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NORMALIZED SYSTEM FOR WAVE AND DUNKL OPERATORS

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Abstract. Normalized systems are constructed with respect to wave and Dunkl operators. Non-trivial solutions can thus be built to the equation $Dv(x) + \lambda v(x) = 0$, where D is either the wave operator or the Dunkl operators and $\lambda \in \mathbb{C}$.

1. Introduction

The notion of normalized system with respect to operators was introduced by Karachik ([2]).

Let L_1 and L_2 be *commuting* linear partial differential operators on function space X such that $L_k X \subset X \quad (k = 1, 2)$. A sequence of functions $\{f_k(x)\}_{k=-1}^{\infty}$ in X is called a f-*normalized* system with respect to L_1 if $f = f_{-1}$ and

$$L_1 f_k = f_{k-1}, \quad k \in \mathbb{N} \cup \{0\}.$$

With normalized system, the differential equations

$$L_1 v - L_2 v = f \tag{1.1}$$

has a formal solution(in [3])

$$v = \sum_{k=0}^{\infty} L_2^k f_k. \tag{1.2}$$

The classical example is that the wave equation in $\mathbb{R}^2$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) u(s, t) = 0$$

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has solutions
$$u(s,t) = \cos(t\frac{\partial}{\partial s})g_1(s) + \cos(t\frac{\partial}{\partial s})g_2(s),$$
since
$$f_k(t,s) = \frac{t^{2k}}{(2k)!}g_1(s) + \frac{t^{2k+1}}{(2k+1)!}\frac{dg_2}{ds}(s), \qquad k \geq 0,$$
presents the 0-normalized system with respect to $L_1 := \frac{\partial^2}{\partial t^2}$.

Generally, when L_1 is the Laplace operator Δ in $\mathbb{R}^n$, Karachik [3] constructed the 0-*normalized* system as

$$f_0(x) = u(x), \qquad f_k(x) = \frac{|x|^{2k}}{4^k k!(k-1)!}\int_0^1 (1-t)^{k-1}t^{n/2-1}u(tx)dt.$$

The main purpose of this article is to construct 0-*normalized* system with respect to the wave operators and Dunkl operators. As applications, we study Riquier problem and the Helmholtz equations with respect to the wave operators and Dunkl operators.

2. Radial Derivative

In $\mathbb{R}^n$, we let

$$[x,x] = {x_1}^2 + ... + {x_p}^2 - {x_{p+1}}^2 - ... - {x_{p+q}}^2.$$

where $n = p+q$. We consider two kinds of generalized Laplacian. One is the wave operator

$$\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial {x_i}^2} - \sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial {x_i}^2}. \tag{2.1}$$

The other is the Dunkl Laplacian. Let G be a Coxeter group associated with a reduced root system R, κ_υ a multiplicity function on R and σ_υ the reflection with respect to the root υ. We denote $\upsilon := \sum_{\upsilon\in R_+} \kappa_\upsilon$ and always assume that Re $\upsilon \geq 0$. Let $\mathcal{D}_j$ be the Dunkl operator attached to the Coxeter group G,

$$\mathcal{D}_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\upsilon\in R_+} \kappa_\upsilon \frac{f(x) - f(\sigma_\upsilon x)}{\langle x, \upsilon\rangle}\upsilon_j. \tag{2.2}$$

Then the Dunkl Laplacian is defined as
$$\Delta_h = \sum_{j=1}^{n} {\mathcal{D}_j}^2$$

For any $k > 0$, we consider the radial derivative and fractional integral operators

$$
\begin{aligned}
R_k f(x) &= kf(x) + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x), \\
J_k f(x) &= \int_0^1 (1-t)^{k-1} t^{n/2-1} f(tx)dt.
\end{aligned} \tag{2.3}
$$

Lemma 2.1. *Let Ω be a starlike domain in $\mathbb{R}^n$ and $f(x) \in C^1(\Omega)$. For any $k > 1$,*

$$
R_{n/2+k-1} J_k f(x) = (k-1) J_{k-1} f(x) \tag{2.4}
$$

Proof. If $f \in C^1(\Omega)$, then

$$
\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f(tx) = t \frac{\partial}{\partial t} f(tx)
$$

for any $t \in [0, 1]$ and $x \in \Omega$. By direct calculation, we have

$$
\begin{aligned}
R_{\frac{n}{2}+k-1} J_k f(x) &= (\frac{n}{2} + k - 1) J_k f(x) + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \int_0^1 (1-t)^{k-1} t^{n/2-1} f(tx)dt \\
&= (\frac{n}{2} + k - 1) J_k f(x) + \int_0^1 (1-t)^{k-1} t^{n/2-1} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f(tx)dt \\
&= (\frac{n}{2} + k - 1) J_k f(x) + \int_0^1 (1-t)^{k-1} t^{n/2-1} t \frac{\partial}{\partial t} f(tx)dt.
\end{aligned}
$$

