi \geqslant \lambda$, so $x \notin (E_\varphi)_a$.
(*iii*) (a) Notice that, by (*iv*) $b_\varphi(t) = \infty$ for μ-a.e. $t \in \Omega \cap supp(x)$, whence $\varphi \circ (ma_\varphi)\chi_{C_m^\Omega}$ has finite values. Suppose conversely there is $m \in \mathbb{N}$ that $\varphi \circ (ma_\varphi)\chi_{C_m^\Omega} \notin E_a$. Then there are $\delta > 0$ and a sequence $(D_n)_{n=1}^\infty \subset C_m^\Omega$ of pairwise disjoint sets with $\|\varphi \circ (ma_\varphi)\chi_{D_n}\| \geqslant \delta$ for each n. Setting $z_n = |x|\, \chi_{D_n}$ we get $0 \leqslant z_n \leqslant |x|$ and $z_n \to 0$ μ-a.e. Moreover, for $l > m^2$

$$I_\varphi(lz_n) \geqslant \left\| \varphi \circ \left(\frac{l}{m} a_\varphi \right) \chi_{D_n} \right\|_E \geqslant \|\varphi \circ (ma_\varphi)\chi_{D_n}\|_E \geqslant \delta,$$

whence $\|z_n\|_\varphi \not\to 0$. Thus $x \notin (E_\varphi)_a$.
(b) Notice first that if for some $m \in \mathbb{N}$ there holds

$$ma_\varphi \chi_D \geqslant b_\varphi \chi_D, \tag{15}$$

where $card(D) = \aleph_0$ and $D \subset C_m^{\mathbb{N}}$, then for every $i \in D$ we have

$$\frac{1}{m} a_\varphi(i) \leqslant |x(i)| \quad \text{and} \quad b_\varphi(i) < \infty$$

whence together with (*v*)

$$ma_\varphi(i) \leqslant m^2\, |x(i)| \quad \text{and} \quad \limsup_{i \in D} \frac{|x(i)|}{b_\varphi(i)} = 0.$$

Thus there is $i_0 = i_0(m)$ that

$$ma_\varphi(i) \leqslant m^2\, |x(i)| < b_\varphi(i)$$

for every $i \geqslant i_0,\ i \in D$, a contradiction with (15).

Assume conversely there is m that $card(C_m^{\mathbb{N}}) = \aleph_0$ and $\varphi \circ (ma_\varphi)\chi_{C_m^{\mathbb{N}} \cap \{i \geqslant i_0\}} \notin E_a$ where $i_0 = i_0(m)$ is such that $ma_\varphi(i) < b_\varphi(i)$ for every $i \geqslant i_0,\ i \in C_m^{\mathbb{N}}$. Then following the reasoning in (a) we conclude that $x \notin (E_\varphi)_a$.

Sufficiency. Let $0 \leqslant x_n \leqslant |x|$, $x_n \to 0$ μ-a.e.. Set $l > 1$. We shall show that $I_\varphi(lx_n) \to 0$ in two steps.

1. We first prove that $I_\varphi \left(lx_n \chi_{supp(x)\, \setminus C} \right) \to 0$. Take $m > l$. Then $\varphi \circ (lx_n \chi_{C_m}) \to 0$ μ-a.e. as $n \to \infty$ and

$$\varphi \circ \left(lx_n \chi_{C_m^\Omega} \right) \leqslant \varphi \circ \left(la_\varphi \chi_{C_m^\Omega} \right) \leqslant \varphi \circ \left(ma_\varphi \chi_{C_m^\Omega} \right) \in E_a.$$

Hence $I_\varphi \left(lx_n \chi_{C_m^\Omega} \right) \to 0$ as $n \to \infty$.

