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PERTURBATION ANALYSIS OF THE EIGENVECTOR MATRIX AND SINGULAR VECTOR MATRICES

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Abstract. Let A be an $n \times n$ Hermitian matrix and $A = U\Lambda U^H$ be its spectral decomposition, where U is a unitary matrix of order n and Λ is a diagonal matrix. In this note we present the perturbation bound and condition number of the eigenvector matrix U of A with distinct eigenvalues. A perturbation bound of singular vector matrices is also given for a real $n \times n$ or $(n+1) \times n$ matrix. The results are illustrated by numerical examples.

1. Introduction

Let A be a $n \times n$ Hermitian matrix with the spectral decomposition

$$A = U\Lambda U^H, \tag{1.1}$$

where U is an $n \times n$ unitary matrix, U^H denotes the conjugate transpose of U and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let B be an $m \times n (m \geq n)$ complex matrix with the singular value decomposition

$$B = W\Sigma V^H, \tag{1.2}$$

where the $m \times m$ left singular vector matrix W and the $n \times n$ right singular vector matrix V are unitary, and

$$\Sigma = \begin{pmatrix} \Sigma_1 \\ 0 \end{pmatrix}, \quad \Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_i \geq 0,\ i = 1, \ldots, n. \tag{1.3}$$

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The spectral decomposition of a Hermitian matrix and the singular value decomposition of a general matrix are very useful tools in many matrix computation problems (see, e.g.[1, 5, 6]).

In this paper we focus on perturbation analysis for the eigenvector matrix U in (1.1) and singular vector matrices W and V in (1.2). As to perturbation of eigenvalues, singular values, eigenspaces and singular subspaces have been studied [1, 2, 4, 9, 10, 12, 15, 16, 17]. We know that eigenspaces and singular subspaces are usually much less sensitive to perturbations compared with the corresponding basis. It is well-known that the maximum eigenspace spanned by column vectors of U and the maximum singular subspaces spanned by column vectors of W and V are is not sensitive at all, but the eigenvector matrix U and the singular vector matrices W and V may be infinitely sensitive. The following simple example illustrates this case.

Example 1.1. Let $A = I_2$(i.e.2×2 identity matrix). Then A has the eigendecomposition (1.1) with $U = I_2, \Lambda = A$. If its perturbed matrix $\widetilde{A}$ has following form

$$\widetilde{A} = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix},$$

where ε is a nonzero real number. It is easy to see that the eigenvector matrix of $\widetilde{A}$ has the following form

$$\widetilde{U} = \begin{pmatrix} \frac{d_1}{\sqrt{2}} & \frac{d_2}{\sqrt{2}} \\ \frac{d_1}{\sqrt{2}} & -\frac{d_2}{\sqrt{2}} \end{pmatrix},$$

where $|d_1| = |d_2| = 1$. Then for any nonzero real number ε, we have

$$\|\widetilde{U} - U\|_F^2 = 4 - \frac{1}{\sqrt{2}}(d_1 + \overline{d}_1 + d_2 + \overline{d}_2) \geq 4 - 2\sqrt{2},$$

where $\overline{a}$ denotes the conjugate of a complex number a. This example shows that the eigenvector matrix U isn't a continuous function of matrix elements.

Based on the technique of splitting operators and Lyapunov majorants, perturbation bounds of four kinds of Schur decompositions have been presented by some authors(see, e.g.[3, 7, 8, 13, 14]). In this paper, we give the absolute and relative perturbation bounds and condition numbers of the eigenvector matrix U of the matrix A with distinct eigenvalues under Hermitian perturbations. We also present absolute and relative perturbation bounds and condition numbers of singular vector matrices W and V for the singular value decomposition of a real $n \times n$ matrix with distinct singular values and a real $(n+1) \times n$ matrix with distinct nonzero singular values, respectively.

We use the following notation: Let A^H and A^T stand for the conjugate transpose and transpose of a matrix A, respectively. I_n is the identity matrix of order n. The Frobenius norm and spectral norm of a matrix A are denoted by $\|A\|_F$ and $\|A\|_2$, respectively. Let $\mathbb{C}^{m\times n}(\mathbb{R}^{m\times n})$, $\mathbb{C}_D^{m\times n}(\mathbb{R}_D^{m\times n})$ and $\mathbb{C}_N^{m\times n}(\mathbb{R}_N^{m\times n})$ denote the sets of complex(real) $m \times n$ matrices, complex(real) $m \times n$ diagonal matrices, and complex(real) $m \times n$ matrices whose diagonal entries are zeros, respectively. Let $X = (x_{ij}) \in \mathbb{C}^{m\times n}(\mathbb{R}^{m\times n})$, we defined X_N and X_D by

$$(X_N)_{ij} = \begin{cases} x_{ij} & i \neq j \\ 0 & i = j \end{cases} \qquad \text{and} \qquad (X_D)_{ij} = \begin{cases} 0 & i \neq j \\ x_{ij} & i = j \end{cases}$$

respectively. It is evident that any $X \in \mathbb{C}^{m\times n}$ can be split uniquely as

$$X = X_N + X_D, \quad X_N \in \mathbb{C}_N^{m\times n}, \quad X_D \in \mathbb{C}_D^{m\times n}.$$

For example we have

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \\ e & f \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \\ 0 & 0 \end{pmatrix}.$$

2. A Perturbation Bound of the Eigenvector Matrix

In this section we study the perturbation bound of the eigenvector matrix U of A in the decomposition (1.1). First we define the linear operator $\mathbf{L}:\ \mathbb{C}_N^{n\times n} \to \mathbb{C}_N^{n\times n}$ by

$$\mathbf{L}\left(\frac{1}{\tau}X\right) = \frac{1}{\nu}(X\Lambda - \Lambda X), \tag{2.1}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and τ, ν are positive parameters.

