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NONLINEAR OPERATORS OF MONOTONE TYPE AND CONVERGENCE THEOREMS WITH EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. Our purpose in this paper is first to discuss nonlinear operators and nonlinear projections in Banach spaces which are related to the resolvents of m-accretive operators and maximal monotone operators. Some of these operators in Banach spaces are new. Next, we discuss some properties for such nonlinear operators and nonlinear projections in Banach spaces. Further, using these properties, we prove strong convergence theorems by hybrid methods for nonlinear operators with equilibrium problems in Banach spaces.

1. INTRODUCTION

Let E be a Banach space and let E^* be the dual space of E. Let $A : E \to 2^E$ be a multi-valued operator with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$. Then, A is called accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$, where J is the duality mapping from E into 2^{E^*} An accretive operator A is m-accretive if and only if $R(I + rA) = E$ for all $r > 0$. If A is m-accretive, then for each $r > 0$ and $x \in E$, we can define the resolvent $J_r\colon R(I + rA) \to D(A)$ by $J_rx = \{z \in E : x \in z + rAz\}$. A multi-valued operator $A\colon E \to 2^{E^*}$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. The monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. Let E be a reflexive, strictly convex and smooth Banach space and let $A\colon E \to 2^{E^*}$ be a monotone operator. Then, A is maximal if and only if $R(J + rA) = E^*$ for all

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$r > 0$; see [47]. If $A : E \to 2^{E^*}$ is a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we can consider the following resolvents:

$$J_\lambda x = \{z \in E : 0 \in J(z - x) + \lambda A(z)\}$$

and

$$Q_\lambda x = \{z \in E : Jx \in Jz + \lambda A(z)\}.$$

Further, if $B : E^* \to 2^E$ be a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we can consider the resolvent

$$R_\lambda x = \{z \in E : x \in z + \lambda BJ(z)\}.$$

These four resolvents are important and have interesting properties.

Let C be a nonempty closed convex subset of a Banach space E and let $f : C \times C \to \mathbb{R}$ be a bifunction. We consider the following equilibrium problem:

$$\text{Find } z \in C \text{ such that } f(z, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of such $z \in C$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{z \in C : f(z, y) \geq 0, \quad \forall y \in C\}.$$

Problem (1.1) is also important in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others; see, for instance, [32] and [6].

Our purpose in this paper is first to discuss nonlinear operators and nonlinear projections in Banach spaces which are related to the resolvents of m-accretive operators and maximal monotone operators. Some of these operators are new. Next, we discuss fixed point theorems and duality theorems for such nonlinear operators and nonlinear projections in Banach spaces. Further, using these properties, we prove strong convergence theorems by hybrid methods for nonlinear operators with equilibrium problems in Banach spaces.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty closed convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call P_C the metric projection of E onto C. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J ia a single valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm to norm continuous. If E is uniformly smooth, J is uniformly norm to norm continuous on each bounded subset of E. For more details, see [46].

We know the following result: Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty closed convex subset of E and let P_C be the metric projection of E onto C. Let $x_0 \in C$ and $x_1 \in E$. Then, $x_0 = P_C(x_1)$ if and only if

$$\langle x_0 - y, J(x_1 - x_0) \rangle \geq 0$$

for all $y \in C$, where J is the duality mapping of E.

Let C be a nonempty subset of E and let T be a mapping of C into E. We denote the set of all fixed points of T by $F(T)$. A mapping $T \colon C \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Let D be a subset of C and let P be a mapping of C into D. Then P is said to be sunny if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into C is said to be a retraction if $P^2 = P$. We denote the closure of the convex hull of D by $\overline{\text{co}}D$.

Let E be a Banach space and let $A : E \to 2^E$ be a multi-valued operator. Then, A is called accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. An accretive operator A is m-accretive if and only if $R(I + rA) = E$ for all $r > 0$. If A is m-accretive, then for each $r > 0$ and $x \in E$, we can define $J_r \colon R(I + rA) \to D(A)$ by $J_r x = \{z \in E : x \in z + rAz\}$. We call such $J_r = (I + rA)^{-1}$ the accretive resolvent of A for $r > 0$.

A multi-valued operator $A \colon E \to 2^{E^*}$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. The following theorems are well known; see, for instance, [47].

Theorem 2.1. *Let E be a reflexive, strictly convex and smooth Banach space and let $A \colon E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Theorem 2.2. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let E be a reflexive, strictly convex and smooth Banach space and let $A : E \to 2^{E^*}$ be a maximal monotone operator. Then, for $\lambda > 0$ and $x \in E$, consider

$$J_\lambda x = \{z \in E : 0 \in J(z - x) + \lambda A(z)\}$$

and

$$Q_\lambda x = \{z \in E : Jx \in Jz + \lambda A(z)\}.$$

We denote J_λ and Q_λ by $J_\lambda = (I + \lambda J^{-1}A)^{-1}$ and $Q_\lambda = (J + \lambda A)^{-1}J$, respectively. We call such J_λ and Q_λ the metric resolvent and the relative resolvent of A for $\lambda > 0$, respectively. We also consider another resolvent of a maximal monotone operator. Let $B : E^* \to 2^E$ be a maximal monotone operator. Then, for $\lambda > 0$ and $x \in E$, consider

$$R_\lambda x = \{z \in E : x \in z + \lambda BJ(z)\}.$$

We denote R_λ by $R_\lambda = (I + \lambda BJ)^{-1}$. We call such R_λ the generalized resolvent of B for $\lambda > 0$.

Let E be a reflexive, strictly convex and smooth Banach space. The function $\phi\colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [20]. We have from the definition of ϕ that

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz - Jy\rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\|^2 - \|y\|^2) \le \phi(x,y)$ for all $x, y \in E$, we can see that $\phi(x,y) \ge 0$. Let $\phi_*\colon E^* \times E^* \to (-\infty, \infty)$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi(x,y) = \phi_*(Jy, Jx) \tag{2.3}$$

for $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x,y) = 0 \iff x = y. \tag{2.4}$$

If C is a nonempty closed convex subset of a smooth, strictly and reflexive Banach space E, then for all $x \in E$ there exists a unique $z \in C$ (denoted by $\Pi_C x$) such that

$$\phi(z,x) = \min_{y \in C} \phi(y,x). \tag{2.5}$$

The mapping Π_C is called the generalized projection from E onto C; see Alber [1], Alber and Reich [2], and Kamimura and Takahashi [20].

The following theorem is well known; see, for instance, [20].

Theorem 2.3. *Let E be a reflexive, strictly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

For a sequence $\{C_n\}$ of nonempty closed convex subsets of a reflexive Banach space E, define s-Li$_n C_n$ and w-Ls$_n C_n$ as follows: $x \in$s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-Ls$_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies that

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.6}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [30] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [30]. We know the following theorem [11].

Theorem 2.4. *Let E be a smooth Banach space and let E^* have a Fréchet differentiable norm. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. If $C_0 =$M-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\Pi_{C_n} x$ converges strongly to $\Pi_{C_0} x$, where Π_{C_n} and Π_{C_0} are the generalized projections of E onto C_n and C_0, respectively.*

Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. For solving the equilibrium problem, let us assume that a bifunction $f : JC \times JC \to \mathbb{R}$ satisfies the following conditons:

(A1) $f(x^*, x^*) = 0, \quad \forall x^* \in JC$;

(A2) f is monotone, i.e., $f(x^*, y^*) + f(y^*, x^*) \leq 0, \quad \forall x^*, y^* \in JC$;

(A3) $\lim_{t\downarrow 0} f(tz^* + (1-t)x^*, y^*) \leq f(x^*, y^*), \quad \forall x^*, y^*, z^* \in JC$;

(A4) for each $x^* \in JC$, $y^* \mapsto f(x^*, y^*)$ is convex and lower semicontinuous.

