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THE WEAKER CONVERGENCE OF NON-STATIONARY MATRIX MULTISPLITTING METHODS FOR ALMOST LINEAR SYSTEMS

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Abstract. In 1999, Arnal et al. [*Numerical linear algebra and its applications*, 6(1999): 79-92] introduced the non-stationary matrix multisplitting algorithms for almost linear systems and studied the convergence of them. In this paper, we generalize Arnal's algorithms and study the non-stationary matrix multisplitting multi-parameters methods for almost linear systems. The parameters can be adjusted suitably so that the convergence property of methods can be substantially improved. Furthermore, the convergence results of our new method in this paper are weaker than those of Arnal's. Finally, numerical examples show that our new convergence results are better and more efficient than Arnal's.

1. INTRODUCTION

For solving almost linear systems of the form

$$Ax + \Phi(x) = b, \tag{1}$$

where $A \in R^{n\times n}$ is a square nonsingular H-matrix, $x, b \in R^n$ and $\Phi : R^n \to R^n$ is a nonlinear diagonal mapping, i.e., the ith component Φ_i of Φ is a function only of x_i, an iterative method is usually considered. The concept of multisplitting for the parallel solution of linear system was introduced by O'Leary and White and further studied by many other authors [5-8, 10, 11, 13, 14].

Since that system (1) usually appears in practice from the discretization of differential equations, which arise in many applied fields such as trajectory calculation

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and the study of oscillatory systems [3, 4], for solving that system (1), White [16] introduced the parallel nonlinear Gauss-Seidel algorithm, based on both the classical nonlinear Gauss-Seidel method [12] and then the multisplitting techniques were used for nonlinear problems. Later, Bai [2] presented a class of algorithms called the parallel nonlinear AOR methods for almost linear systems, which are one generalization of the parallel nonlinear Gauss-Seidel [15]. In 1999, Arnal et al. [1] studied the non-stationary parallel multisplitting methods for almost linear systems. Recently, Kuo, Lin, Shieh and Wang [9] focused on the use of continuation methods to solve the time-independent m-component discrete nonlinear Schr$\ddot{o}$inger equations, proposed a new algorithm that is capable of tracking the local minimal energy solutions along the solution curves and showed how one may compute the ground states of the decoupled discrete nonlinear Schr$\ddot{o}$inger equation with $V(z) = 0$.

Now, we will introduce the concept of multisplitting method and the detailed process of parallel nonlinear iterative method.

$\{M_k, N_k, E_k\}_{k=1}^{\alpha}$ is a $multisplitting$ of A if

(1) $A = M_k - N_k$ is a splitting for $k = 1, 2, ..., \alpha$;

(2) $E_k \geq 0$ is a nonnegative diagonal matrix, called weighting matrix;

(3) $\sum\limits_{k=1}^{\alpha} E_k = I$, where I is the identity matrix.

Let us define $r_i: \ R \rightarrow R, \ 1 \leq i \leq n$, such that

$$r_i(t) = a_{ii}t + \Phi_i(t), \ t \in R,$$

and suppose that there exists the inverse function of each r_i, denoted by r_i^{-1}. Then given an arbitrary initial vector $x^{(0)}$, the parallel nonlinear AOR method produces the following sequence of vectors

$$x^{(l)} = \sum_{k=1}^{\alpha} E_k x^{l,k}, \ l = 1, 2, ... \tag{2}$$

where

$$\begin{aligned} x_i^{l,k} &= \omega \tilde{x}_i^{l,k} + (1-\omega) x_i^{(l-1)}, \\ \tilde{x}_i^{l,k} &= r_i^{-1}(z_i^{l,k}), \ 1 \leq i \leq n, \ \omega \in R, \ \omega \neq 0, \\ z^{l,k} &= \ \mu L_k \tilde{x}^{l,k} + (1-\mu) L_k x^{(l-1)} + U_k x^{(l-1)} + b, \ \mu \in R. \end{aligned} \tag{3}$$

After some algebraic manipulations, the corresponding iterations have two conditions, which are as follows

Condition 1: If $\mu \neq 0$, then

For $i = 1, 2, ..., n$

$$
\begin{aligned}
a_{ii}\tilde{x}_i^{l,k} + \Phi_i(\tilde{x}_i^{l,k}) &= \sum_{j<i} l_{ij}^k y_j^{l,k} + \sum_{j\neq i} u_{ij}^k x_j^{(l-1)} + b_i, \\
y_i^{l,k} &= \mu\tilde{x}_i^{l,k} + (1-\mu)x_i^{(l-1)}, \\
x_{l,k} &= \frac{\omega}{\mu} y^{l,k} + (1-\frac{\omega}{\mu})x^{(l-1)}.
\end{aligned} \tag{4}
$$

Condition 2: If $\mu = 0$, then
For $i = 1, 2, ..., n$

$$
\begin{aligned}
a_{ii}\tilde{x}_i^{l,k} + \Phi_i(\tilde{x}_i^{l,k}) &= -\sum_{j\neq i} a_{ij} x_j^{(l-1)} + b_i, \\
x^{l,k} &= \omega\tilde{x}^{l,k} + (1-\omega)x^{(l-1)}.
\end{aligned} \tag{5}
$$

Remark 1.1. In iteration scheme (2), let $\omega = \mu = 1$, then the parallel nonlinear AOR method reduces to the parallel nonlinear Gauss-Seidel method [16]. Let $\omega = \mu \neq 0$, then the parallel nonlinear AOR method reduces to the parallel nonlinear SOR method [16].