The following lemma, whose proof is omitted, is similar to Theorem 4.4.5 in [6].

Lemma 2.1. *The linear operator* $\mathbf{L}$ *is defined by (2.1). Then* $\mathbf{L}$ *is nonsingular if and only if all eigenvalues of* Λ *are simple , i. e.* $\lambda_i \neq \lambda_j, i \neq j, i, j = 1, 2, \ldots, n$.

Let $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and let $\mathbf{L}$ be the operator defined by (2.1). Now we define the function

$$\eta(\Lambda) = \min\left\{ \left\| \frac{1}{\nu}(X\Lambda - \Lambda X) \right\|_F :\ X \in \mathbb{C}_N^{n\times n},\ \left\| \frac{1}{\tau} X \right\|_F = 1 \right\}.$$

It is easy to see that if $\mathbf{L}$ is nonsingular then

$$\eta(\Lambda) = \|\mathbf{L}^{-1}\|^{-1} = \frac{\tau}{\nu} \min_{i \neq j} |\lambda_i - \lambda_j|, \tag{2.2}$$

where $\mathbf{L^{-1}}$ is defined by

$$\|\mathbf{L}^{-1}\| = \max\{\|\mathbf{L}^{-1}(X)\|_F : X \in \mathbb{C}_N^{n\times n}, \|X\|_F = 1\}.$$

Lemma 2.2. *Let $X \in \mathbb{C}^{n\times n}$. Then we have*

$$\|(X^H X)_N\|_F \le \sqrt{1 - \frac{1}{n}} \|X\|_F^2. \tag{2.3}$$

Proof. Noting that $X^H X$ is a Hermitian matrix, (2.3) follows from (16) in [7]. ■

In this section we have the main theorem below.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix with distinct eigenvalues and have the spectral decomposition (1.1). Let $\widetilde{A} = A + \Delta A$ be a Hermitian matrix, and let*

$$\epsilon = \frac{1}{\eta(\Lambda)} \frac{\|\Delta A\|_F}{\nu}, \quad \alpha = \frac{\tau^2}{\nu \eta(\Lambda)} \left(1 + \sqrt{1 - \frac{1}{n}}\right) \|A\|_2, \tag{2.4}$$

where $\eta(\Lambda)$ is defined by (2.2). If ϵ satisfies the following condition

$$\epsilon \le \frac{1}{2\alpha + \sqrt{\tau^2 + 4\alpha^2}}, \tag{2.5}$$

then $\widetilde{A}$ has a spectral decomposition $\widetilde{A} = \widetilde{U}\widetilde{\Lambda}\widetilde{U}^H$ such that

$$\frac{\|\widetilde{U} - U\|_F}{\tau} \le \frac{\sqrt{2}\epsilon}{\sqrt{1 - 2\alpha\epsilon + \sqrt{1 - 4\alpha\epsilon - \tau^2\epsilon^2}}} \equiv b_u(\epsilon). \tag{2.6}$$

Proof. Notice that the Frobenius norm is unitarily invariant norm. Hence without loss of generality we may assume that $U = I_n$ in (1.1). Write $\Delta\Lambda = \widetilde{\Lambda} - \Lambda$ and $\Delta U = \widetilde{U} - U = \widetilde{U} - I_n$. Then by $\widetilde{A}\widetilde{U} - AU = \widetilde{U}\widetilde{\Lambda} - U\Lambda$, we can obtain

$$\Delta A \widetilde{U} + A \Delta U = \widetilde{U} \Delta\Lambda + \Delta U \Lambda. \tag{2.7}$$

Let

$$E = \widetilde{U}^H \Delta A \widetilde{U}. \tag{2.8}$$

Noting that $\widetilde{U}^H = I_n + \Delta U^H$, then from (2.7) we get

$$\Delta U \Lambda - \Lambda \Delta U = E + \Delta U^H \Lambda \Delta U - \Delta U^H \Delta U \Lambda - \Delta\Lambda. \tag{2.9}$$

By the definition (2.1) of the operator $\mathbf{L}$ and (2.9), we can get

$$\mathbf{L}\left(\frac{1}{\tau}\Delta U_N\right) = \frac{1}{\nu}E_N + \frac{1}{\nu}(\Delta U^H \Lambda \Delta U - \Delta U^H \Delta U \Lambda)_N. \tag{2.10}$$

