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KURZWEIL-HENSTOCK INTEGRATION ON MANIFOLDS

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Abstract. In this paper, we give an alternative proof that the Kurzweil-Henstock integral using partition of unity is equivalent to the Lebesgue integral in the n-dimensional Euclidean space. We also define and discuss the Kurzweil-Henstock integral on manifolds.

1. Introduction

The partition of unity plays an important role in the integral on manifolds. In [2], Kurzweil and Jarnik defined the Kurzweil-Henstock integral using the partition of unity, called the PUL integral. They proved that the PUL integral is equivalent to the Lebesgue integral in the n-dimensional Euclidean space. They indicated that the PUL integral can be used for integration on manifolds without details. The classical integral on manifolds is defined using change of variables formula. The Kurzweil-Henstock integral is defined by Riemann sums. It is an integral of Riemann type, see [4]. In this paper, we shall define and discuss the Kurzweil-Henstock integral using the partition of unity on manifolds.

2. PUL Integral in $\mathbb{R}^n$

In this paper, let E denote a compact interval in $\mathbb{R}^n$ and $|I|$ denote the volume of an interval $I \subseteq \mathbb{R}^n$.

A finite collection $\{\varphi_i\}_{i=1}^m$ of continuously differentiable functions defined on an interval E is said to be a *partial partition of unity* if $\varphi_i(\xi) \geq 0$ for each $\xi \in E$ and for each i, and $\sum_{i=1}^{m} \varphi_i(\xi) \leq 1$ for all $\xi \in E$. If, in addition, $\sum_{i=1}^{m} \varphi_i(\xi) = 1$ for all $\xi \in E$, then $\{\varphi_i\}_{i=1}^m$ is said to be a *partition of unity*.

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Let φ be a continuously differentiable function on E. Let δ be a positive function on E and $\xi \in E$. Then a triple (ξ, I, φ) is said to be δ-fine if the support of the function φ is contained in the compact interval I, which is contained in an open ball with center ξ and radius $\delta(\xi)$, i.e., supp $\varphi \subseteq I \subset B_{\delta(\xi)}(\xi)$, where supp φ is the closure of the set $\{x : \varphi(x) \neq 0\}$. Note that ξ may not be contained in supp φ.

Let $D = \{(\xi_i, I_i, \varphi_i)\}_{i=1}^m$ be a finite collection of triples. Then D is said to be a *δ-fine partial division of E* if $\{\varphi_i\}_{i=1}^m$ is a partial partition of unity and for each i, (ξ_i, I_i, φ_i) is δ-fine. In addition, if $\{\varphi_i\}_{i=1}^m$ is a partition of unity, then D is said to be a *δ-fine division of E*. Note that $\{I_i\}_{i=1}^m$ may be overlapping.

The existence of δ-fine divisions of E can be proved by the open covering theorem and the existence of a partition of unity.

Definition 2.1. [2]. Let $f : E \to \mathbb{R}$. Then f is said to be *PUL integrable* to real number A on E if for every $\epsilon > 0$, there exists a positive function δ defined on E such that for every δ-fine division $D = \{(\xi_i, I_i, \varphi_i)\}_{i=1}^m$ of E, we have

$$|S(f, \delta, D) - A| \leq \epsilon,$$

where

$$S(f, \delta, D) = \sum_{i=1}^{m} f(\xi_i) \int_{I_i} \varphi_i$$

and $\int_{I_i} \varphi_i$ is the Riemann integral of φ_i on I_i. We denote A by $\int_E f$.

The standard and basic properties of integration hold for the PUL integral. The proofs are standard, see [4]. We shall not state them here.

The following Henstock Lemma is proved in [2].

Lemma 2.2. ([Henstock's Lemma]). *Let $f : E \to \mathbb{R}$ be PUL integrable. Then for every $\epsilon > 0$, there exists a positive function δ defined on E such that whenever $D = \{(\xi_i, I_i, \varphi_i)\}_{i=1}^m$ is a δ-fine division of E, we have*

$$\sum_{i=1}^{m} \left| f(\xi_i) \int_{I_i} \varphi_i - \int_{I_i} f\varphi_i \right| \leq \epsilon.$$

Let $F = \prod_{i=1}^{n} [a_i, b_i]$ be an interval in $\mathbb{R}^n$ and β a positive real number. In this note, we denote, for convenience, $F + \beta$ the interval $\prod_{i=1}^{n} [a_i - \beta, b_i + \beta]$ and $F - \beta$ the interval $\prod_{i=1}^{n} [a_i + \beta, b_i - \beta]$.

