TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 1, pp. 207-215, February 2012
This paper is available online at http://tjm.math.ntu.edu.tw

COPIES OF c_0 AND ℓ_∞ INTO A REGULAR OPERATOR SPACE

Yongjin Li, Donghai Ji and Qingying Bu

Abstract. For an Orlicz function φ and a Banach lattice X, let ℓ_φ denote the Orlicz sequence space associated to φ, $\mathcal{L}^r(\ell_\varphi, X)$ denote the space of regular operators from ℓ_φ to X, and $\mathcal{K}^r(\ell_\varphi, X)$ denote the linear span of positive compact operators from ℓ_φ to X. In this paper, we show that if φ and its complementary function φ^* satisfy the Δ_2-condition, then (a) $\mathcal{K}^r(\ell_\varphi, X)$ contains no copy of ℓ_∞ if and only if X contains no copy of ℓ_∞; and (b) $\mathcal{K}^r(\ell_\varphi, X)$ contains no copy of c_0 if and only if $\mathcal{L}^r(\ell_\varphi, X)$ contains no copy of ℓ_∞ if and only if X contains no copy of c_0 and each positive linear operator from ℓ_φ to X is compact.

1. Introduction

The copies of c_0 and ℓ_∞ into the space of bounded linear operators and the space of compact operators on Banach spaces are discussed in many papers, for instance, see papers [6, 7, 8, 9] and reference in these papers. It is also interesting to discuss the copies of c_0 and ℓ_∞ into the space of regular operators and the space of compact regular operators on Banach lattices. When Bu, Buskes, and Lai [1] discussed inheritance of geometric properties of Banach lattices by their positive tensor products, they introduced Banach lattice-valued Orlicz sequence spaces $\ell_\varphi^\varepsilon(X)$ and $\ell_\varphi^{\varepsilon,0}(X)$. Then they related $\ell_\varphi^\varepsilon(X)$ and $\ell_\varphi^{\varepsilon,0}(X)$ to the space of regular operators from an Orlicz sequence space ℓ_φ to a Banach lattice X. In this paper, we will use this relationship to discuss the copies of c_0 and ℓ_∞ into the space of regular operators and the space of compact regular operators from an Orlicz sequence space ℓ_φ to a Banach lattice X.

All vector spaces in this paper are over $\mathbb{R}$, the set of real numbers. For an ordered set X, the usual order on $X^{\mathbb{N}}$ is defined by $(x_i)_i \geq 0 \Longleftrightarrow x_i \geq 0$ for

Received January 14, 2010, accepted October 16, 2010.
Communicated by Bor-Luh Lin.
2010 *Mathematics Subject Classification*: 46B42, 46B20.
Key words and phrases: Orlicz sequence space, Regular operator space, Copies of c_0 and ℓ_∞.
The first author is supported by the NSF of China (10871213), the second author is the corresponding author, and the third author is supported by Shanghai Leading Academic Discipline Project (J50101).

each $i \in \mathbb{N}$. For a Banach lattice X, X^* denotes its topological dual space, B_X denotes its closed unit ball, and X^+ denotes its positive cone. For Banach lattices X and Y, $\mathcal{L}^r(X,Y)$ denotes the space of regular operators from X to Y, and $\mathcal{K}^r(X,Y)$ denotes the linear span of compact positive operators from X to Y. For each $T \in \mathcal{L}^r(X,Y)$, the r-norm of T is given by

$$\|T\|_r = \inf\left\{\|S\| : S \in \mathcal{L}(X,Y)^+, |T(x)| \leq S(x)\ \forall\, x \in X^+\right\}.$$

Then $(\mathcal{L}^r(X,Y), \|\cdot\|_r)$ is a Banach space. Moreover, if Y is Dedekind complete then $(\mathcal{L}^r(X,Y), \|\cdot\|_r)$ is a Banach lattice (see [11, §1.3]).

