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GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES AND BANACH SPACES

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Abstract. In this paper, we deal with a broad class of nonlinear mappings in a Hilbert space and a Banach space called generalized hybrid which contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings. Then, we prove fixed point theorems for these nonlinear mappings in a Hilbert space and a Banach space. Furthermore, we obtain duality theorems for nonlinear mappings in a Banach space.

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty subset of H. Then a mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping F is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [5] and Goebel and Kirk [10]. It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [4] and [8]. Recently, Kohsaka and Takahashi [23], and Takahashi [31] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to H$ is called nonspreading [23] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.1}$$

for all $x, y \in C$. Similarly, a mapping $T : C \to H$ is called hybrid [31] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.2}$$

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for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [20], Iemoto and Takahashi [16] and Takahashi and Yao [34]. Motivated by these mappings and results, Aoyama, Iemoto, Kohsaka and Takahashi [2] introduced a class of nonlinear mappings called λ-hybrid containing the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Kocourek, Takahashi and Yao [21] also introduced a more broad class of nonlinear mappings than the class of λ-hybrid mappings in a Hilbert space. They called such a class the class of generalized hybrid mappings and then proved general fixed point theorems and convergence theorems in a Hilbert space.

In this paper, we deal with a broad class of nonlinear mappings in a Hilbert space and a Banach space called generalized hybrid which contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings. Then, we prove fixed point theorems for these nonlinear mappings in a Hilbert space and a Banach space. Furthermore, we obtain duality theorems for nonlinear mappings in a Banach space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. From [30], we know the following basic equalities. For $x, y, u, v \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

and

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x-z\| = \inf_{y \in C} \|x-y\|$. We denote such a correspondence by $z = P_C x$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \geq 0$$

for all $x \in H$ and $u \in C$; see [30] for more details.

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty closed convex subset of a strictly convex Banach space E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [18]. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.3) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.3) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.3) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm to norm continuous. If E is uniformly smooth, J is uniformly norm to norm continuous on each bounded subset of E. For more details, see [28, 29]. The following results are also in [28, 29].

Theorem 2.1. *Let E be a Banach space and let J be the duality mapping on E. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j \rangle,$$

where $j \in Jy$.

Theorem 2.2. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi \colon E \times E \to (-\infty, \infty)$ is defined by

(2.4) $$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [19]. We have from the definition of ϕ that

(2.5) $$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Further, we can obtain the following equality:

(2.6) $$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w)$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

(2.7) $$\phi(x, y) = 0 \Longleftrightarrow x = y.$$

The following result was proved by Xu [36].

Theorem 2.3. (Xu [36]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For a proof of existence of a Banach limit and its other elementary properties, see [28].

3. Generalized Hybrid Mappings in Hilbert Spaces

Let C be a nonempty subset of a Hilbert space H and let $\lambda \in \mathbb{R}$. Then, a mapping $T : C \to H$ is called λ-hybrid [2] if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 \\ + 2(\lambda - 1)\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. A mapping $T : C \to H$ is also called generalized hybrid [21] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping is called an (α, β)-generalized hybrid mapping. Recently, Hojo, Takahashi and Yao [11] proved the following result.

Lemma 3.1. (Hojo, Takahashi and Yao [11]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let α and β be in $\mathbb{R}$. Then, a mapping $T : C \to H$ is (α, β)-generalized hybrid if and only if it satisfies that*

$$\|Tx - Ty\|^2 \le (\alpha - \beta)\|x - y\|^2 \\ + 2(\alpha - 1)\langle x - Tx, y - Ty\rangle - (\alpha - \beta - 1)\|y - Tx\|^2$$

for all $x, y \in C$.

Using Hojo, Takahashi and Yao [11], we obtain that an (α, β)-generalized hybrid mapping with $\alpha - \beta = 1$ is a λ-hybrid mapping. Furthermore, we have the following result for generalized hybrid mappings in a Hilbert space.