Lemma 2.3. *Let A be a subset of E and χ_A be the characteristic function of A. Then $\int_E \chi_A = 0$ if and only if A is of Lebesgue measure zero.*

Proof. If χ_A is PUL integrable to 0 on E. Then for every $\epsilon > 0$, there exists a positive function δ defined on E such that for every δ-fine division $D = \{(\xi_i, I_i, \varphi_i)\}_{i=1}^m$ of E, we have

$$\left|\sum_{i=1}^m \chi_A(\xi_i) \int_{I_i} \varphi_i\right| \leq \epsilon.$$

Let $D' = \{(\eta_i, F_i)\}_{i=1}^m$ be a δ-fine division on E. Let β be a sufficient small number such that for $i = 1, 2, ..., m$, we have $F_i + \beta \subseteq B_{\delta(\eta_i)}(\eta_i)$ and

$$|F_i + \beta| - |F_i - \beta| \leq \frac{\epsilon}{2m}.$$

Let $\{\varphi_i\}_{i=1}^m$ be a partition of unity defined on E such that, $\varphi_i = 1$ on $F_i - \beta$ and zero outside $F_i + \beta$.

Hence, the division $D'' = \{(\eta_i, F_i + \beta, \varphi_i)\}_{i=1}^m$ forms a δ-fine division of E. Note that, for $i = 1, 2, ..., m$, the difference between $|F_i|$ and $\int_{F_i+\beta} \varphi_i$, say ζ_i, is not more than $\epsilon/2m$. Hence,

$$\begin{aligned}\left|\sum_{i=1}^m \chi_A(\eta_i)|F_i|\right| &= \left|\sum_{i=1}^m \chi_A(\eta_i)\left[\int_{F_i+\beta} \varphi_i + \zeta_i\right]\right| \\ &\leq \sum_{i=1}^m \left|\frac{\epsilon}{2m}\right| + \left|\sum_{i=1}^m \chi_A(\eta_i) \int_{F_i+\beta} \varphi_i\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Therefore,

$$(L)\int_E \chi_A = \int_E \chi_A = 0.$$

Hence A is of Lebesgue measure zero.

Conversely, if A is of Lebesgue measure zero. Then χ_A is Lebesgue integrable to 0 on E. Thus for every $\epsilon > 0$, there exists a positive function δ defined on E such that for every δ-fine division $D = \{(\eta_i, F_i)\}_{i=1}^m$ of E, we have

$$\left|\sum_{i=1}^m \chi_A(\eta_i)|F_i|\right| \leq \epsilon.$$

Let $D' = \{(\xi_i, I_i, \varphi_i)\}_{i=1}^m$ be a δ-fine division of E and $\{I_i\}_{i=1}^m$ a partition of E. Hence, the division $D'' = \{(\xi_i, I_i)\}_{i=1}^m$ forms a δ-fine division of E. Note that

$$0 \leq \int_{I_i} \varphi_i \leq |I_i|.$$

Therefore,

$$\left|\sum_{i=1}^{m} \chi_A(\xi_i) \int_{I_i} \varphi_i\right| \leq \left|\sum_{i=1}^{m} \chi_A(\xi_i)|I_i|\right| \leq \epsilon.$$

So, $\displaystyle\int_E \chi_A = 0$. ∎

Using the above Henstock's Lemma, we can prove the Monotone Convergence theorem for PUL integrals. The proof is standard, see [4, 5]. By the Monotone Convergence theorem, we can prove Fatou's Lemma, Dominated Convergence theorem and the following Mean Convergence theorem as in the theory of Lebesgue integrals.

We shall state the Mean Convergence theorem for PUL integrals without proof.

Theorem 2.4. ([Mean Convergence Theorem]). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of PUL integrable functions on E. If $f_n \to f$ a.e. on E and $\displaystyle\int_a^b |f_n - f_m| \to 0$ as $n, m \to \infty$, then f is PUL integrable on E and*

$$\lim_{n\to\infty} \int_E |f_n - f| = 0.$$

3. Equivalence Theorem

In this section, we shall give an alternative proof of the following Theorem 3.2. The proof is natural and shorter. In [2], the theorem is proved by using lower and upper semi-continuous functions.