In this paper, we extend the non-stationary parallel multisplitting methods (ALNPM-style) [1] to the non-stationary parallel multisplitting multi-parameters (ALNPMM-style) methods for almost linear systems. Furthermore, we study the convergence of our methods when the matrix in question is either H-matrix or M-matrix and the convergence conditions of our new method in this paper are weaker than those of Arnal's in [1].

2. Preliminaries

A matrix $A = (a_{ij})$ is called an *M-matrix* if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of matrix $A = (a_{ij})$ is defined by: $\alpha_{ij} = |a_{ij}|$, if $i = j; \alpha_{ij} = -|a_{ij}|$, if $i \neq j$. A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix. Let $\rho(A)$ denote the spectral radius of A. A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. Let A and B be M-matrices. If $A \leq B$, then $A^{-1} \geq B^{-1}$. Let A be an H-matrix, and $A = D - B, D = \mathrm{diag}(A)$, then $\rho(|D|^{-1}|B|) < 1$. Moreover, D is nonsingular.

Lemma 2.1. ([6]). *Let A be an H-matrix, then A is nonsingular, and* $|A^{-1}| \leq \langle A\rangle^{-1}$.

Lemma 2.2. ([13]). *Let $H^{(1)}, H^{(2)}, ..., H^{(l)}$...be a sequence of nonnegative matrices in $R^{n\times n}$. If there exist a real number $0 \leq \theta < 1$, and a vector $\nu > 0$ in R^n, such that*

$$H^{(l)}\nu \leq \theta\nu,\ l = 1, 2, ...$$

then $\rho(K_l) \leq \theta^l < 1$, *where* $K_l = H^{(l)}H^{(l-1)}...H^{(1)}$, *and therefore* $\lim_{l\to\infty} K_l = 0$.

Lemma 2.3. ([2]). *Let* $A = (a_{ij}) \in R^{n\times n}$ *be an* H*-matrix and let* $\Phi:\ R^n \to R^n$ *be a continuous and diagonal mapping. If sign*$(a_{\,ij})(t-s)(\Phi_i(t)-\Phi_i(s)) \geq 0,\ i = 1, 2, ...n,$ *for all* $t, s \in R$*, then the almost linear system (1) has a unique solution.*

Lemma 2.4. ([19]). *Let* $A = (a_{ij}) \in Z^{n\times n}$ *which has all positive diagonal entries.* A *is an* M*-matrix if and only if* $\rho(B) < 1$*, where* $B = D^{-1}C,\ D = \mathrm{diag}(A),\ A = D - C$.

3. Main Results

Let us define l operators $P_k:\ R^n \to R^n$ such that each of them maps x into y in the same way as $x_{l,k}$ is obtained from $x^{(l-1)}$ in equation (3).

Based iteration scheme (2), we will present the non-stationary parallel multisplitting multi-parameters method (ALNPMM-style) for almost linear systems, which is as follows:

Algorithm 1. (ALNPMM). Given the initial vector $x^{(0)}$
For $l = 0, 1, ...,$ repeat (I) and (II), until convergence.
In k processors, $k = 1, ..., \alpha$, let $x^{l,k,0} = x^{(l-1)}$.

(I) For $m = 1, 2, ..., q(m, k)$, (parallel) solving $x^{l,k,m}$.

$$x^{l,k,m} = P_k(x^{l,k,m-1}).$$

(II) Computing

$$x^{(l)} = \sum_{k=1}^{\alpha} E_k x^{l,k,q(l,k)}.$$

The algorithm 1 may be written as

$$x^{(l)} = \sum_{k=1}^{\alpha} E_k x^{l,k,q(l,k)},\ l = 1, 2, ... \tag{6}$$

where

$$\begin{aligned} x_i^{l,k,m} &= \omega_k \tilde{x}_i^{l,k,m} + (1-\omega_k)x_i^{l,k,m-1},\ k = 1, 2, ..., \alpha,\ \omega_k \in R, \\ \tilde{x}_i^{l,m,k} &= r_i^{-1}(z_i^{l,k,m}),\ 1 \leq i \leq n,\ \omega_k \neq 0,\ m = 1, 2, ..., q(l,k), \\ z^{l,k,m} &= \mu_k L_k \tilde{x}^{l,k,m} + (1-\mu_k)L_k x^{l,k,m-1} + U_k x^{l,k,m-1} + b,\ \mu_k \in R. \end{aligned} \tag{7}$$