The following result is in Takahashi and Zembayashi [57]; see also Blum and Oettli [6] and Aoyama, Kimura and Takahashi [3].

Lemma 2.5. *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Then, for any $r > 0$ and $x \in E$, there exists a unique $z \in C$ such that*

$$f(Jz, Jy) + \frac{1}{r}\langle Jy - Jz, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define $T_r x = \{z \in C : f(Jz, Jy) + \frac{1}{r}\langle Jy - Jz, z - x\rangle \geq 0,\ \forall y \in C\}$ for all $r > 0$ and $x \in H$. Then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly generalized nonexpansive, i.e.,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle x - y, JT_r x - JT_r y\rangle, \quad \forall x, y \in E;$$

(3) $F(T_r) = EP(f)$;

(4) *$JEP(f)$ is closed and convex;*

(5) $\phi(x, T_r x) + \phi(T_r x, q) \leq \phi(x, q), \quad \forall x \in E,\ q \in F(T_r)$.

3. Four Nonlinear Retractions

In this section, we first define nonlinear operators which are deduced from m-accretive operators and maximal monotone operators in a Banach space. If $A : E \to 2^E$ is m-accretive, then for each $\lambda > 0$ and $x \in E$, we can define the accretive resolvent $J_\lambda \colon E \to D(A)$ by $J_\lambda x = \{z \in E : x \in z + \lambda Az\}$. Then, we know that $J_\lambda x$ is always nonempty and a singleton. We denote such J_λ by $J_\lambda = (I + \lambda A)^{-1}$. If J_λ is the accretive resolvent, then we can show that

$$0 \le \langle x - J_\lambda x - (y - J_\lambda y), J(J_\lambda x - J_\lambda y)\rangle$$

for all $x, y \in E$. Let C be a subset of E. Then, a nonlinear operator $T : C \to E$ is called firmly nonexpansive if

$$0 \le \langle x - Tx - (y - Ty), J(Tx - Ty)\rangle \tag{3.1}$$

for all $x, y \in C$. If $A : E \to 2^{E^*}$ is a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we define the metric resolvent $J_\lambda \colon E \to D(A)$ by $J_\lambda x = \{z \in E : 0 \in J(z - x) + \lambda A(z)\}$. Then, we know that $J_\lambda x$ is always nonempty and a singleton. We denote such J_λ by $J_\lambda = (I + \lambda J^{-1}A)^{-1}$. If J_λ is the metric resolvent, then we have

$$0 \le \langle J_\lambda x - J_\lambda y, J(x - J_\lambda x) - J(y - J_\lambda y)\rangle$$

for all $x, y \in E$; see, for instance, [4]. In general, a nonlinear operator $T : C \to E$ is called firmly metric if

$$0 \le \langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \tag{3.2}$$

for all $x, y \in C$. If $A : E \to 2^{E^*}$ is a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we can consider the relative resolvent $Q_\lambda \colon E \to D(A)$ by $Q_\lambda x = \{z \in E : Jx \in Jz + \lambda A(z)\}$. Then, we know that $Q_\lambda x$ is always nonempty and a singleton. We denote such Q_λ by $Q_\lambda = (J + \lambda A)^{-1}J$. If Q_λ is the relative resolvent, then we have

$$0 \le \langle J_\lambda x - J_\lambda y, Jx - JJ_\lambda x - (Jy - JJ_\lambda y)\rangle$$

for all $x, y \in E$. In general, a nonlinear operator $T : C \to E$ is firmly relative nonexpansive if

$$0 \le \langle Tx - Ty, Jx - JTx - (Jy - JTy)\rangle \tag{3.3}$$

for all $x, y \in C$. We can define another nonlinear operator. If $B : E^* \to 2^E$ is a maximal monotone operator, then for $\lambda > 0$ and $x \in E$, we can consider the generalized resolvent $R_\lambda \colon E \to D(A)$ by $R_\lambda x = \{z \in E : x \in z + \lambda BJ(z)\}$.

Then, we know that $R_\lambda x$ is always nonempty and a singleton. We denote such R_λ by $R_\lambda = (I + \lambda BJ)^{-1}$. If R_λ is the generalized resolvent, then we know that

$$0 \le \langle x - J_\lambda x - (y - J_\lambda y), JJ_\lambda x - JJ_\lambda y\rangle$$

for all $x, y \in E$. In general, a nonlinear operator $T : C \to E$ is firmly generalized nonexpansive if

$$0 \le \langle x - Tx - (y - Ty), JTx - JTy\rangle \tag{3.4}$$

for all $x, y \in C$.

Next, we define four projections in a Banach space. Let E be a reflexive, smooth and strictly convex Banach space. We know that $T : C \to E$ is firmly nonexpansive if

$$0 \le \langle x - Tx - (y - Ty),\ J(Tx - Ty)\rangle$$

for all $x, y \in C$. If $F(T)$ is nonempty, then we have that

$$0 \le \langle x - Tx,\ J(Tx - y)\rangle$$

for all $x \in C$ and $y \in F(T)$. Let P be a retraction of E onto C, i.e., $P^2 = P$ and $P(E) = C$. Then a retraction P is called sunny nonexpansive if

$$0 \le \langle x - Px,\ J(Px - y)\rangle \tag{3.5}$$

for all $x \in E$ and $y \in C$. We know that $T : C \to E$ is a firmly metric operator if

$$0 \le \langle Tx - Ty,\ J(x - Tx) - J(y - Ty)\rangle$$

for all $x, y \in C$. If $F(T)$ is nonempty, then we have that

$$0 \le \langle Tx - y,\ J(x - Tx)\rangle$$

for all $x \in C$ and $y \in F(T)$. A retraction P of E onto C is called metric if

$$0 \le \langle Px - y,\ J(x - Px)\rangle \tag{3.6}$$

for all $x \in E$ and $y \in C$. If $T : C \to E$ is firmly relative nonexpansive, then we have

$$0 \le \langle Tx - Ty,\ Jx - JTx - (Jy - JTy)\rangle$$

for all $x, y \in C$. If $F(T)$ is nonempty, then we have that

$$0 \le \langle Tx - y,\ Jx - JTx\rangle$$

for all $x \in C$ and $y \in F(T)$. A retraction Π_C of E onto C is called generalized if

$$0 \leq \langle \Pi_C x - y,\ Jx - J\Pi_C x\rangle \tag{3.7}$$

for all $x \in E$ and $y \in C$. Such a retraction is also the generalized projection; see [1, 2]. If $T : C \to E$ is firmly generalized nonexpansive, we have

$$0 \leq \langle x - Tx - (y - Ty),\ JTx - JTy\rangle$$

for all $x, y \in C$. If $F(T)$ is nonempty, then we have

$$0 \leq \langle x - Tx,\ JTx - Jy\rangle$$

for all $x \in C$ and $y \in F(T)$. A retraction R of E onto C is called sunny generalized nonexpansive if

$$0 \leq \langle x - Rx,\ JRx - Jy\rangle \tag{3.8}$$

for all $x \in E$ and $y \in C$; see also [12].

Kohsaka and Takahashi [24] proved the following theorems.

