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STRONG CONVERGENCE THEOREMS BY A RELAXED EXTRAGRADIENT-LIKE

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Abstract. Very recently, Takahashi and Takahashi [S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Analysis 69 (2008) 1025-1033] suggested and analyzed an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In this paper, we introduce a general system of generalized equilibria with inverse-strongly monotone mappings in a real Hilbert space. First, this system of generalized equilibria is proven to be equivalent to a fixed point problem of nonexpansive mapping. Second, by using the demi-closedness principle for nonexpansive mappings, we prove that under quite mild conditions the iterative sequence defined by the relaxed extragradient-like method converges strongly to a solution of this system of generalized equilibria. In addition, utilizing this result, we provide some applications of the considered problem not just giving a pure extension of existing mathematical problems.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Recall that a mapping $S : C \to C$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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We denote by $F(S)$ the set of fixed points of S and by P_C the metric projection of H onto C. A mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C;$$

see, e.g., [1,4] for more details.

Very recently, Takahashi and Takahashi [17] introduced and considered the following generalized equilibrium problem: Find $x^* \in C$ such that

$$F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $F : C \times C \to \mathbf{R}$ is a bifunction and $A : C \to H$ is a nonlinear mapping. The set of such $z \in C$ is denoted by EP, i.e.,

$$EP = \{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C\}.$$

If $A \equiv 0$, EP is denoted by $EP(F)$. If $F \equiv 0$, the problem (1.1) reduces to the classical variational inequality, denoted by VI(A, C), is to find an $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C.$$

In this case, EP is also denoted by $VI(C, A)$, i.e., the set of solutions of the VI(A, C). The variational inequality has been widely studied in the literature; see, e.g., [4-5,12-16,18,26-29] and the references therein. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, e.g., [23-24].

Recently, Tada and Takahashi [8], and Takahashi and Takahashi [9] considered iterative methods for finding an element of $EP(F) \cap F(S)$. Subsequently, the main Theorem 3.2 of Takahashi and Takahashi [9] is extended to develop several more general results in [20-22].

Very recently, Moudafi [25] introduced an iterative method for finding an element of $EP \cap F(S)$, where $A : C \to H$ is an inverse-strongly monotone mapping and then proved a weak convergence theorem. Motivated by Moudafi [25], Takahashi and Takahashi [17] introduced another iterative method for finding an element of $EP \cap F(S)$, where $A : C \to H$ is also an inverse-strongly monotone mapping and then obtained a strong convergence theorem.

Theorem 1.1 (cf. [17, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying the following conditions:*

$(A1)$ $F(x, x) = 0$, $\forall x \in C$;

$(A2)$ *F is monotone, i.e., $F(x,y)+F(y,x)\le 0,\ \forall x,y\in C$;*

$(A3)$ $\lim_{t\to 0^+} F(tz+(1-t)x,y)\le F(x,y),\ \forall x,y,z\in C$;

$(A4)$ *for each $x\in C,\ y\mapsto F(x,y)$ is convex and lower semicontinuous.*
Let $A: C\to H$ be an α-inverse-strongly monotone mapping and let $S: C\to C$ be a nonexpansive mapping such that $F(S)\cap EP\neq\emptyset$. Let $u\in C$ and $x_1\in C$ and let $\{z_n\}\subset C$ and $\{x_n\}\subset C$ be sequences generated by

$$\begin{cases} F(z_n,y)+\langle Ax_n,y-z_n\rangle+\dfrac{1}{\lambda_n}\langle y-z_n,z_n-x_n\rangle\ge 0, & \forall y\in C,\\ x_{n+1}=\beta_n x_n+(1-\beta_n)S[\alpha_n u+(1-\alpha_n)z_n], & \forall n\in N,\end{cases}$$

where $\{\alpha_n\}\subset[0,1],\{\beta_n\}\subset[0,1]$ and $\{\lambda_n\}\subset[0,2\alpha]$ satisfy

$$0<c\le\beta_n\le d<1,\qquad 0<a\le\lambda_n\le b<2\alpha,$$

$$\lim_{n\to\infty}(\lambda_n-\lambda_{n+1})=0,\ \lim_{n\to\infty}\alpha_n=0 \text{ and } \sum_{n=1}^{\infty}\alpha_n=\infty.$$

Then, $\{x_n\}$ converges strongly to $z=P_{F(S)\cap EP}u$.

On the other hand, in order to find an element of $F(S)\cap VI(C,A)$, Takahashi and Toyoda [13] introduced the following iterative scheme:

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)SP_C(x_n-\lambda_n Ax_n),\quad \forall n\ge 0, \tag{1.2}$$

where $x_0=x\in C,\ \{\alpha_n\}\subset(0,1)$ and $\{\lambda_n\}\subset(0,2\alpha)$.

Furthermore, in order to solve the VI(A,C) in the Euclidean space $\mathbf{R}^n$, Korpelevich [3] introduced the following so-called extragradient method:

$$\begin{cases} x_0=x\in C,\\ y_n=P_C(x_n-\lambda Ax_n),\\ x_{n+1}=P_C(x_n-\lambda Ay_n), & \forall n\ge 0,\end{cases} \tag{1.3}$$

where $\lambda\in(0,1/k)$.

Recently, Nadezhkina and Takahashi [5] and Zeng and Yao [15] proposed some iterative schemes for finding elements in $F(S)\cap VI(C,A)$ by combining (1.2) with (1.3). Further, these iterative schemes are extended in Yao and Yao [14] to develop a new iterative scheme for finding elements in $F(S)\cap VI(C,A)$.

