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A NOTE ON HEAT KERNELS OF GENERALIZED HERMITE OPERATORS

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Abstract. In this note, the author obtains heat kernels for the generalised Hermite operators $L = -\Delta + \langle Bx, x\rangle$ where B is a (not necessarily symmetry) semi-positive definite matrix.

1. Introduction

It is well known that the Hermite operators $L = -\frac{d^2}{dx^2} + \lambda^2 x^2$ and $L = -\frac{d^2}{dx^2} - \lambda^2 x^2$ corresponding to harmonic oscillator and anti-harmonic oscillator play an important role in many mathematical and physical problems (cf. [1, 3, 6, 7, 8]). Hence seeking fundamental solutions of such operators becomes a basic and natural problem.

The purpose of this paper is to consider the heat kernels for the generalised operators taking the form $L = -\Delta + \langle Bx, x\rangle$. In particular, one may concern that B is a positive definite or a negative definite matrix. Recently, [5] obtained the heat kernel for L with any $n \times n$ matrix B by using Hamiltonian formalism. The most striking result they obtained is how the geodesics–solution of the Hamiltonian system–behave for different B in terms of the eigenvalues of $B + B^t$. However, the computation is rather complicated as long as the matrix $B + B^t$ has negative eigenvalues, especially when the dimension n is large.

In some special cases, one can get the explicit heat kernel without solving Hamiltonian system. When B is semi-positive definite, a detailed discussion will be presented in Section 2. In Section 3, resorting to the qualitative conclusion in [5], one also reads off an explicit formulae if B is semi-negative definite.

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2. Heat Kernel for L in $\mathbb{R}^n$ $(B \geq 0)$

One may start with the positive definite case. Consider the generalised Hermite operators of the following form

$$L = -\Delta + \langle Bx, x\rangle$$

where B is a $n \times n$ (not necessarily symmetry) positive definite, $\langle \cdot, \cdot\rangle$ denotes the standard inner product in Euclidean space $\mathbb{R}^n$. The Hamiltonian function associated with L is

$$H(\xi, x) = -\langle \xi, \xi\rangle + \langle Bx, x\rangle$$

hence one obtains the corresponding Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial \xi} = -2\xi \quad \text{and} \quad \dot{\xi} = -\frac{\partial H}{\partial x} = -(B + B^t)x$$

The geodesic $x(s)$ between x_0 and x in $\mathbb{R}^n$ satisfies the boundary problem

$$\begin{cases} \ddot{x} = Ax \\ x(0) = x_0, \;\; x(t) = x \end{cases} \tag{2.1}$$

where $A = 2(B + B^t) > 0$. Since A is a symmetry positive definite matrix, one can find an orthogonal matrix P such that $PAP^t = \text{diag}\{\lambda_1, \ldots, \lambda_n\} =: \Lambda$, where $\lambda_j > 0$ are eigenvalues of A. Set

$$y(s) = Px(s), \;\; y_0 = y(0) = Px(0) = Px_0, \;\; \text{and} \;\; y = y(t) = Px(t) = Px,$$

then problem (2.1) is equivalent to

$$\begin{cases} \ddot{y} = \Lambda y \\ y(0) = y_0, \;\; y(t) = y \end{cases} \tag{2.2}$$

According to [4], the energy function in y-variables is

$$E_y = \sum_{j=1}^{n} \frac{\lambda_j \left[y_j^2 + \left(y_j^0\right)^2 - 2y_j y_j^0 \cosh\left(t\lambda_j^{1/2}\right)\right]}{2\sinh^2\left(t\lambda_j^{1/2}\right)}$$

Noticing that $E = \frac{1}{2}\left(\langle \dot{x}, \dot{x}\rangle - \langle x, \ddot{x}\rangle\right)$, one obtains

$$\begin{aligned} E_x &= \frac{1}{2}\left(\langle P\dot{x}, P\dot{x}\rangle - \langle Px, P\ddot{x}\rangle\right) \\ &= \frac{1}{2}\left(\langle \dot{y}, \dot{y}\rangle - \langle y, \ddot{y}\rangle\right) \\ &= E_y \\ &= \sum_{j=1}^{n} \frac{\lambda_j \left[y_j^2 + \left(y_j^0\right)^2 - 2y_j y_j^0 \cosh\left(t\lambda_j^{1/2}\right)\right]}{2\sinh^2\left(t\lambda_j^{1/2}\right)} \end{aligned}$$

To move on, one needs some properties on the action function.