Notice that (*iii*) implies that if $card\left(C_m^{\mathbb{N}}\right) = \aleph_0$ then there is $i_0 = i_0(m)$ that

$$\varphi \circ \left(lx_n \chi_{C_m^{\mathbb{N}} \cap \{i \geqslant i_0\}}\right) \leqslant \varphi \circ \left(la_\varphi \chi_{C_m^{\mathbb{N}} \cap \{i \geqslant i_0\}}\right) \leqslant \varphi \circ \left(ma_\varphi \chi_{C_m^{\mathbb{N}} \cap \{i \geqslant i_0\}}\right) \in E_a.$$

Since the set $C_m^{\mathbb{N}} \cap \{i < i_0\}$ is finite then pointwise convergence of $\varphi \circ (lx_n)\chi_{C_m^{\mathbb{N}} \cap \{i < i_0\}}$ involves norm convergence. It proceeds also if the set $C_m^{\mathbb{N}}$ is finite. Therefore $I_\varphi\left(lx_n \chi_{C_m^{\mathbb{N}}}\right) \to 0$ as $n \to \infty$.

Additionally, denoting $C_m' = (supp(x) \backslash C) \backslash C_m$ we get

$$I_\varphi\left(lx_n \chi_{C_m'}\right) \leqslant I_\varphi\left(lx \chi_{C_m'}\right) < I_\varphi\left(a_\varphi \chi_{C_m'}\right) = 0.$$

2. We now prove $I_\varphi(lx_n \chi_C) \to 0$. Denote

$$D_1 = C \cap \Omega, \quad D_2 = C \cap \mathbb{N}, \quad D_{21} = D_2 \cap (\mathbb{N} \backslash N_1), \quad D_{22} = D_2 \cap N_1,$$

where N_1 is defined in (14). Notice that (*iv*) involves that $b_\varphi = \infty$ μ-a.e. in $supp(x) \cap \Omega$ whence

$$lx_n \chi_{D_1 \cup D_{21}} \leqslant lx \chi_{D_1 \cup D_{21}} < b_\varphi \chi_{D_1 \cup D_{21}}.$$

By (*ii*) and Lemma 10 we get $\varphi \circ (lx)\chi_{D_1 \cup D_{21}} \in E_a$ thus $I_\varphi\left(lx_n \chi_{D_1 \cup D_{21}}\right) \to 0$.

It is now enough to show $I_\varphi\left(lx_n \chi_{D_{22}}\right) \to 0$ as $n \to \infty$. It is obvious when N_1 is finite. Suppose $card(N_1) = \aleph_0$. (*v*) implies there is $i_0 \in N_1$ that for every $i \geqslant i_0$, $i \in N_1$ we have $\frac{l|x(i)|}{b_\varphi(i)} < 1$. Indeed, otherwise we find a sequence $(i_k) \subset N_1$ satisfying $\frac{l|x(i_k)|}{b_\varphi(i_k)} \geqslant 1$ for all k and in consequence

$$\limsup_{i \in N_1} \frac{|x(i)|}{b_\varphi(i)} \geqslant \lim_{k \to \infty} \frac{|x(i_k)|}{b_\varphi(i_k)} \geqslant \frac{1}{l} > 0,$$

which contradicts with (*v*). Denote

$$D_{22}^1 = \{i \in D_{22} \colon i \geqslant i_0\}, \quad D_{22}^2 = \{i \in D_{22} \colon i < i_0\}.$$

Clearly, $I_\varphi\left(lx_n \chi_{D_{22}^2}\right) \to 0$ since D_{22}^2 is finite and $\varphi \circ (lx_n)\chi_{D_{22}^2}$ converges pointwise to zero. Moreover, $\varphi \circ (lx_n)\chi_{D_{22}^1}$ also converges pointwise and by (*ii*) and Lemma 10 we get

$$\varphi \circ (lx_n)\chi_{D_{22}^1} \leqslant \varphi \circ (lx)\chi_{D_{22}^1} \in E_a.$$

Thus $I_\varphi\left(lx_n \chi_{D_{22}^1}\right) \to 0,\ n \to \infty$ what finishes the proof.