Let C be a nonempty closed convex subset of a real Hilbert space H. let $F,G: C\times C\to\mathbf{R}$ be two bifunctions and let $A,B: C\to H$ be two nonlinear mappings.

In this paper, we consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$(1.4) \qquad \begin{cases} F(x^*, x) + \langle Ay^*, x - x^* \rangle + \dfrac{1}{\lambda}\langle x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ G(y^*, y) + \langle Bx^*, y - y^* \rangle + \dfrac{1}{\mu}\langle y^* - x^*, y - y^* \rangle \geq 0, \ \forall y \in C, \end{cases}$$

which is called a general system of generalized equilibria where $\lambda > 0$ and $\mu > 0$ are two constants.

Special Cases.

(1) If $F = G$ and $A = B$, then problem (1.4) reduces to the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$(1.4)' \quad \begin{cases} F(x^*, x) + \langle Ay^*, x - x^* \rangle + \dfrac{1}{\lambda}\langle x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ F(y^*, y) + \langle Ax^*, y - y^* \rangle + \dfrac{1}{\mu}\langle y^* - x^*, y - y^* \rangle \geq 0, \ \forall y \in C, \end{cases}$$

which is called a new system of generalized equilibria where $\lambda > 0$ and $\mu > 0$ are two constants.

(2) If $F = G,\ A = B,$ and $x^* = y^*$, then problem (1.4) reduces to problem (1.1).

(3) If $F = G = 0$, then problem (1.4) reduces to the following general system of variational inequalities: Find $(x^*, y^*) \in C \times C$ such that

$$(1.5) \qquad \begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, y - y^* \rangle \geq 0, \ \forall y \in C, \end{cases}$$

where $\lambda > 0$ and $\mu > 0$ are two constants, which is introduced and considered by Ceng, Wang and Yao [18];

(4) If $A = B$ in (1.5), then problem (1.5) reduces to the following new system of variational inequalities:

$$(1.6) \qquad \begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, y - y^* \rangle \geq 0, \ \forall y \in C, \end{cases}$$

which is defined by Verma [26] (see also [27]);

(5) If $x^* = y^*$ in (1.6), then problem (1.6) reduces to the classical variational inequality.

Very recently, motivated by the iterative methods in Korpelevich [3], Takahashi and Toyoda [8], Nadezhkina and Takahashi [5], Zeng and Yao [15], and Yao and Yao [14], Ceng, Wang and Yao [18] proposed a relaxed extragradient method for finding solutions of problem (1.5), and derived a strong convergence theorem for problem (1.5).

Theorem 1.2. (cf. [18, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by*

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \end{cases} \tag{1.7}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0, 1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1,\ \forall n \geq 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap\Omega} u$, and $(\bar{x}, \bar{y})$ is a solution of problem (1.5), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$ and Ω is the set of fixed points of the mapping $G : C \to C$ defined by

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C.$$

Inspired by Korpelevich [3], Takahashi and Toyoda [8], Nadezhkina and Takahashi [5], Takahashi and Takahashi [9, 17], Zeng and Yao [15], Yao and Yao [14], and Ceng, Wang and Yao [18], we suggest and analyze a relaxed extragradient-like method for finding solutions of problem (1.4). Let $F, G : C \times C \to \mathbf{R}$ be two bifunctions and the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by

$$\begin{cases} G(y_n, y) + \langle B x_n, y - y_n \rangle + \dfrac{1}{\mu} \langle y_n - x_n, y - y_n \rangle \geq 0,\ \forall y \in C, \\ F(t_n, z) + \langle A y_n, x - t_n \rangle + \dfrac{1}{\lambda} \langle t_n - y_n, x - t_n \rangle \geq 0,\ \forall x \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S t_n, \end{cases}$$

where $\lambda \in (0, 2\alpha]$, $\mu \in (0, 2\beta]$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1,\ \forall n \geq 1$. First, problem (1.4) is proven to be equivalent to a fixed point problem of nonexpansive mapping. Second, by using the demi-closedness principle for nonexpansive mappings, we prove that under quite mild conditions the iterative sequence $\{x_n\}$ converges strongly to some $\bar{x} \in C$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.4), where $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$. In addition, utilizing this result, we provide some applications of the considered problem not just giving a pure extension of existing mathematical problems.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Denote by I the identity mapping of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point of C, denote by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C, i.e.,

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Obviously, this immediately implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \le \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Recall that, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} &\langle x - P_C x, y - P_C x\rangle \le 0, \\ &\|x - y\|^2 \ge \|x - P_C x\|^2 + \|P_C x - y\|^2, \end{aligned} \tag{2.1}$$

for all $x \in H$ and $y \in C$; see Goebel and Kirk [2] for more details.

Recall that, if $S : C \to C$ is nonexpansive, then the set $F(S)$ of fixed points of S is closed and convex. Moreover, if C is bounded, closed and convex, then $F(S)$ is nonempty. Notice also that, if $S : C \to C$ is nonexpansive, then $A = I - S$ is $\frac{1}{2}$-inverse-strongly monotone; see [10] for more details.

We need the following propositions and lemmas for the proof of our main result.