2. Orlicz Sequence Spaces

An function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is called an *Orlicz function* if (i) φ is even, continuous, and convex, (ii) $\varphi(0) = 0$ and $\varphi(u) > 0$ for all $u \neq 0$, and (iii) $\lim_{u\to 0} \varphi(u)/u = 0$ and $\lim_{u\to\infty} \varphi(u)/u = \infty$. Every Orlicz function φ has a right derivative p and

$$\varphi(u) = \int_0^{|u|} p(t)dt.$$

The right derivative p of φ is a right-continuous and non-decreasing function such that $p(0) = 0$, $p(t) > 0$ whenever $t > 0$, and $\lim_{t\to\infty} p(t) = \infty$. The right inverse q of p,

$$q(s) = \sup\{t : p(t) \leq s\}, \qquad s \geq 0,$$

is a right-continuous and non-decreasing function such that $q(0) = 0$, $q(s) > 0$ whenever $s > 0$, and $\lim_{s\to\infty} q(s) = \infty$. Define

$$\varphi^*(v) = \int_0^{|v|} q(s)ds.$$

Then φ^* is also an Orlicz function and q is its right derivative. φ^* is called the *complementary function* of φ. Obviously, φ is the complementary function of φ^*, i.e., $\varphi^{**} = \varphi$. An Orlicz function φ is said to satisfy the Δ_2*-condition* (*at zero*) if there exist $K > 2$ and $u_0 > 0$ such that $\varphi(2u) \leq K\varphi(u)$ whenever $|u| \leq u_0$.

An *Orlicz sequence space* ℓ_φ associated to an Orlicz function φ is a sequence space defined by

$$\ell_\varphi = \left\{a = (a_i)_i \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} \varphi(|\lambda a_i|) < \infty \text{ for some } \lambda > 0\right\}.$$

Let h_φ denote the order continuous part of ℓ_φ, i.e.,

$$h_\varphi = \left\{a = (a_i)_i \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} \varphi(|\lambda a_i|) < \infty \text{ for all } \lambda > 0\right\}.$$

Then $\ell_\varphi = h_\varphi$ if and only if φ satisfies the Δ_2-condition. The *Luxemburg norm* and the *Orlicz norm* on ℓ_φ are, respectively, defined to be

$$\|a\|_\varphi = \inf\left\{\lambda > 0 : \sum_{i=1}^{\infty} \varphi(|a_i/\lambda|) \le 1\right\}, \qquad a = (a_i)_i \in \ell_\varphi$$

and

$$\|a\|_{o\varphi} = \inf\left\{\frac{1}{\lambda}\left(1 + \sum_{i=1}^{\infty} \varphi(|\lambda a_i|)\right) : \lambda > 0\right\}, \qquad a = (a_i)_i \in \ell_\varphi.$$

Then the space ℓ_φ with both two norms are Banach spaces, denoted by ℓ_φ and $\ell_{o\varphi}$ respectively. Moreover,

$$\|a\|_\varphi \le \|a\|_{o\varphi} \le 2\|a\|_\varphi, \qquad a = (a_i)_i \in \ell_\varphi,$$

and

$$\langle a, b\rangle := \sum_{i=1}^{\infty} a_i b_i \le \|a\|_\varphi \cdot \|b\|_{o\varphi^*}, \qquad a = (a_i)_i \in \ell_\varphi,\ b = (b_i)_i \in \ell_{\varphi^*}.$$

It is known that h_φ is a closed subspace of ℓ_φ under both Luxemburg norm and Orlicz norm and the standard unit vectors $\{e_n\}_1^\infty$ form an unconditional basis of h_φ. Moreover, $(h_\varphi, \|\cdot\|_\varphi)^* = \ell_{o\varphi^*}$ and $(h_\varphi, \|\cdot\|_{o\varphi})^* = \ell_{\varphi^*}$ isometrically. About Orlicz functions φ and Orlicz sequence spaces ℓ_φ, we refer to [10, chapter 4] and [4, chapter 1].