Remark 3.1. The functions $P_k(\cdot)$ in Algorithm 1 is actually given in equation (3) for solving $\tilde{x}^{l,k}$, i.e., $\tilde{x}_i^{l,k} = r_i^{-1}(z_i^{l,k}),\ 1 \leq i \leq n,\ \omega \in R,\ \omega \neq 0,$ where

r_i is defined as: $R \to R,\ 1 \le i \le n$, such that $r_i(t) = a_{ii}t + \Phi_i(t),\ t \in R$. Then, based on the difference of $\mu \neq 0$ or $\mu = 0$, we may solve these nonlinear equations using two equations $a_{ii}\tilde{x}_i^{l,k} + \Phi_i(\tilde{x}_i^{l,k}) = \sum_{j<i} l_{ij}^k y_j^{l,k} + \sum_{j\neq i} u_{ij}^k x_j^{(l-1)} + b_i$ or $a_{ii}\tilde{x}_i^{l,k} + \Phi_i(\tilde{x}_i^{l,k}) = \sum_{j<i} l_{ij}^k y_j^{l,k} + \sum_{j\neq i} u_{ij}^k x_j^{(l-1)} + b_i$.

Remark 3.2. Here we note $q(l,k)$ as the numbers of local iteration in step (I) of the Algorithm 1, i.e., each processor excuses different numbers of local iteration.

Remark 3.3. From Table 1, we obviously see that ALNPM-AOR method in [1] used the same parameters μ, ω in different processors, but ALNPMM-AOR method in this paper used different parameters $\mu_k, \omega_k (k = 1, 2, ..., \alpha)$ in different processors. Moreover, the convergence results in [1] are $0 \le \mu \le \omega < \frac{2}{1+\rho}, \omega \neq 0, q(l,k) \ge 1$, but the convergence results in this paper are $0 \le \mu_k < \frac{2}{1+\rho}, 0 < \omega_k < \frac{2}{1+\rho}, q(l,k) \ge 1$. So, our methods are not only the generalization of [1,8,12,16,20], but also convergence results of our new method are weaker than those of Arnal's. In ALNPMM-style methods, we may choose proper E_k to balance the load of each processor and avoid synchronization.

Table 1: different non-stationary multisplitting multi-parameters methods and corresponding convergence results

$\omega_k, \mu_k, q(l,k)$	$\Phi(x)$	Method	Ref
$\mu_k = \omega_k = 1, q(l,k) \ge 1$	0	NPM-G-S	[8]
$\mu_k = \omega_k \neq 0, q(l,k) \ge 1$	0	NPM-SOR	[8]
$0 \le \mu_k \le \omega_k < \frac{2}{1+\rho}, \omega_k \neq 0, q(l,k) \ge 1$	0	NPMM-AOR	[20]
$\mu_k = \omega_k = 1, q(l,k) = 1$	$\neq 0$	ALPM-G-S	[12]
$\mu_k = \omega_k \neq 0, q(l,k) = 1$	$\neq 0$	ALPM-SOR	[16]
$0 \le \mu(\mu_k) \le \omega(\omega_k) < \frac{2}{1+\rho}, \omega \neq 0, q(l,k) \ge 1$	$\neq 0$	ALNPM-AOR	[1]
$0 \le \mu_k < \frac{2}{1+\rho}, 0 < \omega_k < \frac{2}{1+\rho}, q(l,k) \ge 1$	$\neq 0$	ALNPMM-AOR	this paper

In 1999, based the non-stationary parallel multisplitting methods for almost linear systems, Arnal et al. obtained the following results.

Theorem 3.1. [1] *Let $A = D - L_k - U_k = D - B,\ k = 1, 2, ..., \alpha$ be an H-matrix, where $D = \mathrm{diag(A)}$ and L_k are strictly lower triangular matrices. Assume that $|B| = |L_k| + |U_k|,\ k = 1, 2, ..., \alpha$. Let Φ be a continuous and diagonal mapping satisfying*

$$\mathrm{sign}(a_{ii})(t-s)(\Phi_i(t) - \Phi_i(s)) \ge 0,\ i = 1, 2, ..., n,\ \text{for all } t, s \in R.$$

If $0 \le \mu \le \omega < \frac{2}{1+\rho}$, with $\omega \neq 0$, where $\rho = \rho(|D|^{-1}|B|)$, and $q(l,k) \ge 1,\ l = 1, 2, ...,\ k = 1, 2, ..., \alpha$, then ALNPM method is well-defined and converges to the unique solution of the almost linear system (1), for every initial vector $x^{(0)} \in R^n$.

The following results show that our methods are the generalization of Arnal's [1] and our convergence results are weaker than those of Arnal's. The parameters can be adjusted suitably so that the convergence property of methods can be substantially improved. Furthermore, we primely improve Arnal's [1] convergence results.