Lemma 2.1. (see Osilike and Igbokwe [6]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 = {}& \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 \\ & -\alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \end{aligned}$$

Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbf{R}$ be a bifunction. We say that F satisfies (A1), (A2), (A3) and (A4) if there hold the following:

$(A1)$ $F(x,x)=0,\ \forall x\in C$;

$(A2)$ F is monotone, i.e., $F(x,y)+F(y,x)\leq 0,\ \forall x,y\in C$;

$(A3)$ $\lim_{t\to 0^+} F(tz+(1-t)x,y)\leq F(x,y),\ \forall x,y,z\in C$;

$(A4)$ for each $x\in C,\ y\mapsto F(x,y)$ is convex and lower semicontinuous.

We know the following Lemmas 2.2 and 2.3; see, e.g., [23,28].

Lemma 2.2. (cf. [17, Lemma 2.2]). *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C\times C$ into $\mathbf{R}$ satisfying $(A1),(A2),(A3)$ and $(A4)$. Then, for any $r>0$ and $x\in H$, there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\quad \forall y\in C.$$

Further, if $T_r^F x=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C\}$, then the following hold:

(1) T_r^F *is single-valued;*

(2) T_r^F *is firmly nonexpansive, i.e.,*

$$\|T_r^F x-T_r^F y\|^2\leq\langle T_r^F x-T_r^F y,x-y\rangle,\quad\forall x,y\in H;$$

(3) $F(T_r^F)=EP(F)$*;*

(4) $EP(F)$ *is closed and convex.*

Lemma 2.3. (cf. [17, Lemma 2.3]). *Let C,H,F and $T_r^F x$ be as in Lemma 2.2. Then the following holds:*

$$\|T_s^F x-T_t^F x\|^2\leq\frac{s-t}{s}\langle T_s^F x-T_t^F x,T_s^F x-x\rangle$$

for all $s,t>0$ and $x\in H$.

Proposition 2.1. *Let C be a nonempty closed convex subset of H, and let F and G be two bifunctions from $C\times C$ into $\mathbf{R}$ satisfying (A1), (A2), (A3) and (A4). For given $x^*,y^*\in C,\ (x^*,y^*)$ is a solution of problem (1.4) if and only if x^* is a fixed point of the mapping $\Gamma: C\to C$ defined by*

$$\Gamma(x)=T_\lambda^F[T_\mu^G(x-\mu Bx)-\lambda AT_\mu^G(x-\mu Bx)],\quad\forall x\in C,$$

where $y^*=T_\mu^G(x^*-\mu Bx^*)$.

Proof. Observe that

$$\begin{cases} F(x^*,x)+\langle Ay^*,x-x^*\rangle+\dfrac{1}{\lambda}\langle x^*-y^*,x-x^*\rangle\geq 0,\ \forall x\in C,\\ G(y^*,y)+\langle Bx^*,y-y^*\rangle+\dfrac{1}{\mu}\langle y^*-x^*,y-y^*\rangle\geq 0,\ \forall y\in C \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x^*=T_\lambda^F(y^*-\lambda Ay^*),\\ y^*=T_\mu^G(x^*-\mu Bx^*) \end{cases}$$

$$\Updownarrow$$

$$x^*=T_\lambda^F[T_\mu^G(x^*-\mu Bx^*)-\lambda AT_\mu^G(x^*-\mu Bx^*)].$$

Corollary 2.1. (see [18, Lemma 2.1]). *For given $x^*,y^*\in C$, (x^*,y^*) is a solution of problem (1.5) if and only if x^* is a fixed point of the mapping $\Phi:C\to C$ defined by*

$$\Phi(x)=P_C[P_C(x-\mu Bx)-\lambda AP_C(x-\mu Bx)],\quad \forall x\in C,$$

where $y^*=P_C(x^*-\mu Bx^*)$.

Proof. Putting $F=G=0$ and utilizing Lemma 2.2, we deduce that $T_\lambda^F=T_\mu^G=P_C$. Thus, from Proposition 2.1 we obtain the desired result.

Remark 2.1. In terms of the proof of Theorem 3.1, we know that if $F,G:C\times C\to\mathbf{R}$ are two bifunctions satisfying (A1), (A2), (A3) and (A4), and the mappings $A,B:C\to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, then $\Gamma:C\to C$ is a nonexpansive mapping provided $\lambda\in(0,2\alpha]$ and $\mu\in(0,2\beta]$.

Throughout this paper, the set of fixed points of the mapping Γ is denoted by $\mho$.

Proposition 2.2. (see Suzuki [7]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$. Suppose $x_{n+1}=(1-\beta_n)y_n+\beta_nx_n$ for all integers $n\geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0$. Then, $\lim_{n\to\infty}\|y_n-x_n\|=0$.*

Lemma 2.4. (cf. [11, Lemma 2.1]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1}\leq(1-\gamma_n)a_n+\delta_n,\quad \forall n\geq 1,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(*i*) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(*ii*) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 (Goebel and Kirk [2]) (Demi-closedness Principle). *Assume that* T *is a nonexpansive self-mapping of a nonempty closed convex subset* C *of a real Hilbert space* H. *If* T *has a fixed point, then* $I - T$ *is demi-closed; that is, whenever* $\{x_n\}$ *is a sequence in* C *converging weakly to some* $x \in C$ *(for short,* $x_n \rightharpoonup x \in C$*), and the sequence* $\{(I-T)x_n\}$ *converges strongly to some* y *(for short,* $(I-T)x_n \to y$*), it follows that* $(I-T)x = y$.