Theorem 3.1. (Kohsaka and Takahashi [24]). *Let E be a smooth, strictly convex and reflexive Banach space and let C_* be a nonempty closed convex subset of E^*. Suppose that Π_{C_*} is the generalized projection of E^* onto C_*. Then, R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C_*$.*

Theorem 3.2. (Kohsaka and Takahashi [24]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty subset of E. Then, the following conditions are equivalent*

(1) *D is a sunny generalized nonexpansive retract of E;*

(2) *D is a generalized nonexpansive retract of E;*

(3) *JD is closed and convex.*

In this case, D is closed.

Theorem 3.3. (Kohsaka and Takahashi [24]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction R of E onto D and let $(x, z) \in E \times C$. Then, the following conditions are equivalent*

(1) *$z = Rx$;*

(2) *$\phi(x, z) = \min_{y \in D} \phi(x, y)$.*

Ibaraki and Takahashi [12] also proved the following theorems.

Theorem 3.4. (Ibaraki and Takahashi [12]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Then, a sunny generalized nonexpansive retraction of E onto D is uniquely determined.*

Theorem 3.5. (Ibaraki and Takahashi [12]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction R of E onto D and let $(x, z) \in E \times C$. Then, the following hold:*

(1) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz\rangle \leq 0,\ \forall y \in D$;

(2) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

4. Four Nonlinear Operators

Let E be a reflexive, smooth and strictly convex Banach space. Let C be a nonempty subset of E. If $T : C \to E$ is a firmly nonexpansive mapping, then we have that for any $x, y \in C$,

$$\begin{aligned} &0 \leq \langle x - Tx - (y - Ty), J(Tx - Ty)\rangle \\ &\iff \|Tx - Ty\|^2 \leq \langle x - y,\ J(Tx - Ty)\rangle \\ &\iff 2\|Tx - Ty\|^2 \leq 2\langle x - y,\ J(Tx - Ty)\rangle \\ &\iff 2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty\|^2 - \phi(x - y, Tx - Ty), \end{aligned}$$

where

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$. So, from a firmly nonexpansive mapping $T : C \to E$, we can define a nonexpansive mapping. That is, a mapping $T : C \to E$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{4.1}$$

for all $x, y \in C$. In the case that E is a Hilbert space, an operator $T : C \to E$ is nonexpansive if and only if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$; see [52, 56].

An operator $T : C \to E$ is firmly metric if

$$0 \leq \langle Tx - Ty,\ J(x - Tx) - J(y - Ty)\rangle$$

for all $x, y \in C$. Using

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w)$$

for $x, y, z, w \in E$, we have that for any $x, y \in C$,

$$\begin{aligned}
&0 \le \langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \\
&\quad \Longleftrightarrow 0 \le 2\langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \\
&\quad \Longleftrightarrow 2\langle x - Tx - (y - Ty),\ J(x - Tx) - J(y - Ty)\rangle \\
&\qquad\quad \le 2\langle x - y,\ J(x - Tx) - J(y - Ty)\rangle \\
&\quad \Longleftrightarrow\ \phi(x - Tx, y - Ty) + \phi(y - Ty, x - Tx) \\
&\qquad\quad \le \phi(x, y - Ty) + \phi(y, x - Tx) - \phi(x, x - Tx) - \phi(y, y - Ty) \\
&\quad \Longrightarrow\ \phi(x - Tx, y - Ty) + \phi(y - Ty, x - Tx) \\
&\qquad\quad \le \phi(x, y - Ty) + \phi(y, x - Tx).
\end{aligned}$$

So, from a firmly metric operator, we can define a metric operator. That is, $T : C \to E$ is called a metric operator if

$$\phi(x - Tx, y - Ty) + \phi(y - Ty, x - Tx) \le \phi(x, y - Ty) + \phi(y, x - Tx) \tag{4.2}$$

for all $x, y \in C$. In the case that E is a Hilbert space, $T : C \to E$ is firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \langle Tx - Ty,\ x - y\rangle$$

for all $x, y \in C$. Further, $T : C \to E$ is a metric operator if for any $x, y \in C$,

$$2\|x - Tx - (y - Ty)\|^2 \le \|x - (y - Ty)\|^2 + \|y - (x - Ty)\|^2.$$

This inequality is equivalent to

$$\|Tx - Ty\|^2 \le 2\langle x - y, Tx - Ty\rangle + 2\langle Tx,\ Ty\rangle.$$

An operator $T : C \to E$ is firmly relatively nonexpansive if

$$0 \le \langle Tx - Ty,\ Jx - JTx - (Jy - JTy)\rangle$$

for all $x, y \in C$. Then, we have that for any $x, y \in C$,

$$\begin{aligned}
&0\ \le \langle Tx - Ty, Jx - JTx - (Jy - JTy)\rangle \\
&\quad \Longleftrightarrow\ \langle Tx - Ty,\ JTx - JTy\rangle \le \langle Tx - Ty,\ Jx - Jy\rangle \\
&\quad \Longleftrightarrow\ \phi(Tx, Ty) + \phi(Ty, Tx) \\
&\quad \le \phi(Tx, y) + \phi(Ty, x) - \phi(Tx, x) - \phi(Ty, y).
\end{aligned}$$

So, from a firmly relatively nonexpansive operator, we can define a nonspreading operator. That is, $T : C \to E$ is a nonspreading operator [26] if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x) \tag{4.3}$$

for all $x, y \in C$. In the case that E is a Hilbert space, an operator $T : C \to E$ is firmly nonexpansive if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 - \|Tx - x\|^2 - \|Ty - y\|^2$$

for all $x, y \in C$. Further, an operator $T : C \to E$ is nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. This inequality is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$; see [14].

An operator $T : C \to E$ is firmly generalized nonexpansive if

$$0 \leq \langle x - Tx - (y - Ty),\ JTx - JTy\rangle$$

for all $x, y \in C$. Then, we have that for any $x, y \in C$,

$$\begin{aligned}
0 \ &\leq \langle x - Tx - (y - Ty), JTx - JTy\rangle \\
&\iff \langle Tx - Ty,\ JTx - JTy\rangle \leq \langle x - y,\ JTx - JTy\rangle \\
&\iff \phi(Tx, Ty) + \phi(Ty, Tx) \\
&\leq \phi(x, Ty) + \phi(y, Tx) - \phi(x, Tx) - \phi(y, Ty).
\end{aligned}$$

So, from a firmly generalized nonexpansive operator, we can define a generalized nonexpansive type operator. That is, $T : C \to E$ is a generalized nonexpansive type operator if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx) \tag{4.4}$$

for all $x, y \in C$.

The following is Kohsaka and Takahashi's fixed point theorem [26].

Theorem 4.1. (Kohsaka and Takahashi [26]). *Let E be a smooth, strictly convex, and reflexive Banach space and let C is a closed convex subset of E. Suppose that T: $C \to C$ is nonspreading, i.e., for all $x, y \in C$,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x).$$

Then the following are equivalent:

(1) *There exists $x \in C$ such that $\{T^n x\}$ is bounded;*

(2) *$F(T)$ is nonempty.*

In the case that E is a Hilbert space, we have the following theorem.

Theorem 4.2. (Kohsaka and Takahashi [26]). *Let H be a Hilbert space and let C be a closed convex subset of H. Suppose that $T\colon C \to C$ is nonspreading, i.e., for all $x, y \in C$,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2.$$

Then the following are equivalent:

(1) *There exists $x \in C$ such that $\{T^n x\}$ is bounded;*

(2) *$F(T)$ is nonempty.*

In the remainder of this section, we deal with nonlinear operators with fixed points in a Banach space. Let E be a reflexive, smooth and strictly convex Banach space. Let C be a nonempty subset of E. A mapping $T : C \to E$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{4.5}$$

for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\| \tag{4.6}$$

for all $x \in C$ and $y \in F(T)$. If C is a closed convex subset of E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [16].