Proposition. *For action function $S = -\int E dt$, the following equalities hold:*

$$|\nabla S|^2 = \langle Ax, x\rangle + 2E_x \tag{2.3}$$

$$\Delta S = \frac{1}{t} tr\left[\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)\right] \tag{2.4}$$

where $\varphi(M)$denotes functional calculus of continuous function φ on the symmetry positive definite matrix M, and $tr(M)$ denotes the trace of matrix M.

Proof. A direct computation shows that

$$\begin{aligned}
S &= -\int E dt \\
&= \frac{1}{2t}\sum_{j=1}^{n}\left(t\lambda_j^{1/2}\right)\coth\left(t\lambda_j^{1/2}\right)y_j^2 + \frac{1}{2t}\sum_{j=1}^{n}\left(t\lambda_j^{1/2}\right)\coth\left(t\lambda_j^{1/2}\right)\left(y_j^0\right)^2 \\
&\quad - \frac{1}{t}\sum_{j=1}^{n}\frac{t\lambda_j^{1/2}}{\sinh\left(t\lambda_j^{1/2}\right)}y_j y_j^0 \\
&= \frac{1}{2t}\left\langle\sqrt{\left(t\Lambda^{1/2}\right)\coth\left(t\Lambda^{1/2}\right)}y, \sqrt{\left(t\Lambda^{1/2}\right)\coth\left(t\Lambda^{1/2}\right)}y\right\rangle \\
&\quad + \frac{1}{2t}\left\langle\sqrt{\left(t\Lambda^{1/2}\right)\coth\left(t\Lambda^{1/2}\right)}y_0, \sqrt{\left(t\Lambda^{1/2}\right)\coth\left(t\Lambda^{1/2}\right)}y_0\right\rangle \\
&\quad - \frac{1}{t}\left\langle\sqrt{\frac{t\Lambda^{1/2}}{\sinh\left(t\Lambda^{1/2}\right)}}y, \sqrt{\frac{t\Lambda^{1/2}}{\sinh\left(t\Lambda^{1/2}\right)}}y_0\right\rangle \\
&= \frac{1}{2t}\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x, x\right\rangle + \frac{1}{2t}\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x_0, x_0\right\rangle \\
&\quad - \frac{1}{t}\left\langle\frac{tA^{1/2}}{\sinh\left(tA^{1/2}\right)}x_0, x\right\rangle
\end{aligned}$$

hence,

$$\begin{aligned}
\partial_{x_j}S &= \frac{1}{t}\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x, e_j\right\rangle - \frac{1}{t}\left\langle\frac{tA^{1/2}}{\sinh\left(tA^{1/2}\right)}x_0, e_j\right\rangle \\
|\nabla S|^2 &= \sum_{j=1}^{n}\left(\partial_{x_j}S\right)^2 \\
&= \frac{1}{t^2}\left[\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x, \left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x\right\rangle\right. \\
&\quad + \left\langle\frac{tA^{1/2}}{\sinh\left(tA^{1/2}\right)}x_0, \frac{tA^{1/2}}{\sinh\left(tA^{1/2}\right)}x_0\right\rangle
\end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&-2\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x,\frac{tA^{1/2}}{\sinh\left(tA^{1/2}\right)}x_0\right\rangle\Bigg]\\
&=\frac{1}{t^2}\left\{\sum_{j=1}^{n}\left[\left(t\lambda_j^{1/2}\right)\coth\left(t\lambda_j^{1/2}\right)y_j\right]^2+\sum_{j=1}^{n}\left(\frac{t\lambda_j^{1/2}}{\sinh\left(t\lambda_j^{1/2}\right)}y_j^0\right)^2\right.\\
&\left.-2\sum_{j=1}^{n}\frac{t^2\lambda_j\coth\left(t\lambda_j^{1/2}\right)}{\sinh\left(t\lambda_j^{1/2}\right)}y_jy_j^0\right\}\\
&=\sum_{j=1}^{n}\frac{\lambda_j\cosh^2\left(t\lambda_j^{1/2}\right)y_j^2}{\sinh^2\left(t\lambda_j^{1/2}\right)}+\sum_{j=1}^{n}\frac{\lambda_j\left(y_j^0\right)^2}{\sinh^2\left(t\lambda_j^{1/2}\right)}-\sum_{j=1}^{n}\frac{\lambda_j\left[y_j^2+\left(y_j^0\right)^2\right]}{\sinh^2\left(t\lambda_j^{1/2}\right)}\\
&+\sum_{j=1}^{n}\frac{\lambda_j\left[y_j^2+\left(y_j^0\right)^2\right]}{\sinh^2\left(t\lambda_j^{1/2}\right)}-2\sum_{j=1}^{n}\frac{\lambda_j\coth\left(t\lambda_j^{1/2}\right)}{\sinh\left(t\lambda_j^{1/2}\right)}y_jy_j^0\\
&=\sum_{j=1}^{n}\lambda_jy_j^2+2E_x\\
&=\langle Ax,x\rangle+2E_x
\end{aligned}
$$