Theorem 3.2. *Let* $A = D - L_k - U_k = D - B,\ k = 1, 2, ..., \alpha$ *be an* H*-matrix, where* $D = \mathrm{diag(A)}$ *and* L_k *are strictly lower triangular matrices. Assume that* $|B| = |L_k| + |U_k|,\ k = 1, 2, ..., \alpha$. *Let* Φ *be a continuous and diagonal mapping satisfying*

$$\mathrm{sign}(a_{ii})(t - s)(\Phi_i(t) - \Phi_i(s)) \geq 0,\ i = 1, 2, ..., n,\ \text{for all } t, s \in R. \tag{8}$$

If

$$0 \leq \mu_k < \frac{2}{1+\rho},\ 0 < \omega_k < \frac{2}{1+\rho}, \tag{9}$$

where $\rho = \rho(|D|^{-1}|B|)$*, and* $q(l, k) \geq 1,\ l = 1, 2, ...,\ k = 1, 2, ..., \alpha$, *then AL-NPMM method is well-defined and converges to the unique solution of the almost linear system (1), for every initial vector* $x^{(0)} \in R^n$.

Proof. From (8), we can easily obtain that each $r_i,\ i = 1, 2, ..., n$ has an inverse function defined in all of R and thus iteration scheme (6) is well-defined. Furthermore, by Lemma 2.3, almost linear system (1) has a unique solution x^*. Let $\epsilon^{(l)} = x^{(l)} - x^*$ denote the error vector, then

$$|\epsilon^{(l)}| \leq \sum_{k=1}^{\alpha} E_k |x^{l,k,q(l,k)} - x^*|. \tag{10}$$

To analyze $|\epsilon^{(l)}|$ of (10) we need to study the expressions $|x^{l,k,m} - x^*|,\ m = 1, 2, ..., q(l, k)$. At first, we will prove that $|a_{ii}||y - \tilde{y}| \leq |r_i(y) - r_i(\tilde{y})|,\ y, \tilde{y} \in R$, where $r_i(t) = a_{ii}t + \Phi_i(t),\ t \in R$. Then

$$\begin{aligned} |r_i(y) - r_i(\tilde{y})| &= |a_{ii}y + \Phi_i(y) - a_{ii}\tilde{y} - \Phi_i(\tilde{y})| \\ &= |a_{ii}(y - \tilde{y}) + (\Phi_i(y) - \Phi_i(\tilde{y}))|. \end{aligned} \tag{11}$$

From (8), we can get $\mathrm{sign}(a_{ii})(y - \tilde{y})(\Phi_i(y) - \Phi_i(\tilde{y})) \geq 0,\ i = 1, 2, ..., n,$ for all $y, \tilde{y} \in R$, then $a_{ii}(y - \tilde{y})$ and $\Phi_i(y) - \Phi_i(\tilde{y})$ have the same sign. So

$$|r_i(y) - r_i(\tilde{y})| \geq |a_{ii}(y - \tilde{y})| = |a_{ii}||y - \tilde{y}|.$$

$$|a_{ii}||r_i^{-1}(z) - r_i^{-1}(\tilde{z})| \leq |z - \tilde{z}|,\ z, \tilde{z} \in R.$$

Since $x_i^* = r_i^{-1}(z_i^k),\ z^k = \mu_k L_k x^* + (1-\mu_k)L_k x^* + U_k x^* + b$, then we have, for each $i = 1, 2, ..., n$

$$
\begin{aligned}
|a_{ii}||\tilde{x}_i^{l,k,m} - x_i^*| &= |a_{ii}||r_i^{-1}(z_i^{l,k,m}) - r_i^{-1}(z_i^k)| \le |z_i^{l,k,m} - z_i^k| \\
&= |[\mu_k L_k(\tilde{x}^{l,k,m} - x^*) + (1-\mu_k)L_k(x^{l,k,m-1} - x^*) \\
&\quad + U_k(x^{l,k,m-1} - x^*)]_i|.
\end{aligned} \tag{12}
$$

Due to the randomicity of i, we can obtain

$$
\begin{aligned}
|D||\tilde{x}^{l,k,m} - x^*| \le\ & |\mu_k L_k(\tilde{x}^{l,k,m} - x^*) + (1-\mu_k)L_k(x^{l,k,m-1} - x^*) \\
& + U_k(x^{l,k,m-1} - x^*)|.
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
|D||x^{l,k,m} - x^*| &\le \omega_k|D||\tilde{x}^{l,k,m} - x^*| + |1-\omega_k||D||\tilde{x}^{l,k,m-1} - x^*| \\
&\le \omega_k|\mu_k L_k(\tilde{x}^{l,k,m} - x^*) + (1-\mu_k)L_k(x^{l,k,m-1} - x^*) \\
&\quad + U_k(x^{l,k,m-1} - x^*)| + |1-\omega_k||D||x^{l,k,m-1} - x^*|.
\end{aligned} \tag{14}
$$

Since $(|D| - \mu_k|L_k|)^{-1} \ge 0$ and $\tilde{x}^{l,k,m} = \frac{1}{\omega_k}x^{l,k,m} + (1 - \frac{1}{\omega_k})x^{l,k,m-1}$, then we may get

$$
\begin{aligned}
|x^{l,k,m} - x^*| \le\ & (|D| - \mu_k|L_k|)^{-1}(|\omega_k - \mu_k||L_k| + \omega_k|U_k| \\
& + |1-\omega_k||D|)|x^{l,k,m-1} - x^*|,\ m = 1, 2, ..., q(l,k).
\end{aligned} \tag{15}
$$