A mapping $T : C \to E$ is metric if

$$\phi(x - Tx, y - Ty) + \phi(y - Ty, x - Tx) \leq \phi(x, y - Ty) + \phi(y, x - Tx) \tag{4.7}$$

for all $x, y \in C$. A mapping $T : C \to E$ is quasi-metric if $F(T) \neq \emptyset$ and

$$2\|x - Tx\|^2 \leq \|x\|^2 + \phi(y, x - Tx) \tag{4.8}$$

for all $x \in C$ and $y \in F(T)$.

A mapping $T : C \to E$ is nonspreading if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \tag{4.9}$$

for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonspreading or quasi-relatively nonexpansive if $F(T) \neq \emptyset$ and

$$\phi(y, Tx) \leq \phi(y, x) \tag{4.10}$$

for all $x \in C$ and $y \in F(T)$.

A mapping $T : C \to E$ is generalized nonexpansive type if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx) \tag{4.11}$$

for all $x, y \in C$. A mapping $T : C \to E$ is generalized nonexpansive [12] or quasi-generalized nonexpansive type if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y) \tag{4.12}$$

for all $x \in C$ and $y \in F(T)$.

5. Duality Theorems

Let E be a reflexive, smooth and strictly convex Banach space. Let C be a nonempty subset of E. Let $T : C \to C$ be a mapping. Then, $p \in C$ is called an asymptotic fixed point of T [37] if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. Matsushita and Takahashi [28] also gave the following definition: An operator $T : C \to E$ is relatively nonexpansive if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and

$$\phi(y, Tx) \leq \phi(y, x)$$

for all $x \in C$ and $y \in F(T)$.

The following theorems are in Kohsaka and Takahashi [26].

Theorem 5.1. (Kohsaka and Takahashi [26]). *Let E be a smooth and strictly convex Banach space and let C be a closed convex subset of E. Suppose that T: $C \to C$ is nonspreading, i.e.,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. Then, $F(T)$ is closed and convex.

Theorem 5.2. (Kohsaka and Takahashi [26]). *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Suppose $T : C \to C$ is nonspreading, i.e.,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. Then, $\hat{F}(T) = F(T)$. Further, if $F(T)$ is nonempty, then, $T : C \to C$ is relatively nonexpansive.

Let E be a smooth Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a mapping. Then, $p \in C$ is called a generalized asymptotic fixed point of

T [13] if there exists $\{x_n\} \subset C$ such that $Jx_n \rightharpoonup Jp$, $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$. We denote by $\check{F}(T)$ the set of generalized asymptotic fixed points of T.

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into itself. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the duality mapping of T; see also [9]. It is easy to show that T^* is a mapping of JC into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have

$$T^*x^* = JTJ^{-1}x^* \in JC.$$

Then, T^* is a mapping of JC into itself. Further, we define the duality mapping T^{**} of T^* as follows:

$$T^{**}x = J^{-1}T^*Jx, \quad \forall x \in C.$$

It is easy to show that T^{**} is a mapping of C into itself. In fact, for $x \in C$, we have

$$T^{**}x = J^{-1}T^*Jx = J^{-1}JTJ^{-1}Jx = Tx \in C.$$

So, T^{**} is a mapping of C into itself.

Now, we obtain the following duality theorems in a Banach space.

Theorem 5.3. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into itself and let T^* be the duality mapping of JC into itself. Then the following hold:*

(1) $JF(T) = F(T^*)$;
(2) $J\hat{F}(T) = \check{F}(T^*)$;
(3) $J\check{F}(T) = \hat{F}(T^*)$.

Proof. (1) We have from the definition of T^* that

$$\begin{aligned} x^* \in JF(T) &\Longleftrightarrow J^{-1}x^* \in F(T) \\ &\Longleftrightarrow TJ^{-1}x^* = J^{-1}x^* \\ &\Longleftrightarrow JTJ^{-1}x^* = JJ^{-1}x^* \\ &\Longleftrightarrow T^*x^* = x^* \\ &\Longleftrightarrow x^* \in F(T^*). \end{aligned}$$

This implies that $JF(T) = F(T^*)$.

(2) Let $x^* \in J\hat{F}(T)$. Then $J^{-1}x^* \in \hat{F}(T)$. Since $J^{-1}x^*$ is an asymptotic fixed point of T, there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup J^{-1}x^*$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Put $x_n^* = Jx_n$ for each $n \in \mathbb{N}$. Then, we have that $J^{-1}x_n^* = x_n \rightharpoonup J^{-1}x^*$ and

$$\begin{aligned}\|J^{-1}x_n^* - J^{-1}T^*x_n^*\| &= \|J^{-1}Jx_n - J^{-1}JTJ^{-1}Jx_n\| \\ &= \|x_n - Tx_n\| \to 0.\end{aligned}$$

Since the duality mapping J_* on E^* is J^{-1}, we have $x^* \in \check{F}(T^*)$. This implies that $J\hat{F}(T) \subset \check{F}(T^*)$. Conversely, let $x^* \in \check{F}(T^*)$. Then, there exists a sequence $\{x_n^*\} \subset JC$ such that $J^{-1}x_n^* \rightharpoonup J^{-1}x^*$ and $\lim_{n\to\infty} \|J^{-1}x_n^* - J^{-1}T^*x_n^*\| = 0$. Put $x_n = J^{-1}x_n^*$ for each $n \in \mathbb{N}$. Then, we have that $x_n \rightharpoonup J^{-1}x^*$ and

$$\begin{aligned}\|x_n - Tx_n\| &= \|J^{-1}x_n^* - J^{-1}JTJ^{-1}x_n^*\| \\ &= \|J^{-1}x_n^* - J^{-1}T^*x_n^*\| \to 0.\end{aligned}$$

Hence, we have $J^{-1}x^* \in \hat{F}(T)$. So, we have $x^* \in J\hat{F}(T)$. This implies that $\check{F}(T^*) \subset J\hat{F}(T)$. Then, $J\hat{F}(T) = \check{F}(T^*)$.

(3) Let $x^* \in J\check{F}(T)$. Then, $J^{-1}x^* \in \check{F}(T)$. So, there exists a sequence $\{x_n\} \subset C$ such that $Jx_n \rightharpoonup JJ^{-1}x^* = x^*$ and $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$. Put $x_n^* = Jx_n$ for each $n \in \mathbb{N}$. Then, we have that $x_n^* \rightharpoonup x^*$ and

$$\begin{aligned}\|x_n^* - T^*x_n^*\| &= \|Jx_n - JTJ^{-1}Jx_n\| \\ &= \|Jx_n - JTx_n\| \to 0.\end{aligned}$$

Hence, we have $x^* \in \hat{F}(T^*)$. This implies that $J\check{F}(T) \subset \hat{F}(T^*)$. Conversely, let $x^* \in \hat{F}(T^*)$. Then, there exists a sequence $\{x_n^*\} \subset JC$ such that $x_n^* \rightharpoonup x^*$ and $\lim_{n\to\infty} \|x_n^* - T^*x_n^*\| = 0$. Put $x_n = J^{-1}x_n^*$ for each $n \in \mathbb{N}$. Then, we have that $Jx_n \rightharpoonup x^* = JJ^{-1}x^*$ and

$$\begin{aligned}\|Jx_n - JTx_n\| &= \|JJ^{-1}x_n^* - JTJ^{-1}x_n^*\| \\ &= \|x_n^* - T^*x_n^*\| \to 0.\end{aligned}$$