Lemma 3.1. *Let s be a step function defined on E. Then*

$$\int_E s = (L)\int_E s,$$

where (L) denotes the Lebesgue integral.

Proof. By the additive property of the integral and Lemma 2.3. It is sufficient to prove the case where $s(x) = 1$ all $x \in E$, i.e., $\displaystyle\int_E s = |E|$.

Let $D = \{(\xi_i, I_i, \varphi_i)\}_{i=1}^{m}$ be a δ-fine division of E. Then

$$\sum_{i=1}^{m} s(\xi_i)\int_{I_i} \varphi_i = \sum_{i=1}^{m} \int_{I_i} \varphi_i = \sum_{i=1}^{m} \int_{E} \varphi_i = \int_E \sum_{i=1}^{m} \varphi_i = \int_E 1 = |E|.$$

Therefore $\displaystyle\int_E s = |E| = (L)\int_E s$. ∎

Theorem 3.2. *Let $f : E \to \mathbb{R}$. Then f is PUL integrable on E if and only if f is Lebesgue integrable on E.*

Proof. If f is Lebesgue integrable on E, then there exists a sequence $\{s_i\}_{i=1}^{\infty}$ of step functions such that $s_i \to f$ a.e. on E and $(L)\int_E |s_i - f| \to 0$ as $n \to \infty$. Note that $(L)\int_E |s_i - s_j| = \int_E |s_i - s_j|$. Thus, by the Mean Convergence Theorem for PUL integrals, f is PUL integrable on E.

Conversely, if f is PUL integrable on E. Then for each $\epsilon > 0$, there exists a positive function δ defined on E such that for every δ-fine division $D = \{(\xi_i, I_i, \varphi_i)\}_{i=1}^{n}$ of E, we have

$$\left|\sum_{i=1}^{n} f(\xi_i)\int_{I_i}\varphi_i - \int_E f\right| \leq \frac{\epsilon}{2}.$$

Now, let $D' = \{(\eta_i, F_i)\}_{i=1}^{m}$ be a δ-fine division on E. Let β be a sufficient small number such that for $i = 1, 2, ..., m$, we have $F_i + \beta \subseteq B_{\delta(\eta_i)}(\eta_i)$ and

$$|F_i + \beta| - |F_i - \beta| \leq \frac{\epsilon}{2mM},$$

where $M = \max_{i=1,\dots,m}\{|f(\xi_i)|\}$.

Let $\{\varphi_i\}_{i=1}^{m}$ be a partitions of unity defined on E such that, $\varphi_i = 1$ on $F_i - \beta$ and zero outside $F_i + \beta$.

Hence, the division $D'' = \{(\eta_i, F_i + \beta, \varphi_i)\}_{i=1}^{m}$ forms a δ-fine division of E. Note that, for $i = 1, 2, ..., m$, the difference between $|F_i|$ and $\int_{F_i+\beta}\varphi_i$, say ζ_i, is not more than $\epsilon/2mM$.

Therefore,

$$\begin{aligned}\left|\sum_{i=1}^{m} f(\eta_i)|F_i| - \int_E f\right| &= \left|\sum_{i=1}^{m} f(\eta_i)\left[\int_{F_i+\beta}\varphi_i + \zeta_i\right] - \int_E f\right| \\ &\leq \sum_{i=1}^{m}\left|M\frac{\epsilon}{2mM}\right| + \left|\sum_{i=1}^{m} f(\eta_i)\int_{F_i+\beta}\varphi_i - \int_E f\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So, f is Lebesgue integrable on E. ■

4. Change of Variable

Let A be a closed or an open set contained in a compact interval E^* in $\mathbb{R}^r$. Let $f : A \to \mathbb{R}$. Suppose $f\chi_A$ is integrable on E^*. Then write

$$\int_{E^*} f\chi_A = \int_A f.$$

The following lemma is well-known, see [1, p211].

Lemma 4.1. *Let U be an open subset of a compact interval E^* in $\mathbb{R}^r$ and $\psi : U \to \psi(U)$ be C^1-diffeomorphism, i.e., ψ is one to one and both ψ and ψ^{-1} are of C^1-class. Suppose $E \subseteq \psi(U) \subseteq \mathbb{R}^n$ and $\varphi : E \to \mathbb{R}$ is continuous. Then*

$$(R)\int_E \varphi = (R)\int_{\psi^{-1}(E)} (\varphi o \psi)\left|\det D\psi\right|,$$

where $(R)\displaystyle\int_E \varphi$ represents the Riemann integral of φ on an interval E and $|\det D\psi|$ or $|D\psi|$ represents the euclidean norm of the partial derivative of ψ at x.