Theorem 3.2. *Let C be a nonempty subset of a Hilbert space H and let T be a generalized hybrid mapping of C into H, i.e., there are $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{3.1}$$

for all $x, y \in C$. Then, the following hold:

(*i*) *If $\alpha + \beta < 1$, then $T = I$, where $Ix = x$ for all $x \in C$;*

(*ii*) *if $\alpha = 0$ and $\beta = 1$, then T satisfies that $\|Tx - y\| = \|Ty - x\|$ for all $x, y \in C$;*

(*iii*) *if $\alpha = 0$ and $\beta > 1$, then T satisfies that*

$$2\|x - y\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$;

(*iv*) *if $\beta = t\alpha + 1$, $-1 \le t < \infty$ and $\alpha > 0$, then T satisfies that*

$$2\|Tx - Ty\|^2 + 2t\|x - y\|^2 \le (t + 1)\|Tx - y\|^2 + (t + 1)\|Ty - x\|^2$$

for all $x, y \in C$. In particular, T is nonexpansive for $t = -1$, nonspreading for $t = 0$, and hybrid for $t = -\frac{1}{2}$;

(v) *if* $\beta = t\alpha + 1$, $-\infty < t < -1$ *and* $\alpha < 0$, *then* T *satisfies that*

$$2\|Tx - Ty\|^2 + 2t\|x - y\|^2 \geq (t+1)\|Tx - y\|^2 + (t+1)\|Ty - x\|^2$$

for all $x, y \in C$.

Proof.

(i) Putting $x = y$ in the inequality (3.1), we have $(1 - \alpha - \beta)\|x - Tx\|^2 \leq 0$. So, from $\alpha + \beta < 1$ we have $Tx = x$ for all $x \in C$ and hence $T = I$.

(ii) Let $\alpha = 0$ and $\beta = 1$. Then we get that $\|x - Ty\|^2 \leq \|Tx - y\|^2$ for all $x, y \in C$. Replace x, y by y, x, respectively. We also have $\|y - Tx\|^2 \leq \|Ty - x\|^2$. This implies that $\|Tx - y\| = \|Ty - x\|$ for all $x, y \in C$.

(iii) Let $\alpha = 0$. Then we have that

$$\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Changing the role of x and y again, we also have

$$\|y - Tx\|^2 \leq \beta\|Ty - x\|^2 + (1 - \beta)\|x - y\|^2.$$

Summing these two inequalities and then dividing by $1 - \beta$, we have

$$2\|x - y\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$.

(iv) Let $\beta = t\alpha + 1$. Then we have that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq (t\alpha + 1)\|Tx - y\|^2 - t\alpha\|x - y\|^2$$

for all $x, y \in C$. Changing the role of x and y again, we also have

$$\alpha\|Ty - Tx\|^2 + (1 - \alpha)\|y - Tx\|^2 \leq (t\alpha + 1)\|Ty - x\|^2 - t\alpha\|y - x\|^2.$$

Summing these two inequalities, we have

$$2\alpha\|Tx - Ty\|^2 + 2t\alpha\|x - y\|^2 \leq (t+1)\alpha\|Tx - y\|^2 + (t+1)\alpha\|Ty - x\|^2.$$

Dividing by $\alpha > 0$, we have

$$2\|Tx - Ty\|^2 + 2t\|x - y\|^2 \leq (t+1)\|Tx - y\|^2 + (t+1)\|Ty - x\|^2$$

for all $x, y \in C$. In particular, T is nonexpansive for $t = -1$, nonspreading for $t = 0$, and hybrid for $t = -\frac{1}{2}$.

(v) By the same argument as in (iv), we have

$$2\|Tx - Ty\|^2 + 2t\|x - y\|^2 \geq (t+1)\|Tx - y\|^2 + (t+1)\|Ty - x\|^2$$

if $-\infty < t < -1$ and $\alpha < 0$. This completes the proof. ∎

4. Generalized Hybrid Mappings in Banach Spaces

Let E be a Banach space and let C be a nonempty subset of E. Then, a mapping $T : C \to E$ is said to be firmly nonexpansive [6] if

$$\|Tx - Ty\|^2 \leq \langle x - y, j \rangle,$$

for all $x, y \in C$, where $j \in J(Tx - Ty)$. It is known that the resolvent of an accretive operator in a Banach space is a firmly nonexpansive mapping; see [6] and [7]. Using Theorem 2.1, we have that for any $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned}
\|Tx-Ty\|^2 &\leq \langle x - y, j \rangle \\
&\iff 0 \leq 2\langle x - Tx - (y - Ty), j \rangle \\
&\Longrightarrow 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 \\
&\iff \|Tx - Ty\|^2 \leq \|x - y\|^2 \\
&\iff \|Tx - Ty\| \leq \|x - y\|.
\end{aligned}$$