The following lemma is an immediate consequence of the inner product properties.

Lemma 2.6. *In a real Hilbert space* H*, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

3. Main Results

We are now in a position to prove the main result of this paper.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $F, G : C \times C \to \mathbf{R}$ *be two bifunctions satisfying (A1), (A2), (A3) and (A4). Let the mappings* $A, B : C \to H$ *be* α*-inverse-strongly monotone and* β*-inverse-strongly monotone, respectively. Let* $S : C \to C$ *be a nonexpansive mapping such that* $F(S) \cap \mho \neq \emptyset$. *Suppose* $x_1 = u \in C$ *and* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = T_\mu^G(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S T_\lambda^F(y_n - \lambda A y_n), \end{cases} \tag{3.1}$$

where $\lambda \in (0, 2\alpha],\ \mu \in (0, 2\beta]$*, and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0,1]$ *such that*

(*i*) $\alpha_n + \beta_n + \gamma_n = 1,\ \forall n \ge 1$;

(*ii*) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.
Then $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(S)\cap\mho} u$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem (1.4), where* $\bar{y} = T_\mu^G(\bar{x} - \mu B\bar{x})$.

Proof. First, on account of (iii), there exist an integer $n_0 \geq 1$ and some $[a,b] \subset (0,1)$ such that $\beta_n \in [a,b]$ for all $n \geq n_0$. Without loss of generality, we may assume that there exists some $[a,b] \subset (0,1)$ such that $\beta_n \in [a,b]$ for all $n \geq 1$.

Let $x^* \in F(S) \cap \mho$. Then $x^* = Sx^*$ and

$$x^* = T_\lambda^F[T_\mu^G(x^* - \mu Bx^*) - \lambda AT_\mu^G(x^* - \mu Bx^*)].$$

Put $y^* = T_\mu^G(x^* - \mu Bx^*)$ and $t_n = T_\lambda^F(y_n - \lambda Ay_n)$. Then $x^* = T_\lambda^F(y^* - \lambda Ay^*)$ and

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n St_n.$$

Observe that

$$\begin{aligned} &\|(I-\lambda A)y_n - (I-\lambda A)y^*\|^2 \\ &= \|y_n - y^*\|^2 - 2\lambda\langle y_n - y^*, Ay_n - Ay^*\rangle + \lambda^2\|Ay_n - Ay^*\|^2 \\ &\leq \|y_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2 \\ &\leq \|y_n - y^*\|^2, \end{aligned} \tag{3.2}$$

and similarly,

$$\begin{aligned} \|(I-\mu B)x_n - (I-\mu B)x^*\|^2 &\leq \|x_n - x^*\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{3.3}$$

Hence, utilizing Lemma 2.2 we have from (3.2) and (3.3)

$$\begin{aligned} \|t_n - x^*\| &= \|T_\lambda^F(y_n - \lambda Ay_n) - T_\lambda^F(y^* - \lambda Ay^*)\| \\ &\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| \\ &\leq \|y_n - y^*\| \\ &= \|T_\mu^G(x_n - \mu Bx_n) - T_\mu^G(x^* - \mu Bx^*)\| \\ &\leq \|(x_n - \mu Bx_n) - (x^* - \mu Bx^*)\| \\ &\leq \|x_n - x^*\| \end{aligned} \tag{3.4}$$

which implies that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + \gamma_n St_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|t_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\} \\ &= \|u - x^*\|. \end{aligned}$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{t_n\}, \{y_n\}, \{St_n\}, \{Ay_n\}$ and $\{Bx_n\}$ are also bounded. Also, utilizing Lemma 2.2 we have

$$\begin{aligned}\|t_{n+1} - t_n\| &= \|T_\lambda^F(y_{n+1} - \lambda Ay_{n+1}) - T_\lambda^F(y_n - \lambda Ay_n)\| \\ &\le \|(y_{n+1} - \lambda Ay_{n+1}) - (y_n - \lambda Ay_n)\| \\ &\le \|y_{n+1} - y_n\| \\ &= \|T_\mu^G(x_{n+1} - \mu Bx_{n+1}) - T_\mu^G(x_n - \mu Bx_n)\| \\ &\le \|(x_{n+1} - \mu Bx_{n+1}) - (x_n - \mu Bx_n)\| \\ &\le \|x_{n+1} - x_n\|.\end{aligned} \tag{3.5}$$

Let $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$. Then, we obtain

$$\begin{aligned}z_{n+1} - z_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}St_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n St_n}{1 - \beta_n} \\ &= \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right) u + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(St_{n+1} - St_n) \\ &\quad + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right) St_n.\end{aligned} \tag{3.6}$$

Combining (3.5) with (3.6) we have

$$\begin{aligned}&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ \le{}& |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}|\|u\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| \\ &+|\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}|\|St_n\| - \|x_{n+1} - x_n\| \\ \le{}& |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}|(\|u\| + \|St_n\|).\end{aligned}$$

This implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, utilizing Proposition 2.2 we get $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|z_n - x_n\| = 0. \tag{3.7}$$

From (3.5) and (3.7) it follows that $\|t_{n+1} - t_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