4. Applications

4.1. Calderón-Lozanovskiĭ spaces

Corollary 12. ([12, Theorem 11]). *Let E be a Köthe space, φ be an Orlicz function and $x \in B(E_\varphi)$. Then $x \in (E_\varphi)_a$ if and only if:*

(*i'*) $\varphi \circ x \in E_a$*;*

(*ii'*) $\varphi \in \Delta_2^E(x\chi_C)$*, where* $C = \{t \in supp(x)\colon a_\varphi < |x(t)|\}$*;*

(*iii'*) $\chi_{A_m} \in E_a$ *for every* $m \in \mathbb{N}$*, where*

$$A_m = \left\{t \in supp(x)\colon \frac{1}{m} \leqslant |x(t)| \leqslant a_\varphi\right\};$$

(*iv'*) $\|x\chi_\Omega\|_\varphi > 0$ *implies* $b_\varphi = \infty$*;*

(*v'*) *If* $\|x\chi_\Omega\|_\varphi = 0$ *and* $b_\varphi < \infty$ *then* $|x(i)| \to 0$ *as* $i \to \infty$*.*

Proof. Notice that it is enough to prove the equivalence of conditions of the Corollary with the conditions of Theorem 11. Conditions (*i'*) and (*ii'*) are the same as (*i*) and (*ii*) correspondingly.

Let us first show that conditions (*iii'*) to (*v'*) follow from (*iii*) to (*v*).
(*iv'*) Notice that if $\|x\chi_\Omega\|_\varphi > 0$ then $\mu(supp(x) \cap \Omega) > 0$ and by (*iv*) we have that $\mu(supp(x) \cap D) = 0$. Therefore $\mu(D) = 0$ whence $b_\varphi = \infty$.
(*v'*) Let $\|x\chi_\Omega\|_\varphi = 0$ and $b_\varphi < \infty$. By (*v*) we get $N_1 = \mathbb{N}$ and $\limsup_{i \in \mathbb{N}} \frac{|x(i)|}{b_\varphi} = 0$ implies $|x(i)| \to 0$ as $i \to \infty$.
(*iii'*) Observe that if $\|x\chi_\Omega\|_\varphi = 0$ and $b_\varphi < \infty$ then by (*v'*) the set A_m is finite for every $m \in \mathbb{N}$, whence $\chi_{A_m} \in E_a$. Thus together with (*iv'*) we may assume that $b_\varphi = \infty$. Let $m \in \mathbb{N}$ and take $n \in \mathbb{N}$ such that $n > ma_\varphi$, so $\frac{a_\varphi}{n} < \frac{1}{m}$. Hence

$$\frac{a_\varphi}{n} < \frac{1}{m} \leqslant |x(t)| \leqslant a_\varphi \quad \text{for every} \quad t \in A_m.$$

Therefore (*iii*) implies $A_m \subset C_n$ and

$$0 < \varphi \circ \frac{n}{m}\chi_{A_m} \leqslant \varphi \circ \frac{n}{m}\chi_{C_n} < \varphi \circ (na_\varphi)\chi_{C_n} \in E_a.$$

Finally $\chi_{A_m} \in E_a$.

Assume conditions (*iii'*) to (*v'*). We shall show they imply conditions (*iii*) to (*v*).
(*iii*) Observe that if $\|x\chi_\Omega\|_\varphi = 0$ and $b_\varphi < \infty$ then by (*v'*) the set A_m is finite for every $m \in \mathbb{N}$ whence together with (*iv'*) we can restrict to the case of $b_\varphi = \infty$. Let $n \in \mathbb{N}$ and take $m \in \mathbb{N}$ that $m > \max\left\{n, \frac{n}{a_\varphi}\right\}$. Then

$$\frac{1}{m} < \frac{a_\varphi}{n} \leqslant |x(i)| \leqslant a_\varphi$$

for every $t \in C_n = C_n^{\Omega} \cup C_n^{\mathbb{N}}$, whence (*iii'*) involve $C_n \subset A_m$ and