Choose a spectral decomposition of $\widetilde{A}$ so that the diagonal elements of $\widetilde{U}$ are real. Then by $\Delta U + \Delta U^H + \Delta U^H \Delta U = 0$, we get

$$\Delta U_D = -\frac{1}{2}\text{diag}(\Delta U^H \Delta U). \tag{2.11}$$

From Lemma 2.1, we know that $\mathbf{L}$ is nonsingular. Hence (2.10)-(2.11) can be rewritten as a continuous mapping $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ expressed by

$$\begin{aligned} \frac{1}{\tau}\Delta U_N \;&=\; \frac{1}{\nu}\mathbf{L}^{-1}(E_N) + \frac{1}{\nu}\mathbf{L}^{-1}\left((\Delta U^H \Lambda \Delta U - \Delta U^H \Delta U \Lambda)_N\right), \\ \Delta U_D \;&=\; -\frac{1}{2}\text{diag}(\Delta U^H \Delta U). \end{aligned} \tag{2.12}$$

Since $(\Delta U^H \Lambda \Delta U - \Delta U^H \Delta U \Lambda)_N = (\Delta U^H \Lambda \Delta U)_N - (\Delta U^H \Delta U)_N \Lambda$, from Lemma 2.2 we have

$$\begin{aligned} \|(\Delta U^H \Lambda \Delta U - \Delta U^H \Delta U \Lambda)_N\|_F \;&\le\; \|(\Delta U^H \Lambda \Delta U)_N\|_F + \|(\Delta U^H \Delta U)_N \Lambda\|_F \\ &\le\; \tau^2\left(1+\sqrt{1-\frac{1}{n}}\right)\|A\|_2 \left\|\frac{1}{\tau}\Delta U\right\|_F^2. \end{aligned}$$

Notice that

$$\|\text{diag}(\Delta U^H \Delta U)\|_F \le \|\Delta U\|_F^2, \quad \|E_N\|_F \le \|\Delta A\|_F.$$

Hence the mapping Φ expressed by (2.12) satisfies

$$\left\|\frac{1}{\tau}\Delta U_N\right\|_F \le \epsilon + \alpha\left\|\frac{1}{\tau}\Delta U\right\|_F^2, \quad \left\|\frac{1}{\tau}\Delta U_D\right\|_F \le \frac{\tau}{2}\left\|\frac{1}{\tau}\Delta U\right\|_F^2, \tag{2.13}$$

where ϵ, α are defined by (2.4). Let $z = (c_1, c_2)^T \in \mathbb{C}^2$. Consider the system

$$c_1 = \epsilon + \alpha(c_1^2 + c_2^2), \quad c_2 = \frac{\tau}{2}(c_1^2 + c_2^2). \tag{2.14}$$

From (2.14) we get the equation

$$(\frac{4\alpha^2}{\tau} + \tau)c_2^2 + 2(2\alpha\epsilon - 1)c_2 + \tau\epsilon^2 = 0. \tag{2.15}$$

If ϵ satisfies (2.5), then

$$c_2^* = \frac{\tau\epsilon^2}{1 - 2\alpha\epsilon + \sqrt{1 - 4\alpha\epsilon - \tau^2\epsilon^2}} \tag{2.16}$$

is a solution to (2.15). By $c_1 = \epsilon + \frac{2\alpha}{\tau} c_2$ and (2.16), we get a solution $c^* = (c_1^*, c_2^*)^T$ to the system (2.14).

Let

$$\mathcal{X}_{c^*} = \{X: \ X \in \mathbb{C}^{n\times n}, \|X_N\|_F \le \tau c_1^*, \ \|X_D\|_F \le \tau c_2^*\}.$$

Then, $\mathcal{X}_{c^*}$ is a bounded closed convex set of $\mathbb{C}^{n\times n}$, and the relation (2.13) shows that the continuous mapping Φ maps $\mathcal{X}_{c^*}$ into $\mathcal{X}_{c^*}$. By the Brouwer fixed-point theorem, the mapping Φ has a fixed point $\Delta U^* \in \mathcal{X}_{c^*}$ such that

$$\left\| \frac{1}{\tau} \Delta U^* \right\|_F \le \|c^*\|_2 = \sqrt{\frac{2}{\tau} c_2^*},$$

which yields the desired result (2.6). ∎

Remark 2.1. Taking $\tau = \nu = 1$ and $\nu = \|A\|_F, \tau = \|U\|_F = \sqrt{n}$, the bound in (2.6) is called the absolute and relative perturbation bound, respectively.

Remark 2.2. For small ϵ the upper bound $b_u(\epsilon)$ defined by (2.6) has the Taylor expansion

$$b_u(\epsilon) = \epsilon + \alpha\epsilon^2 + \left(\frac{\tau^2}{8} + 2\alpha^2\right)\epsilon^3 + O(\epsilon^4), \quad \epsilon \to 0. \tag{2.17}$$

Combining (2.17) with (2.4) we see that the quantities $\frac{1}{\eta(\Lambda)} = \frac{1}{\min\limits_{1\le i\ne j\le n} |\lambda_i - \lambda_j|}$ for $\nu = \tau = 1$ and $\frac{1}{\eta(\Lambda)} = \frac{\|A\|_F}{\sqrt{n}\min\limits_{1\le i\ne j\le n} |\lambda_i - \lambda_j|}$ for $\nu = \|A\|_F, \tau = \|U\|_F = \sqrt{n}$ can be regarded as absolute and relative condition numbers of the eigenvector matrix U of the spectral decomposition (1.1), respectively.