Since

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \gamma_n(St_n - x_n),$$

this together with (ii) and (3.7) implies that $\|x_n - St_n\| \to 0$ as $n \to \infty$. Since $x^* \in F(S) \cap \mho$, from Lemma 2.1 and (3.4) we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n u + \beta_n x_n + \gamma_n St_n - x^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|t_n - x^*\|^2 \\
&= \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|T_\lambda^F(y_n - \lambda Ay_n) - T_\lambda^F(y^* - \lambda Ay^*)\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n[\|y_n - y^*\|^2 \\
&\quad +\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2] \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n[\|x_n - x^*\|^2 \\
&\quad +\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2] \\
&\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n u + \beta_n x_n + \gamma_n St_n - x^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|t_n - x^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - y^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|(x_n - \mu Bx_n) - (x^* - \mu Bx^*)\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n[\|x_n - x^*\|^2 \\
&\quad +\mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2] \\
&\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n\mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&-\gamma_n\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&= \alpha_n\|u-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)(\|x_n-x^*\|-\|x_{n+1} - x^*\|) \\
&\leq \alpha_n\|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&-\gamma_n\mu(\mu-2\beta)\|Bx_n-Bx^*\|^2\\
&\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&=\alpha_n\|u-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)(\|x_n-x^*\|-\|x_{n+1}-x^*\|)\\
&\le \alpha_n\|u-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|,
\end{aligned}\tag{3.9}$$

Since $\alpha_n \to 0$ and $\|x_n-x_{n+1}\|\to 0$ as $n\to\infty$, from (3.8) and (3.9) we derive

$$\lim_{n\to\infty}\|Ay_n-Ay^*\|=0 \quad \text{and} \quad \lim_{n\to\infty}\|Bx_n-Bx^*\|=0.$$

Utilizing Lemma 2.2, we have

$$\begin{aligned}
\|y_n-y^*\|^2 &= \|T_\mu^G(x_n-\mu Bx_n)-T_\mu^G(x^*-\mu Bx^*)\|^2\\
&\le \langle (x_n-\mu Bx_n)-(x^*-\mu Bx^*), y_n-y^*\rangle\\
&= \frac{1}{2}[\|(x_n-\mu Bx_n)-(x^*-\mu Bx^*)\|^2+\|y_n-y^*\|^2\\
&\quad -\|(x_n-\mu Bx_n)-(x^*-\mu Bx^*)-(y_n-y^*)\|^2]\\
&\le \frac{1}{2}[\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&\quad -\|(x_n-y_n)-\mu(Bx_n-Bx^*)-(x^*-y^*)\|^2]\\
&= \frac{1}{2}[\|x_n-x^*\|^2+\|y_n-y^*\|^2-\|(x_n-y_n)-(x^*-y^*)\|^2\\
&\qquad +2\mu\langle (x_n-y_n)-(x^*-y^*), Bx_n-Bx^*\rangle-\mu^2\|Bx_n-Bx^*\|^2].
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|y_n-y^*\|^2 &\le \|x_n-x^*\|^2-\|(x_n-y_n)-(x^*-y^*)\|^2\\
&\qquad +2\mu\langle (x_n-y_n)-(x^*-y^*), Bx_n-Bx^*\rangle-\mu^2\|Bx_n-Bx^*\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n u+\beta_n x_n+\gamma_n St_n-x^*\|^2\\
&\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|t_n-x^*\|^2\\
&\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|y_n-y^*\|^2\\
&\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\gamma_n\|(x_n-y_n)-(x^*-y^*)\|^2\\
&\quad +2\gamma_n\mu\langle (x_n-y_n)-(x^*-y^*), Bx_n-Bx^*\rangle-\gamma_n\mu^2\|Bx_n-Bx^*\|^2\\
&\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\gamma_n\|(x_n-y_n)-(x^*-y^*)\|^2\\
&\quad +2\gamma_n\mu\|(x_n-y_n)-(x^*-y^*)\|\|Bx_n-Bx^*\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\gamma_n\|(x_n-y_n)-(x^*-y^*)\|^2\\
&\leq \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\quad +2\gamma_n\mu\|(x_n-y_n)-(x^*-y^*)\|\|Bx_n-Bx^*\| \qquad (3.10)\\
&\leq \alpha_n\|u-x^*\|^2+2\gamma_n\mu\|(x_n-y_n)-(x^*-y^*)\|\|Bx_n-Bx^*\|\\
&\quad +\|x_n-x_{n+1}\|(\|x_n-x^*\|-\|x_{n+1}-x^*\|).
\end{aligned}$$

Since $\alpha_n \to 0,\ \|x_n-x_{n+1}\|\to 0$ and $\|Bx_n-Bx^*\|\to 0$ as $n\to\infty$, from (3.10) we get $\|(x_n-y_n)-(x^*-y^*)\|\to 0$ as $n\to\infty$.

Now, utilizing Lemma 2.6 and the firm nonexpansivity of T_λ^F we have

$$\begin{aligned}
&\|(y_n-t_n)+(x^*-y^*)\|^2\\
&=\|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)-[T_\lambda^F(y_n-\lambda Ay_n)\\
&\quad -T_\lambda^F(y^*-\lambda Ay^*)]+\lambda(Ay_n-Ay^*)\|^2\\
&\leq \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)-[T_\lambda^F(y_n-\lambda Ay_n)-T_\lambda^F(y^*-\lambda Ay^*)]\|^2\\
&\quad +2\lambda\langle Ay_n-Ay^*,(y_n-t_n)+(x^*-y^*)\rangle\\
&\leq \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|^2-\|T_\lambda^F(y_n-\lambda Ay_n)-T_\lambda^F(y^*-\lambda Ay^*)\|^2\\
&\quad +2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|\\
&\leq \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|^2-\|ST_\lambda^F(y_n-\lambda Ay_n)-ST_\lambda^F(y^*-\lambda Ay^*)\|^2\\
&\quad +2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\| \qquad (3.11)\\
&=\|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|^2-\|St_n-Sx^*\|^2\\
&\quad +2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|\\
&\leq \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)-(St_n-x^*)\|\\
&\quad \times(\|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|+\|St_n-x^*\|)\\
&\quad +2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|\\
&=\|x_n-St_n+x^*-y^*-(x_n-y_n)-\lambda(Ay_n-Ay^*)\|\\
&\quad \times(\|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|+\|St_n-x^*\|)\\
&\quad +2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|.
\end{aligned}$$