$$0 \leqslant \varphi \circ (na_\varphi)\chi_{C_n} < \varphi \circ (na_\varphi)\chi_{A_m} \in E_a$$

for every $n \in \mathbb{N}$. Therefore $\varphi \circ (na_\varphi)\chi_{C_n} \in E_a$.
(*iv*) Let $\|x\chi_\Omega\|_\varphi > 0$. By (*iv'*) we get $b_\varphi = \infty$, whence $\mu(D) = 0$. Finally $\mu(supp(x) \cap D) = 0$.
(*v*) If $card(N_1) = \aleph_0$ then $b_\varphi < \infty$, so by (*iv'*) we get $\|x\chi_\Omega\|_\varphi = 0$. Thus by (*v'*) $x \in c_0$, whence $\limsup\limits_{i \in N_1} \frac{|x(i)|}{b_\varphi(i)} = 0$.

4.2. Orlicz-Lorentz spaces

If E is a *Lorentz function (sequence) space* Λ_ω (λ_ω), then E_φ is the corresponding *Orlicz-Lorentz function (sequence) space* $(\Lambda_\omega)_\varphi$ ($(\lambda_\omega)_\varphi$) equipped with the Luxemburg-Nakano norm. We shall write shortly $\Lambda_{\varphi,\omega}$ ($\lambda_{\varphi,\omega}$). Recall that the function $\omega\colon [0,\gamma) \to \mathbb{R}_+$ with $\gamma = \mu(T)$ is called a *weight function* if it is nonnegative, nonincreasing and locally integrable function with the Lebesgue measure m not identically equal to zero. The space Λ_ω consists of all functions $x\colon [0,\gamma) \to \mathbb{R}$ measurable with respect to m for which $\|x\|_\omega = \int_0^\gamma x^*(t)\omega(t)dt < \infty$, where x^* is the *nonincreasing rearrangement of* x, i.e.

$$x^*(t) = \inf\{\tau\colon d_x(\tau) \leqslant t\}.$$

Recall that d_x is a *distribution function* of x, i.e.

$$d_x(\tau) = \mu(\{t \in [0,\gamma)\colon |x(t)| > \tau\}), \quad \tau \geqslant 0.$$

The Lorentz sequence space λ_ω consists of all sequences $x = (x(i))_{i=1}^\infty$ such that $\sum\limits_{i=1}^\infty x^*(i)\omega(i) < \infty$, where $\omega = (\omega(i))_{i=1}^\infty$ is a *weight sequence*, a nonincreasing sequence of nonnegative real numbers.

Remark 13. ([11]). The Lorentz function (sequence) space is order continuous if and only if $\int_0^\infty \omega = \infty$ $\left(\sum\limits_i \omega(i) = \infty\right)$.

Hence it is enough to discuss below only the case $\int_0^\infty \omega < \infty$ $\left(\sum\limits_i \omega(i) < \infty\right)$.

Lemma 14. *Let* Λ_ω *be a Lorentz function space with* $\int_0^\infty \omega(t)dt < \infty$ *and* $x \in \Lambda_\omega$. *Then* $x \in (\Lambda_\omega)_a$ *if and only if* $d_x(\tau) < \infty$ *for every* $\tau > 0$.

Proof. Necessity. Suppose for the contrary that there is $\tau > 0$ that $d_x(\tau) = \infty$. Denoting $C = \{t\colon |x(t)| > \tau\}$ we get $m(C) = \infty$. There is a sequence $(C_n)_{n=1}^{\infty}$ of subsets of C that $C = \sum_{n=1}^{\infty} C_n$, $C_n \cap C_m = \emptyset$, $n \neq m$ and $m(C_n) = \infty$ for every $n \in \mathbb{N}$. Take a sequence $x_n = x\chi_{C_n}$. Then $0 \leqslant x_n \leqslant x$ and $x_n \to 0$ μ-a.e. On the other hand

$$\|x_n\|_\omega = \|x\chi_{C_n}\|_\omega > \tau \|\chi_{C_n}\|_\omega = \tau \int_0^\infty (\chi_{C_n})^*(t)\omega(t)dt = \tau \int_0^\infty \omega(t)dt = \delta > 0.$$

Hence $x \notin (\Lambda_\omega)_a$, a contradiction.