Remark 2.3. Let $A \in \mathbb{C}^{n\times n}$ have the Schur decomposition $A = UTU^H$, where U is an $n \times n$ unitary matrix and T is an $n \times n$ upper triangular matrix. Let Y be strictly lower triangular matrix and the quantity η_A is defined by

$$\eta_A = \min\left\{\|\mathbf{low}(TY - YT)\|_F : \ \|Y\|_F = 1\right\}.$$

Konstantinov, Petkov and Christov[7] show that absolute and relative condition numbers of the Schur factor U are $\frac{\sqrt{2}}{\eta_A}$ and $\sqrt{\frac{2}{n}\frac{\|A\|_F}{\eta_A}}$, respectively. When both A and its perturbed matrix are Hermitian, from Remark 2.2 it is obvious that condition numbers of the eigenvector matrix U in (1.1) improve ones of the Schur factor by improving a numerical factor $\sqrt{2}$.

3. A Perturbation Bound of Singular Vector Matrices

In this section we investigate perturbation analysis of singular vector matrices for a $m\times n$ real matrix. Let $B \in \mathbb{R}^{m\times n}(m \geq n)$ have the singular value decomposition (1.2)-(1.3). Since B is a real matrix, V, Σ, and W may all be taken to be real.

First we define the linear operator $\mathbf{P}: \mathbb{R}_N^{m\times m} \times \mathbb{R}_N^{n\times n} \to \mathbb{R}_N^{n\times m} \times \mathbb{R}_N^{m\times n}$ by

$$\mathbf{P}\left(\frac{1}{\delta}X, \frac{1}{\omega}Y\right) = \left(\frac{1}{\mu}(\Sigma^T X - Y\Sigma^T),\ \frac{1}{\mu}(\Sigma Y - X\Sigma)\right), \tag{3.1}$$

where Σ is defined by (1.3), and δ, ω, μ are positive parameters. Let $T = (t_{ij}) \in \mathbb{R}_N^{n\times m}, R = (r_{ij}) \in \mathbb{R}_N^{m\times n}(m \geq n)$. For $i = 1, 2, \ldots, n$ we define row vectors l_i with respect to T and R by

$$l_i = (t_{i1}, r_{i1}, \ldots, t_{i,i-1}, r_{i,i-1}, t_{i,i+1}, r_{i,i+1}, \ldots, t_{in}, r_{in}), \quad i = 1, 2, \ldots, n. \tag{3.2}$$

The following lemma determine when the operator $\mathbf{P}$ is nonsingular.

Lemma 3.1. *Let* $\mathbf{P}$ *be the linear operator defined by (3.1). (1) If* $m = n$*, then* $\mathbf{P}$ *is nonsingular if and only if all the singular values of* B *are simple, i. e.* $\sigma_i \neq \sigma_j,\ \forall i \neq j,\ i, j = 1, \ldots, n$. *(2) If* $m = n + 1$*, then* $\mathbf{P}$ *is nonsingular if and only if all the singular values of* B *are simple and* $\sigma_i > 0, i = 1, 2, \ldots, n$. *(3) If* $m > n + 1$*, then* $\mathbf{P}$ *is singular.*

Proof. Let

$$d_{ij} = \frac{1}{\mu}\begin{pmatrix} \delta\sigma_i & -\omega\sigma_j \\ -\delta\sigma_j & \omega\sigma_i \end{pmatrix},\ i \neq j,\ i, j = 1, \ldots, n \tag{3.3}$$

and

$$P = \mathrm{diag}(d_{12}, \ldots, d_{1n}, \ldots, d_{i1}, \ldots, d_{i,i-1}, d_{i,i+1}, \ldots, d_{in} \ldots, d_{n1}, \ldots, d_{n,n-1}). \tag{3.4}$$

(1) If $m = n$, we define a column vector **nvec**(T, R) of $T = (t_{ij}), R = (r_{ij}) \in \mathbb{C}^{n\times n}$ by

$$\mathbf{nvec}(T, R) = (l_1, l_2, \ldots, l_n)^T,$$

where l_i are defined by (3.2). Then we have

$$\mathbf{nvec}\left(\frac{1}{\mu}(\Sigma^T X - Y\Sigma^T),\ \frac{1}{\mu}(\Sigma Y - X\Sigma)\right) = P\ \mathbf{nvec}\left(\frac{1}{\delta}X, \frac{1}{\omega}Y\right). \tag{3.5}$$

Hence the operator $\mathbf{P}$ is nonsingular if and only if P is a nonsingular matrix. Obviously, P is nonsingular if and only if $\sigma_i \neq \sigma_j, i \neq j, i, j = 1, \ldots, n$.