Since $\|St_n-x_n\|\to 0,\ \|(x_n-y_n)-(x^*-y^*)\|\to 0$ and $\|Ay_n-Ay^*\|\to 0$, it follows from (3.11) that $\|(y_n-t_n)+(x^*-y^*)\|\to 0$ as $n\to\infty$. Also, observe that

$$\|St_n-t_n\|\leq \|St_n-x_n\|+\|(x_n-y_n)-(x^*-y^*)\|+\|(y_n-t_n)+(x^*-y^*)\|.$$

Thus, we get $\|St_n - t_n\| \to 0$ as $n \to \infty$.

Next, let us show that

$$\limsup_{n\to\infty} \langle u - \bar{x}, x_n - \bar{x}\rangle \leq 0,$$

where $\bar{x} = P_{F(S)\cap\mho} u$.

Indeed, since $\{t_n\}$ and $\{St_n\}$ are two bounded sequences in C, we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z \in C$ and

$$\limsup_{n\to\infty} \langle u - \bar{x}, St_n - \bar{x}\rangle = \lim_{i\to\infty} \langle u - \bar{x}, St_{n_i} - \bar{x}\rangle.$$

Since $\lim_{n\to\infty} \|St_n - t_n\| = 0$, we obtain , $St_{n_i} \rightharpoonup z$ as $i \to \infty$. Now we claim that $z \in F(S) \cap \mho$. First by Lemma 2.5 it is easy to see that $z \in F(S)$. Second, utilizing Lemma 2.2 we have for all $x, y \in C$

$$\begin{aligned}
\|\Gamma(x) - \Gamma(y)\|^2 &= \|T_\lambda^F[T_\mu^G(x - \mu Bx) - \lambda A T_\mu^G(x - \mu Bx)] \\
&\quad - T_\lambda^F[T_\mu^G(y - \mu By) - \lambda A T_\mu^G(y - \mu By)]\|^2 \\
&\leq \|T_\mu^G(x - \mu Bx) - \lambda A T_\mu^G(x - \mu Bx) \\
&\quad - [T_\mu^G(y - \mu By) - \lambda A T_\mu^G(y - \mu By)]\|^2 \\
&= \|T_\mu^G(x - \mu Bx) - T_\mu^G(y - \mu By) \\
&\quad - \lambda(A T_\mu^G(x - \mu Bx) - A T_\mu^G(y - \mu By))\|^2 \\
&\leq \|T_\mu^G(x - \mu Bx) - T_\mu^G(y - \mu By)\|^2 \\
&\quad + \lambda(\lambda - 2\alpha)\|A T_\mu^G(x - \mu Bx) - A T_\mu^G(y - \mu By)\|^2 \\
&\leq \|T_\mu^G(x - \mu Bx) - T_\mu^G(y - \mu By)\|^2 \\
&\leq \|x - \mu Bx - (y - \mu By)\|^2 \\
&\leq \|x - y\|^2 + \mu(\mu - 2\beta)\|Bx - By\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

This shows that $\Gamma : C \to C$ is nonexpansive. Since $\|St_n - t_n\| \to 0$, $\|St_n - x_n\| \to 0$ and

$$\|t_n - x_n\| \leq \|St_n - t_n\| + \|St_n - x_n\|,$$

we conclude that $\|t_n - x_n\| \to 0$ as $n \to \infty$. Furthermore, note that

$$\begin{aligned}
\|t_n - \Gamma(t_n)\| &= \|T_\lambda^F[T_\mu^G(x_n - \mu Bx_n) - \lambda A T_\mu^G(x_n - \mu Bx_n)] - \Gamma(t_n)\| \\
&= \|\Gamma(x_n) - \Gamma(t_n)\| \\
&\leq \|x_n - t_n\|.
\end{aligned}$$

Thus $\limsup_{n\to\infty} \|t_n - \Gamma(t_n)\| = 0$. According to Lemma 2.5 we obtain $z \in \mho$. Therefore there holds $z \in F(S) \cap \mho$.