Sufficiency. Let $d_x(\tau) < \infty$ for all $\tau \geqslant 0$. Take a sequence (x_n) such that $0 \leqslant x_n \leqslant |x|$ and $x_n \to 0$ μ-a.e. Properties of nonincreasing rearrangement (see [14]) imply that $x_n^*(\tau) \to 0$ for all τ and

$$x_n^*(\tau)\omega(\tau) \leqslant x^*(\tau)\omega(\tau) \in L^1 \in (OC).$$

Therefore $x_n^*(\tau)\omega(\tau) \to 0$ for every τ, whence

$$\|x_n\|_\omega = \int_0^\infty x_n^*(t)\omega(t)dt \to 0, \text{ as } n \to \infty$$

what finishes the proof.

Remark 15. Let E be a symmetric Köthe sequence space. If $x \in E_a$ then $x^*(i) \to 0$ as $i \to \infty$.

Proof. Assume for the contrary that $x^*(i) \not\to 0$, i.e. there is $\delta > 0$ and a subsequence $(i_k)_{k=1}^{\infty}$ that

$$|x(i_k)| \geqslant \delta \quad \text{for every} \quad k \in \mathbb{N}\,.$$

Define $x_n = |x|\,\chi_{\{i_n, i_{n+1}, \ldots\}}$ and notice that $0 \leqslant x_n \leqslant |x|$ for every $n \in \mathbb{N}$ and $x_n \to 0$ pointwisely. On the other hand observe that

$$x_n^*(i) \geqslant \delta \quad \text{for all} \quad n, i \in \mathbb{N}\,,$$

whence $\|x_n\|_E = \|x_n^*\|_E \not\to 0$, so $x \notin E_a$.

Lemma 16. *Let λ_ω be a Lorentz sequence space with $\sum_{i=1}^{\infty} \omega(i) < \infty$ and $x \in \lambda_\omega$. Then $x \in (\lambda_\omega)_a$ if and only if $x^*(i) \to 0$ as $i \to \infty$.*

Proof. *Necessity.* Follows from Remark 15.

Sufficiency. Take a sequence (x_n) such that $0 \leqslant x_n \leqslant |x|$ and $x_n \to 0$ pointwisely, i.e. for every $i \in \mathbb{N}$

$$x_n(i) \to 0 \quad \text{as} \quad n \to \infty. \tag{16}$$

Notice also, that $x^*(i) \to 0$ implies

$$x(i) \to 0 \quad \text{as} \quad i \to \infty. \tag{17}$$

Indeed, otherwise, if $x \notin c_0$ then there is $\delta > 0$ and infinite sequence (i_k) that $|x(i_k)| \geqslant \delta$ for every $k \in \mathbb{N}$. Thus $x^*(i) > \delta$ for every $i \in \mathbb{N}$ which contradicts with the assumption $x^* \in c_0$.

In addition $0 \leqslant x_n \leqslant x$ implies that

$$0 \leqslant x_n^* \leqslant x^* \quad \text{for every} \quad n \in \mathbb{N}. \tag{18}$$