By integration by part, the last integral equals

$$
\begin{aligned}
&= -\int_0^1 f(tx) \left(-(1-t)^{k-2}(k-1)t^{n/2} + \frac{n}{2} t^{n/2-1}(1-t)^{k-1} \right) dt \\
&= -(\frac{n}{2} + k - 1) J_k f(x) + \int_0^1 f(tx)(k-1)(1-t)^{k-2} t^{n/2-1} dt \\
&= -(\frac{n}{2} + k - 1) J_k f(x) + (k-1) J_{k-1} f.
\end{aligned}
$$

Combining the above identities, we obtain the desired result. ■

Lemma 2.2. *If $g(x)$ is twice continuously differentiable in a region in $\mathbb{R}^n$ and $\Box g(x) = 0$, then*

$$
\Box([x, x]^k g(x)) = 4k[x, x]^{k-1} R_{\frac{n+2k-2}{2}} g(x) \tag{2.5}
$$

Proof. We first prove that for any $f, g \in C^2(\Omega)$

$$\Box(fg) = (\Box f)g + 2\langle \tilde{\nabla} f, \nabla g\rangle + f(\Box g). \tag{2.6}$$

Here we denote by Δ_k and ∇_k the usual Laplacian and the gradient with respect to the variables $x_1, \cdots, x_p$ when $k = 1$ or $x_{p+1}, \cdots, x_{p+q}$ when $k = 2$, and we denote

$$\Box = \Delta_1 - \Delta_2, \qquad \nabla = (\nabla_1, \nabla_2), \qquad \tilde{\nabla} = (\nabla_1, -\nabla_2).$$

Indeed,

$$\begin{aligned}
\Box(fg) &= \Delta_1(fg) - \Delta_2(fg) \\
&= (\Delta_1 f)g + 2\nabla_1 f \nabla_1 g + f(\Delta_1 g) - (\Delta_2 f)g - 2\nabla_2 f \nabla_2 g - f(\Delta_2 g) \\
&= (\Box f)g + 2\nabla_1 f \nabla_1 g - 2\nabla_2 f \nabla_2 g + f(\Box g) \\
&= (\Box f)g + 2\tilde{\nabla} f \nabla g + f(\Box g).
\end{aligned}$$

Let $1 \le i \le p$ and $1 \le j \le q$. Recall

$$[x, x]^k = (x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2)^k.$$

Then

$$\begin{aligned}
\frac{\partial}{\partial x_i}[x, x]^k &= 2k[x, x]^{k-1} x_i \\
\frac{\partial}{\partial x_{p+j}}[x, x]^k &= -2k[x, x]^{k-1} x_{p+j},
\end{aligned}$$

so that

$$\tilde{\nabla}[x, x]^k = 2k[x, x]^{k-1} x.$$

We now calculate the second derivatives

$$\begin{aligned}
\frac{\partial^2}{\partial x_i^2}[x, x]^k &= 2k x_i \frac{\partial}{\partial x_i}[x, x]^{k-1} + 2k[x, x]^{k-1} \\
&= 4k(k-1) x_i^2 [x, x]^{k-2} + 2k[x, x]^{k-1}
\end{aligned}$$

as well as

$$\begin{aligned}
\frac{\partial^2}{\partial x_{p+j}^2}[x, x]^k &= -2k x_{p+j} \frac{\partial}{\partial x_{p+j}}[x, x]^{k-1} - 2k[x, x]^{k-1} \\
&= 4k(k-1) x_{p+j}^2 [x, x]^{k-2} - 2k[x, x]^{k-1}.
\end{aligned}$$

Therefore

$$\Delta_1[x,x]^k = 4k(k-1)(x_1^2+\ldots+x_p^2)[x,x]^{k-2}+2kp[x,x]^{k-1}$$

$$\Delta_2[x,x]^k = 4k(k-1)(x_{p+1}^2+\ldots+x_{p+q}^2)[x,x]^{k-2}-2kq[x,x]^{k-1}.$$

