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FIXED POINT THEOREMS AND ERGODIC THEOREMS FOR NONLINEAR MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first consider classes of nonlinear mappings containing the class of firmly nonexpansive mappings which can be deduced from an equilibrium problem in a Hilbert space. Further, we deal with fixed point theorems and ergodic theorems for these nonlinear mappings.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that if C is a bounded closed convex subset of H and $T : C \to C$ is nonexpansive, then the set $F(T)$ of fixed points of T is nonempty. Further, from Baillon [1] we know the first nonlinear ergodic theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of H and let $T : C \to C$ be nonexpansive. Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping F is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$; see, for instance, Browder [3], Goebel and Kirk [5], Goebel and Reich [6], Reich and Shoikhet [11] and Takahashi [13]. It is known that a mapping $F : C \to H$ is firmly nonexpansive if and only if

$$\|Fx - Fy\|^2 + \|(I - F)x - (I - F)y\|^2 \leq \|x - y\|^2$$

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for all $x, y \in C$, where I is the identity mapping on H. It is also known that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space as follows: Let C be a nonempty closed convex subset of H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

$(A1)$ $f(x, x) = 0, \quad \forall x \in C$;

$(A2)$ f is monotone, i.e., $f(x, y) + f(y, x) \le 0, \quad \forall x, y \in C$;

$(A3)$ $\lim_{t\downarrow 0} f(tz + (1-t)x, y) \le f(x, y), \quad \forall x, y, z \in C$;

$(A4)$ for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

We know the following lemma; see, for instance, [2] and [4].

Lemma 1.1. *Let C be a nonempty closed convex subset of H and let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Further, if $T_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\}$, then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H.$$

Recently, Kohsaka and Takahashi [9] introduced the following nonlinear mapping: Let E be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of E and let C be a nonempty closed convex subset of E. Then, a mapping $S : C \to E$ is said to be nonspreading if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \le \phi(Sx, y) + \phi(Sy, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. They considered such a mapping to study the resolvents of a maximal monotone operator in the Banach space. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping S in a Hilbert space H is defined as follows:

$$2\|Sx - Sy\|^2 \le \|Sx - y\|^2 + \|x - Sy\|^2$$

for all $x, y \in C$. On the other hand, Takahashi [16] found another new nonlinear mapping called a hybrid mapping which is deduced from a firmly nonexpansive mapping.

In this paper, we first discuss classes of nonlinear mappings containing the class of firmly nonexpansive mappings which can be deduced from a firmly nonexpansive mapping in a Hilbert space. Further, we deal with fixed point theorems and ergodic theorems for these nonlinear mappings.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$, respectively. In a Hilbert space, it is known that

(1) $$\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha) \|y\|^2 - \alpha(1-\alpha) \|x-y\|^2$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see, for instance, [15]. Further, in a Hilbert space, we have that

(2) $$2 \langle x-y, z-w \rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2$$

for all $x, y, z, w \in H$. Indeed, we have that

$$\begin{aligned} 2 \langle x-y, z-w \rangle &= 2 \langle x, z \rangle - 2 \langle x, w \rangle - 2 \langle y, z \rangle + 2 \langle y, w \rangle \\ &= (-\|x\|^2 + 2 \langle x, z \rangle - \|z\|^2) + (\|x\|^2 - 2 \langle x, w \rangle + \|w\|^2) \\ &\quad + (\|y\|^2 - 2 \langle y, z \rangle + \|z\|^2) + (-\|y\|^2 + 2 \langle y, w \rangle - \|w\|^2) \\ &= \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 . \end{aligned}$$

Let C be a closed convex subset of H and let T be a mapping of C into H. We denote by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{z \in C : Tz = z\}$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. A mapping $T : C \to H$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $F : C \to H$ is firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$. We know that a firmly nonexpansive mapping $S : C \to H$ is nonexpansive. The following lemma is in [13].