This implies that T is nonexpansive. We also have that for any $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned}
\|Tx-Ty\|^2 &\leq \langle x - y, j \rangle \\
&\iff 0 \leq 2\langle x - Tx - (y - Ty), j \rangle \\
&\iff 0 \leq 2\langle x - Tx, j \rangle + 2\langle Ty - y, j \rangle \\
&\Longrightarrow 0 \leq \|x - Ty\|^2 - \|Tx - Ty\|^2 + \|Tx - y\|^2 - \|Tx - Ty\|^2 \\
&\iff 0 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\
&\iff 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.
\end{aligned}$$

This implies that T is a nonspreading mapping in the sense of (1.1). Furthermore we have that for any $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned}
\|Tx-Ty\|^2 &\leq \langle x - y, j \rangle \\
&\iff 0 \leq 4\langle x - Tx - (y - Ty), j \rangle \\
&\iff 0 \leq 2\langle x - Tx - (y - Ty), j \rangle + 2\langle x - Tx - (y - Ty), j \rangle \\
&\Longrightarrow 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 + \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\
&\iff 3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - Ty\|^2 + \|y - Tx\|^2.
\end{aligned}$$

This implies that T is a hybrid mapping in the sense of (1.2). Thus, it is natural that we extend a generalized hybrid mapping in a Hilbert space by Kocourek, Takahashi and Yao [21] to that of a Banach space as follows: Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{4.1}$$

for all $x, y \in C$. We may also call such a mapping an (α, β)-generalized hybrid mapping in a Banach space. We note that an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$.

On the other hand, Kocourek, Takahashi and Yao [22] extended a generalized hybrid mapping in a Hilbert space to that of a Banach space as follows: Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called generalized nonspreading [22] if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\leq \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{4.2}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an ($\alpha, \beta, \gamma, \delta$)-generalized nonspreading mapping. If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for $x, y \in E$. So, we obtain the following:

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 + \gamma\{\|Tx - Ty\|^2 - \|x - Ty\|^2\} \\ &\leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 + \delta\{\|Tx - y\|^2 - \|x - y\|^2\} \end{aligned} \tag{4.3}$$

for all $x, y \in C$. This implies that

$$\begin{aligned} &(\alpha+\gamma)\|Tx - Ty\|^2 + \{1 - (\alpha+\gamma)\}\|x - Ty\|^2 \\ &\leq (\beta+\delta)\|Tx - y\|^2 + \{1 - (\beta+\delta)\}\|x - y\|^2 \end{aligned} \tag{4.4}$$

for all $x, y \in C$. That is, T is a generalized hybrid mapping in a Hilbert space. The following is Kocourek, Takahashi and Yao fixed point theorem [22].

Theorem 4.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let T be a generalized nonspreading mapping of C into itself. Then, the following are equivalent:*
(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

5. Fixed Point Theorems

In this section, we prove a fixed point theorem for generalized hybrid mappings in a Banach space. For proving the theorem, we need the following lemma; see, for instance, [32] and [28].

Lemma 5.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, let $\{x_n\}$ be a bounded sequence in E and let μ be a mean on l^∞. If $g : E \to \mathbb{R}$ is defined by*

$$g(z) = \mu_n \|x_n - z\|^2, \quad \forall z \in E,$$

then there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Using Lemma 5.1, we can prove the following theorem.

Theorem 5.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a mapping of C into itself. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on l^∞. If*

$$\mu_n \|x_n - Ty\|^2 \leq \mu_n \|x_n - y\|^2$$

for all $y \in C$, then T has a fixed point in C.