Subtract the two identities to yield

$$\begin{aligned}\Box[x,x]^k &= \Delta_1[x,x]^k-\Delta_2[x,x]^k\\ &= 4k(k-1)[x,x]^{k-1}+2k(p+q)[x,x]^{k-1}\\ &= 4k((k-1)+(p+q)/2)[x,x]^{k-1}.\end{aligned}$$

Finally, by taking $f(x)=[x,x]^k$ in (2.6), we obtain

$$\begin{aligned}&\Box([x,x]^k g(x))\\ &= 4k((k-1)+(p+q)/2)[x,x]^{k-1}g(x)+2\cdot 2k[x,x]^{k-1}x\nabla g(x)+[x,x]^k g(x)\\ &= 4k[x,x]^{k-1}(((k-1)+(p+q)/2)g(x)+x\nabla g(x))+[x,x]^k g(x)\\ &= 4k[x,x]^{k-1}R_{\frac{2(k-1)+N}{2}}g(x)+[x,x]^k g(x).\end{aligned}$$ ■

Lemma 2.3. [4]. *If $u(x)$ is twice continuously differentiable in a region in $\mathbb{R}^n$ and $\Delta_h u(x)=0$ in this region, then*

$$\Delta_h(|x|^\lambda g(x)) = 2\lambda|x|^{\lambda-2}R_{\frac{n+\lambda-2}{2}+v}g(x) \tag{2.7}$$

where λ is an integer larger than 1.

3. Wave Operator

In this section we give the 0-normalized systems with respect to $\Box$ in a starlike domain of $\mathbb{R}^n$.

Let Ω be a starlike domain in $\mathbb{R}^n$. Assume $u(x)\in C^1(\Omega)$ and

$$\Box u(x)=0, \qquad x\in\Omega,$$

Recall the definition of operator J_k in (2.3). Put

$$\begin{aligned}G_{-1}(x;u) &= 0,\\ G_0(x;u) &= u(x),\\ G_k(x;u) &= \frac{1}{4^k k!(k-1)!}[x,x]^{2k}J_k u.\end{aligned} \tag{3.1}$$

Theorem 3.1. *$G_k(x;u), k \geq -1$, is the 0-normalized system with respect to the operator $\Box$.*

Proof. Since $\Box u(x) = 0$, by (2.4) and (2.5) we have

$$\begin{aligned}\Box G_k(x;u) &= C_k \Box([x,x]^k J_k u)\\ &= 4kC_k[x,x]^{k-1} R_{\frac{n+2k-2}{2}} J_k u\\ &= 4kC_k[x,x]^{k-1}(k-1)J_{k-1}u\\ &= G_{k-1}(x;u),\end{aligned}$$

which shows that $G_k(x;u)$ is a 0-normalized system.

As an application of Theorem 3.1, we can now obtain non-trivial solutions to the equation

$$\Box v(x) + \lambda v(x) = 0, \quad \forall\ x \in \Omega. \tag{3.2}$$

Theorem 3.2. *Let Ω be a starlike domain in $\mathbb{R}^n$. $\lambda \in \mathbb{C}$. If $u(x) \in C^2(\Omega)$ such that $\Box u(x) = 0$ in Ω, then equation (3.2) has solution*

$$v(x) = u(x) + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{4^k k!(k-1)!}[x,x]^{2k} J_k u.$$

Proof. Take $G_k(x,u)$ to be the 0-normalized system with respect to D as in (3.1). Then setting $L_1 = \Box$, $L_2 = -\lambda$ and $f(x) = 0$ in equation (1.1), we obtain solutions to equation (3.2), with

$$v(x) = \sum_{k=0}^{\infty}(-\lambda)^k G_k(x,u) = u(x) + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{4^k k!(k-1)!}[x,x]^{2k} J_k u$$

for any function $u(x)$ in Ω such that $\Box u(x) = 0$. ■

Next, we apply the normalized system to Riquier 's problem

$$\begin{cases} \Box^m u(x) = 0, \\ \Box^k u\,|_{\partial\Omega} = f_k(s), \quad s \in \partial\Omega, \quad k = 0, 1, \ldots, m-1. \end{cases} \tag{3.3}$$

Theorem 3.3. *Let Ω be a starlike domain in $\mathbb{R}^n$. If for any $f(s) \in C(\partial\Omega)$ the Dirichlet problem*

$$\begin{cases} \Box u(x) = 0, \\ u\,|_{\partial\Omega} = f(s), \end{cases} \tag{3.4}$$

has a solution, then the Riquier 's problem (3.3) with $f_k(x)$ being continuous on $\partial\Omega$ has a solution.