Differentiating equation(2.5) on x_j once more, one has

$$\partial_{x_j}^2S=\frac{1}{t}\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)e_j,e_j\right\rangle.$$

Consequently,

$$\Delta S=\sum_{j=1}^{n}\partial_{x_j}^2S=\frac{1}{t}tr\left[\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)\right]=\sum_{j=1}^{n}\lambda_j^{1/2}\coth\left(t\lambda_j^{1/2}\right)\qquad\blacksquare$$

One expects to find the heat kernel of L in the following form

$$K\left(x_0,x,t\right)=V\left(t\right)e^{\alpha S\left(x_0,x,t\right)}$$

where α is a real number to be chosen later. Then

$$\partial_tK=e^{\alpha S}\left[V'\left(t\right)-\alpha EV\left(t\right)\right]\tag{2.6}$$

and by use of the Proposition

$$
\begin{aligned}
\Delta e^{\alpha S}&=\alpha e^{\alpha S}\left(\alpha|\nabla S|^2+\Delta S\right)\\
&=\alpha e^{\alpha S}\left[\alpha\langle Ax,x\rangle+2\alpha E+\sum_{j=1}^{n}\lambda_j^{1/2}\coth\left(t\lambda_j^{1/2}\right)\right]
\end{aligned}\tag{2.7}
$$

On the one hand, kernel $K(x_0, x, t)$ of the Heat operator $P = \partial_t + L = \partial_t - \Delta + \langle Bx, x\rangle$ satisfies $PK = 0$ for any $t > 0$; on the other hand, noticing that $\langle Ax, x\rangle = 4\langle Bx, x\rangle$,

$$\begin{aligned} &PK \\ &= \partial_t K - V(t)\,\Delta e^{\alpha S} + \langle Bx, x\rangle V(t)\,e^{\alpha S} \\ &= K\left[\frac{V'(t)}{V(t)} - \alpha(1+2\alpha)E - \alpha^2\langle Ax, x\rangle + \langle Bx, x\rangle - \alpha\sum_{j=1}^{n}\lambda_j^{1/2}\coth\left(t\lambda_j^{1/2}\right)\right] \\ &= K\left[\frac{V'(t)}{V(t)} - \alpha(1+2\alpha)E + \left(1-4\alpha^2\right)\langle Bx, x\rangle - \alpha\sum_{j=1}^{n}\lambda_j^{1/2}\coth\left(t\lambda_j^{1/2}\right)\right]. \end{aligned}$$

Let $\alpha = -\frac{1}{2}$ and $V(t)$ satisfy the transport equation

$$\frac{V'(t)}{V(t)} = -\frac{1}{2}\sum_{j=1}^{n}\lambda_j^{1/2}\coth\left(t\lambda_j^{1/2}\right)$$

Integration yields $V(t) = \prod_{j=1}^{n}\frac{C_j}{\sinh^{1/2}\left(t\lambda_j^{1/2}\right)}$. Hence kernel K has the form

$$\begin{aligned} K(x_0, x, t) = &\left(\prod_{j=1}^{n}\frac{C_j}{\sinh^{1/2}\left(t\lambda_j^{1/2}\right)}\right) \\ &\times e^{-\frac{1}{4t}\left[\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x,x\right\rangle+\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x_0,x_0\right\rangle-2\left\langle\frac{tA^{1/2}}{\sinh\left(tA^{1/2}\right)}x_0,x\right\rangle\right]} \end{aligned}$$