Lemma 2.1. *Let C be a nonempty closed convex subset of H and let $f : C \to (-\infty, \infty]$ be a proper convex lower semicontinuous function such that $f(z_m) \to \infty$ as $\|z_m\| \to \infty$. Then there exists an element $z_0 \in C$ such that*

$$f(z_0) = \min\{f(z) : z \in C\}.$$

Let $\mathbb{N}$ be the set of positive integers and let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞; see [13] for more details.

3. Nonlinear Mappings

Let H be a Hilbert space. Let C be a nonempty closed convex subset of H and let T be a mapping of C into H. Then, from [16], we have the following equality:

$$\|Tx - Ty\|^2 = \|x - y - (Tx - Ty)\|^2 - \|x - y\|^2 + 2\langle x - y, Tx - Ty\rangle \tag{3}$$

for all $x, y \in C$. We have also from (2) that

$$2\,\langle x - y, Tx - Ty\rangle = \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2. \tag{4}$$

Further, we have that

$$\|x - y - (Tx - Ty)\|^2 = \|x - Tx\|^2 + \|y - Ty\|^2 - 2\langle x - Tx, y - Ty\rangle. \tag{5}$$

If $T : C \to H$ is firmly nonexpansive, then

$$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle.$$

So, we have from (3) that

$$\begin{aligned} 2\|Tx - Ty\|^2 &\le 2\langle x - y, Tx - Ty\rangle \\ &= \|Tx - Ty\|^2 - \|x - y - (Tx - Ty)\|^2 + \|x - y\|^2 \\ &\le \|Tx - Ty\|^2 + \|x - y\|^2. \end{aligned}$$

Then, we have

$$\|Tx - Ty\|^2 \le \|x - y\|^2$$

and hence

$$\|Tx - Ty\| \le \|x - y\|.$$

Such a mapping is called a nonexpansive mapping. Thus, we can get new classes of nonlinear operators which contain the class of firmly nonexpansive mappings in a Hilbert space. For example, Kohsaka and Takahahi [9] obtained a *nonspreading mapping*, i.e.,

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$$

for all $x, y \in C$. We know that the class of nonspreading mappings contains the class of firmly nonexpansive mappings; see [16]. From Iemoto and Takahashi [7], we know the following lemma.

Lemma 3.1. *Let C be a nonempty closed convex subset of H. Then a mapping $S : C \to H$ is nonspreading if and only if*

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\langle x - Sx, y - Sy\rangle$$

for all $x, y \in C$.

Further, Takahashi [16] defined the following *hybrid mapping*, i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. We also know that the class of hybrid mappings contains the class of firmly nonexpansive mappings; see [16]. From Takahashi [16], we know the following lemma.

Lemma 3.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to H$ is hybrid if and only if*

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2$$

for all $x, y \in C$.

So, a hybrid mapping $T : C \to H$ is different from a nonspreading mapping.

4. Generalized Fixed Point Theorem

In this section, we prove a generalized fixed point theorem in a Hilbert space. Before proving the theorem, we show the following lemma.

Lemma 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H, let $\{x_n\}$ be a bounded sequence in H and let μ be a Banach limit. If $g : C \to \mathbb{R}$ is defined by*

$$g(z) = \mu_n\|x_n - z\|^2, \quad \forall z \in C,$$

then there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Proof. Let $z, y \in C$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. Then, for any $n \in \mathbb{N}$ we have from (1)

$$\|x_n - (\alpha z + \beta y)\|^2 \leq \alpha \|x_n - z\|^2 + \beta \|x_n - y\|^2.$$

Since μ is a Banach limit, we have

$$\begin{aligned} g(\alpha z + \beta y) &= \mu_n \|x_n - (\alpha z + \beta y)\|^2 \\ &\leq \alpha \mu_n \|x_n - z\|^2 + \beta \mu_n \|x_n - y\|^2 \\ &= \alpha g(z) + \beta g(y). \end{aligned}$$