3. Banach Lattice-valued Orlicz Sequence Spaces

For a Banach lattice X, let

$$\ell_\varphi^\varepsilon(X) = \left\{\bar{x} = (x_i)_i \in X^{\mathbb{N}} : \big(x^*(|x_i|)\big)_i \in \ell_\varphi,\ \forall\, x^* \in X^{*+}\right\}.$$

The Luxemburg norm and the Orlicz norm on $\ell_\varphi^\varepsilon(X)$ are, respectively, defined to be

$$\|\bar{x}\|_{\ell_\varphi^\varepsilon(X)} = \sup\left\{\left\|\big(x^*(|x_i|)\big)_i\right\|_\varphi : x^* \in B_{X^{*+}}\right\}, \qquad \bar{x} = (x_i)_i \in \ell_\varphi^\varepsilon(X)$$

and

$$\|\bar{x}\|_{\ell_{o\varphi}^\varepsilon(X)} = \sup\left\{\left\|\big(x^*(|x_i|)\big)_i\right\|_{o\varphi} : x^* \in B_{X^{*+}}\right\}, \qquad \bar{x} = (x_i)_i \in \ell_\varphi^\varepsilon(X).$$

Then $\ell_\varphi^\varepsilon(X)$ with both two norms are Banach lattices (see [1]), denoted by $\ell_\varphi^\varepsilon(X)$ and $\ell_{o\varphi}^\varepsilon(X)$ respectively. Let

$$\ell_\varphi^{\varepsilon,0}(X) = \left\{ (x_i)_i \in \ell_\varphi^\varepsilon(X) : \lim_n \|(0, \cdots, 0, x_n, x_{n+1}, \cdots)\|_{\ell_\varphi^\varepsilon(X)} = 0 \right\}.$$

Then $\ell_\varphi^{\varepsilon,0}(X)$ is a closed sublattice of $\ell_\varphi^\varepsilon(X)$. Let

$$K = \inf\{\lambda > 0 : \varphi(1/\lambda) \le 1\}.$$

Then it is easy to see that $\|e_n\|_\varphi = K$ for every $n \in \mathbb{N}$ and $\|(0, \cdots, 0, x, 0, 0, \cdots)\|_{\ell_\varphi^\varepsilon(X)} = K\|x\|$ for every $x \in X$. We need the following two propositions to obtain our main result in next section.

Proposition 1. ([1]). *If φ satisfies the Δ_2-condition, then $\ell_\varphi^\varepsilon(X)$ is isometrically isomorphic and lattice homomorphic to $\mathcal{L}^r((h_{\varphi^*}, \|\cdot\|_{o\varphi^*}), X)$ under the mapping: $\bar{x} \longrightarrow T_{\bar{x}}$, where $T_{\bar{x}}$ is defined by $T_{\bar{x}}(t) = \sum_{i=1}^\infty t_i x_i$ for each $t = (t_i)_i \in h_{\varphi^*}$ and each $\bar{x} = (x_i)_i \in \ell_\varphi^\varepsilon(X)$. Moreover, $T_{\bar{x}} \in \mathcal{K}^r(h_{\varphi^*}, X)$ if and only if $\bar{x} \in \ell_\varphi^{\varepsilon,0}(X)$.*

Proposition 2. ([2]).*Assume that φ^* satisfies the Δ_2-condition. Let $\bar{x}^{(n)} = (x_i^{(n)})_i, \bar{x}^{(0)} = (x_i^{(0)})_i \in \ell_\varphi^{\varepsilon,0}(X)$ for each $n \in \mathbb{N}$. Then $\lim_n \bar{x}^{(n)} = \bar{x}^{(0)}$ weakly in $\ell_\varphi^{\varepsilon,0}(X)$ if and only if $\lim_n x_i^{(n)} = x_i^{(0)}$ weakly in X for all $i \in \mathbb{N}$ and $\sup_n \|\bar{x}^{(n)}\|_{\ell_\varphi^\varepsilon(X)} < \infty$.*

4. Main Results

Recall that we say that a Banach space contains a copy of c_0 (or ℓ_∞) if it contains a subspace isomorphic to c_0 (or ℓ_∞). Note that if a Banach lattice X contains a subspace isomorphic to c_0, by [11, p. 104, Theorem 2.5.6], X is not a KB-space, and hence, by [11, p. 92, Theorem 2.4.12], X contains a sublattice isomorphic to c_0. By the proof of [11, p. 92, Theorem 2.4.12] and the proof of [11, p. 82, Lemma 2.3.10], this isomorphism is also a lattice homomorphism. We summarize this fact as follows.