Now we shall prove the following change of variable. A proof is also given in [2] in a different setting.

Theorem 4.2. ([Change of variable]). *Let $f : E \to \mathbb{R}$ be PUL integrable. Let U be an open subset of a compact interval E^* in $\mathbb{R}^r$. Let $\psi : U \to \psi(U)$ be C^1-diffeomorphism, and $E \subseteq \psi(U) \subseteq \mathbb{R}^n$. Then $(fo\psi)\left|\det D\psi\right| \chi_{\psi^{-1}(E)}$ is PUL integrable on E^* and*

$$\int_E f = \int_{\psi^{-1}(E)} (fo\psi)\left|\det D\psi\right|.$$

Proof. Let $\epsilon > 0$ be given. Then there exists a positive function δ' define on E such that for every δ'-fine division $D = \{(\xi_i, I_i, \varphi_i)\}_{i=1}^m$ of E, we have

$$\left|\sum_{i=1}^m f(\xi_i)\int_{I_i}\varphi_i - \int_E f\right| \le \frac{\epsilon}{2}.$$

We may assume that when $\xi \in \psi(U)$, $B_{\delta'(\xi)}(\xi) \subset \psi(U)$.

Now let δ be a positive function defined on E^* such that when $y \in \psi^{-1}(E)$, we have

$$B_{\delta(y)}(y) \subset \psi^{-1}\left(B_{\frac{\delta'(\psi(y))}{2\sqrt{n}}}(\psi(y))\right); \tag{1}$$

when $y' \in B_{\delta(y)}(y)$, we have

$$\left|\,\left|\det D\psi(y')\right| - \left|\det D\psi(y)\right|\,\right| \le \frac{\epsilon}{2|E^*|(1+|f(\psi(y))|)}. \tag{2}$$

It can be done since $|\det D\psi|$ is continuous; and when $y \in E^* \setminus \psi^{-1}(E)$, we have

$$B_{\delta(y)}(y) \cap E^* \subset E^* \setminus \psi^{-1}(E). \tag{3}$$

Let $D' = \{(y_j, I_j, \sigma_j)\}_{j=1}^q$ be a δ-fine division of E^*. Let $D' = D'_1 \cup D'_2$ where $D'_1 = \{(y, I, \sigma) \in D' : y \in \psi^{-1}(E)\}$ and $D'_2 = D' \setminus D'_1$. We may assume that $D'_1 = \{(y_j, I_j, \sigma_j)\}_{j=1}^p$ and $D'_2 = \{(y_j, I_j, \sigma_j)\}_{j=p+1}^q$. Note, for $j = p+1, ..., q$, supp $\sigma_j \cap \psi^{-1}(E) = \emptyset$. Thus for any $y \in \psi^{-1}(E)$,

$$\sum_{j=1}^{q} \sigma_j(y) = \sum_{j=1}^{p} \sigma_j(y) + \sum_{j=p+1}^{q} \sigma_j(y) = \sum_{j=1}^{p} \sigma_j(y).$$

Let $x_j = \psi(y_j)$ and $\omega_j = \sigma_j o \psi^{-1}$ for $j = 1, 2, ..., p$. Then $\omega_j : \psi(U) \to \mathbb{R}^n$. Recall $E \subset \psi(U)$. For $j = 1, 2, ..., p$, there exists an interval J_j such that

$$\text{supp } \omega_j \chi_E \subset B_{\frac{\delta'(\psi(y))}{2\sqrt{n}}}(\psi(y)) \cap E \subset J_j \subset B_{\delta'(\psi(y))}(\psi(y)) \cap E.$$

It is clear that $\omega_j \chi_E$ is continuously differentiable function on E and $\displaystyle\sum_{j=1}^{p} \omega_j \chi_E(x) = 1$ for all $x \in E$, i.e., $\{\omega_j \chi_E\}_{j=1}^p$ forms a partition of unity on E.