So, we can obtain

$$
\begin{aligned}
|\epsilon^{(l)}| &\le \sum_{k=1}^{\alpha} E_k |x^{l,k,q(l,k)} - x^*| \\
&\le \sum_{k=1}^{\alpha} E_k((|D| - \mu_k|L_k|)^{-1} \\
&\qquad (|\omega_k - \mu_k||L_k| + \omega_k|U_k| + |1-\omega_k||D|))^{q(l,k)}|x^{(l-1)} - x^*| \\
&= \sum_{k=1}^{\alpha} E_k H_k^{(l)} |\epsilon^{(l-1)}| \le ... \\
&\le \sum_{k=1}^{\alpha} E_k H_k^{(l)} ... \sum_{k=1}^{\alpha} E_k H_k^{(1)} |\epsilon^{(0)}|,
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
H^{(l)} &= \sum_{k=1}^{\alpha} E_k H_k^{(l)}, \\
H_k^{(1)} &= ((|D| - \mu_k|L_k|)^{-1}((\omega_k - \mu_k)|L_k| + \omega_k|U_k| + (1-\omega_k)|D|))^{q(l,k)}.
\end{aligned}
$$

Now, we will prove that there exist a real constant $0 \leq \theta < 1$ and a positive vector ν such that $H^{(l)}\nu \leq \theta\nu$. Then, from Lemma 2.2 we can get that $\lim_{l\to\infty} H^{(l)}H^{(l-1)}...H^{(1)} = 0$ and $\lim_{l\to\infty} \epsilon^{(l)} = 0$. Define

$$
\begin{aligned}
M_k &= |D| - \mu_k|L_k|. \\
N_k &= |\omega_k - \mu_k||L_k| + \omega_k|U_k| + |1-\omega_k||D|. \\
H^{(l)} &\leq \sum_{k=1}^{\alpha} E_k(M_k^{-1}N_k)^{q(l,k)}.
\end{aligned}
\tag{17}
$$

Case 1. $0 \leq \mu_k \leq 1,\ 0 < \omega_k \leq 1$. Define

$$A_k = M_k - N_k = (|D| - \mu_k|L_k|) - (|\omega_k - \mu_k||L_k| + \omega_k|U_k| + |1-\omega_k||D|). \tag{18}$$

Subcase 1. $\mu_k \leq \omega_k$. From (18), we have

$$A_k = M_k - N_k = \omega_k|D| - \omega_k|L_k| - \omega_k|U_k| = \omega_k\langle A\rangle,\ k = 1, 2, ..., \alpha. \tag{19}$$

Let e denote the vector $e = (1, 1, ..., 1)^T \in R^n$, and for each $k,\ \nu_1 = \omega_k^{-1}\langle A\rangle^{-1}e$. Since $\omega_k^{-1}\langle A\rangle^{-1} \geq 0$ and no row of $\omega_k^{-1}\langle A\rangle^{-1}$ can have all null entries, we can obtain $\nu_1 > 0$ and $M_k^{-1}e > 0$. So

$$\nu_1 - M_k^{-1}N_k\nu_1 = M_k^{-1}\omega_k\langle A\rangle\nu_1 = M_k^{-1}e > 0.$$

Then $M_k^{-1}N_k\nu_1 < \nu_1$, thus there exists a real constant $0 \leq \theta_1 < 1$, such that $M_k^{-1}N_k \leq \theta_1\nu_1 < \nu_1$, for all $k = 1, 2, ..., \alpha$. So

$$H^{(l)}\nu_1 \leq \sum_{k=1}^{\alpha} E_k(M_k^{-1}N_k)^{q(l,k)}\nu_1 \leq \theta_1\nu_1,\ l = 0, 1, ...,$$

Subcase 2. $\mu_k \geq \omega_k$. From (18), we have

$$A_k = M_k - N_k = \omega_k|D| - (2\mu_k - \omega_k)|L_k| - \omega_k|U_k|,\ k = 1, 2, ..., \alpha. \tag{20}$$

Since $\mu_k \geq \omega_k$, then $-(2\mu_k - \omega_k) \leq -\omega_k$, for each k. So

$$A_k = M_k - N_k \geq \omega_k|D| - (2\mu_k - \omega_k)|B|,\ k = 1, 2, ..., \alpha.$$

Obviously, the matrices $\omega_k|D| - (2\mu_k - \omega_k)|B|$ are Z-matrices and have positive diagonal entries. Moreover, $0 \leq \mu_k \leq 1,\ 0 < \omega_k \leq 1$, and $\frac{(2\mu_k-\omega_k)\rho}{\omega_k} < 1$ for each k, then from Lemma 2.4 we get that $\omega_k|D| - (2\mu_k - \omega_k)|B|$ are M-matrices.