Lemma 3. *A Banach lattice contains a subspace isomorphic to c_0 if and only if it contains a sublattice isomorphic and lattice homomorphic to c_0.*

To get the main result in this section, we need a characterization of non-containment of a copy of ℓ_∞ in Banach spaces which was due to Rosenthal [12] and was summarized by Cembranos and Mendoza in [3, p. 12, Theorem 1.3.1] as follows.

Lemma 4. *Let Z be a Banach space. Then the following statements are equivalent:*

(a) *Z contains a copy of ℓ_∞.*

(b) *There exists a bounded linear operator $T : \ell_\infty \longrightarrow Z$ such that $\lim_n T(e_n) \neq 0$ in Z.*

(c) *There exists a bounded linear operator* $T : \ell_\infty \longrightarrow Z$ *which is not weakly compact.*

For an infinite subset M of $\mathbb{N}$, let $\ell_\infty(M)$ denote the subspace of ℓ_∞ consisting of all $(\xi_n)_n \in \ell_\infty$ with $\xi_n = 0$ for $n \notin M$. It is known from [3, p. 13, Remark 1.3.2] that if an operator $T : \ell_\infty \longrightarrow Z$ is weakly compact, then for all $\xi = (\xi_n)_n \in \ell_\infty$, the series $\sum_n \xi_n T(e_n)$ converges in Z. But its limit $\sum_{n=1}^\infty \xi_n T(e_n)$ and $T(\xi)$ may not coincide. To get the main result in this section, we also need the following result due to Drewnowski [6] (also see [3, p. 14, Corollary 1.3.3]).

Lemma 5. ([6]). *Let* $T_i : \ell_\infty \longrightarrow Z$ *be weakly compact operators for each* $i \in \mathbb{N}$. *Then there exists an infinite subset* M *of* $\mathbb{N}$ *such tat* $T_i(\xi) = \sum_{n=1}^\infty \xi_n T_i(e_n)$ *for all* $\xi = (\xi_n)_n \in \ell_\infty(M)$ *and all* $i \in \mathbb{N}$.

Theorem 6. *If* φ^* *satisfies the* Δ_2*-condition, then* $\ell_\varphi^{\varepsilon,0}(X)$ *contains no copy of* ℓ_∞ *if and only if* X *contains no copy of* ℓ_∞.

Proof. Since X is a closed subspace of $\ell_\varphi^{\varepsilon,0}(X)$, $\ell_\varphi^{\varepsilon,0}(X)$ contains a copy of ℓ_∞ whenever X contains a copy of ℓ_∞. Now assume that X contains no copy of ℓ_∞. We want to show that $\ell_\varphi^{\varepsilon,0}(X)$ contains no copy of ℓ_∞. Suppose that $\ell_\varphi^{\varepsilon,0}(X)$ contains a copy of ℓ_∞, that is, there is an isomorphism $T : \ell_\infty \longrightarrow T(\ell_\infty) \hookrightarrow \ell_\varphi^{\varepsilon,0}(X)$. For each $i \in \mathbb{N}$, define a bounded linear operator $T_i : \ell_\infty \longrightarrow X$ by $T_i(\xi) = T(\xi)_i$ for each $\xi \in \ell_\infty$, where $T(\xi)_i$ denotes the i-th coordinate of $T(\xi)$. Since X contains no copy of ℓ_∞, by Lemma 4, each T_i is weakly compact and hence, by Lemma 5, there exists an infinite subset M of $\mathbb{N}$ such that for all $\xi = (\xi_n)_n \in \ell_\infty(M)$,

$$T(\xi)_i = T_i(\xi) = \sum_{n=1}^\infty \xi_n T_i(e_n) = \sum_{n=1}^\infty \xi_n T(e_n)_i, \qquad \forall i \in \mathbb{N}.$$