(2) If $m = n + 1$, we define $\mathbf{nvec2}(T, R)$ of $T = (t_{ij}) \in \mathbb{R}^{n\times(n+1)}$ and $R = (r_{ij}) \in \mathbb{R}^{(n+1)\times n}$ by

$$\mathbf{nvec2}(T, R) = (l_1, \ldots, l_n, t_{1,n+1}, \ldots, t_{n,n+1}, r_{n+1,1}, \ldots, r_{n+1,n})^T$$

and also define $\mathbf{nvec1}(T, R)$ of $T = (t_{ij}) \in \mathbb{R}^{(n+1)\times(n+1)}$ and $R = (r_{ij}) \in \mathbb{R}^{n\times n}$ by

$$\mathbf{nvec1}(T, R) = (l_1, \ldots, l_n, t_{1,n+1}, \ldots, t_{n,n+1}, t_{n+1,1}, \ldots, t_{n+1,n})^T,$$

where $l_i, i = 1, 2, \ldots, n$ are defined by (3.2). Hence we have

$$\mathbf{nvec2}\left(\frac{1}{\mu}(\Sigma^T X - Y\Sigma^T),\ \frac{1}{\mu}(\Sigma Y - X\Sigma)\right) = Q\ \mathbf{nvec1}\left(\frac{1}{\delta}X, \frac{1}{\omega}Y\right), \tag{3.6}$$

where

$$Q = \text{diag}\left(P,\ \frac{\delta}{\mu}\Sigma_1,\ \frac{\delta}{\mu}\Sigma_1\right). \tag{3.7}$$

So the operator $\mathbf{P}$ is nonsingular if and only if Q is a nonsingular matrix. Obviously, Q is nonsingular if and only if $\sigma_i \neq \sigma_j, i \neq j, i, j = 1, \ldots, n$ and $\sigma_i > 0, i = 1, 2, \ldots, n$.

(3) If $m > n+1$, we take $X = (x_{ij}) \in \mathbb{R}_N^{m\times m}$ and $Y \in \mathbb{R}_N^{n\times n}$ as the following form

$$x_{ij} = \begin{cases} 0, & (i,j) \neq (n+2, n+1) \\ 1, & (i,j) = (n+2, n+1) \end{cases}, \quad Y = 0.$$

It is easy to verify that $\mathbf{P}$ annihilates the nonzero matrix pair $\left(\frac{1}{\delta}X, \frac{1}{\omega}Y\right)$ and must be singular. The proof is complete. ■

Let $\mathbf{P}$ be the operator defined by (3.1). Now we define the function

$$\eta(\Sigma) = \min\left\{\left\|\left(\frac{1}{\mu}(\Sigma^T X - Y\Sigma^T),\ \frac{1}{\mu}(\Sigma Y - X\Sigma)\right)\right\|_F : X \in \mathbb{R}_N^{m\times m}, Y \in \mathbb{R}_N^{n\times n}, \left\|\left(\frac{1}{\delta}X, \frac{1}{\omega}Y\right)\right\|_F = 1\right\}.$$

When the operator $\mathbf{P}$ is nonsingular, from (3.3)-(3.7) we can obtain

$$\eta(\Sigma) = \|\mathbf{P}^{-1}\|^{-1} = \begin{cases} \frac{1}{\mu} \min\limits_{1\leq i\neq j\leq n} \varphi(\sigma_i, \sigma_j), & m = n, \\ \frac{1}{\mu} \min\left\{\min\limits_{1\leq i\neq j\leq n} \varphi(\sigma_i, \sigma_j), \min\limits_{i=1,2,\ldots,n} \delta\sigma_i\right\}, & m = n+1, \end{cases} \tag{3.8}$$

where $\|\mathbf{P}^{-1}\|$ is defined by

$$\|\mathbf{P}^{-1}\| = \max\{\|\mathbf{P}^{-1}(X,Y)\|_F :\ X \in \mathbb{R}_N^{n\times m}, Y \in \mathbb{R}_N^{m\times n}, \|(X,Y)\|_F = 1\}$$

and

$$\varphi(\sigma_i,\sigma_j) = \frac{\sqrt{2}\delta\omega(\sigma_i+\sigma_j)|\sigma_i-\sigma_j|}{\sqrt{(\delta^2+\omega^2)(\sigma_i^2+\sigma_j^2)+\sqrt{(\sigma_i^2+\sigma_j^2)^2(\delta^2-\omega^2)^2+16\delta^2\omega^2\sigma_i^2\sigma_j^2}}}. \tag{3.9}$$

Now we give the following main result.

Theorem 3.2. *Let* $m = n$ *or* $m = n+1$ *and* $B \in \mathbb{R}^{m\times n}$ *have the singular value decomposition (1.2)-(1.3) with for* $m = n$, $\sigma_i \neq \sigma_j, 1 \le i \neq j \le n$ *and for* $m = n+1$, $\sigma_i \neq \sigma_j, \sigma_i > 0, 1 \le i \neq j \le n$. *Moreover, let* $\widetilde{B} = B + \Delta B \in \mathbb{R}^{m\times n}$, *and*

$$\begin{aligned} \varepsilon &= \frac{\sqrt{2}}{\eta(\Sigma)}\frac{\|\Delta B\|_F}{\mu}, \quad \zeta = \max\{\delta,\ \omega\}, \\ \gamma &= \frac{1}{\mu\eta(\Sigma)}\left(\max\{\delta^2,\omega^2\}\sqrt{1-\frac{1}{m}}+\delta\omega\right)\|B\|_2, \end{aligned} \tag{3.10}$$

where $\eta(\Sigma)$ *is defined by (3.8). If* ε *satisfies the following condition*