Then $\{(x_j, J_j, \omega_j \chi_E)\}_{j=1}^p$ is a δ'-fine division of E. Note that supp $(\omega_j o \psi) =$ supp $\sigma_j \subset I_j$. Then, by Lemma 4.1,

$$\int_{J_j} \omega_j = \int_{\psi^{-1}(J_j)} (\omega_j o \psi) |\det D\psi| = \int_{I_j} \sigma_j |\det D\psi|.$$

Hence

$$\begin{aligned}
&\left| \sum_{j=1}^{q} f(\psi(y_j)) |\det D\psi(y_j)| \chi_{\psi^{-1}(E)}(y_j) \int_{I_j} \sigma_j - \sum_{j=1}^{p} f(x_i) \int_{J_j} \omega_j \chi_E \right| \\
=& \left| \sum_{j=1}^{p} f(\psi(y_j)) |\det D\psi(y_j)| \int_{I_j} \sigma_j - \sum_{j=1}^{p} f(x_i) \int_{J_j} \omega_j \right| \\
=& \left| \sum_{j=1}^{p} f(\alpha(y_j)) |\det D\psi(y_j)| \int_{I_j} \sigma_j - \sum_{j=1}^{p} f(\alpha(y_j)) \int_{\psi^{-1}(J_j)} \omega_j o \psi |\det D\psi| \right| \\
=& \left| \sum_{j=1}^{p} f(\alpha(y_j)) |\det D\psi(y_j)| \int_{I_j} \sigma_j - \sum_{j=1}^{p} f(\alpha(y_j)) \int_{I_j} \sigma_j |\det D\psi| \right| \\
\le& \sum_{j=1}^{p} |f(\alpha(y_j))| \left| \int_{I_j} \sigma_j \big| |\det D\psi(y_j)| - |\det D\psi| \big| \right| \\
\le& \sum_{j=1}^{p} |f(\alpha(y_j))| \frac{\epsilon}{2|E^*|(1 + |f(\alpha(y_j))|)} \int_{I_j} \sigma_j \\
\le& \frac{\epsilon}{2|E^*|} \sum_{j=1}^{p} \int_{I_j} \sigma_j \le \frac{\epsilon}{2}.
\end{aligned}$$

Therefore,

$$\left|\sum_{j=1}^{q} f(\psi(y_j))|\det D\psi(y_j)|\chi_{\psi^{-1}(E)}(y_j)\int_{I_j}\sigma_j - \int_E f\right| \le \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Thus $(fo\psi)\,|\det D\psi|\,\chi_{\psi^{-1}(E)}$ is integrable to $\displaystyle\int_E f$. ■

5. Integral on Manifolds

In this section, $\mathbb{H}^n$ denotes the upper half space in $\mathbb{R}^n$, which consists of those $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ for which $x_n \ge 0$.

Definition 5.1. A non-empty subset M of $\mathbb{R}^n$ is said to be an r-manifold if for each $p \in M$ there exist an open subset V of M containing p, an open subset U of $\mathbb{R}^r$ (or $\mathbb{H}^r$) and a homeomorphism mapping $\alpha : U \to V$ (i.e., α is bijection and both α and α^{-1} are continuous) and $D\alpha(x)$ has rank r for each $x \in U$. Such α is called a chart.

If, in addition, the mapping $\alpha : U \to V$ is C^1-diffeomorphism, i.e., α is bijection and both α and α^{-1} are of C^1-class, then M is said to be a differentiable r-manifold.

Definition 5.2. Let M be a manifold. A finite collection $\Theta = \{\alpha_j\}$ of charts, where $\alpha_j : U_j \to V_j$, is said to be an atlas if $M = \displaystyle\bigcup_j V_j$.

Definition 5.3. M is said to be a compact r-manifold if M is compact.

Definition 5.4. Let $M \subseteq \mathbb{R}^n$ be an r-manifold, $\alpha : U \to V$ be a chart and $I \subseteq U$ be an interval in $\mathbb{R}^r$. Let $I^\alpha = \alpha(I)$, which is called a tile. Here I^α can be viewed as a distorted r-dimensional interval.

Let $D = \{(x_i, I_i^{\alpha_{s_i}}, \varphi_i)\}_{i=1}^m$ be a finite collection of point-distorted interval-function triples. Then D is said to be a division of M if $\{\varphi_i\}_{i=1}^m$ is a partition of unity such that supp $\varphi_i \subseteq I_i^{\alpha_{s_i}}$, where α_{s_i} is a chart in atlas Θ. Note x_i may not be contained in supp φ_i and $I_i^{\alpha_{s_i}}$.