Hence, we have $J^{-1}x^* \in \check{F}(T)$. So, we have $x^* \in J\check{F}(T)$. This implies that $\hat{F}(T^*) \subset J\check{F}(T)$. Then, $J\check{F}(T) = \hat{F}(T^*)$. ∎

Theorem 5.4. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a nonspreading mapping of C into itself and let T^* be the duality mapping of T. Then T^* is a generalized nonexpansive type mapping of JC into itself.*

Proof. Since $T^* = JTJ^{-1}$, we have that for $x, y \in C$, $x^* = Jx$ and $y^* = Jy$,

$$\begin{aligned}
&\phi_*(T^*x^*, T^*y^*) + \phi_*(T^*y^*, T^*x^*) \\
&= \phi_*(JTJ^{-1}Jx, JTJ^{-1}Jy) + \phi_*(JTJ^{-1}Jy, JTJ^{-1}Jx) \\
&= \phi_*(JTx, JTy) + \phi_*(JTy, JTx) \\
&= \|JTx\|^2 - 2\langle JTx, J^{-1}JTy\rangle + \|JTy\|^2 \\
&\quad + \|JTy\|^2 - 2\langle JTy, J^{-1}JTx\rangle + \|JTx\|^2 \\
&= \|Tx\|^2 - 2\langle JTx, Ty\rangle + \|Ty\|^2 + \|Ty\|^2 - 2\langle JTy, Tx\rangle + \|Tx\|^2 \\
&= \phi(Ty, Tx) + \phi(Tx, Ty)
\end{aligned}$$

and

$$\begin{aligned}
&\phi_*(y^*, T^*x^*) + \phi_*(x^*, T^*y^*) \\
&= \phi_*(Jy, JTJ^{-1}Jx) + \phi_*(Jx, JTJ^{-1}Jy) \\
&= \phi_*(Jy, JTx) + \phi_*(Jx, JTy) \\
&= \|Jy\|^2 - 2\langle Jy, J^{-1}JTx\rangle + \|JTx\|^2 \\
&\quad + \|Jx\|^2 - 2\langle Jx, J^{-1}JTy\rangle + \|JTy\|^2 \\
&= \|y\|^2 - 2\langle Jy, Tx\rangle + \|Tx\|^2 + \|x\|^2 - 2\langle Jx, Ty\rangle + \|Ty\|^2 \\
&= \phi(Tx, y) + \phi(Ty, x).
\end{aligned}$$

Since T be a nonspreading mapping, we have

$$\begin{aligned}
&\phi_*(T^*x^*, T^*y^*) + \phi_*(T^*y^*, T^*x^*) \\
&= \phi(Ty, Tx) + \phi(Tx, Ty) \\
&\leq \phi(Tx, y) + \phi(Ty, x) \\
&= \phi_*(y^*, T^*x^*) + \phi_*(x^*, T^*y^*).
\end{aligned}$$

So, T^* is a generalized nonexpansive type mapping. ∎

Theorem 5.5. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a generalized nonexpansive type mapping of C into itself and let T^* be the duality mapping of T. Then T^* is a nonspreading mapping of JC into itself.*

Proof. As in the proof of Theorem 5.4, we obtain Theorem 5.5. ∎

Using such duality theorems, we prove the following theorem which will be used in Section 6.

Theorem 5.6. *Let E be a smooth and reflexive Banach space and E^* has a uniformly Gâteaux differentiable norm. Let C be a closed subset of E such that JC is closed and convex and let $T : C \to C$ be a gneralized nonexpansive type mapping, i.e.,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in C$. Then, the following hold:

(1) $\check{F}(T) = F(T)$*;*

(2) $JF(T)$ *is closed and convex;*

(3) $F(T)$ *is closed.*

Proof. Since T is a mapping of C into itself, we can define the duality mapping T^* of JC into itself. From Theorem 5.5, we have that T^* is a nonspreading mapping of JC into itself.

(1) From Theorem 5.2, we know $\hat{F}(T^*) = F(T^*)$. Since $JF(T) = F(T^*)$ and $J\check{F}(T) = \hat{F}(T^*)$ from Theorem 5.3, we have that $JF(T) = J\check{F}(T)$. Since the duality mapping J is one-to-one and onto, we have $\check{F}(T) = F(T)$.

(2) Since $JF(T) = F(T^*)$ from Theorem 5.3 and $F(T^*)$ is closed and convex from Theorem 5.1, we have that $JF(T)$ is closed and convex.

(3) Since E is a smooth Banach space, the duality mapping J is norm-to-weak* continuous. Since a closed convex set is weakly closed and $JF(T)$ is closed and convex from (2), $JF(T)$ is wealky closed. So, we obtain that $F(T) = J^{-1}JF(T)$ is closed. ∎

6. Strong Convergence Theorems

In this section, using the hybrid method by Nakajo and Takahashi [33], we first prove a strong convergence theorem for generalized nonexpansive type mappings with equilibrium problems in a Banach space.

Theorem 6.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a closed subset of E such that JC is closed and convex. Let $f : JC \times JC \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3) and (A4) and let S be a generalized nonexpansive type mapping of C into itself such that $EP(f) \cap F(S) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} f(Jz_n, Jy) + \frac{1}{\lambda_n} \langle z_n - x_n, Jy - Jz_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S z_n, \\ C_n = \{z \in C : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C : \langle Jx_n - Jz, x - x_n \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \quad \forall n \in \{0\} \cup \mathbb{N}, \end{cases}$$

where $R_{C_n \cap Q_n}$ *is the sunny generalized nonexpansive retraction of* E *onto* $C_n \cap Q_n$*, and* $\{\alpha_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset [0,\infty)$ *satisfy*

$$0 \le \alpha_n \le a < 1 \quad and \quad 0 < b \le \lambda_n$$

for some $a, b \in \mathbb{R}$*. Then,* $\{x_n\}$ *converges strongly to* $z_0 = R_{F(S)\cap EP(f)}x$*, where* $R_{F(S)\cap EP(f)}$ *is the sunny generalized nonexpansive retraction of* E *onto* $F(S) \cap EP(f)$*.*

Proof. We first show that $F(S)\cap EP(f)$ is a sunny generalized nonexpansive retract of E. Using Lemma 2.5 and Theorem 5.6, we have that $JF(S)$ and $JEP(f)$ are closed and convex. Sinece E is uniformly convex, J is injective and hence

$$J(F(S) \cap EP(f)) = JF(S) \cap JEP(f),$$

which is also closed and convex. Using Theorem 3.2, we have that $F(S)\cap EP(f)$ is a sunny generalized nonexpansive retract of E. Since E is reflexive, J is surjective and hence

$$JQ_n = \{z^* \in JC : \langle x - x_n, z^* - Jx_n\rangle \le 0\}$$

and

$$JC_n = \{z^* \in JC : \phi_*(z^*, Jy_n) \le \phi_*(z^*, Jx_n)\}$$

for all $n \in \{0\} \cup \mathbb{N}$. We can see that JC_n is convex since

$$\begin{aligned} &\phi(y_n, z) \le \phi(x_n, z) \\ \Longleftrightarrow &\|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, Jz\rangle \le 0. \end{aligned}$$