Thus the series $\sum_n \xi_n T(e_n)_i$ converges to $T(\xi)_i$ in X and hence, weakly in X for each $i \in \mathbb{N}$. Note that for each $m \in \mathbb{N}$,

$$\begin{aligned}\Big\| \sum_{n=1}^m \xi_n T(e_n) \Big\|_{\ell_\varphi^\varepsilon(X)} &= \Big\| T\Big((\xi_1, \cdots, \xi_m, 0, 0, \cdots)\Big) \Big\|_{\ell_\varphi^\varepsilon(X)} \\ &\le \|T\| \cdot \|(\xi_1, \cdots, \xi_m, 0, 0, \cdots)\|_{\ell_\infty} \\ &\le \|T\| \cdot \|\xi\|_{\ell_\infty}.\end{aligned}$$

By Proposition 2, the series $\sum_n \xi_n T(e_n)$ converges to $T(\xi)$ weakly in $\ell_\varphi^{\varepsilon,0}(X)$ for all $\xi \in \ell_\infty(M)$. It follows that the series $\sum_{n\in M} T(e_n)$ is weakly subseries convergent and hence subseries convergent in $\ell_\varphi^{\varepsilon,0}(X)$. Thus $T(e_n) \longrightarrow 0$ in $\ell_\varphi^{\varepsilon,0}(X)$ as $n \in M$ and $n \to \infty$. But for each $n \in \mathbb{N}$, $\|T(e_n)\|_{\ell_\varphi^\varepsilon(X)} \ge \|e_n\|_{\ell_\infty} / \|T^{-1}\| = 1/\|T^{-1}\|$. This contradiction shows that $\ell_\varphi^{\varepsilon,0}(X)$ contains no copy of ℓ_∞. ∎

Lemma 7. *If $\ell_\varphi^\varepsilon(X)$ contains no copy of ℓ_∞, then both X and $\ell_\varphi^{\varepsilon,0}(X)$ contain no copy of c_0.*

Proof. For each $\xi = (\xi_i)_i \in \ell_\infty$ and each $\eta = (\eta_i)_i \in \ell_1^+$,

$$\sum_{i=1}^{\infty} \Big\| \langle |\xi_i e_i|, \eta\rangle e_i \Big\|_{\ell_\varphi} = \sum_{i=1}^{\infty} \langle |\xi_i e_i|, \eta\rangle \|e_i\|_{\ell_\varphi} = K \cdot \sum_{i=1}^{\infty} |\xi_i|\eta_i < \infty.$$

Thus $(\langle |\xi_i e_i|, \eta\rangle)_i = \sum_{i=1}^\infty \langle |\xi_i e_i|, \eta\rangle e_i \in \ell_\varphi$ and hence, $(\xi_i e_i)_i \in \ell_\varphi^\varepsilon(c_0)$. Define $T : \ell_\infty \longrightarrow \ell_\varphi^\varepsilon(c_0)$ by $T(\xi) = (\xi_i e_i)_i$ for each $\xi = (\xi_i)_i \in \ell_\infty$. Then

$$\begin{aligned} \Big\|T(\xi)\Big\|_{\ell_\varphi^\varepsilon(c_0)} &= \sup\left\{ \Big\|(\langle |\xi_i e_i|, \eta\rangle)_i\Big\|_{\ell_\varphi} : \ \eta = (\eta_i)_i \in B_{\ell_1^+} \right\} \\ &= \sup\left\{ \Big\| \sum_{i=1}^{\infty} \langle |\xi_i e_i|, \eta\rangle e_i\Big\|_{\ell_\varphi} : \ \eta = (\eta_i)_i \in B_{\ell_1^+} \right\} \\ &\leq \sup\left\{ K \cdot \sum_{i=1}^{\infty} |\xi_i|\eta_i : \ \eta = (\eta_i)_i \in B_{\ell_1^+} \right\} \\ &\leq K \cdot \|\xi\|_{\ell_\infty} \end{aligned}$$

and hence, T is a bounded linear operator. Moreover,

$$\|T(e_n)\|_{\ell_\varphi^\varepsilon(c_0)} = \|(0, \cdots, 0, e_n, 0, 0, \cdots)\|_{\ell_\varphi^\varepsilon(c_0)} = K \cdot \|e_n\|_{c_0} = K.$$