Proof. Using the mean μ on l^∞, we can define $g : E \to \mathbb{R}$ as follows:

$$g(y) = \mu_n \|x_n - y\|^2, \quad \forall y \in E.$$

From Lemma 5.1, there exists a unique $z \in C$ such that

$$g(z) = \min\{g(y) : y \in C\}.$$

So, we have

$$g(Tz) = \mu_n \|x_n - Tz\|^2 \leq \mu_n \|x_n - z\|^2 = g(z).$$

Since a minimizer in C concerning the function g is unique and $Tz \in C$, we have $Tz = z$ and then z is a fixed point of T. This completes the proof. ∎

In the case when E is a Hilbert space, we can also show the following fixed point theorem for non-self mappings by using Lemma 5.1.

Theorem 5.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a mapping of C into H such that for any $x \in C$,*

$$Tx \in \{x + t(y - x) : y \in C,\ t \geq 1\}.$$

Let $\{x_n\}$ be a bounded sequence of H and let μ be a mean on l^∞. If

$$\mu_n \|x_n - Ty\|^2 \leq \mu_n \|x_n - y\|^2$$

for all $y \in C$, then T has a fixed point in C.

Proof. Using the mean μ on l^∞, we can define $g : H \to \mathbb{R}$ as follows:

$$g(y) = \mu_n \|x_n - y\|^2, \quad \forall y \in H.$$

From Lemma 5.1, there exists a unique $z \in C$ such that

$$g(z) = \min\{g(y) : y \in C\}.$$

So, we have

$$g(Tz) = \mu_n \|x_n - Tz\|^2 \le \mu_n \|x_n - z\|^2 = g(z).$$

From $Tz \in \{z + t(y - z) : y \in C,\ t \ge 1\}$, there are $y \in C$ and $t \ge 1$ such that $Tz = z + t(y - z)$. If $t = 1$, then we have $Tz = y \in C$. Since z is a unique minimizer in C of the function $g : C \to \mathbb{R}$, we have $z = y$. So, we have $Tz = z$. In the case of $t > 1$, we have from (2.1) that

$$\begin{aligned}
\mu_n \|x_n - Tz\|^2 &= \mu_n \|x_n - (z + t(y - z))\|^2 \\
&= \mu_n \|x_n - (ty + (1 - t)z)\|^2 \\
&= \mu_n \|t(x_n - y) + (1 - t)(x_n - z)\|^2 \\
&= \mu_n \{t\|x_n - y\|^2 + (1 - t)\|x_n - z\|^2 - t(1 - t)\|y - z\|^2\} \\
&= t\mu_n \|x_n - y\|^2 + (1 - t)\mu_n \|x_n - z\|^2 - t(1 - t)\mu_n \|y - z\|^2 \\
&\ge t\mu_n \|x_n - z\|^2 + (1 - t)\mu_n \|x_n - z\|^2 - t(1 - t)\|y - z\|^2 \\
&= \mu_n \|x_n - z\|^2 - t(1 - t)\|y - z\|^2
\end{aligned}$$

and hence

$$-t(1 - t)\|y - z\|^2 \le \mu_n \|x_n - Tz\|^2 - \mu_n \|x_n - z\|^2.$$

From $\mu_n \|x_n - Tz\|^2 \le \mu_n \|x_n - z\|^2$, we have that $-t(1 - t)\|y - z\|^2 \le 0$. From $t > 1$, we have $\|y - z\|^2 \le 0$. This means $y = z$ and hence $Tz = z + t(y - z) = z$. This completes the proof. ∎

Using Theorem 5.2, we prove a fixed point theorem for generalized hybrid mappings in a Banach space.