Since J is injective,

$$J(C_n \cap Q_n) = JC_n \cap JQ_n.$$

Thus, JC_n, JQ_n and $J(C_n \cap Q_n)$ are closed and convex for all $n \in \{0\} \cup \mathbb{N}$. Using Theorem 3.2, we have that each $C_n \cap Q_n$ is a sunny generalized nonexpansive retract of E. We next show that $C_n \cap Q_n$ is nonempty. Let $z \in F(S) \cap EP(f)$. Put $z_n = T_{\lambda_n} x_n$ for each $n \in \{0\} \cup \mathbb{N}$. From $z = T_{\lambda_n} z$ and Lemma 2.5, we have that for any $n \in \{0\} \cup \mathbb{N}$,

$$\begin{aligned} \phi(z_n, z) &= \phi(T_{\lambda_n} x_n, z) \\ &\le \phi(x_n, z). \end{aligned} \tag{6.1}$$

Since S is of generalized nonexpansive type, we have

$$\begin{aligned} \phi(y_n, z) &= \phi(\alpha_n x_n + (1-\alpha_n) S z_n, z) \\ &\le \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(z_n, z) \\ &\le \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(x_n, z) \\ &= \phi(x_n, z). \end{aligned}$$

So, we have $z \in C_n$ and hence $F(S) \cap EP(f) \subset C_n$ for all $n \in \{0\} \cup \mathbb{N}$. Next, we show by induction that $F(S) \cap EP(f) \subset C_n \cap Q_n$ for all $n \in \{0\} \cup \mathbb{N}$. It is obvious that $F(S) \cap EP(f) \subset C_0 \cap Q_0$. Suppose that $F(S) \cap EP(f) \subset C_k \cap Q_k$ for some k. From $x_{k+1} = R_{C_k \cap Q_k} x$, we have

$$\langle Jx_{k+1} - Jz, x - x_{k+1} \rangle \geq 0, \quad \forall z \in C_k \cap Q_k.$$

Since $F(S) \cap EP(f) \subset C_k \cap Q_k$, we also have

$$\langle Jx_{k+1} - Jz, x - x_{k+1} \rangle \geq 0, \quad \forall z \in F(S) \cap EP(f).$$

This implies $F(S) \cap EP(f) \subset Q_{k+1}$. So, we have $F(S) \cap EP(f) \subset C_{k+1} \cap Q_{k+1}$. By induction, we have $F(S) \cap EP(f) \subset C_n \cap Q_n$ for all $n \in \{0\} \cup \mathbb{N}$. This means that $\{x_n\}$ and $\{z_n\}$ are well-defined.

Since $x_n = R_{Q_n} x$ by Theorem 3.5 (1) and $x_{n+1} = R_{C_n \cap Q_n} x \subset Q_n$, we have from Theorem 3.5 and (2.2) that

$$\begin{aligned} 0 &\leq 2\langle x - x_n, Jx_n - Jx_{n+1} \rangle \\ &= \phi(x, x_{n+1}) - \phi(x, x_n) - \phi(x_n, x_{n+1}) \\ &\leq -\phi(x, x_n) + \phi(x, x_{n+1}). \end{aligned}$$

So, we get that

$$\phi(x, x_n) \leq \phi(x, x_{n+1}). \tag{6.2}$$

Further, since $x_n = R_{Q_n} x$ and $z \in F(S) \cap EP(f) \subset C_n$, from Theorem 3.3 we have

$$\phi(x, x_n) \leq \phi(x, z). \tag{6.3}$$

So, we have that $\lim_{n\to\infty} \phi(x, x_n)$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{z_n\}$ and $\{Sz_n\}$ are also bounded. From Theorem 3.5, we have

$$\begin{aligned} \phi(x_n, x_{n+1}) &= \phi(R_{Q_n} x, x_{n+1}) \\ &\leq \phi(x, x_{n+1}) - \phi(x, R_{Q_n} x) \\ &= \phi(x, x_{n+1}) - \phi(x, x_n) \to 0. \end{aligned}$$

So, we have that

$$\phi(x_n, x_{n+1}) \to 0. \tag{6.4}$$

From $x_{n+1} \in C_n$, we have that $\phi(y_n, x_{n+1}) \leq \phi(x_n, x_{n+1})$. So, we get that $\phi(y_n, x_{n+1}) \to 0$. From Theorem 2.3, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

So, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{6.5}$$

From $\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Sz_n\| = (1-\alpha_n)\|x_n - Sz_n\|$, we also have that

$$\|Sz_n - x_n\| \to 0. \tag{6.6}$$

Let $z \in F(S) \cap EP(f)$. Using $z_n = T_{\lambda_n} x_n$ and Lemma 2.5, we have that

$$\begin{aligned}\phi(x_n, z) &\geq \phi(x_n, T_{\lambda_n} x_n) + \phi(T_{\lambda_n} x_n, z)\\ &= \phi(x_n, z_n) + \phi(z_n, z)\end{aligned}$$

and hence

$$\phi(x_n, z_n) \leq \phi(x_n, z) - \phi(z_n, z).$$

From the definition of ϕ, we have $\phi(y_n, z) \leq \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(z_n, z)$ and hence

$$\phi(z_n, z) \geq \frac{\phi(y_n, z) - \alpha_n \phi(x_n, z)}{1-\alpha_n}.$$

Therefore, we have

$$\begin{aligned}\phi(x_n, z_n) &\leq \phi(x_n, z) - \frac{\phi(y_n, z) - \alpha_n \phi(x_n, z)}{1-\alpha_n}\\ &= \frac{\phi(x_n, z) - \phi(y_n, z)}{1-\alpha_n}.\end{aligned}$$

We also have

$$\begin{aligned}\phi(x_n, z) - \phi(y_n, z) &= \|x_n\|^2 - 2\langle x_n, Jz\rangle + \|z\|^2 - \|y_n\|^2 + 2\langle y_n, Jz\rangle - \|z\|^2\\ &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, Jz\rangle\\ &\leq |\|x_n\|^2 - \|y_n\|^2| + 2|\langle x_n - y_n, Jz\rangle|\\ &\leq \|x_n - y_n\|(\|x_n\| + \|y_n\|) + 2\|x_n - y_n\|\|Jz\|.\end{aligned}$$

Since $\phi(x_n, z) - \phi(y_n, z) \geq 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we have

$$\lim_{n\to\infty} (\phi(x_n, z) - \phi(y_n, z)) = 0. \tag{6.7}$$

Since $0 \leq \alpha_n \leq a < 1$, from (6.7) we have $\lim_{n\to\infty} \phi(x_n, z_n) = 0$. From Theorem 2.3, we have

$$\|x_n - z_n\| \to 0. \tag{6.8}$$

Since $y_n = \alpha_n x_n + (1-\alpha_n)Sz_n$, we have $y_n - Sz_n = \alpha_n(x_n - Sz_n)$. So, from (6.6) we have

$$\|y_n - Sz_n\| = \alpha_n\|x_n - Sz_n\| \to 0. \tag{6.9}$$

Since

$$\|z_n - Sz_n\| \le \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Sz_n\|,$$

from (6.5), (6.8) and (6.9) we have

$$\|z_n - Sz_n\| \to 0. \tag{6.10}$$

Since E is uniformly smooth, J is norm-to-norm continuous. So, we have

$$\|Jz_n - JSz_n\| \to 0. \tag{6.11}$$

Since $\{Jx_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $Jx_{n_i} \rightharpoonup z^*$. Since J is uniformly norm-to-norm continuous on bounded sets, we have from (6.8) that