Let e denote the vector $e = (1, 1, ..., 1)^T \in R^n$, and for each $k,\ \nu_2 = [\omega_k|D| - (2\mu_k - \omega_k)|B|]^{-1}e$. Since $[\omega_k|D| - (2\mu_k - \omega_k)|B|]^{-1} \geq 0$ and no row of $[\omega_k|D| -$

$(2\mu_k - \omega_k)|B|]^{-1}$ can have all null entries, we can obtain $\nu_2 > 0$ and $M_k^{-1}e > 0$. So

$$\nu_2 - M_k^{-1}N_k\nu_2 \geq M_k^{-1}[\omega_k|D| - (2\mu_k - \omega_k)|B|]\nu_2 = M_k^{-1}e > 0.$$

Then $M_k^{-1}N_k\nu_2 < \nu_2$, thus there exists a real constant $0 \leq \theta_2 < 1$, such that $M_k^{-1}N_k \leq \theta_2\nu_2 < \nu_2$, for all $k = 1, 2, ..., \alpha$. So

$$H^{(l)}\nu_2 \leq \sum_{k=1}^{\alpha} E_k(M_k^{-1}N_k)^{q(l,k)}\nu_2 \leq \theta_1\nu_2,\ l = 0, 1, ...,$$

Case 2. $0 \leq \mu_k \leq 1,\ 1 < \omega_k < \frac{2}{1+\rho}$. From (18), we have

$$A_k = M_k - N_k = (2 - \omega_k)|D| - \omega_k|B|,\ k = 1, 2, ..., \alpha. \tag{21}$$

Obviously, the matrices $(2 - \omega_k)|D| - \omega_k|B|$ are Z-matrices and have positive diagonal entries. Moreover, $0 \leq \mu_k \leq 1,\ 1 < \omega_k < \frac{2}{1+\rho}$, and $\frac{\omega_k\rho}{2-\omega_k} < 1$ for each k, then from Lemma 2.4 we get that $(2 - \omega_k)|D| - \omega_k|B|$ are M-matrices.

Consider the vector $\nu_3 = [(2 - \omega_k)|D| - \omega_k|B|]^{-1}e,\ e = (1, 1, ..., 1)^T$. Similar to Case 1, there exists $0 \leq \theta_3 < 1$ such that $M_k^{-1}N_k\nu_3 \leq \theta_3\nu_3 < \nu_3$. Then, $H^{(l)}\nu_3 \leq \theta_3\nu_3$.

Case 3. $1 \leq \mu_k < \frac{2}{1+\rho},\ 0 < \omega_k \leq 1$. From (18), we have

$$A_k = M_k - N_k = \omega_k|D| - (2\mu_k - \omega_k)|L_k| - \omega_k|U_k|,\ k = 1, 2, ..., \alpha. \tag{22}$$

Since $\mu_k \geq \omega_k$, then $-(2\mu_k - \omega_k) \leq -\omega_k$, for each k. So

$$A_k = M_k - N_k \geq \omega_k|D| - (2\mu_k - \omega_k)|B|,\ k = 1, 2, ..., \alpha.$$

Similar to Subcase 2 of Case 1, then from Lemma 2.4 we get that $\omega_k|D| - (2\mu_k - \omega_k)|B|$ are M-matrices.

Consider the vector $\nu_4 = [\omega_k|D| - (2\mu_k - \omega_k)|B|]^{-1}e,\ e = (1, 1, ..., 1)^T$. Similar to Case 1, there exists $0 \leq \theta_4 < 1$ such that $M_k^{-1}N_k\nu_4 \leq \theta_4\nu_4 < \nu_4$. Then, $H^{(l)}\nu_4 \leq \theta_4\nu_4$.

Case 4. $1 < \mu_k < \frac{2}{1+\rho},\ 1 < \omega_k < \frac{2}{1+\rho}$.

Subcase 1. $\mu_k \geq \omega_k$. From (18), we have

$$A_k = M_k - N_k = (2 - \omega_k)|D| - (2\mu_k - \omega_k)|L_k| - \omega_k|U_k|,\ k = 1, 2, ..., \alpha. \tag{23}$$

Since $\mu_k \geq \omega_k$, then $-(2\mu_k - \omega_k) \leq -\omega_k$, for each k. So

$$A_k = M_k - N_k = (2 - \omega_k)|D| - (2\mu_k - \omega_k)|B|.$$

Obviously, the matrices $(2-\omega_k)|D|-(2\mu_k-\omega_k)|B|$ are Z-matrices and have positive diagonal entries. Moreover, $1 < \mu_k < \frac{2}{1+\rho},\ 1 < \omega_k < \frac{2}{1+\rho}$, and $\frac{(2\mu_k-\omega_k)\rho}{2-\omega_k} < 1$ for each k, then from Lemma 2.4 we get that $(2-\omega_k)|D|-(2\mu_k-\omega_k)|B|$ are M-matrices.