Proof. Let $G_k(x,u)$ be the normalized system with respect to $\Box$ as in (3.1). Take $u_{(k)}$ to be the solution of Dirichlet problem

$$\begin{cases} \Box\, u_{(k)}(x) = 0, \qquad x \in \Omega. \\ u\,|_{\partial\Omega} = f_k(s) - \displaystyle\sum_{i=1}^{m-k-1} G_i(s, u_{(i+k)}). \end{cases}$$

We claim that the function

$$u(x) = \sum_{k=0}^{m-1} G_k(x; u_{(k)})$$

satisfies the Riquier 's problem (3.3).

By definition, it is clear that $u(x) \in C^{2m}(\Omega)$ and $\Box^m G_k(x,v) = 0$ for v such that $\Box v = 0$ and $0 \le k \le m-1$. Therefore if we take $0 \le v \le m-1$,then by the property of $G_k(x; u_{(k)})$, we get

$$\Box^{\,v} u(x) = \sum_{k=v}^{m-1} G_{k-v}(x; u_{(k)}) = u_{(v)} + \sum_{i=1}^{m-v-1} G_i(x; u_{(i+v)})$$

Letting $x \to \partial\Omega$ and take $k = v$ we obtain $\Box^v u(x)|_{\partial\Omega} = f_v(x)$.

4. Dunkl Laplacian

In this section we give the 0-normalized systems with respect to Δ_h in a starlike domain of $\mathbb{R}^n$.

Let Ω be a starlike domain in $\mathbb{R}^n$. Assume $u(x) \in C^2(\Omega)$ and

$$\Delta_h u(x) = 0, \qquad x \in \Omega,$$

Recall the definition of operator J_k in (2.3). Put

$$\begin{aligned} G_{-1}(x;u) &= 0 \\ G_0(x;u) &= J_\upsilon u(x), \\ G_k(x;u) &= \frac{1}{4^k k!(\upsilon+1)_{k-1}}|x|^{2k}J_{k+\upsilon}u(x). \end{aligned} \tag{4.1}$$

Here $(\upsilon+1)_{k-1} := (\upsilon+1)(\upsilon+2)\cdots(\upsilon+k-1)$ for $k>1$ and $(\upsilon+1)_0 := \upsilon$.

Theorem 4.1. *$G_k(x;u)$, $k\geq -1$, is the 0-normalized system with respect to Δ_h.*

Proof. Since $\Delta_h u(x)=0$, by (2.4) and (2.7) we have

$$\begin{aligned} \Delta_h G_k(x;u) &= C_k\Delta_h(|x|^{2k}J_{k+\upsilon}u(x) \\ &= 4kC_k|x|^{2k-2}R_{\frac{n+2k-2}{2}+\upsilon}J_{k+\upsilon}u(x) \\ &= 4kC_k|x|^{2k-2}(k-1+\upsilon)J_{k-1+\upsilon}u(x) \\ &= C_{k-1}|x|^{2k-2}J_{k-1+\upsilon}u(x) \\ &= G_{k-1}(x;u) \end{aligned}$$

which completes the proof. ■

As an application of Theorem 4.1, we can now obtain non-trivial solutions to the equation

$$\Delta_h u(x)+\lambda u(x)=0, \quad \forall\ x\in\Omega. \tag{4.2}$$

Theorem 4.2. *Let Ω be a starlike domain in $\mathbb{R}^n$, $\lambda\in\mathbb{C}$. If $u(x)\in C^2(\Omega)$ such that $\Delta_h u(x)=0$ in Ω, then equation (4.2) has solution*

$$v(x)=J_\upsilon u(x)+\sum_{k=1}^{\infty}\frac{(-\lambda)^k}{4^k k!(\upsilon+1)_{k-1}}|x|^{2k}J_{k+\upsilon}u(x).$$

Proof. Take $G_k(x,u)$ to be the 0-normalized system with respect to D as in (4.1). Then setting $L_1=\Delta_h$, $L_2=-\lambda$ and $f(x)=0$ in equation (1.1), we obtain solutions to equation (4.2), with

$$v(x)=\sum_{k=0}^{\infty}(-\lambda)^k G_k(x,u)=J_\upsilon u(x)+\sum_{k=1}^{\infty}\frac{(-\lambda)^k}{4^k k!(\upsilon+1)_{k-1}}|x|^{2k}J_{k+\upsilon}u(x). \quad ■$$

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