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0.$$

From $Jx_{n_i} \rightharpoonup z^*$, we have $Jz_{n_i} \rightharpoonup z^*$. From (6.11), we have $J^{-1}z^* \in \check{F}(S)$. Putting $z = J^{-1}z^*$, from Theorem 5.6 we have $z \in F(S)$. Next, let us show $z \in EP(f)$. Since $z_n = T_{\lambda_n}x_n$, we have, for any $y \in C$,

$$f(Jz_n, Jy) + \frac{1}{\lambda_n}\langle Jy - Jz_n, z_n - x_n\rangle \ge 0.$$

From (A2), we have

$$\frac{1}{\lambda_n}\langle Jy - Jz_n, z_n - x_n\rangle \ge f(Jy, Jz_n).$$

From $0 < b \le \lambda_n$ and (6.8), we have

$$\lim_{n\to\infty} \frac{z_n - x_n}{\lambda_n} = 0.$$

So, from (A4) we have

$$0 \ge f(Jy, z^*). \tag{6.12}$$

Put $z_t^* = tJy + (1-t)z^*$ for all $t \in (0,1]$ and $y \in C$. Since JC is convex, we have $z_t^* \in JC$. From (A1), (A4) and (6.12), we have

$$\begin{aligned} 0 = f(z_t^*, z_t^*) &\le tf(z_t^*, Jy) + (1-t)f(z_t^*, z^*) \\ &\le tf(z_t^*, Jy) \end{aligned}$$

and hence

$$0 \leq f(z_t^*, Jy).$$

Letting $t \to 0$, from (A3) we have that for each $y \in C$,

$$0 \leq f(z^*, Jy). \tag{6.13}$$

This implies $z \in EP(f)$. So, we have $z \in F(S) \cap EP(f)$. Put $z_0 = R_{F(S)\cap EP(f)}x$. Since $z_0 = R_{F(S)\cap EP(f)}x \subset C_n \cap Q_n$ and $x_{n+1} = R_{C_n \cap Q_n}x$, we have that

$$\phi(x, x_{n+1}) \leq \phi(x, z_0). \tag{6.14}$$

Since $\|\cdot\|^2$ is weakly lower semicontinuous, from $Jx_{n_i} \rightharpoonup Jz$ we have that

$$\begin{aligned}
\phi(x, z) &= \|x\|^2 - 2\langle x, Jz\rangle + \|Jz\|^2 \\
&\leq \liminf_{i\to\infty}(\|x\|^2 - 2\langle x, Jx_{n_i}\rangle + \|Jx_{n_i}\|^2) \\
&= \liminf_{i\to\infty} \phi(x, x_{n_i}) \\
&\leq \phi(x, z_0).
\end{aligned}$$

From the definition of z_0, we have $z = z_0$. Thus, we obtain $z^* = Jz = Jz_0$. So, we obtain $Jx_n \rightharpoonup Jz_0$. We finally show that $x_n \to z_0$. From (2.2), we have

$$\phi(z_0, x_n) = \phi(z_0, x) + \phi(x, x_n) + 2\langle z_0 - x, Jx - Jx_n\rangle, \quad \forall n \in \{0\} \cup \mathbb{N}.$$

Since $x_n = R_{Q_n}x$ and $z_0 \in F(S) \cap EP(f) \subset Q_n$, we have

$$\phi(x, x_n) \leq \phi(x, z_0) \tag{6.15}$$

and hence

$$\begin{aligned}
\limsup_{n\to\infty} \phi(z_0, x_n) &= \limsup_{n\to\infty}(\phi(z_0, x) + \phi(x, x_n) + 2\langle z_0 - x, Jx - Jx_n\rangle) \\
&\leq \limsup_{n\to\infty}(\phi(z_0, x) + \phi(x, z_0) + 2\langle z_0 - x, Jx - Jx_n\rangle) \\
&= \limsup_{n\to\infty}(\phi(z_0, x) + \phi(x, z_0) + 2\langle z_0 - x, Jx - Jz_0\rangle) \\
&= \phi(z_0, z_0) = 0.
\end{aligned}$$

Thus, we get $\lim_{n\to\infty} \phi(z_0, x_n) = 0$. So, from Theorem 2.3 we obtain $\lim_{n\to\infty} \|z_0 - x_n\| = 0$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. ■

Next, using Theorem 2.4, we prove a strong convergence theorem by the shrinking projection method [53].

Theorem 6.2. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a closed subset of E such that JC is closed and convex. Let $f : JC \times JC \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3) and (A4) and let S be a generalized nonexpansive type mapping of C into itself such that $EP(f) \cap F(S) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} f(Jz_n, Jy) + \frac{1}{\lambda_n} \langle z_n - x_n, Jy - Jz_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S z_n, \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}, and $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, \infty)$ are sequences such that

$$0 \leq \alpha_n \leq a < 1 \quad \textit{and} \quad 0 < b \leq \lambda_n$$

for some $a, b \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 = R_{F(S) \cap EP(f)} x$, where $R_{F(S) \cap EP(f)}$ is the sunny generalized nonexpansive retraction of E onto $F(S) \cap EP(f)$.

Proof. Put $z_n = T_{\lambda_n} x_n$ for each $n \in \mathbb{N}$ and take $z \in F(S) \cap EP(f)$. From $z = T_{\lambda_n} z$ and Lemma 2.5, we have that for any $n \in \mathbb{N}$,

$$\begin{aligned} \phi(z_n, z) &= \phi(T_{\lambda_n} x_n, z) \\ &\leq \phi(x_n, z). \end{aligned} \tag{6.16}$$

We shall show that JC_n are closed and convex, and $F(S) \cap EP(f) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from the assumption that $JC_1 = JC$ is closed and convex, and $F(S) \cap EP(f) \subset C_1$. Suppose that JC_k is closed and convex, and $F(S) \cap EP(f) \subset C_k$. From the definition of ϕ, we know that for $z \in C_k$,

$$\begin{aligned} &\phi(y_k, z) \leq \phi(x_k, z) \\ \Longleftrightarrow &\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, Jz \rangle \leq 0. \end{aligned}$$

So, JC_{k+1} is closed and convex. If $z \in F(S) \cap EP(f) \subset C_k$, then we have from (6.16) that

$$\begin{aligned} \phi(y_n, z) &= \phi(\alpha_n x_n + (1 - \alpha_n) S z_n, z) \\ &\leq \alpha_n \phi(x_n, z) + (1 - \alpha_n) \phi(z_n, z) \\ &\leq \alpha_n \phi(x_n, z) + (1 - \alpha_n) \phi(x_n, z) \\ &= \phi(x_n, z). \end{aligned}$$

Hence, we have $z \in C_{k+1}$. By induction, we have that JC_n are closed and convex, and $F(S) \cap EP(f) \subset C_n$ for all $n \in \mathbb{N}$. Since JC_n is closed and convex, from Lemma 3.2 there exists a unique sunny generalized nonexpansive retraction R_{C_n} of E onto C_n. We also know from Lemma 3.1 that R_{C_n} is denoted by $J^{-1}\Pi_{JC_n}J$, where J is the duality mapping and Π_{JC_n} is the generalized projection of E^* onto JC_n. Thus, $\{x_n\}$ is well-defined.

Since $\{JC_n\}$ is a nonincreasing sequence of nonempty closed convex subsets of E^* with respect to inclusion, it follows that

$$\emptyset \neq JF(S) \cap JEP(f) \subset \text{M-}\lim_{n\to\infty} JC_n = \bigcap_{n=1}^{\infty} JC_n. \tag{6.17}$$

Put $C_0^* = \bigcap_{n=1}^{\infty} JC_n$. Then, by Theorem 2.4 we have that $\{\Pi_{JC_{n+1}}Jx\}$ converges strongly to $x_0^* = \Pi_{C_0^*}Jx$. Since E^* has a Fréchet differentiable norm, J^{-1} is continuous. So, we have

$$x_{n+1} = J^{-1}\Pi_{JC_{n+1}}Jx \to J^{-1}x_0^*.$$

To complete the proof, it is sufficient to show that $J^{-1}x_0^* = R_{F(S)\cap EP(f)}x$.