Therefore $x^*(i) \to 0$ involves $x_n^*(i) \to 0$ as $i \to \infty$ for all n, whence for every $n \in \mathbb{N}$

$$x_n(i) \to 0 \quad \text{as} \quad i \to \infty. \tag{19}$$

We shall show that $x_n^*(i) \to 0$ as $n \to \infty$ for every $i \in \mathbb{N}$. Suppose conversely that there is i_0 that $x_n^*(i) \not\to 0$. Then there is $\delta > 0$ and a subsequence $\left(x_{n_k}^*(i_0)\right)_{k=1}^{\infty}$ that $x_{n_k}^*(i_0) \geqslant \delta$ for every $k \in \mathbb{N}$. By (19), for every $k \in \mathbb{N}$ we can find $i \in \mathbb{N}$ that $|x_{n_k}(i)| = x_{n_k}^*(i_0)$. Denote $I_k = \left\{i \in \mathbb{N} : |x_{n_k}(i)| = x_{n_k}^*(i_0)\right\}$ and $I = \bigcup_k I_k$. Consider two cases. If the set I is countably infinite then

$$0 < \delta \leqslant x_{n_k}(i) \leqslant |x(i)| \quad \text{for every} \quad i \in I,$$

which contradicts with (17). If I is a finite set then there is $j_0 \in I$ that

$$x_{n_k}(j_0) \geqslant \delta$$

for infinitely many k which contradicts with (16). Therefore $x_n^* \to 0$ poinwisely and so does $x_n^*\omega$ for all n.

Observe that by (18), for every $n \in \mathbb{N}$

$$x_n^*\omega \leqslant x^*\omega \in l^1 \in (OC).$$

Finally $\|x_n\|_\omega = \|x_n^*\|_\omega = \sum_{i=1}^{\infty} x_n^*(i)\omega(i) \to 0.$

Notice, that sufficiency of Lemma 16 does not hold for a symmetric Köthe sequence space in general. It is shown in the following example.

Example 17. Recall that given a weight sequence ω, the Marcinkiewicz sequence space m_ω is defined by

$$m_\omega = \left\{ x \in l^0 \colon \ \|x\| = \sup_i \omega(i) x^{**}(i) < \infty \right\},$$

where $x^{**}(i) = \frac{1}{i} \sum_{k=1}^{i} x^*(k)$.

Consider the space m_ω with $\omega(i) = \sqrt{i}$ for $i \in \mathbb{N}$. Then for every $i \in \mathbb{N}$ we have $\omega(i)x^{**}(i) = \frac{1}{\sqrt{i}} \sum_{k=1}^{i} x^*(k)$. Consider now a sequence $x \in m_\omega$ defined by $x(i) = \sqrt{i} - \sqrt{i-1}$ for $i \geqslant 1$. Then $x^* = x$ by concavity of function $f(u) = \sqrt{u}$. Clearly,

$$x^*(i) \to 0 \quad \text{as} \quad i \to \infty. \tag{20}$$

Take a sequence $x_n = |x| \chi_{\{i \geqslant n\}}$. Then $0 \leqslant x_n \leqslant |x|$, $x_n \to 0$ poinwisely and

$$x_n^{**}(i) = \frac{1}{i} \sum_{k=1}^{i} x_n^*(k) = \frac{1}{i} \sum_{k=n}^{i+n} x^*(k) = \frac{1}{i} \left(-\sqrt{n-1} + \sqrt{i+n} \right)$$

for $n \in \mathbb{N}$, whence

$$\|x_n\| = \sup_i \frac{1}{\sqrt{i}} \left(\sqrt{i+n} - \sqrt{n-1} \right) \geqslant \lim_{i \to \infty} \frac{\sqrt{i+n} - \sqrt{n-1}}{\sqrt{i}} = 1$$

for $n \in \mathbb{N}$. Therefore $x \notin (m_\omega)_a$ and $x^*(i) \to 0$ as $i \to \infty$.

Corollary 18. *Let E be a Lorentz function space* Λ_ω *and* $x \in B(\Lambda_{\varphi,\omega})$. *Then* $|x| \in (\Lambda_{\varphi,\omega})_a$ *if and only if:*

(i) *If* $\int_0^\infty \omega(t)dt < \infty$ *then* $d_{\varphi \circ x}(\tau) < \infty$ *for every* $\tau \geqslant 0$*;*

(ii) $\varphi \in \Delta_2^{\Lambda_\omega}(x\chi_C)$*, where* $C = \{t \in supp(x) \colon a_\varphi < |x(t)|\}$*;*

(iii) $m(A_k) < \infty$ *for every* $k \in \mathbb{N}$*, where* $A_k = \left\{ t \in supp(x) \colon \frac{1}{k} \leqslant |x(t)| \leqslant a_\varphi \right\}$*;*

(iv) $b_\varphi = \infty$.