Consider the vector $\nu_5 = [(2-\omega_k)|D|-(2\mu_k-\omega_k)|B|]^{-1}e,\ e = (1, 1, ..., 1)^T$. Similar to Case 1, there exists $0 \le \theta_5 < 1$ such that $M_k^{-1}N_k\nu_5 \le \theta_5\nu_5 < \nu_5$. Then, $H^{(l)}\nu_5 \le \theta_5\nu_5$.

Subcase 2. $\mu_k \le \omega_k$. From (18), we have

$$A_k = M_k - N_k = (2-\omega_k)|D| - \omega_k|B|,\ k = 1, 2, ..., \alpha. \tag{24}$$

Obviously, the matrices $(2-\omega_k)|D|-\omega_k|B|$ are Z-matrices and have positive diagonal entries. Moreover, $1 < \mu_k < \frac{2}{1+\rho},\ 1 < \omega_k < \frac{2}{1+\rho}$, and $\frac{\omega_k\rho}{2-\omega_k} < 1$ for each k, then from Lemma 2.4 we get that $(2-\omega_k)|D|-\omega_k|B|$ are M-matrices.

Consider the vector $\nu_6 = [(2-\omega_k)|D|-\omega_k|B|]^{-1}e,\ e = (1, 1, ..., 1)^T$. Similar to Case 1, there exists $0 \le \theta_6 < 1$ such that $M_k^{-1}N_k\nu_6 \le \theta_6\nu_6 < \nu_6$. Then, $H^{(l)}\nu_6 \le \theta_6\nu_6$. ■

Corollary 3.3. *Let $A \in R^{n\times n}$ satisfy one of the following conditions*

(a) *A is an M-matrix;*

(b) *A is a strictly or an irreducibly diagonally dominant matrix;*

(c) *A is a symmetric positive definite L-matrix. Moreover, Let $A = D - L_k - U_k = D - B,\ k = 1, 2, ..., \alpha$, where $D = \mathrm{diag(A)}$ and L_k are strictly lower triangular matrices. Assume that $|B| = |L_k| + |U_k|,\ k = 1, 2, ..., \alpha$. Let Φ be a continuous and diagonal mapping satisfying*

$$\mathrm{sign}(a_{ii})(t-s)(\Phi_i(t)-\Phi_i(s)) \ge 0,\ i = 1, 2, ..., n, \text{ for all } t, s \in R.$$

If

$$0 \le \mu_k < \frac{2}{1+\rho},\ 0 < \omega_k < \frac{2}{1+\rho},$$

where $\rho = \rho(|D|^{-1}|B|)$, and $q(l, k) \ge 1,\ l = 1, 2, ...,\ k = 1, 2, ..., \alpha$, then ALNPMM method is well-defined and converges to the unique solution of the almost linear system (1), for every initial vector $x^{(0)} \in R^n$.

Remark 3.4. It is obvious that the conditions of Theorem 3.2 and Corollary 3.3 in this paper are weaker than those of Theorem 3.1 and Corollary 3.1 in [1].

Remark 3.5. Using similar proving course of this paper, $0 \le \mu \le \omega < \frac{2}{1+\rho}$ $(\omega \ne 0)$ of Corollary 3.1, Theorem 4.2 and Corollary 4.1 based ALNPM method in [1] may also be changed into $0 \le \mu_k < \frac{2}{1+\rho},\ 0 < \omega_k < \frac{2}{1+\rho}$ based ALNPMM method.

4. Numerical Examples

In this section, we compare the previously established ALNPMM method with ALNPM method by using the following example of the system of nonlinear equation (1).

$$A = \begin{bmatrix} 8 & -1 & 0 & 0 & 0 & 0 \\ -1 & 8 & -1 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -1 & 8 & -1 \\ 0 & 0 & 0 & 0 & -1 & 8 \end{bmatrix} \in R^{n\times n},\ D = \mathrm{diag}(8, 8, ..., 8),$$

$$\Phi(x) = \begin{bmatrix} x_1 e^{x_1} \\ x_2 e^{2x_2} \\ \vdots \\ x_{n-1} e^{(n-1)x_{n-1}} \\ x_n e^{nx_n} \end{bmatrix} : R^n \to R^n,\ b = \begin{bmatrix} 3 \\ 3 \\ \vdots \\ 3 \\ 3 \end{bmatrix} \in R^n.$$

We take $\alpha = 2$,

$$S_1 = \{1, 2, ..., m_1\},\ S_2 = \{m_2, m_2 + 1, ..., n\},$$

with m_1, m_2 being positive integers satisfying $1 \le m_2 \le m_1 \le n$, and the particular multisplitting $(D - L_k, U_k, E_k),\ k = 1, 2$ of the coefficient matrix $A \in R^{n\times n}$:

$$L_k = (l_{ij}^{(k)}),\ l_{ij}^{(k)} = \begin{cases} 1, & \text{if } j = i - 1,\ i, j \in S_k, \\ 0, & \text{otherwise}, \end{cases}$$

$$U_k = (u_{ij}^{(k)}),\ u_{ij}^{(k)} = \begin{cases} 0, & \text{if } j = i, \\ -(a_{ij} + l_{ij}^{(k)}), & \text{otherwise}, \end{cases}$$

$$e_j^{(1)} = \begin{cases} 1, & \text{if } 1 \le j < m_2, \\ \frac{1}{2}, & \text{if } m_2 \le j \le m_1, \\ 0, & \text{if } m_1 < j \le n, \end{cases} \quad e_j^{(2)} = \begin{cases} 0, & \text{if } 1 \le j < m_2, \\ \frac{1}{2}, & \text{if } m_2 \le j \le m_1, \\ 1, & \text{if } m_1 < j \le n, \end{cases}$$