This implies that $g : C \to \mathbb{R}$ is a convex function. Let $z \in C$ and let $\{z_m\}$ be a sequence in C such that $z_m \to z$. Then, for any $n, m \in N$ we have

$$\begin{aligned} \|x_n - z_m\|^2 - \|x_n - z\|^2 &\leq |\|x_n - z_m\| - \|x_n - z\||(\|x_n - z_m\| + \|x_n - z\|) \\ &\leq M_1 \|z_m - z\|, \end{aligned}$$

where $M_1 = \sup_{n,m \in N}(\|x_n - z_m\| + \|x_n - z\|)$. So, we have

$$g(z_m) - g(z) \leq M_1 \|z_m - z\|.$$

Similarly, we have

$$g(z) - g(z_m) \leq M_1 \|z_m - z\|.$$

Therefore, we have

$$|g(z_m) - g(z)| \leq M_1 \|z_m - z\|.$$

This implies that $g : C \to \mathbb{R}$ is a continuous function. Suppose that $\{z_m\}$ is a sequence in C such that $\|z_m\| \to \infty$. Then, we have

$$\begin{aligned} \|z_m\|^2 &= \|z_m - x_n + x_n\|^2 \\ &= \|z_m - x_n\|^2 + \|x_n\|^2 + 2\langle z_m - x_n, x_n \rangle \\ &\leq \|z_m - x_n\|^2 + \|x_n\|^2 + 2(\|z_m\| + \|x_n\|)\|x_n\| \\ &\leq \|z_m - x_n\|^2 + M_2^2 + 2(\|z_m\| + M_2)M_2, \end{aligned}$$

where $M_2 = \sup_{n \in \mathbb{N}} \|x_n\|$. Hence, we have

$$\|z_m\|(\|z_m\| - 2M_2) - 3M_2^2 \leq \|z_m - x_n\|^2.$$

So, we have

$$\|z_m\|(\|z_m\| - 2M_2) - 3M_2^2 \leq \mu_n \|z_m - x_n\|^2.$$

This implies that $g(z_m) \to \infty$ as $\|z_m\| \to \infty$. Therefore, we have from Lemma 2.1 that there exists an element $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Let z_0 and z_1 be elements in C such that $z_0 \neq z_1$ and

$$g(z_0) = g(z_1) = \min\{g(z) : z \in C\} = r.$$

From (1), we have

$$\|x_n - (\frac{1}{2}z_0 + \frac{1}{2}z_1)\|^2 = \frac{1}{2}\|x_n - z_0\|^2 + \frac{1}{2}\|x_n - z_1\|^2 - \frac{1}{4}\|z_0 - z_1\|^2.$$

So, using μ, we have

$$\begin{aligned} g(\frac{1}{2}z_0 + \frac{1}{2}z_1) &= \frac{1}{2}g(z_0) + \frac{1}{2}g(z_1) - \frac{1}{4}\|z_0 - z_1\|^2 \\ &= r - \frac{1}{4}\|z_0 - z_1\|^2. \end{aligned}$$

This is a contradiction. So, we have $z_0 = z_1$. ∎

Theorem 4.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself. Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and*

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2, \quad \forall y \in C$$

for some Banach limit μ. Then, T has a fixed point in C.

Proof. Using a Banach limit μ on l^∞, we can define $g : C \to \mathbb{R}$ as follows:

$$g(z) = \mu_n \|T^n x - z\|^2, \quad \forall z \in C.$$

From Lemma 4.1, there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

So, we have

$$g(Tz_0) = \mu_n \|T^n x - Tz_0\|^2 \leq \mu_n \|T^n x - z_0\|^2 = g(z_0).$$

Since Tz_0 is in C and $z_0 \in C$ is a unique element such that

$$g(z_0) = \min\{g(z) : z \in C\},$$

we have $Tz_0 = z_0$. This completes the proof. ∎

5. Some Fixed Point Theorems

In this section, we obtain some fixed point theorems by using Theorem 4.1. The following is the well-known fixed point theorem for nonexpansive mappings in a Hilbert space; see, for instance, [15].