It follows from Lemma 4 that $\ell_\varphi^\varepsilon(c_0)$ contains a copy of ℓ_∞.

If X contains a copy of c_0, then by Lemma 3, X contains a sublattice isomorphic and lattice homomorphic to c_0. Thus $\ell_\varphi^\varepsilon(X)$ contains a sublattice isomorphic and lattice homomorphic to $\ell_\varphi^\varepsilon(c_0)$ and hence, $\ell_\varphi^\varepsilon(X)$ contains a copy of ℓ_∞. This contradiction shows that X contains no copy of c_0.

Now suppose that $\ell_\varphi^{\varepsilon,0}(X)$ contains a copy of c_0. By Lemma 3, $\ell_\varphi^{\varepsilon,0}(X)$ contains a sublattice isomorphic and lattice homomorphic to c_0. That is, there is an isomorphism and lattice homomorphism $\psi : c_0 \longrightarrow \psi(c_0) \hookrightarrow \ell_\varphi^{\varepsilon,0}(X)$. Note that the series $\sum_n e_n$ is a weakly unconditionally Cauchy series in c_0. So the series $\sum_n \psi(e_n)$ is a weakly unconditionally Cauchy series in $\ell_\varphi^{\varepsilon,0}(X)$. Thus for each $i \in \mathbb{N}$, the series $\sum_n \psi(e_n)_i$ is a weakly unconditionally Cauchy series in X. It is known from the first part that X contains no copy of c_0. Therefore, the series $\sum_n \psi(e_n)_i$ is an unconditionally convergent series in X and hence, for every $\xi = (\xi_n)_n \in \ell_\infty$, the series $\sum_n \xi_n \psi(e_n)_i$ converges in X.

Take any $(t_i)_i \in h_{\varphi^*}^+$ and any $x^* \in X^{*+}$. Then $(t_i x^*)_i \in \ell_\varphi^{\varepsilon,0}(X)^*$. Note that each $\psi(e_n)$ is positive. We have

$$\begin{aligned}
\sum_{i=1}^{\infty} t_i \langle x^*, |\sum_{n=1}^{\infty} \xi_n \psi(e_n)_i|\rangle &\le \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} |\xi_n| \langle t_i x^*, \psi(e_n)_i\rangle \\
&= \sum_{n=1}^{\infty} |\xi_n| \langle (t_i x^*)_i, \psi(e_n)\rangle \\
&\le \|\xi\|_{\ell_\infty} \sum_{n=1}^{\infty} \langle (t_i x^*)_i, \psi(e_n)\rangle < \infty.
\end{aligned}$$

Thus $(\langle x^*, |\sum_{n=1}^{\infty} \xi_n \psi(e_n)_i|\rangle)_i \in (h_{\varphi^*})^* = \ell_\varphi$ and hence, $(\sum_{n=1}^{\infty} \xi_n \psi(e_n)_i)_i \in \ell_\varphi^\varepsilon(X)$. Define $T : \ell_\infty \longrightarrow \ell_\varphi^\varepsilon(X)$ by $T(\xi) = (\sum_{n=1}^{\infty} \xi_n \psi(e_n)_i)_i$. Then