$$\varepsilon \le \frac{1}{2\gamma+\sqrt{\zeta^2+4\gamma^2}}, \tag{3.11}$$

then $\widetilde{B}$ *has the singular value decomposition*

$$\widetilde{B} = \widetilde{W}\widetilde{\Sigma}\widetilde{V}^T, \tag{3.12}$$

where $\widetilde{W}$ *and* $\widetilde{V}$ *are* $m\times m$ *and* $n \times n$ *orthogonal matrices, respectively, and*

$$\widetilde{\Sigma} = \begin{pmatrix} \widetilde{\Sigma}_1 \\ 0 \end{pmatrix}, \widetilde{\Sigma}_1 = \text{diag}(\tilde{\sigma}_1,\tilde{\sigma}_2,\ldots,\tilde{\sigma}_n), \tilde{\sigma}_i \ge 0, i = 1,2,\ldots,n$$

such that

$$\sqrt{\frac{\|\widetilde{W}-W\|_F^2}{\delta^2}+\frac{\|\widetilde{V}-V\|_F^2}{\omega^2}} \le \frac{\sqrt{2}\varepsilon}{\sqrt{1-2\gamma\varepsilon+\sqrt{1-4\gamma\varepsilon-\zeta^2\varepsilon^2}}} \equiv b_{w,v}(\varepsilon). \tag{3.13}$$

Proof. Notice that the Frobenius norm is unitarily invariant norm. Hence without loss of generality we may assume that $W = I_m$ and $V = I_n$ in (1.2).

By (1.2) and (3.12) we obtain

$$(\widetilde{B}-B)\widetilde{V} + B(\widetilde{V}-I_n) = \widetilde{W}(\widetilde{\Sigma}-\Sigma) + (\widetilde{W}-I_m)\Sigma. \tag{3.14}$$

Let

$$F=\widetilde{W}^T\Delta B\widetilde{V},\quad \Delta\Sigma=\widetilde{\Sigma}-\Sigma,\quad \Delta W=\widetilde{W}-I_m,\quad \Delta V=\widetilde{V}-I_n. \tag{3.15}$$

(3.14) can be written as the following form

$$F+\widetilde{W}^T\Sigma\Delta V=\Delta\Sigma+\widetilde{W}^T\Delta W\Sigma. \tag{3.16}$$

Similarly, from $\widetilde{W}^T(\widetilde{B}-B)+(\widetilde{W}^T-I_m)B=(\widetilde{\Sigma}-\Sigma)\widetilde{V}^T+\Sigma(\widetilde{V}^T-I_n)$ we have

$$F+\Delta W^T\Sigma\widetilde{V}=\Delta\Sigma+\Sigma\Delta V^T\widetilde{V}. \tag{3.17}$$

Since $\widetilde{W}=I_m+\Delta W$ and $\widetilde{V}=I_n+\Delta V$, from (3.16)-(3.17) we can obtain

$$\begin{aligned}\Sigma^T\Delta W-\Delta V\Sigma^T &= -F^T+\Delta\Sigma^T+\Delta V^T\Delta V\Sigma^T-\Delta V^T\Sigma^T\Delta W,\\ \Sigma\Delta V-\Delta W\Sigma &= -F+\Delta\Sigma+\Delta W^T\Delta W\Sigma-\Delta W^T\Sigma\Delta V.\end{aligned} \tag{3.18}$$

Moreover, the matrices ΔW and ΔV satisfy

$$\Delta W+\Delta W^T+\Delta W^T\Delta W=0,\quad \Delta V+\Delta V^T+\Delta V^T\Delta V=0. \tag{3.19}$$

In terms of the definition (3.1) of the operator $\mathbf{P}$, we can obtain from (3.18)

$$\begin{aligned}&\mathbf{P}\left(\frac{1}{\delta}\Delta W_N,\frac{1}{\omega}\Delta V_N\right)\\ &=\left(\frac{1}{\mu}(\Delta V^T\Delta V\Sigma^T–\Delta V^T\Sigma^T\Delta W)_N,\ \frac{1}{\mu}(\Delta W^T\Delta W\Sigma–\Delta W^T\Sigma\Delta V)_N\right)\\ &\quad-\frac{1}{\mu}((F^T)_N,\ F_N).\end{aligned} \tag{3.20}$$

From (3.19) we have

$$(\Delta W_D,\ \Delta V_D)=-\frac{1}{2}\left((\Delta W^T\Delta W)_D,\ (\Delta V^T\Delta V)_D\right). \tag{3.21}$$

Since the operator $\mathbf{P}$ is nonsingular, then (3.20) can be rewritten as

$$\begin{aligned}&\left(\frac{1}{\delta}\Delta W_N,\frac{1}{\omega}\Delta V_N\right)\\ &=\frac{1}{\mu}\mathbf{P}^{-1}\Big((\Delta V^T\Delta V\Sigma^T–\Delta V^T\Sigma^T\Delta W)_N,(\Delta W^T\Delta W\Sigma–\Delta W^T\Sigma\Delta V)_N\Big)\\ &\quad-\frac{1}{\mu}\mathbf{P}^{-1}\left((F^T)_N,\ F_N\right).\end{aligned} \tag{3.22}$$