Theorem 5.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. Let μ be a Banach limit on l^∞. For any $n \in \mathbb{N}$ and $y \in C$, we have

$$\|T^{n+1}x - Ty\|^2 \leq \|T^n x - y\|^2.$$

So, we have

$$\mu_n \|T^n x - Ty\|^2 = \mu_n \|T^{n+1}x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2$$

for all $y \in C$. By Theorem 4.1, T has a fixed point in C. ∎

The following is a fixed point theorem for nonspreading mappings in a Hilbert space.

Theorem 5.2. ([9]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. Let μ be a Banach limit on l^∞. For any $n \in \mathbb{N}$ and $y \in C$, we have

$$2\|T^{n+1}x - Ty\|^2 \leq \|T^{n+1}x - y\|^2 + \|T^n x - Ty\|^2.$$

So, we have

$$\begin{aligned} 2\mu_n \|T^n x - Ty\|^2 &= 2\mu_n \|T^{n+1}x - Ty\|^2 \\ &\leq \mu_n \|T^{n+1}x - y\|^2 + \mu_n \|T^n x - Ty\|^2 \\ &= \mu_n \|T^n x - y\|^2 + \mu_n \|T^n x - Ty\|^2 \end{aligned}$$

and hence

$$\mu_n \|T^n x - Ty\|^2 \le \mu_n \|T^n x - y\|^2.$$

By Theorem 4.1, T has a fixed point in C. ■

The following is a fixed point theorem for hybrid mappings in a Hilbert space.

Theorem 5.3. ([16]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a hybrid mapping, i.e.,*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. Let μ be a Banach limit on l^∞. We know from Lemma 3.2 that a mapping $T : C \to C$ is hybrid if and only if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

So, for any $n \in \mathbb{N}$ and $y \in C$, we have

$$3\|T^{n+1} x - Ty\|^2 \le \|T^n x - y\|^2 + \|T^{n+1} x - y\|^2 + \|T^n x - Ty\|^2.$$

So, we have

$$\begin{aligned} 3\mu_n \|T^n x - Ty\|^2 &= 3\mu_n \|T^{n+1} x - Ty\|^2 \\ &\le 2\mu_n \|T^n x - y\|^2 + \mu_n \|T^n x - Ty\|^2 \end{aligned}$$

and hence

$$\mu_n \|T^n x - Ty\|^2 \le \mu_n \|T^n x - y\|^2.$$

By Theorem 4.1, T has a fixed point in C. ■

We can also prove the following fixed point theorem in a Hilbert space.

Theorem 5.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping such that*

$$2\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. Let μ be a Banach limit on l^∞. For any $n \in \mathbb{N}$ and $y \in C$, we have

$$2\|T^{n+1}x - Ty\|^2 \le \|T^n x - y\|^2 + \|T^{n+1}x - y\|^2.$$

So, we have

$$\begin{aligned} 2\mu_n\|T^n x - Ty\|^2 &= 2\mu_n\|T^{n+1}x - Ty\|^2 \\ &\le 2\mu_n\|T^n x - y\|^2 \end{aligned}$$

and hence

$$\mu_n\|T^n x - Ty\|^2 \le \mu_n\|T^n x - y\|^2.$$

By Theorem 4.1, T has a fixed point in C. ■

We can also discuss the demiclosedness of our nonlinear mappings in a Hilbert space. The following result is well known; see [13].

Theorem 5.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T is demiclosed, i.e., $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ imply $u \in F(T)$.*

The following result is in [7].

Theorem 5.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonspreadind mapping of C into itself. Then T is demiclosed, i.e., $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ imply $u \in F(T)$.*

From Takahashi [16], we also know the following result.

Theorem 5.7. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a hybrid mapping of C into itself. Then T is demiclosed, i.e., $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ imply $u \in F(T)$.*

We can further prove the following result.