Theorem 5.4. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let α and β be in $\mathbb{R}$. Let $T : C \to C$ be a generalized hybrid mapping. Then the following are equivalent:*

(a) *$F(T) \neq \emptyset$;*

(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Let $T : C \to C$ be a generalized hybrid mapping, i.e., there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{5.1}$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then $\{T^n z\} = \{z\}$ for $z \in F(T)$. So, $\{T^n z\}$ is bounded. We show the reverse. Take $z \in C$ such that $\{T^n z\}$ is bounded. Let μ be a Banach limit. Then, we have that for any $y \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned}\alpha\|T^{n+1}z - Ty\|^2 &+ (1-\alpha)\|T^n z - Ty\|^2 \\ &\leq \beta\|T^{n+1}z - y\|^2 + (1-\beta)\|T^n z - y\|^2.\end{aligned}$$

Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Then, we have

$$\begin{aligned}&\mu_n(\alpha\|T^{n+1}z - Ty\|^2 + (1-\alpha)\|T^n z - Ty\|^2) \\ &\qquad \leq \mu_n(\beta\|T^{n+1}z - y\|^2 + (1-\beta)\|T^n z - y\|^2).\end{aligned}$$

So, we obtain

$$\begin{aligned}\alpha\mu_n\|T^{n+1}z - Ty\|^2 &+ (1-\alpha)\mu_n\|T^n z - Ty\|^2 \\ &\leq \beta\mu_n\|T^{n+1}z - y\|^2 + (1-\beta)\mu_n\|T^n z - y\|^2\end{aligned}$$

and hence

$$\begin{aligned}\alpha\mu_n\|T^n z - Ty\|^2 &+ (1-\alpha)\mu_n\|T^n z - Ty\|^2 \\ &\leq \beta\mu_n\|T^n z - y\|^2 + (1-\beta)\mu_n\|T^n z - y\|^2.\end{aligned}$$

This implies

$$\mu_n\|T^n z - Ty\|^2 \leq \mu_n\|T^n z - y\|^2$$

for all $y \in C$. By Theorem 5.2, we have a fixed point in C. ■

Using Theorem 5.4, we can also prove the following fixed point theorems in a Banach space.

Theorem 5.5. *Let* E *be a uniformly convex Banach space and let* C *be a nonempty closed convex subset of* E. *Let* $T : C \to C$ *be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ *such that* $\{T^n x\}$ *is bounded. Then,* T *has a fixed point in* C.

Proof. In Theorem 5.4, a $(1, 0)$-generalized hybrid mapping of C into itself is nonexpansive. By Theorem 5.4, T has a fixed point in C. ■

Theorem 5.6. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 5.4, a $(2, 1)$-generalized hybrid mapping of C into itself is nonspreading. By Theorem 5.4, T has a fixed point in C. ■

Theorem 5.7. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a hybrid mapping, i.e.,*

$$3\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 5.4, a $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping of C into itself is hybrid. By Theorem 5.4, T has a fixed point in C. ■

6. Duality Theorems

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into itself. Define a mapping T^* as follows:

$$T^* x^* = JTJ^{-1} x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. The mapping T^* is called the duality mapping of T; see [35] and [12]. It is easy to show that T^* is a mapping of JC into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have

$$T^* x^* = JTJ^{-1} x^* \in JC.$$

Then, T^* is a mapping of JC into itself. Further, we define the duality mapping T^{**} of T^* as follows:

$$T^{**} x = J^{-1} T^* J x, \quad \forall x \in C.$$

It is easy to show that T^{**} is a mapping of C into itself. In fact, for $x \in C$, we have

$$T^{**} x = J^{-1} T^* J x = J^{-1} JTJ^{-1} J x = Tx \in C.$$

So, T^{**} is a mapping of C into itself. We know the following result in a Banach space; see [9] and [35].

Lemma 6.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into itself and let T^* be the duality mapping of JC into itself. Then, the following hold:*
(1) $JF(T) = F(T^*)$;
(2) $\|T^n x\| = \|(T^*)^n Jx\|$ *for each* $x \in C$ *and* $n \in \mathbb{N}$.