Proof. *Necessity.* Condition (i) follows from Corollary 12, Remark 13 and Lemma 14. Conditions (ii) and (iv) follows from Corollary 12.

(iii) Let $k \in \mathbb{N}$. By Corollary 12 we have $\chi_{A_k} \in (\Lambda_\omega)_a$. In case of $\int_0^\infty \omega = \infty$ we get $m(A_k) < \infty$ since otherwise $(\chi_{A_k})^* = \chi_{[0,\infty]} \notin \Lambda_\omega$. If $\int_0^\infty \omega < \infty$ we also obtain $m(A_k) < \infty$ because in opposite case, taking $\tau = \frac{1}{2}$ we get $d_{\chi_{A_k}}(\frac{1}{2}) = \infty$, a contradiction with Lemma 14.

Sufficiency. We apply Corollary 12. It is enough to discuss (*i'*) and (*iii'*).

(*i'*) If $\int\limits_0^\infty \omega < \infty$ then we apply Lemma 14. In case of $\int\limits_0^\infty \omega = \infty$ we apply Remark 13.

(*iii'*) Notice that $m(A_k) < \infty$ implies that $d_{\chi_{A_k}}(\tau) < \infty$ for every $\tau > 0$. Then by Lemma 14 we obtain $\chi_{A_k} \in (\Lambda_\omega)_a$.

Corollary 19. *Let E be a Lorentz sequence space* λ_ω *and* $x \in B(\lambda_{\varphi,\omega})$. *Then* $|x| \in (\lambda_{\varphi,\omega})_a$ *if and only if:*

(i) *If* $\sum\limits_{i=0}^{\infty} \omega(i) < \infty$ *then* $(\varphi \circ x)^*(i) \to 0$ *as* $i \to \infty$*;*

(ii) $\varphi \in \Delta_2^{\lambda_\omega}(x\chi_C)$, *where* $C = \{i \in supp(x) \colon a_\varphi < |x(i)|\}$*;*

(iii) *If* $b_\varphi = \infty$ *then* $x\chi_{\{i\in\mathbb{N}\,:\,x(i)\leqslant a_\varphi\}} \in c_0$*;*

(iv) *If* $b_\varphi < \infty$ *then* $x \in c_0$.

Proof. *Necessity.* Condition (i) follows from Corollary 12, Remark 13 and Lemma 16. Conditions (ii) and (iv) follows from Corollary 12.
(iii) By Corollary 12 we have $\chi_{A_m} \in (\Lambda_\omega)_a$, hence together with Lemma 16 we get that $(\chi_{A_m})^* \in c_0$, whence $m(A_m) < \infty$ for each $m \in \mathbb{N}$. Thus $x\chi_{\{i\in\mathbb{N}\,:\,x(i)\leqslant a_\varphi\}} \in c_0$.

Sufficiency. We apply Corollary 12 in analogous way as in the proof of Corollary 18.

(*i'*) If $\sum\limits_{i=0}^{\infty} \omega(i) < \infty$ then Lemma 16 implies that $\varphi \circ x \in (\lambda_\omega)_a$. If $\sum\limits_{i=0}^{\infty} \omega(i) = \infty$ then we apply Remark 13.

(*iii'*) Notice that (iii) implies that for every $m \in \mathbb{N}$ there is $i_0 \in \mathbb{N}$ that $|x(i)| < \frac{1}{m}$ for all $i \geqslant i_0$, whence the set A_m is finite and $\chi_{A_m} \in E_a$.

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