$$\begin{aligned}
&\Big\| T(\xi) \Big\|_{\ell_\varphi^\varepsilon(X)} \\
&= \sup\left\{ \Big\| \Big(\langle x^*, |\sum_{n=1}^{\infty} \xi_n \psi(e_n)_i|\rangle \Big)_i \Big\|_{\ell_\varphi} : x^* \in B_{X^{*+}} \right\} \\
&= \sup\left\{ \sum_{i=1}^{\infty} t_i \langle x^*, |\sum_{n=1}^{\infty} \xi_n \psi(e_n)_i|\rangle : x^* \in B_{X^{*+}}, (t_i)_i \in B_{h^+_{o\varphi^*}} \right\} \\
&\le \sup\left\{ \sum_{n=1}^{\infty} |\xi_n| \langle (t_i x^*)_i, \psi(e_n)\rangle : x^* \in B_{X^{*+}}, (t_i)_i \in B_{h^+_{o\varphi^*}} \right\} \\
&= \sup\left\{ \sum_{n=1}^{m} |\xi_n| \langle (t_i x^*)_i, \psi(e_n)\rangle : x^* \in B_{X^{*+}}, (t_i)_i \in B_{h^+_{o\varphi^*}}, m \in \mathbb{N} \right\} \\
&= \sup\left\{ \langle (t_i x^*)_i, \psi(\theta)\rangle : x^* \in B_{X^{*+}}, (t_i)_i \in B_{h^+_{o\varphi^*}}, m \in \mathbb{N} \right\} \\
&\le \sup\left\{ \Big\| (t_i x^*)_i \Big\|_{\ell_\varphi^{\varepsilon,0}(X)^*} \cdot \Big\| \psi(\theta) \Big\|_{\ell_\varphi^{\varepsilon,0}(X)} : x^* \in B_{X^{*+}}, (t_i)_i \in B_{h^+_{o\varphi^*}}, m \in \mathbb{N} \right\} \\
&\le \sup\{\|\psi\| \cdot \|\theta\|_{c_0} : m \in \mathbb{N}\} \\
&= \|\psi\| \cdot \|\xi\|_{\ell_\infty}, \qquad \text{where } \theta = (|\xi_1|, \cdots, |\xi_m|, 0, 0, \cdots),
\end{aligned}$$

and hence, T is a bounded linear operator. Note that $\lim_n e_n \neq 0$ in c_0 and ψ is an isomorphism. So $\lim_n T(e_n) = \lim_n \psi(e_n) \neq 0$ in $\ell_\varphi^\varepsilon(X)$. It follows from Lemma 4 that $\ell_\varphi^\varepsilon(X)$ contains a copy of ℓ_∞. This contradiction shows that $\ell_\varphi^{\varepsilon,0}(X)$ contains no copy of c_0. ∎

Theorem 8. *If φ^* satisfies the Δ_2-condition, then the following statements are equivalent.*

(*i*) *$\ell_\varphi^\varepsilon(X)$ contains no copy of ℓ_∞.*

(ii) $\ell_\varphi^{\varepsilon,0}(X)$ *contains no copy of* c_0.

(iii) X *contains no copy of* c_0 *and* $\ell_\varphi^{\varepsilon}(X) = \ell_\varphi^{\varepsilon,0}(X)$.

Proof. (iii) $\Longrightarrow$ (i). It follows from Theorem 6.

(i) $\Longrightarrow$ (ii). It follows from Lemma 7.

(ii) $\Longrightarrow$ (iii). Since X is a closed subspace of $\ell_\varphi^{\varepsilon,0}(X)$, X contains no copy of c_0. Take any $\bar{x} = (x_i)_i \in \ell_\varphi^{\varepsilon}(X)$. For each $i \in \mathbb{N}$, let $\bar{x}(i) = (0, \cdots, 0, x_i, 0, 0, \cdots)$. Then for each $(t_i)_i \in c_0$, $t_i\bar{x}(i) \in \ell_\varphi^{\varepsilon,0}(X)$ and for each $n \in \mathbb{N}$,