On the other hand, it follows from (2.1) that

$$\begin{aligned} \limsup_{n\to\infty} \langle u - \bar{x}, x_n - \bar{x}\rangle &= \limsup_{n\to\infty} \langle u - \bar{x}, St_n - \bar{x}\rangle \\ &= \lim_{i\to\infty} \langle u - \bar{x}, St_{n_i} - \bar{x}\rangle \\ &= \langle u - \bar{x}, z - \bar{x}\rangle \\ &\le 0. \end{aligned} \tag{3.12}$$

Hence we have

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &= \langle \alpha_n u + \beta_n x_n + \gamma_n St_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &= \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n \langle x_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\quad + \gamma_n \langle St_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\le \frac{1}{2}\beta_n(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\quad + \frac{1}{2}\gamma_n(\|t_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) \\ &\le \frac{1}{2}(1 - \alpha_n)(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle, \end{aligned}$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \le (1 - \alpha_n)\|x_n - \bar{x}\|^2 + 2\alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle.$$

Consequently, according to (3.12) and Lemma 2.4, we deduce that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. ∎

Example 3.1. Let $H = \mathbf{R}$ and $C = [-\pi/2, \pi/2]$. Define the mappings $S : C \to C$, $A, B : C \to H$ and $F, G : C \times C \to \mathbf{R}$ as follows:

$$S(x) = \sin x,\ A(x) = x - (\sin x)/2,\ B(x) = x - (\sin x)/3,$$
$$F(x, y) = -x^2 + y^2 \quad \text{and} \quad G(x, y) = -|x| + |y|,$$

for all $x, y \in C$. Then it is clear that S is nonexpansive, A is $2/9$-inverse-strongly monotone and B is $3/8$-inverse-strongly monotone. In this case we have $F(S) \cap \mho = \{0\}$. In terms of Theorem 3.1, we choose the parameters λ, μ. Then the sequence $\{x_n\}$ generated from $x_1 = u \in C$ by the iterative scheme (3.1) converges to $0 = P_{F(S)\cap\mho} u$ and $(\bar{x}, \bar{y}) = (0, 0)$ is a solution of problem (1.4), where $\bar{y} = T^G_\mu(\bar{x} - \mu B\bar{x})$.

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying $(A1), (A2), (A3)$ and $(A4)$. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \mho \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by*

$$\begin{cases} y_n = T_\mu^F(x_n - \mu A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S T_\lambda^F(y_n - \lambda A y_n), \end{cases}$$

where $\lambda, \mu \in (0, 2\alpha]$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ such that

(*i*) $\alpha_n + \beta_n + \gamma_n = 1, \ \forall n \geq 1$;

(*ii*) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap\mho} u$ and $(\bar{x}, \bar{y})$ is a solution of problem $(1.4)'$, where $\bar{y} = T_\mu^F(\bar{x} - \mu A \bar{x})$.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F, G : C \times C \to \mathbf{R}$ be two bifunctions satisfying $(A1), (A2), (A3)$ and $(A4)$. Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, such that $\mho \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by*

$$\begin{cases} y_n = T_\mu^G(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T_\lambda^F(y_n - \lambda A y_n), \end{cases}$$

where $\lambda \in (0, 2\alpha], \ \mu \in (0, 2\beta]$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ such that

(*i*) $\alpha_n + \beta_n + \gamma_n = 1, \ \forall n \geq 1$;

(*ii*) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\bar{x} = P_\mho u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.4), where $\bar{y} = T_\mu^G(\bar{x} - \mu B \bar{x})$.

Recall that a mapping $T : C \to C$ is called strictly pseudocontractive if there exists some k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

For recent convergence result for strictly pseudocontractive mappings, we refer to Zeng, Wong and Yao [19]. Put $A = I - T$. Then we have

$$\|(I-A)x-(I-A)y\|^2 \le \|x-y\|^2 + k\|Ax-Ay\|^2.$$

On the other hand,

$$\|(I-A)x-(I-A)y\|^2 = \|x-y\|^2 + \|Ax-Ay\|^2 - 2\langle x-y, Ax-Ay\rangle.$$

Hence we have

$$\langle x-y, Ax-Ay\rangle \ge \frac{1-k}{2}\|Ax-Ay\|^2.$$

Consequently, if $T : C \to C$ is a strictly pseudocontractive mapping with constant k, then the mapping $A = I - T$ is $(1-k)/2$-inverse-strongly monotone.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F, G : C \times C \to \mathbf{R}$ be two bifunctions satisfying (A1), (A2), (A3) and (A4). Let the mappings $A, B : C \to H$ be $\frac{1-k}{2}$-inverse-strongly monotone and $\frac{1-l}{2}$-inverse-strongly monotone, respectively, where $A = I - T,\ B = I - V$, and $T, V : C \to C$ are strictly pseudocontractive with constant k and strictly pseudocontractive with constant l, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \mho \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by*

$$\text{(3.13)} \qquad \begin{cases} y_n = T_\mu^G((1-\mu)x_n + \mu V x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S T_\lambda^F((1-\lambda)y_n + \lambda T y_n), \end{cases}$$

where $\lambda \in (0, 1-k],\ \mu \in (0, 1-l]$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1,\ \forall n \ge 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap\mho} u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.4), where $\bar{y} = T_\mu^G((1-\mu)\bar{x} + \mu V \bar{x})$.

Proof. Since $A = I - T,\ B = I - V,\ \lambda \in (0, 1-k]$ and $\mu \in (0, 1-l]$, we have

$$T_\lambda^F(y_n - \lambda A y_n) = T_\lambda^F((1-\lambda)y_n + \lambda T y_n),$$
$$T_\mu^G(x_n - \mu B x_n) = T_\mu^G((1-\mu)x_n + \mu V x_n).$$

Thus the iterative schemes (3.1) and (3.13) are equivalent. Therefore, the conclusion follows immediately from Theorem 3.1. ∎

4. Applications

Using Theorem 3.1, we prove two results in a real Hilbert space.