Since kernel K becomes Gaussian $\frac{1}{(4\pi t)^{n/2}}e^{-\frac{1}{4t}|x-x_0|^2}$ if $B \to 0$, one may compare the volume element $V(t)$ with $\frac{1}{(4\pi t)^{n/2}}$ to establish the parameters

$$C_j = \left(\frac{\lambda_j}{16\pi^2}\right)^{1/4}.$$

Theorem. *Let B be a $n\times n$ positive definite matrix, then $A=2\left(B+B^t\right)$ is a symmetry positive definite matrix whose Jordan normal form is denoted by* diag$\{\lambda_1, \ldots, \lambda_n\}$ *with $\lambda_j > 0$. The kernel of the heat operator $P = \partial_t - \Delta + \langle Bx, x\rangle$ is*

$$\begin{aligned} K(x_0, x, t) = &\frac{1}{(4\pi t)^{n/2}}\left(\prod_{j=1}^{n}\frac{t\lambda_j^{1/2}}{\sinh\left(t\lambda_j^{1/2}\right)}\right)^{1/2} \\ &\times e^{-\frac{1}{4t}\left[\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x,x\right\rangle+\left\langle\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)x_0,x_0\right\rangle-2\left\langle\frac{tA^{1/2}}{\sinh\left(tA^{1/2}\right)}x_0,x\right\rangle\right]}. \end{aligned} \tag{2.8}$$

Remark. The form of K is still valid if $B \geq 0$, but the terms $\frac{t\lambda_j^{1/2}}{\sinh\left(t\lambda_j^{1/2}\right)}$, $\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right)$ and $\frac{tA^{1/2}}{\sinh\left(tA^{1/2}\right)}$ in equation (2.8) should be replaced by their corresponding limiting forms. Specifically, if

$$PAP^t = \Lambda = \text{diag}\left\{\lambda_1, \ldots, \lambda_m, 0, \ldots, 0\right\}$$

with $\lambda_j > 0, j = 1, \ldots, m$ and $\lambda_l = 0, l = m+1, \ldots, n$, then the volume element

$$\prod_{j=1}^{n} \frac{t\lambda_j^{1/2}}{\sinh\left(t\lambda_j^{1/2}\right)} = \prod_{j=1}^{m} \frac{t\lambda_j^{1/2}}{\sinh\left(t\lambda_j^{1/2}\right)},$$

and

$$\begin{aligned}
&\left(tA^{1/2}\right)\coth\left(tA^{1/2}\right) \\
&= P^t\text{diag}\left\{\left(t\lambda_1^{1/2}\right)\coth\left(t\lambda_1^{1/2}\right), \ldots, \left(t\lambda_m^{1/2}\right)\coth\left(t\lambda_m^{1/2}\right), 1, \ldots, 1\right\} P; \\
&\frac{tA^{1/2}}{\sinh\left(tA^{1/2}\right)} = P^t\text{diag}\left\{\frac{t\lambda_1^{1/2}}{\sinh\left(t\lambda_1^{1/2}\right)}, \ldots, \frac{t\lambda_m^{1/2}}{\sinh\left(t\lambda_m^{1/2}\right)}, 1, \ldots, 1\right\} P.
\end{aligned}$$

3. Open Question

If B has negative eigenvalues, the energy function E is not well defined on singular regions which are of zero measure in $2n+1$ dimensional Lebesgue measure (cf. [5]). So far the only way to establish the singular regions is to solve the Hamiltonian system, which requires heavy computations for large n .

Here one conjectures that the kernel has the following form. In particular, for the same reason mentioned in the Remark above, it is sufficient to formulate the case $B < 0$.

Conjecture. *Let $B < 0$ and $A = 2\left(B + B^t\right) \sim$ diag $\left\{\lambda_1, \ldots, \lambda_n\right\}$ with $\lambda_j < 0$. Then one kernel of the heat operator $P = \partial_t - \Delta + \langle Bx, x\rangle$ is*

$$K\left(x_0, x, t\right) = \frac{1}{(4\pi t)^{n/2}} \left(\prod_{j=1}^{n} \frac{t\left(-\lambda_j\right)^{1/2}}{\sin\left(t\left(-\lambda_j\right)^{1/2}\right)}\right)^{1/2} e^{\left\langle \frac{(-A)^{1/2}}{2\sin\left(t(-A)^{1/2}\right)} x_0, x\right\rangle}$$
$$\times\, e^{-\frac{1}{4t}\left[\left\langle\left(t(-A)^{1/2}\right)\cot\left(t(-A)^{1/2}\right)x, x\right\rangle + \left\langle\left(t(-A)^{1/2}\right)\cot\left(t(-A)^{1/2}\right)x_0, x_0\right\rangle\right]}$$

a.e. $\left(x_0, x, t\right) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$.

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