Let E be a smooth Banach space, let J be the duality mapping from E into E^* and let C be a nonempty subset of E. A mapping $T : C \to E$ is called skew-generalized nonspreading if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) + \gamma\{\phi(Tx, Ty) - \phi(x, Ty)\} \\ &\leq \beta\phi(y, Tx) + (1-\beta)\phi(y, x) + \delta\{\phi(Tx, y) - \phi(x, y)\} \end{aligned} \tag{6.1}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an $(\alpha, \beta, \gamma, \delta)$-skew-generalized nonspreading mapping. Let T be an $(\alpha, \beta, \gamma, \delta)$-skew-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(Ty, u) \leq \phi(y, u)$ for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (6.1), we obtain

$$\phi(Ty, u) + \gamma\{\phi(u, Ty) - \phi(u, Ty)\} \leq \phi(y, u) + \delta\{\phi(u, y) - \phi(u, y)\}.$$

So, we have that

$$\phi(Ty, u) \leq \phi(y, u) \tag{6.2}$$

for all $u \in F(T)$ and $y \in C$. Further, if E is a Hilbert space, then $\phi(x, y) = \|x-y\|^2$ for $x, y \in E$. So, from (6.1) we obtain the following:

$$\begin{aligned} &\alpha\|Ty - Tx\|^2 + (1-\alpha)\|Ty - x\|^2 + \gamma\{\|Tx - Ty\|^2 - \|x - Ty\|^2\} \\ &\leq \beta\|y - Tx\|^2 + (1-\beta)\|y - x\|^2 + \delta\{\|Tx - y\|^2 - \|x - y\|^2\} \end{aligned} \tag{6.3}$$

for all $x, y \in C$. This implies that

$$\begin{aligned} &(\alpha+\gamma)\|Tx - Ty\|^2 + \{1 - (\alpha+\gamma)\}\|Ty - x\|^2 \\ &\leq (\beta+\delta)\|y - Tx\|^2 + \{1 - (\beta+\delta)\}\|y - x\|^2 \end{aligned} \tag{6.4}$$

for all $x, y \in C$. That is, T is a generalized hybrid mapping [21] in a Hilbert space. Now, we prove a fixed point theorem for skew-generalized nonspreading mappings in a Banach space. Before proving the theorem, we need the following definition: Let $\phi_*\colon E^* \times E^* \to (-\infty, \infty)$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{6.5}$$

for $x, y \in E$.

Theorem 6.2. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let T be a skew-generalized nonspreading mapping of C into itselt. Then, the following are equivalent:*

(*a*) $F(T) \neq \emptyset$;

(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Let T be a skew-generalized nonspreading mapping of C into itselt. Then, there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Ty, Tx)+&(1-\alpha)\phi(Ty, x) + \gamma\{\phi(Tx, Ty) - \phi(x, Ty)\} \\ &\leq \beta\phi(y, Tx) + (1-\beta)\phi(y, x) + \delta\{\phi(Tx, y) - \phi(x, y)\}\end{aligned}$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then $\phi(Ty, u) \leq \phi(y, u)$ for all $u \in F(T)$ and $y \in C$. So, if u is a fixed point in C, then we have $\phi(T^n x, u) \leq \phi(x, u)$ for all $n \in \mathbb{N}$ and $x \in C$. This implies (a) $\Longrightarrow$ (b). Let us show (b) $\Longrightarrow$ (a). Suppose that there exists $x \in C$ such that $\{T^n x\}$ is bounded. Then for any $x^*, y^* \in JC$ with $x^* = Jx$ and $y^* = Jy$ and $T^* = JTJ^{-1}$, we have from (6.5) that

$$\begin{aligned}\alpha\phi_*(T^*x^*, T^*y^*) &+ (1-\alpha)\phi_*(x^*, T^*y^*) + \gamma\{\phi_*(T^*y^*, T^*x^*) - \phi_*(T^*y^*, x^*)\} \\ &= \alpha\phi_*(JTx, JTy)+(1-\alpha)\phi_*(Jx, JTy)+\gamma\{\phi_*(JTy, JTx)-\phi_*(JTy, Jx)\} \\ &= \alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) + \gamma\{\phi(Tx, Ty) - \phi(x, Ty)\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\beta\phi_*(T^*x^*, y^*) &+ (1-\beta)\phi_*(x^*, y^*) + \delta\{\phi_*(y^*, T^*x^*) - \phi_*(y^*, x^*)\} \\ &= \beta\phi_*(JTx, Jy) + (1-\beta)\phi_*(Jx, Jy) + \delta\{\phi_*(Jy, JTx) - \phi_*(Jy, Jx)\} \\ &= \beta\phi(JTx, Jy) + (1-\beta)\phi(y, x) + \delta\{\phi(Tx, y) - \phi(x, y)\}.\end{aligned}$$