Theorem 5.8. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping such that*

$$2\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

Then T is demiclosed, i.e., $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ imply $u \in F(T)$.

Proof. Let $\{x_n\} \subset C$ be a sequence such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ as $n \to \infty$. Then the sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded. Suppose that $u \neq Tu$. From Opial's theorem [10], we have

$$\begin{aligned}\liminf_{n\to\infty}\|x_n-u\|^2 &< \liminf_{n\to\infty}\|x_n-Tu\|^2\\ &= \liminf_{n\to\infty}\|x_n-Tx_n+Tx_n-Tu\|^2\\ &= \liminf_{n\to\infty}\|Tx_n-Tu\|^2\\ &\le \liminf_{n\to\infty}\frac{1}{2}(\|x_n-u\|^2+\|Tx_n-u\|^2)\\ &= \liminf_{n\to\infty}\frac{1}{2}(\|x_n-u\|^2+\|Tx_n-x_n+x_n-u\|^2)\\ &= \liminf_{n\to\infty}\|x_n-u\|^2.\end{aligned}$$

This is a contradiction. Hence we get the conclusion. ■

6. Nonlinear Erdodic Theorems

Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space.

Theorem 6.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

We can also prove the following nonlinear ergodic theorem for our nonlinear operators in a Hilbert space.

Theorem 6.2. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself such that $F(T)$ is nonempty. Suppose that T satisfies one of the following conditions:*

(*i*) *T is nonspreading;*
(*ii*) *T is hybrid;*
(*iii*) $2\|Tx-Ty\|^2 \le \|x-y\|^2+\|Tx-y\|^2, \quad \forall x,y \in C.$

Then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

Proof. Let us prove the case of (i). We first show that $F(T)$ is closed and convex. It follows from Theorem 5.6 that $F(T)$ is closed. In fact, let $\{x_n\} \subset F(T)$ and $x_n \to z$. Then, we have $x_n \rightharpoonup z$ and $x_n - Tx_n = 0$. So, from Theorem 5.6 we have $z = Tz$. Let us show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\alpha \in [0, 1]$ and put $z = \alpha x + (1 - \alpha)y$. Then, we have from (1) that

$$\begin{aligned}
\|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\
&= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha\|Tx - Tz\|^2 + (1-\alpha)\|Ty - Tz\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&\leq \alpha(1-\alpha)^2\|x - y\|^2 + (1-\alpha)\alpha^2\|x - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(1-\alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 \\
&= 0.
\end{aligned}$$

So, we have $Tz = z$. Let $x \in C$ and let P be the metric projection of H onto $F(T)$. Then, we have

$$\begin{aligned}
\|PT^n x - T^n x\| &\leq \|PT^{n-1}x - T^n x\| \\
&= \|TPT^{n-1}x - T^n x\| \\
&\leq \|PT^{n-1}x - T^{n-1}x\|.
\end{aligned}$$

This implies that $\{\|PT^n x - T^n x\|\}$ is nonincreasing. We also know that for any $v \in C$ and $u \in F(T)$,

$$\langle v - Pv, Pv - u\rangle \geq 0$$

and hence

$$\|v - Pv\|^2 \leq \langle v - Pv, v - u\rangle.$$

So, we get

$$\begin{aligned}
\|Pv - u\|^2 &= \|Pv - v + v - u\|^2 \\
&= \|Pv - v\|^2 - 2\langle Pv - v, u - v\rangle + \|v - u\|^2 \\
&\leq \|v - u\|^2 - \|Pv - v\|^2.
\end{aligned}$$

Let $m, n \in \mathbb{N}$ with $m \geq n$. Putting $v = T^m x$ and $u = PT^n x$, we have

$$\begin{aligned}
&\|PT^m x - PT^n x\|^2 \\
&\leq \|T^m x - PT^n x\|^2 - \|PT^m x - T^m x\|^2 \\
&\leq \|T^n x - PT^n x\|^2 - \|PT^m x - T^m x\|^2.
\end{aligned}$$