The computations are proceeded with $m_1 = [\frac{4n}{5}],\ m_2 = [\frac{n}{5}]$. Let $\mu_k = \mu, \omega_k = \omega$, we will show the validity of Theorem 3.2 and Corollary 3.3 in this paper and compare the spectral radius of $H^{(l)}$ with Theorem 3.1 in [1]. Let Ω_1 and Ω_2 denote the convergence domain of Arnal's and this paper, respectively. Let $'-'$ denote

Table 2: Comparisons of convergence domain and spectral radii about ALNPM and ALNPMM methods

n	$\frac{2}{1+\rho}$	$q(l,k)$	(μ_k, ω_k)	$\rho(H^{(l)})$	Ω_1	Ω_2
10	1.6131	1,1	(0.4,0.2)	0.8397	$-$	$+$
10	1.6131	1,1	(0.2,0.4)	0.8794	$+$	$+$
10	1.6131	1,1	(1.4,0.5)	0.5258	$-$	$+$
10	1.6131	1,1	(0.5,1.4)	0.7345	$+$	$+$
10	1.6131	1,1	(1.6,1.3)	0.3369	$-$	$+$
10	1.6131	1,1	(1.3,1.6)	0.6087	$+$	$+$
100	1.6002	1,1	(0.4,0.2)	0.8415	$-$	$+$
100	1.6002	1,1	(0.2,0.4)	0.8841	$+$	$+$
100	1.6002	1,1	(1.3,0.6)	0.4315	$-$	$+$
100	1.6002	1,1	(0.6,1.3)	0.6412	$+$	$+$
100	1.6002	1,1	(1.5,1.1)	0.1935	$-$	$+$
100	1.6002	1,1	(1.1,1.5)	0.3529	$+$	$+$
10	1.6131	2,1	(0.4,0.2)	0.8237	$-$	$+$
10	1.6131	2,1	(0.2,0.4)	0.8539	$+$	$+$
10	1.6131	2,1	(1.4,0.5)	0.5072	$-$	$+$
10	1.6131	2,1	(0.5,1.4)	0.6693	$+$	$+$
10	1.6131	2,1	(1.6,1.3)	0.2892	$-$	$+$
10	1.6131	2,1	(1.3,1.6)	0.5235	$+$	$+$
100	1.6002	2,1	(0.4,0.2)	0.8408	$-$	$+$
100	1.6002	2,1	(0.2,0.4)	0.8823	$+$	$+$
100	1.6002	2,1	(1.3,0.6)	0.4309	$-$	$+$
100	1.6002	2,1	(0.6,1.3)	0.6369	$+$	$+$
100	1.6002	2,1	(1.5,1.1)	0.1921	$-$	$+$
100	1.6002	2,1	(1.1,1.5)	0.3495	$+$	$+$

(μ_k, ω_k) be not included in the set of Ω and $'+'$ denote (μ_k, ω_k) be included in the set of Ω, where Ω is a set of (μ_k, ω_k).

From Table 2, on the one hand, we may know that when convergence results [1] cannot be used, our results can be used and $\rho(H^{(l)}) < 1$. On the other hand, we may obtain that, to satisfy the convergence domain [1] when we exchange μ_k and ω_k, spectral radius of $H^{(l)}$ always increases.

Remark 4.1. The above numerical examples not only clearly show the validity of Theorem 3.2 and Corollary 3.3 in this paper, but also show that our new convergence domain is wider than Arnal's [1] convergence domain. Furthermore, in some conditions $\mu_k \geq \omega_k,\ k = 1, 2, ..., \alpha$ have better convergence rate.

5. Conclusion

In this paper, we extend the non-stationary parallel multisplitting methods (ALNPM-style) [1] to the non-stationary parallel multisplitting multi-parameters (ALNPMM-style) methods for almost linear systems, then our methods are the generalization of [1,8,12,16,20]. Furthermore, we study the convergence of our methods when the matrices in question are either H-matrices or M-matrices and the convergence results of our new method in this paper are weaker than those of Arnal's [1]. The parameters can be adjusted suitably so that the convergence property of methods can be substantially improved. Finally, numerical examples show that our new convergence results are weaker than Arnal's.

Particularly, one may discuss how choose the set of relaxed parameters in order to really accelerate the convergence of the considered method. Furthermore, The optimal choice of this set of relaxed parameters is valuably studied.

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