Let δ be a positive function on M and $\xi \in M$. Then a triple (x, I^α, φ) is said to be δ-fine if $I^\alpha \subseteq B_{\delta(x)}(x)$.

A division $D = \{(x_i, I_i^{\alpha_{s_i}}, \varphi_i)\}_{i=1}^m$ of M is said to be a δ-fine division of M if each $(x_i, I_i^{\alpha_{s_i}}, \varphi_i)$ is δ-fine, i.e., supp $\varphi_i \subseteq I_i^{\alpha_{s_i}} \subset B_{\delta(x_i)}(x_i)$.

Using the open covering theorem and the existence of a partition of unity, we can prove the following lemma.

Lemma 5.5. *Let M be a compact manifold and δ a positive function defined on M. Then there is a δ-fine division of M.*

Definition 5.6. Let M be a compact differentiable r-manifold and $\Theta = \{\alpha_j\}$ an atlas of M. A function $f : M \to \mathbb{R}$ is said to be *PUL integrable* to real number A on M if for every $\epsilon > 0$, there exists a positive function δ defined on M such that for every δ-fine division $D = \{(x_i, I_i^{\alpha_{s_i}}, \varphi_i)\}_{i=1}^m$ of M, we have

$$|S(f, \delta, D) - A| \le \epsilon,$$

where

$$S(f, \delta, D) = \sum_{i=1}^m f(x_i) \int_{I_i^{\alpha_{s_i}}} \varphi_i.$$

We denote A by $\displaystyle\int_M f$.

Note that in the definition, the integral is defined using an atlas Θ. We will show that the integral does not depend on the atlas Θ.

Lemma 5.7. [2]. *Let $D = \{(\xi_i, I_i^{\alpha_{s_i}}, \varphi_i)\}_{i=1}^m$ be a partial δ-fine division of M, $\varphi(x) = \displaystyle\sum_{i=1}^m \varphi_i(x)$ and $D_1 = \{(\eta_j, J_j^{\alpha_{s_j}}, \psi_j)\}_{j=1}^p$ be a δ-fine division of M. Then $D \cup D_2$ is a δ-fine division of M, where $D_2 = \{(\eta_j, J_j^{\alpha_{s_j}}, (1 - \varphi)\psi_j)\}_{j=1}^p$.*

Proof. A proof is given in [2, p121]. Here we write down the proof for easy reference.

For $x \in M$, we have

$$\sum_{i=1}^m \varphi_i(x) + \sum_{j=1}^p (1 - \varphi)\psi_j(x) = \varphi(x) + 1 - \varphi(x) = 1,$$

that is, $\{\varphi_i\}_{i=1}^m \cup \{(1 - \varphi)\psi_j)\}_{j=1}^p$ is a partition of unity of M.

Obviously, for $i = 1, 2, ..., m$ and $j = 1, 2, ..., p$, we have

supp $\varphi_i \subseteq I_i^{\alpha_{s_i}} \subset B_{\delta(\xi_i)}(\xi_i)$ and supp $(1 - \varphi)\psi_j \subseteq$ supp $\psi_j \subseteq J_j^{\alpha_{s_j}} \subset B_{\delta(\eta_j)}(\eta_j)$.

Hence $D \cup D_2$ is a δ-fine division of M. ∎

Theorem 5.8 (Change of variable). *Let M be a compact differentiable r-manifold with atlas Θ. Let $f : M \to \mathbb{R}$. Suppose there exists a chart α in Θ such that $\alpha : U \to V \supset$ supp f. If f is PUL integrable on M, then $(f o \alpha)|D\alpha|\chi_U$ is integrable on a compact interval E, where $U \subseteq E \subseteq \mathbb{R}^n$ and*

$$\int_M f = \int_E (f o\alpha)|D\alpha|\chi_U$$

Proof. Let β be a sufficient small number such that $U^* := \alpha^{-1}(\text{supp } f)+\beta \subset U$, where $\alpha^{-1}(\text{supp } f) + \beta = \left\{ y \in U : \min_{y' \in \alpha^{-1}(\text{supp } f)} |y - y'| \le \beta \right\}$.