Theorem 4.1. *Let H be a real Hilbert space and let $F : H \times H \to \mathbf{R}$ be a bifunction satisfying (A1), (A2), (A3) and (A4). Let $A : H \to H$ be an α-inverse-strongly monotone mapping and let $S : H \to H$ be a nonexpansive mapping such that $F(S) \cap \mho \neq \emptyset$. Suppose $x_1 = u \in H$ and $\{x_n\}$ is generated by*

$$(4.1)\qquad \begin{cases} y_n = T_\lambda^F(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S T_\lambda^F(y_n - \lambda A y_n), \end{cases}$$

where $\lambda \in (0, 2\alpha)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ such that

(*i*) $\alpha_n + \beta_n + \gamma_n = 1,\ \forall n \geq 1$;

(*ii*) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap\mho} u$ and $(\bar{x}, \bar{y})$ is a solution of problem $(1.4)'$ with $C = H$, where $\bar{y} = T_\lambda^F(\bar{x} - \lambda A \bar{x})$. In particular, if $F = 0$, then $\bar{x} = P_{F(S)\cap A^{-1}0} u$.

Proof. We have $\lambda = \mu,\ C = H,\ G = F,\ B = A$, and

$$\Gamma(x) = T_\lambda^F[T_\lambda^F(x - \lambda A x) - \lambda A T_\lambda^F(x - \lambda A x)], \quad \forall x \in H.$$

In this case, (3.1) reduces to (4.1). Hence, utilizing Theorem 3.1 we know that $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap\mho} u$ and $(\bar{x}, \bar{y})$ is a solution of problem $(1.4)'$ with $C = H$, where $\bar{y} = T_\lambda^F(\bar{x} - \lambda A \bar{x})$.

Furthermore, whenever $F = 0$, it is easy to see that $A^{-1}0 = \mho$ and $T_\lambda^F = P_H = I$. In this case, there holds the following:

$$\text{problem } (1.4)' \ \Leftrightarrow \ \text{problem } (1.6) \ \Leftrightarrow \ \mathrm{VI}(A, H).$$

Indeed, it is sufficient to show that problem (1.6) $\Rightarrow \mathrm{VI}(A, H)$. Suppose that there is $(x^*, y^*) \in H \times H$ such that

$$\begin{cases} \langle A y^*, x - x^* \rangle + \dfrac{1}{\lambda}\langle x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in H, \\ \langle A x^*, y - y^* \rangle + \dfrac{1}{\lambda}\langle y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in H. \end{cases}$$

Then we have

$$\begin{cases} x^* = P_H(y^* - \lambda A y^*), \\ y^* = P_H(x^* - \lambda A x^*); \end{cases}$$

that is,

$$(4.2) \qquad \begin{cases} x^* = y^* - \lambda A y^*, \\ y^* = x^* - \lambda A x^*. \end{cases}$$

We claim that $x^* = y^*$. Otherwise, from (4.2) it follows that $Ax^* \neq 0,\ Ay^* \neq 0$ and $Ax^* + Ay^* = 0$. Again from (4.2) we obtain

$$\begin{aligned} \|x^* - y^*\|^2 &= \|y^* - x^* - \lambda(Ay^* - Ax^*)\|^2 \\ &\leq \|y^* - x^*\|^2 + \lambda(\lambda - 2\alpha)\|Ay^* - Ax^*\|^2 \\ &< \|y^* - x^*\|^2, \end{aligned}$$

which hence leads to a contradiction. This shows that $x^* = y^*$. Thus, problem (1.6) $\Rightarrow \mathrm{VI}(A, H)$. By Theorem 3.1, we obtain the desired result. ∎

Theorem 4.2. *Let H be a real Hilbert space and let $F : H \times H \to \mathbf{R}$ be a bifunction satisfying (A1), (A2), (A3) and (A4). Let $A : H \to H$ be an α-inverse-strongly monotone mapping and let $B : H \to 2^H$ be a maximal monotone mapping such that $B^{-1}0 \cap \mho \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$. Suppose $x_1 = u \in H$ and $\{x_n\}$ is generated by*

$$(4.3) \qquad \begin{cases} y_n = T_\lambda^F(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_r^B T_\lambda^F(y_n - \lambda A y_n), \end{cases}$$

where $\lambda \in (0, 2\alpha)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ such that

(*i*) $\alpha_n + \beta_n + \gamma_n = 1,\ \forall n \geq 1$;

(*ii*) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^\infty \alpha_n = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{B^{-1}0\cap\mho}u$ and $(\bar{x}, \bar{y})$ is a solution of problem $(1.4)'$ with $C = H$, where $\bar{y} = T_\lambda^F(\bar{x} - \lambda A\bar{x})$. In particular, if $F = 0$, then $\bar{x} = P_{A^{-1}0\cap B^{-1}0}u$.

Proof. We have $F(J_r^B) = B^{-1}0$. Putting $S = J_r^B$, by Theorem 4.1 we know that $\{x_n\}$ converges strongly to $\bar{x} = P_{B^{-1}0\cap\mho}u$ and $(\bar{x}, \bar{y})$ is a solution of problem $(1.4)'$ with $C = H$, where $\bar{y} = T_\lambda^F(\bar{x} - \lambda A\bar{x})$. In particular, if $F = 0$, we obtain that $T_\lambda^F = P_H = I$. Therefore, $A^{-1}0 = \mho$ and hence $\bar{x} = P_{A^{-1}0\cap B^{-1}0}u$. ∎

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