By Lemma 2.2 we have

$$\begin{aligned}
&\|(\Delta V^T\Delta V\Sigma^T)_N, (\Delta W^T\Delta W\Sigma)_N\|_F \\
&= \|((\Delta V^T\Delta V)_N\Sigma^T, (\Delta W^T\Delta W)_N\Sigma)\|_F \\
&\leq \max\{\delta^2, \omega^2\}\|\Sigma\|_2 \left\|\left(\frac{1}{\omega^2}(\Delta V^T\Delta V)_N, \frac{1}{\delta^2}(\Delta W^T\Delta W)_N\right)\right\|_F \\
&\leq \max\{\delta^2, \omega^2\}\|B\|_2\sqrt{1-\frac{1}{m}}\left\|\left(\frac{1}{\delta}\Delta W, \frac{1}{\omega}\Delta V\right)\right\|_F^2
\end{aligned}$$

and

$$\|(\Delta V^T\Sigma\Delta W)_N,\ (\Delta W^T\Sigma\Delta V)_N\|_F \leq \delta\omega\|B\|_2 \left\|\left(\frac{1}{\delta}\Delta W,\ \frac{1}{\omega}\Delta V\right)\right\|_F^2.$$

Hence we get

$$\begin{aligned}
&\left\|\left((\Delta V^T\Delta V\Sigma^T - \Delta V^T\Sigma\Delta W)_N,\ (\Delta W^T\Delta W\Sigma - \Delta W^T\Sigma\Delta V)_N\right)\right\|_F \\
&\leq \left(\max\{\delta^2, \omega^2\}\sqrt{1-\frac{1}{m}} + \delta\omega\right)\|B\|_2\left\|\left(\frac{1}{\delta}\Delta W, \frac{1}{\omega}\Delta V\right)\right\|_F^2. \qquad (3.23)
\end{aligned}$$

From (3.15) we have

$$\|((F^T)_N,\ F_N)\|_F \leq \sqrt{2}\|\Delta B\|_F. \qquad (3.24)$$

Hence by (3.22)-(3.24), we get

$$\left\|\left(\frac{1}{\delta}\Delta W_N,\ \frac{1}{\omega}\Delta V_N\right)\right\|_F \leq \varepsilon + \gamma\left\|\left(\frac{1}{\delta}\Delta W, \frac{1}{\omega}\Delta V\right)\right\|_F^2. \qquad (3.25)$$

By (3.21) we get

$$\left\|\left(\frac{1}{\delta}\Delta W_D,\ \frac{1}{\omega}\Delta V_D\right)\right\|_F \leq \frac{1}{2}\zeta\left\|\left(\frac{1}{\delta}\Delta W,\ \frac{1}{\omega}\Delta V\right)\right\|_F^2. \qquad (3.26)$$

where ζ is defined by (3.10). Next in terms of (3.25) and (3.26), we take the similar proof as Theorem 2.3 to get the bound (3.13). The proof is complete. ■

Remark 3.1. Taking $\delta = \omega = \mu = 1$ and $\mu = \|B\|_F, \delta = \|W\|_F, \omega = \|V\|_F$, the bound in (3.13) is called the absolute and relative perturbation bounds, respectively.

Remark 3.2. For small ε the upper bound $b_{w,v}(\varepsilon)$ defined by (3.13) has the Taylor expansion

$$b_{w,v}(\varepsilon) = \varepsilon + \gamma\varepsilon^2 + \left(\frac{\zeta^2}{8} + 2\gamma^2\right)\varepsilon^3 + O(\varepsilon^4), \quad \varepsilon \to 0. \qquad (3.27)$$

Combining (3.27) with (3.8)-(3.9) we see that the quantity $\frac{\sqrt{2}}{\eta(\Sigma)}$ can be regarded as a condition number of the left singular vector matrix W and the right singular vector matrix V of the singular value decomposition (1.2) of a real $n \times n$ or $(n+1) \times n$ matrix B. Taking $\mu = \delta = \omega = 1$, the quantity

$$\frac{\sqrt{2}}{\eta(\Sigma)} = \begin{cases} \frac{\sqrt{2}}{\min\limits_{1\le i\ne j\le n} |\sigma_i - \sigma_j|}, & m = n, \\ \max\left\{ \frac{\sqrt{2}}{\min\limits_{1\le i\ne j\le n} |\sigma_i - \sigma_j|}, \frac{\sqrt{2}}{\min\limits_{i=1,2,\dots,n} \sigma_i} \right\}, & m = n+1 \end{cases}$$

is regarded as the absolute condition number; Taking $\mu = \|B\|_F, \delta = \|W\|_F, \omega = \|V\|_F$, the quantity

$$\frac{\sqrt{2}}{\eta(\Sigma)} = \begin{cases} \frac{\sqrt{2}\|B\|_F}{\sqrt{n}\min\limits_{1\le i\ne j\le n} |\sigma_i - \sigma_j|}, & m = n, \\ \sqrt{2}\|B\|_F \max\left\{ \frac{1}{\min\limits_{1\le i\ne j\le n} \varphi(\sigma_i,\sigma_j)}, \frac{1}{\min\limits_{i=1,2,\dots,n} \sqrt{n+1}\sigma_i} \right\}, & m = n+1, \end{cases}$$

where

$$\varphi(\sigma_i, \sigma_j) = \frac{\sqrt{2n(n+1)}(\sigma_i + \sigma_j)|\sigma_i - \sigma_j|}{\sqrt{(2n+1)(\sigma_i^2 + \sigma_j^2) + \sqrt{(\sigma_i^2 + \sigma_j^2)^2 + 16n(n+1)\sigma_i^2\sigma_j^2}}}$$

regarded as the relative condition number.