Let $\epsilon > 0$ be given. Then there exists a positive function δ' defined on M such that for every δ'-fine division $D = \{(\xi_i, P_i^{\alpha_{s_i}}, \varphi_i)\}_{i=1}^m$ of M, we have

$$\left| \sum_{i=1}^m f(\xi_i) \int_{P_i^{\alpha_{s_i}}} \varphi_i - \int_M f \right| \le \frac{\epsilon}{2}.$$

We may assume if $\xi \in V$, then $B_{\delta'(\xi)}(\xi) \subset V$; and if $\xi \in$ supp f, then $B_{\delta'(\xi)}(\xi) \subset \alpha(U^*)$.

Now let δ be a positive function defined on E such that when $y \in U^*$, we have

$$B_{\delta(y)}(y) \subset \alpha^{-1}\left(B_{\delta'(\alpha(y))}(\alpha(y))\right); \tag{4}$$

when $y' \in B_{\delta(y)}(y) \cap U$, we have

$$\left| \left|\det D\alpha(y')\right| - \left|\det D\alpha(y)\right| \right| \le \frac{\epsilon}{2|E|(1 + |f(\alpha(y))|)}. \tag{5}$$

It can be done since $|\det D\alpha|$ is continuous; and when $y \in E \setminus U^*$, we have

$$B_{\delta(y)}(y) \cap E \subset E \setminus U^*. \tag{6}$$

Let $D' = \{(y_j, I_j, \sigma_j)\}_{j=1}^q$ be a δ-fine division of E. Let $D' = D_1' \cup D_2'$ where $D_1' = \{(y, I, \sigma) \in D' : y \in U^*\}$ and $D_2' = D' \setminus D_1'$. We may assume that $D_1' = \{(y_j, I_j, \sigma_j)\}_{j=1}^p$ and $D_2' = \{(y_j, I_j, \sigma_j)\}_{j=p+1}^q$. Thus for any $y \in U^*$,

$$\sum_{j=1}^p \sigma_j(y) = \sum_{j=1}^p \sigma_j(y) + \sum_{j=p+1}^q \sigma_j(y) = \sum_{j=1}^q \sigma_j(y) = 1.$$

Note that $fo\alpha(y_j) = 0$ for $j = p+1, p+2, ..., q$, since $y_i \notin U^* \supset \alpha^{-1}(\text{supp } f)$. That is

$$\sum_{j=1}^q fo\alpha(y_j)|\det D\alpha(y_j)|\chi_U(y_j) \int_{I_j} \sigma_j = \sum_{j=1}^p fo\alpha(y_j)|\det D\alpha(y_j)|\chi_U(y_j) \int_{I_j} \sigma_j.$$

Let $x_j = \alpha(y_j)$ and $\omega_j = \sigma_j o \alpha^{-1}$ for $j = 1, 2, ..., p$. Then $\omega_j : V \to \mathbb{R}$ and continuously differentiable on V. Note that supp $\omega_j \subset V$. Hence we can extend ω_j to the manifold M by letting $\omega_j(y) = 0$ if $y \in M \setminus V$. The extended ω_j is still continuously differentiable. Let $D^\alpha = \{(x_j, I_j^\alpha, \omega_j)\}_{j=1}^p$. Let $\omega(x) = \sum_{j=1}^p \omega_j(x)$ for all $x \in M$. Note that $\omega(x) = 1$ for all $x \in \alpha(U^*)$ and $\omega(x) = 0$ if $x \in M \setminus V$.

D^α is a partial δ'-fine division of M. It is not a full δ'-fine division of M. Let $D'' = \{(\xi_k, P_k^{\alpha_{s_k}}, \kappa_k)\}_{k=1}^m$ be a δ'-fine division of M. By Lemma 5.7, $\bar{D} = D^\alpha \cup \{(\xi_k, P_k^{\alpha_{s_k}}, (1-\omega)\kappa_k)\}_{k=1}^m$ forms a full δ'-fine division of M. From the definition of δ', we know that for any $\xi_k \in$ supp f,

$$f(\xi_k)\int_{P_k^{\alpha_{s_k}}}(1-\omega)\kappa_k = 0,$$

since $1-\omega$ is zero on $P_k^{\alpha_{s_k}}$. Therefore

$$\sum_{k=1}^m f(\xi_k)\int_{P_k^{\alpha_{s_k}}}(1-\omega)\kappa_k = 0,$$

that is,

$$\sum_{j=1}^p f(x_j)\int_{I_j^\alpha}\omega_j+\sum_{k=1}^m f(\xi_k)\int_{P_k^{\alpha_{s_k}}}(1-\omega)\kappa_k=\sum_{j=1}^p f(x_j)\int_{I_j^\alpha}\omega_j+0=\sum_{j=1}^p f(x_j)\int_{I_j^\alpha}\omega_j.$$