Since $x_n = R_{C_n}x$ and $x_{n+1} = R_{C_{n+1}}x \in C_{n+1} \subset C_n$, we have from Theorem 3.5 and (2.2) that

$$\begin{aligned} 0 &\leq 2\langle x - x_n, Jx_n - Jx_{n+1}\rangle \\ &= \phi(x, x_{n+1}) - \phi(x, x_n) - \phi(x_n, x_{n+1}) \\ &\leq -\phi(x, x_n) + \phi(x, x_{n+1}). \end{aligned}$$

So, we get that

$$\phi(x, x_n) \leq \phi(x, x_{n+1}). \tag{6.18}$$

Further, since $x_n = R_{C_n}x$ and $z \in F(S) \cap EP(f) \subset C_n$, from Lemma 3.3 we have

$$\phi(x, x_n) \leq \phi(x, z). \tag{6.19}$$

So, we have that $\lim_{n\to\infty} \phi(x, x_n)$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{z_n\}$ and $\{Sz_n\}$ are also bounded. From Theorem 3.5, we have

$$\begin{aligned} \phi(x_n, x_{n+1}) &= \phi(R_{C_n}x, x_{n+1}) \\ &\leq \phi(x, x_{n+1}) - \phi(x, R_{C_n}x) \\ &= \phi(x, x_{n+1}) - \phi(x, x_n) \to 0. \end{aligned}$$

So, we have that

$$\phi(x_n, x_{n+1}) \to 0. \tag{6.20}$$

From $x_{n+1} \in C_{n+1}$, we also have that $\phi(y_n, x_{n+1}) \leq \phi(x_n, x_{n+1})$. So, we get that $\phi(y_n, x_{n+1}) \to 0$. Using Theorem 2.3, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

So, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{6.21}$$

Since $\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Sz_n\| = (1-\alpha_n)\|x_n - Sz_n\|$ and $0 \leq \alpha_n \leq a < 1$, we also have that

$$\|Sz_n - x_n\| \to 0. \tag{6.22}$$

Let $z \in F(S) \cap EP(f)$. Using $z_n = T_{\lambda_n} x_n$ and Lemma 2.5, we have that

$$\begin{aligned}\phi(x_n, z) &\geq \phi(x_n, T_{\lambda_n} x_n) + \phi(T_{\lambda_n} x_n, z)\\ &= \phi(x_n, z_n) + \phi(z_n, z)\end{aligned}$$

and hence

$$\phi(x_n, z_n) \leq \phi(x_n, z) - \phi(z_n, z).$$

From the definition of ϕ, we have $\phi(y_n, z) \leq \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(z_n, z)$ and hence

$$\phi(z_n, z) \geq \frac{\phi(y_n, z) - \alpha_n \phi(x_n, z)}{1-\alpha_n}.$$

Therefore, we have

$$\begin{aligned}\phi(x_n, z_n) &\leq \phi(x_n, z) - \frac{\phi(y_n, z) - \alpha_n \phi(x_n, z)}{1-\alpha_n}\\ &= \frac{\phi(x_n, z) - \phi(y_n, z)}{1-\alpha_n}.\end{aligned}$$

We also have

$$\begin{aligned}\phi(x_n, z) - \phi(y_n, z) &= \|x_n\|^2 - 2\langle x_n, Jz\rangle + \|z\|^2 - \|y_n\|^2 + 2\langle y_n, Jz\rangle - \|z\|^2\\ &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, Jz\rangle\\ &\leq |\|x_n\|^2 - \|y_n\|^2| + 2|\langle x_n - y_n, Jz\rangle|\\ &\leq \|x_n - y_n\|(\|x_n\| + \|y_n\|) + 2\|x_n - y_n\|\|Jz\|.\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - y_n\| = 0$ from (6.21), we have

(6.23) $$\lim_{n\to\infty} (\phi(x_n, z) - \phi(y_n, z)) = 0.$$

Since $0 \le \alpha_n \le a < 1$, we have $\lim_{n\to\infty} \phi(x_n, z_n) = 0$. So, from Theorem 2.3, we have

(6.24) $$\|x_n - z_n\| \to 0.$$

From $y_n = \alpha_n x_n + (1-\alpha_n)Sz_n$, we have $y_n - Sz_n = \alpha_n(x_n - Sz_n)$. So, from (6.22) we have

(6.25) $$\|y_n - Sz_n\| = \alpha_n \|x_n - Sz_n\| \to 0.$$

Since
$$\|z_n - Sz_n\| \le \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Sz_n\|,$$
from (6.21), (6.24) and (6.25) we have

(6.26) $$\|z_n - Sz_n\| \to 0.$$

Since E is uniformly smooth, J is norm-to-norm continuous. So, we have

(6.27) $$\|Jz_n - JSz_n\| \to 0.$$

Since $Jx_n = \Pi_{JC_n} x \to x_0^* = JJ^{-1}x_0^*$, we have $Jz_n \to x_0^*$. So, from (6.27) and Theorem 5.6 we have $J^{-1}x_0^* \in F(S)$. Next, let us show $J^{-1}x_0^* \in EP(f)$. From $x_n \to J^{-1}x_0^*$ and (6.24), we have $z_n \to J^{-1}x_0^*$. We have from $z_n = T_{\lambda_n} x_n$ that for any $y \in C$,

$$f(Jz_n, Jy) + \frac{1}{\lambda_n} \langle Jy - Jz_n, z_n - x_n \rangle \ge 0.$$

From (A2), we have

$$\frac{1}{\lambda_n} \langle Jy - Jz_n, z_n - x_n \rangle \ge f(Jy, Jz_n).$$

From $0 < b \le \lambda_n$ and (6.24), we know

$$\lim_{n\to\infty} \frac{z_n - x_n}{\lambda_n} = 0.$$

So, we have

(6.28) $$0 \ge f(Jy, x_0^*).$$

Put $z_t^* = tJy + (1-t)x_0^*$ for all $t \in (0,1]$ and $y \in C$. Since JC is convex, we have $z_t^* \in JC$. From (A1), (A4) and (6.28), we have

$$\begin{aligned} 0 = f(z_t^*, z_t^*) &\leq tf(z_t^*, Jy) + (1-t)f(z_t^*, x_0^*) \\ &\leq tf(z_t^*, Jy) \end{aligned}$$

and hence

$$0 \leq f(z_t^*, Jy).$$

Letting $t \to 0$, we have from (A3) that for each $y \in C$,

$$0 \leq f(x_0^*, Jy). \tag{6.29}$$

This implies $J^{-1}x_0^* \in EP(f)$. So, we have that $J^{-1}x_0^* \in F(S) \cap EP(f)$. Put $z_0 = R_{F(S)\cap EP(f)}x$. Since $z_0 = R_{F(S)\cap EP(f)}x \subset C_{n+1}$ and $x_{n+1} = R_{C_{n+1}}x$, we have that

$$\phi(x, x_{n+1}) \leq \phi(x, z_0). \tag{6.30}$$

So, we have that

$$\begin{aligned} \phi(x, J^{-1}x_0^*) &= \lim_{n\to\infty} \phi(x, x_n) \\ &\leq \phi(x, z_0). \end{aligned}$$

So, we get $z_0 = J^{-1}x_0^*$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. ∎

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