4. Numerical Examples

In this section we use two simple examples to illustrate the results of previous two sections. All computations were performed by using MATLAB 6.5. The relative machine precision is 2.22×10^{-16}.

Example 4.1. Let $A = \text{diag}(1, 2, 2.001)$ and its perturbed matrix

$$\Delta A = \begin{pmatrix} 1 & -0.2 & 5 \\ -0.2 & -1 & 3 \\ 5 & 3 & 4 \end{pmatrix} \times 10^{-9}.$$

Taking $U = I_2, \Lambda = A$, then $A + \Delta A$ has the spectral decomposition $A + \Delta A = \widetilde{U}\widetilde{\Lambda}\widetilde{U}^H$, where

$$\widetilde{U} = \begin{pmatrix} 1.00000000000000 & -0.00000000200150 & 0.00000004994999 \\ 0.00000000200000 & 0.99999999955005 & 0.00002999849993 \\ -0.00000004995005 & -0.00002999849993 & 0.99999999955004 \end{pmatrix}$$

and

$$\widetilde{\Lambda} = \text{diag}(1.00000001000000, 1.99999998999910, 2.00100004000090).$$

Computation gives

$$\|\widetilde{U}-U\|_F = 4.242625362801784 \times 10^{-6}, \ \frac{\|\widetilde{U}-U\|_F}{\|U\|_F} = 2.449480895284344 \times 10^{-6}.$$

Taking $\nu = \tau = 1$ in Theorem 2.3, we obtain the following absolute perturbation bound

$$b_u(\epsilon) = 9.613884541958045 \times 10^{-6};$$

Taking $\nu = \|A\|_F, \tau = \|U\|_F = \sqrt{3}$ in Theorem 2.3, we obtain the following relative perturbation bound

$$b_u(\epsilon) = 5.550578828211496 \times 10^{-6}.$$

The numerical results shows that the upper bound of (2.6) is fairly sharp.

Example 4.2. Let

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^{-3} \\ 0 & 0 & 0 \end{pmatrix}, \quad \Delta B = \begin{pmatrix} 2 & -1 & 4 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 2 & 7 \end{pmatrix} \times 10^{-9},$$

and $\widetilde{B} = B + \Delta B$. Let $B = W\Sigma V^T$ be the singular value decomposition of B, where $W = I_4, V = I_3$. Then $\widetilde{B}$ has the singular value decomposition $\widetilde{B} = \widetilde{W}\widetilde{\Sigma}\widetilde{V}^T$, where

$$\widetilde{W} =$$

$$\begin{pmatrix} 1.00000000000000 & -0.00000000100000 & -0.00000000050100 & 0.00000000000000 \\ 0.00000000100000 & 1.00000000000000 & -0.00000000100301 & -0.00000000199999 \\ 0.00000000050100 & 0.00000000100300 & 0.99999999997550 & -0.00000699999299 \\ 0.00000000000000 & 0.00000000200000 & 0.00000699999299 & 0.99999999997550 \end{pmatrix},$$

$$\widetilde{\Sigma} =$$

$$\begin{pmatrix} 2.00000000200000 & 0 & 0 \\ 0 & 1.00000000100000 & 0 \\ 0 & 0 & 0.00010000100024 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\widetilde{V} = \begin{pmatrix} 1.00000000000000 & -0.00000000000000 & -0.00000000200025 \\ 0.00000000000000 & 1.00000000000000 & -0.00000000300100 \\ 0.00000000200025 & 0.00000000300100 & 1.00000000000000 \end{pmatrix}.$$

Computation gives

$$\begin{aligned}\sqrt{\|\widetilde{U}-U\|_F^2+\|\widetilde{V}-V\|_F^2} &= 9.899486974420930\times 10^{-6},\\ \sqrt{\frac{\|\widetilde{U}-U\|_F^2}{\|U\|_F^2}+\frac{\|\widetilde{V}-V\|_F^2}{\|V\|_F^2}} &= 4.949743706195210\times 10^{-6}.\end{aligned}$$

Taking $\mu=\delta=\omega=1$ in Theorem 3.2, we obtain the absolute perturbation bound

$$b_{w,v}(\varepsilon) = 1.424840778689057\times 10^{-5};$$

Taking $\mu=\|B\|_F, \delta=\|U\|_F=2, \omega=\|V\|_F=\sqrt{3}$ in Theorem 3.2, we obtain the relative perturbation bound

$$b_{w,v}(\varepsilon)=7.094032685948358\times 10^{-6}.$$

The numerical results show that the upper bound of (3.13) is correct.

5. Conclusion

In this paper we have presented perturbation bounds of the eigenvector matrix of a Hermitian matrix and the singular vector matrices of a $n\times n$ or $(n+1)\times n$ real matrix. The analysis is based on techniques proposed in [3, 7, 8, 13, 14]. Note that for the singular value decomposition $B=W\Sigma V^H$ of a complex matrix B, the diagonal elements of W and V may not be real at the same time. Hence for complex matrices, (3.21) maybe not holds. It remains an open problem how the result in Theorem 3.2 is extended to the complex case.

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