Then, by Lemma 4.1, we have

$$\begin{aligned}
&\left|\sum_{j=1}^p f(\alpha(y_j))|\det D\alpha(y_j)|\int_{I_j}\sigma_j-\sum_{j=1}^p f(x_j)\int_{I_j^\alpha}\omega_j\right| \\
=&\left|\sum_{j=1}^p f(\alpha(y_j))|\det D\alpha(y_j)|\int_{I_j}\sigma_j-\sum_{j=1}^p f(\alpha(y_j))\int_{I_j}\omega_j o\alpha|\det D\alpha|\right| \\
=&\left|\sum_{j=1}^p f(\alpha(y_j))|\det D\alpha(y_j)|\int_{I_j}\sigma_j-\sum_{j=1}^p f(\alpha(y_j))\int_{I_j}\sigma_j|\det D\alpha|\right| \\
\le&\sum_{j=1}^p |f(\alpha(y_j))|\left|\int_{I_j}\sigma_j\big||\det D\alpha(y_j)|-|\det D\alpha|\big|\right| \\
\le&\sum_{j=1}^p |f(\alpha(y_j))|\frac{\epsilon}{1+|f(\alpha(y_j))|}\int_{I_j}\sigma_j \\
\le&\frac{\epsilon}{2|E|}\sum_{j=1}^p\int_{I_j}\sigma_j \le \frac{\epsilon}{2}.
\end{aligned}$$

Therefore,

$$\left|\sum_{j=1}^q fo\alpha(y_j)|\det D\alpha(y_j)|\chi_U(y_j)\int_{I_j}\sigma_j-\int_M f\right|\le\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Hence, $(fo\alpha)|D\alpha|\chi_U$ is integrable to $\int_M f$ on E. ■

Corollary 5.9. *Let M be a compact differentiable r-manifold. If Θ and Θ' are two atlas of M, then f is integrable on M associated with atlas Θ if and only if f is integrable on M associated with atlas Θ'. Furthermore, the values of these two integrals are equal.*

Proof. Suppose that supp f can be parameterised by one chart in both atlas, say $\alpha \in \Theta$ and $\alpha' \in \Theta'$, where $\alpha : U \to V \supset$ supp f and $\alpha' : U' \to V' \supset$ supp f. It is clear that supp $f \subset V \cap V'$. Let U^* be an intersection of U and U' and E a compact interval containing U^*. By Theorem 5.8,

$$\int_{M,\Theta} f = \int_E fo\alpha|D\alpha|\chi_{U^*}.$$

By Theorem 4.2,

$$\int_E fo\alpha|D\alpha|\chi_{U^*} = \int_E fo\alpha o(\alpha^{-1}o\alpha')|D\alpha||D(\alpha^{-1}o\alpha')|\chi_{U^*} = \int_E fo\alpha'|D\alpha'|\chi_{U^*}.$$

Therefore,

$$\int_{M,\Theta} f = \int_{M,\Theta'} f.$$

In general supp f may not be parameterised by one chart. We will use the partition of unity to overcome the difficulty. Let $\Theta = \{\alpha_i : U_i \to V_i\}_{i=1}^m$ and $\Theta' = \{\alpha'_j : U'_j \to V'_j\}_{j=1}^p$ be two atlas of M. Let $V_{ij} = V_i \cap V'_j$ for all i and j. Let $\{\varphi_{ij}\}_{i,j=1}^{m,p}$ be a partition of unity on M such that, for all i and j, supp $\varphi_{ij} \subset V_{ij}$. Therefore,

$$\int_{M,\Theta} f = \sum_{i=1}^m \sum_{j=1}^n \int_{M,\Theta} f\varphi_{ij} = \sum_{i=1}^m \sum_{j=1}^n \int_{M,\Theta'} f\varphi_{ij} = \int_{M,\Theta'} f. \qquad ■$$

The above corollary shows that the PUL integral does not depend on the parameterisation of the manifold.

Remark 5.10. In definitions 2.1 and 5.6, if, in addition, we assume $\xi_i \in I_i$ and $x_i \in I_i^{\alpha_{s_i}}$, respectively, then the integral is called the PU integral. Theorems 4.2 and 5.8 still hold true for the PU integral.

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