Since T is skew-generalized nonspreading, we have that

$$\begin{aligned}\alpha\phi_*(T^*x^*, T^*y^*) &+ (1-\alpha)\phi_*(x^*, T^*y^*) + \gamma\{\phi_*(T^*y^*, T^*x^*) - \phi_*(T^*y^*, x^*)\} \\ &\leq \beta\phi_*(T^*x^*, y^*) + (1-\beta)\phi_*(x^*, y^*) + \delta\{\phi_*(y^*, T^*x^*) - \phi_*(y^*, x^*)\}.\end{aligned}$$

This implies that T^* is a generalized nonspreading mapping of JC into itself. We know from Lemma 6.1 and Theorem 4.1 that T^* has a fixed point in JC. We also have from Lemma 6.1 that $F(T^*) = JF(T)$. Therefore $F(T)$ is nonempty. This completes the proof. ∎

Using Theorem 6.2, we have the following fixed point theorems in a Banach space.

Theorem 6.3. (Dhompongsa, Fupinwong, Takahashi and Yao [9]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a skew-nonspreading mapping, i.e.,*

$$\phi(Ty, Tx) + \phi(Tx, Ty) \leq \phi(y, Tx) + \phi(x, Ty)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $F(T) \neq \emptyset$;

(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha = \beta = \gamma = 1$ and $\delta = 0$ in (6.1), we obtain that

$$\phi(Ty, Tx) + \phi(Tx, Ty) \leq \phi(y, Tx) + \phi(x, Ty)$$

for all $x, y \in C$. So, we have the desired result from Theorem 6.2. ∎

Theorem 6.4. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a mapping such that*

$$2\phi(Ty, Tx) + \phi(Tx, Ty) \leq \phi(y, Tx) + \phi(x, Ty) + \phi(y, x)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $F(T) \neq \emptyset$;

(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha = 1$, $\beta = \gamma = \frac{1}{2}$ and $\delta = 0$ in (6.1), we obtain that

$$2\phi(Ty, Tx) + \phi(Tx, Ty) \leq \phi(y, Tx) + \phi(x, Ty) + \phi(y, x)$$

for all $x, y \in C$. So, we have the desired result from Theorem 6.2. ∎

Theorem 6.5. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a mapping such that*

$$\alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) \leq \beta\phi(y, Tx) + (1-\beta)\phi(y, x)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $F(T) \neq \emptyset$;

(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\gamma = \delta = 0$ in (6.1), we obtain that

$$\alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) \leq \beta\phi(y, Tx) + (1-\beta)\phi(y, x)$$

for all $x, y \in C$. So, we have the desired result from Theorem 6.2. ∎

As a direct consequence of Theorem 6.5, we have Kocourek, Takahashi and Yao fixed point theorem [21] in a Hilbert space.

Theorem 6.6. (Kocourek, Takahashi and Yao [21]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a generalized hybrid mapping, i.e., there are $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{6.6}$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $F(T) \neq \emptyset$*;*

(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$*.*

Proof. We know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in C$ in Theorem 6.5. So, we have the desired result from Theorem 6.5. ∎

7. Some Properties of Skew-generalized Nonspreading Mappings

Let E be a smooth Banach space. Let C be a nonempty subset of E. Let $T : C \to C$ be a mapping. Then, $p \in C$ is called an asymptotic fixed point of T [26] if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. Matsushita and Takahashi [25] also gave the following definition: An operator $T : C \to C$ is relatively nonexpansive if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and

$$\phi(y, Tx) \leq \phi(y, x)$$

for all $x \in C$ and $y \in F(T)$. The following theorems are also in Kocourek, Takahashi and Yao [22].