$$\begin{aligned}\Big\|\sum_{i=n}^{\infty} t_i\bar{x}(i)\Big\|_{\ell_\varphi^{\varepsilon}(X)} &= \Big\|(0, \cdots, 0, t_n x_n, t_{n+1}x_{n+1}, \cdots)\Big\|_{\ell_\varphi^{\varepsilon}(X)} \\ &\le \sup_{i\ge n} |t_i| \cdot \big\|\bar{x}\big\|_{\ell_\varphi^{\varepsilon}(X)} \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Thus the series $\sum_i t_i\bar{x}(i)$ converges in $\ell_\varphi^{\varepsilon,0}(X)$ for each $(t_i)_i \in c_0$. It follows from [5, p.44, Theorem 6] that $\sum_i \bar{x}(i)$ is a weakly unconditionally Cauchy series in $\ell_\varphi^{\varepsilon,0}(X)$. Note that $\ell_\varphi^{\varepsilon,0}(X)$ contains no copy of c_0. By Bessaga-Pelczynski Theorem (see [5, p.45, Theorem 8], $\sum_i \bar{x}(i)$ is an unconditionally convergent series in $\ell_\varphi^{\varepsilon,0}(X)$ and hence $\bar{x} = \lim_n \sum_{i=1}^n \bar{x}(i) \in \ell_\varphi^{\varepsilon,0}(X)$. Thus (iii) follows. ■

By Proposition 1, we have our main result of this section as follows.

Theorem 9. *Let φ be an Orlicz function and φ^* be its complementary function such that both φ and φ^* satisfy the Δ_2-condition (in this case, ℓ_φ is reflexive). Then we have the following statements* (a) *and* (b)*.*

(a) $\mathcal{K}^r(\ell_\varphi, X)$ *contains no copy of* ℓ_∞ *if and only if* X *contains no copy of* ℓ_∞.

(b) *The following assertions are equivalent:*

(i) $\mathcal{L}^r(\ell_\varphi, X)$ *contains no copy of* ℓ_∞.

(ii) $\mathcal{K}^r(\ell_\varphi, X)$ *contains no copy of* c_0.

(iii) X *contains no copy of* c_0 *and each positive linear operator from* ℓ_φ *to* X *is compact.*

References

1. Q. Bu, G. Buskes and W. K. Lai, The Radon-Nikodym property for tensor products of Banach lattices II, *Positivity*, **12** (2008), 45-54.

2. Q. Bu, M. Craddock and D. Ji, Reflexivity and the Grothendieck property for positive tensor products of Banach lattices-II, *Quaest. Math.*, **32** (2009), 339-350.

3. P. Cembranos and J. Mendoza, Banach Spaces of Vector-Valued Functions, Springer-Verlag, 1997.

4. S. Chen, Geometry of Orlicz Spaces, *Dissertaions Math.*, **356**, Warszawa, 1996.

5. J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, 1984.

6. L. Drewnowski, Copies of ℓ_∞ in an operator space, *Math. Proc. Camb. Phil. Soc.*, **108** (1990), 523-526.

7. G. Emmanuele, A remark on the containment of c_0 in spaces of compact operators, *Math. Proc. Cambridge Philos. Soc.*, **111** (1992), 331-335.

8. I. Ghenciu and P. Lewis, The embeddability of c_0 in spaces of operators, *Bull. Pol. Acad. Sci. Math.*, **56** (2008), 239-256.

9. N. Kalton, Spaces of compact operators, *Math. Ann.*, **208** (1974), 267-278.

10. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Sequence Spaces, Springer-Verlag, 1977.

11. P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, 1991.

12. H. P. Rosenthal, On relatively disjoint families of measures with some applications to Banach space theory, *Studia Math.*, **37** (1970), 13-36.

Yongjin Li
Department of Mathematics
Sun Yat-sen University
Guangzhou 510275
P. R. China
E-mail: stslyj@mail.sysu.edu.cn

Donghai Ji
Department of Mathematics
Harbin University of Science and Technology
Harbin 150080
P. R. China
E-mail: jidonghai@126.com

Qingying Bu
Department of Mathematics
University of Mississippi
University, MS 38677
U.S.A.
E-mail: qbu@olemiss.edu