So, $\{PT^n x\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{PT^n x\}$ converges strongly to an element p of $F(T)$. Take $u \in F(T)$. Then we obtain, for any $n \in \mathbb{N}$,

$$\|S_n x - u\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x - u\| \leq \|x - u\|.$$

So, $\{S_n x\}$ is bounded and hence there exists a weakly convergent subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$. If $S_{n_i} x \rightharpoonup v$, then we have $v \in F(T)$. In fact, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned}
\|T^{k+1}x - Ty\|^2 &\leq \|T^k x - y\|^2 + 2\langle T^k x - T^{k+1}x, y - Ty\rangle \\
&= \|T^k x - y\|^2 + \|T^k x - Ty\|^2 + \|T^{k+1}x - y\|^2 \\
&\quad - \|T^k x - y\|^2 - \|T^{k+1}x - Ty\|^2 \\
&= \|T^k x - Ty\|^2 + 2\langle T^k x - Ty, Ty - y\rangle + \|Ty - y\|^2 \\
&\quad + \|T^k x - Ty\|^2 + \|T^{k+1}x - y\|^2 - \|T^k x - y\|^2 - \|T^{k+1}x - Ty\|^2.
\end{aligned}$$

So, we obtain that

$$\begin{aligned}
2\|T^{k+1}x - Ty\|^2 \leq\ & 2\|T^k x - Ty\|^2 + 2\langle T^k x - Ty, Ty - y\rangle \\
& + \|Ty - y\|^2 + \|T^{k+1}x - y\|^2 - \|T^k x - y\|^2.
\end{aligned}$$

Summing these inequalities with respect to $k = 0, 1, \ldots, n-1$, we have

$$\begin{aligned}
2\|T^n x - Ty\|^2 \leq\ & 2\|x - Ty\|^2 + 2\Big\langle \sum_{k=0}^{n-1} T^k x - nTy, Ty - y\Big\rangle \\
& + n\|Ty - y\|^2 + \|T^n x - y\|^2 - \|x - y\|^2.
\end{aligned}$$

Deviding this inequality by n, we have

$$\begin{aligned}
\frac{2}{n}\|T^n x - Ty\|^2 \leq\ & \frac{2}{n}\|x - Ty\|^2 + 2\langle S_n x - Ty, Ty - y\rangle \\
& + \|Ty - y\|^2 + \frac{1}{n}\|T^n x - y\|^2 - \frac{1}{n}\|x - y\|^2,
\end{aligned}$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Replacing n by n_i and letting $n_i \to \infty$, we obtain from $S_{n_i} x \rightharpoonup v$ that

$$0 \leq \|Ty - y\|^2 + 2\langle v - Ty, Ty - y\rangle.$$

Putting $y = v$, we have

$$0 \leq \|Tv - v\|^2 + 2\langle v - Tv, Tv - v\rangle.$$

So, we have $0 \leq -\|Tv - v\|^2$ and hence $Tv = v$. To complete the proof of (i), it is sufficient to show that if $S_{n_i}x \rightharpoonup v$, then $v = p$. We have, for any $u \in F(T)$,

$$\langle T^k x - PT^k x, PT^k x - u\rangle \geq 0.$$

Since $\{\|T^k x - PT^k x\|\}$ is nonincreasing, we have

$$\begin{aligned}\langle u - p, T^k x - PT^k x\rangle &\leq \langle PT^k x - p, T^k x - PT^k x\rangle \\ &\leq \|PT^k x - p\| \cdot \|T^k x - PT^k x\| \\ &\leq \|PT^k x - p\| \cdot \|x - Px\|.\end{aligned}$$

Adding these inequalities from $k = 0$ to $k = n - 1$ and dividing n, we have

$$\langle u - p, S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x\rangle \leq \frac{\|x - Px\|}{n}\sum_{k=0}^{n-1}\|PT^k x - p\|.$$

Since $S_{n_i}x \rightharpoonup v$ and $PT^k x \to p$, we have

$$\langle u - p, v - p\rangle \leq 0.$$

We know $v \in F(T)$. So, putting $u = v$, we have $\langle v - p, v - p\rangle \leq 0$ and hence $\|v - p\|^2 \leq 0$. So, we obtain $v = p$. This completes the proof of (i).