Theorem 7.1. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a generalized nonspreading mapping of C into itself. Then $\hat{F}(T) = F(T)$.*

Theorem 7.2. *Let E be a smooth and strictly convex Banach space, let C be a nonempty closed convex subset of E and let T be a generalized nonspreading mapping of C into itself such that $F(T)$ is nonempty. Then $F(T)$ is closed and convex.*

Theorem 7.3. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a generalized nonspreading mapping of C into itself such that $F(T)$ is nonempty. Then, T is relatively nonexpansive.*

Let E be a smooth Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a mapping. Then, $p \in C$ is called a generalized asymptotic fixed point of T [15] if there exists $\{x_n\} \subset C$ such that $Jx_n \rightharpoonup Jp$ and $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$. We denote by $\check{F}(T)$ the set of generalized asymptotic fixed points of T. From Takahashi and Yao [35], we also know the following result.

Theorem 7.4. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into itself and let T^* be the duality mapping of JC into itself. Then the following hold:*

(*1*) $J\hat{F}(T) = \check{F}(T^*)$;

(*2*) $J\check{F}(T) = \hat{F}(T^*)$.

Using Theorem 7.1, we have the following result.

Theorem 7.5. *Let E be a smooth, strictly convex and reflexive Banach space such that E^* has a uniformly Gâteaux differentiable norm, let C be a nonempty closed subset of E such that JC is closed and convex and let T be a skew-generalized nonspreading mapping of C into itself. Then $\check{F}(T) = F(T)$.*

Proof. The inclusion $F(T) \subset \check{F}(T)$ is obvious. Thus we only need to show $\check{F}(T) \subset F(T)$. Let $u \in \check{F}(T)$ be given. Then we have a sequence $\{x_n\}$ of C such that $Jx_n \rightharpoonup Ju$ and $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$. Since $T : C \to C$ is a skew-generalized nonspreading mapping, as in the proof of Theorem 6.2, $T^* = JTJ^{-1}$ is a generalized nonspreading mapping of JC into itself. Putting $x_n^* = Jx_n$ and $u^* = Ju$, we have from $Jx_n \rightharpoonup Ju$ and $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$ that $x_n^* \to u^*$ and $\lim_{n\to\infty} \|x_n^* - T^*x_n^*\| = 0$. Then, we have $u^* \in \hat{F}(T^*)$. We know from Theorem 7.1 that $\hat{F}(T^*) = F(T^*)$. So, we have $u^* \in F(T^*)$ and hence $u^* = T^*u^*$. This implies that $Ju = JTJ^{-1}Ju$. So, we have $u = Tu$ and hence $u \in F(T)$. Therefore, $\check{F}(T) = F(T)$. This completes the proof. ∎

From Inthakon, Dhompongsa and Takahashi [17], we also know the following result; see also Ibaraki and Takahashi [15].

Theorem 7.6. (Inthakon, Dhompongsa and Takahashi [17]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. If $T : C \to C$ is a generalized nonexpansive mapping such that $F(T)$ is nonempty, then $F(T)$ is closed and $JF(T)$ is closed and convex.*

Using Theorem 7.6 and (6.2), we have the following result.

Theorem 7.7. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed subset of E such that JC is closed and convex and let T be a skew-generalized nonspreading mapping of C into itself such that $F(T)$ is nonempty. Then T is generalized nonexpansive. Furthermore, $F(T)$ is closed and $JF(T)$ is closed and convex.*

Proof. We have from (6.2) that $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. So, T is generalized nonexpansive. From Theorem 7.6, $F(T)$ is closed and $JF(T)$ is closed and convex. ∎

Using Theorems 7.5 and 7.7, we have the following result.

Theorem 7.8. (Takahashi and Yao [35]). *Let E be a smooth and reflexive Banach space and E^* has a uniformly Gâteaux differentiable norm. Let C be a closed subset of E such that JC is closed and convex and let $T : C \to C$ be a skew-nonspreading mapping, i.e.,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in C$. If $F(T)$ is nonempty, then the following hold:

(1) $\check{F}(T) = F(T)$;

(2) $JF(T)$ *is closed and convex;*

(3) $F(T)$ *is closed;*

(4) T *is generalized nonexpansive.*

Proof. An $(\alpha, \beta, \gamma, \delta)$-skew-generalized nonspreading mapping T of C into itself such that $\alpha = \beta = \gamma = 1$ and $\delta = 0$ is a skew-nonspreading mapping. From Theorems 7.5 and 7.7, we have the desired result. ∎

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