Let us prove the case of (ii). It follows from Theorem 5.7 that $F(T)$ is closed. As in the proof of (i), we can show that $F(T)$ is convex. Let $x \in C$ and let P be the metric projection of H onto $F(T)$. Then, as in the proof of (i), we can have that $\{PT^n x\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{PT^n x\}$ converges strongly to an element p of $F(T)$. Take $u \in F(T)$. Then we obtain, for any $n \in \mathbb{N}$,

$$\|S_n x - u\| \leq \frac{1}{n}\sum_{k=0}^{n-1}\|T^k x - u\| \leq \|x - u\|.$$

So, $\{S_n x\}$ is bounded and hence there exists a weakly convergent subsequence $\{S_{n_i}x\}$ of $\{S_n x\}$. If $S_{n_i}x \rightharpoonup v$, then we have $v \in F(T)$. In fact, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned}2\|T^{k+1}x - Ty\|^2 &\leq 2\|T^k x - y\|^2 + 2\langle T^k x - T^{k+1}x, y - Ty\rangle \\ &= 2\|T^k x - y\|^2 + \|T^k x - Ty\|^2 + \|T^{k+1}x - y\|^2 \\ &\quad - \|T^k x - y\|^2 - \|T^{k+1}x - Ty\|^2 \\ &= 2\|T^k x - Ty\|^2 + 4\langle T^k x - Ty, Ty - y\rangle + 2\|Ty - y\|^2 \\ &\quad + \|T^k x - Ty\|^2 + \|T^{k+1}x - y\|^2 - \|T^k x - y\|^2 - \|T^{k+1}x - Ty\|^2.\end{aligned}$$

So, we obtain that

$$3\|T^{k+1}x - Ty\|^2 \leq 3\|T^k x - Ty\|^2 + 4\langle T^k x - Ty, Ty - y\rangle$$
$$+ 2\|Ty - y\|^2 + \|T^{k+1}x - y\|^2 - \|T^k x - y\|^2.$$

Summing these inequalities with respect to $k = 0, 1, \ldots, n-1$, we have

$$3\|T^n x - Ty\|^2 \leq 3\|x - Ty\|^2 + 4\left\langle \sum_{k=0}^{n-1} T^k x - nTy, Ty - y \right\rangle$$
$$+ 2n\|Ty - y\|^2 + \|T^n x - y\|^2 - \|x - y\|^2.$$

Deviding this inequality by n, we have

$$\frac{3}{n}\|T^n x - Ty\|^2 \leq \frac{3}{n}\|x - Ty\|^2 + 4\langle S_n x - Ty, Ty - y\rangle$$
$$+ 2\|Ty - y\|^2 + \frac{1}{n}\|T^n x - y\|^2 - \frac{1}{n}\|x - y\|^2,$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Replacing n by n_i and letting $n_i \to \infty$, we obtain from $S_{n_i} x \rightharpoonup v$ that

$$0 \leq 2\|Ty - y\|^2 + 4\langle v - Ty, Ty - y\rangle.$$

Putting $y = v$, we have

$$0 \leq 2\|Tv - v\|^2 + 4\langle v - Tv, Tv - v\rangle.$$

So, we have $0 \leq -2\|Tv - v\|^2$ and hence $Tv = v$. To complete the proof of (ii), it is sufficient to show that if $S_{n_i} x \rightharpoonup v$, then $v = p$. As in the proof of (i), we can prove $v = p$. This completes the proof of (ii).

As in the proofs of (i) and (ii), we can prove the case of (iii). ■

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