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PERIODIC SOLUTIONS OF DISCRETE RAYLEIGH EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. Perodic solutions of discrete Rayleigh equations with variable delays can be related to steady state solutions of discrete time neural networks. By means of Mawhin's continuation theorem, and sharp a priori estimates, existence of periodic solutions is established.

1. INTRODUCTION

To motivate what follows, let us consider ω neuron units placed on the vertices of a regular ω polygon (naturally $\omega \geq 2$). Let $x_n^{(t)}$ denote the state value of the n-th neuron during the time period $t \in \{0, 1, 2, ...\}$. Assume that each neuron unit is activated by its two neighbors so that the change of state values between two consecutive time periods is given by the superposition of portions of the gradients $x_{n-1}^{(t)} - x_n^{(t)}$ and $x_{n+1}^{(t)} - x_n^{(t)}$. Then

$$x_n^{(t+1)} - x_n^{(t)} = \alpha \left(x_{n-1}^{(t)} - x_n^{(t)} \right) + \alpha \left(x_{n+1}^{(t)} - x_n^{(t)} \right) = \alpha \Delta^2 x_{n-1}^{(t)},$$

where α is a proportionality constant not equal to 0, and Δ is the forward difference. If we assume further that bias mechanism is triggered, then a term $Q\left(n, x_{n-\alpha_n}^{(t)}, x_{n-\beta_n}^{(t)}, x_{n-\gamma_n}^{(t)}\right)$ may be added on the right hand side, where $\alpha_n, \beta_n, \gamma_n$ are integers for each $n = 1, 2, ..., \omega$. Similarly, there may be a control mechanism P_n for each neuron. Thus we may end up with the dynamical system

$$x_n^{(t+1)} - x_n^{(t)} = \alpha \Delta^2 x_{n-1}^{(t)} + Q\left(n, x_{n-\alpha_n}^{(t)}, x_{n-\beta_n}^{(t)}, x_{n-\gamma_n}^{(t)}\right) + P_n,\ n = 1, 2, ..., \omega.$$

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It is possible that the subscripts $n-1$, $n+1$, $n-\alpha_n, n-\beta_n$ and $n-\gamma_n$ fall outside the range $\{1, 2, ..., \omega\}$. If this is the case, they will be taken to mean the numbers that are respectively congruent to them modulo ω. In this setting, $\{\alpha_n\}_{n\in Z}$, $\{\beta_n\}_{n\in Z}$ and $\{\gamma_n\}_{n\in Z}$ are ω-periodic sequences, and Q is ω-periodic in the first variable.

In order to understand the dynamics of the "neural network" model described above, it is of interest to seek "steady state" solutions $\left\{\left(x_1^{(t)}, ..., x_\omega^{(t)}\right)^\dagger\right\}_{t=0}^\infty$ such that $x_n^{(t)} = x_n$ for $n \in \{1, ..., \omega\}$ and $t \geq 0$. This then leads us to finding solutions of the steady state system

$$\Delta^2 x_{n-1} + \frac{1}{\alpha} Q\left(n, x_{n-\alpha_n}, x_{n-\beta_n}, x_{n-\gamma_n}\right) + \frac{1}{\alpha} P_n = 0,\ n = 1, 2, ..., \omega,$$

or equivalently, finding ω-periodic solutions $\{x_n\}_{n=-\infty}^{\infty}$ of

$$\begin{aligned} &\Delta^2 x_{n-1} + \frac{1}{\alpha} Q\left(n, x_{n-\alpha_n}, x_{n-\beta_n}, x_{n-\gamma_n}\right) + \frac{1}{\alpha} P_n = 0, \\ &n \in Z = \{..., -2, -1, 0, 1, 2, 3, ...\}. \end{aligned}$$

In this paper, we consider the existence of ω-periodic solutions of one particular class of equations of the form

$$\Delta^2 x_{n-1} + F(n, \Delta x_{n-\sigma_n}) + G(n, x_{n-\tau_n}) = p_n,\ n \in Z, \tag{1}$$

where $F(n, x)$ and $G(n, x)$ are real continuous functions defined on $Z \times R$ such that $F(n, x) = F(n+\omega, x)$ and $G(n, x) = G(n+\omega, x)$ for all x, $F(n, 0) = 0$ for $n \in Z$, $\{\sigma_n\}_{n\in Z}$, $\{\tau_n\}_{n\in Z}$ are integer ω-periodic sequences and $\{p_n\}_{n\in Z}$ is a real ω-periodic sequence. The assumption

$$\sum_{n=1}^{\omega} p_n = 0$$

is also imposed. We remark that this is imposed for the sake of convenience. Indeed, if $\sum_{n=1}^{\omega} p_n \neq 0$, then by setting $C = \frac{1}{\omega}\sum_{n=1}^{\omega} p_n$, $\overline{p}_n = p_n - C$ and $\overline{G}(n, x_{n-\tau_n}) = G(n, x_{n-\tau_n}) - C$, we see that (1) is equivalent to the following equation

$$\Delta^2 x_{n-1} + F(n, \Delta x_{n-\sigma_n}) + \overline{G}(n, x_{n-\tau_n}) = \overline{p}_n,$$

where $\sum_{n=1}^{\omega} \overline{p}_n = 0$.

One reason why we consider this particular class is that (1) can be regarded as a discrete analogue of the Rayleigh differential equation with deviating arguments

$$x''(t) + F(t, x'(t - \sigma(t))) + G(t, x(t - \tau(t))) = p(t), \tag{2}$$

where $F(t,x)$ and $G(t,x)$ are real continuous functions defined on R^2 with positive period T in the first variable, $F(t,0)=0$ for $t \in Z$, $\sigma(t), \tau(t)$ and $p(t)$ are real continuous functions defined on R with period T.

Although many excellent results have been worked out for the existence of periodic solutions for the special case

$$\Delta^2 x_{n-1} + f(n, x_n) = 0,\; n \in Z,$$

(see for example [1-5]) using variational principles and critical theory, there does not seem to be any result for (1) which has variable "delays". One possible reason is that a variational principle for (1) is difficult to build. Fortunately, the existence of periodic solutions of (2) has been obtained by other means (see for examples [6-8]), and therefore we may try these alternate means for our discrete time equation.

In this note, existence criteria for periodic solutions of (1) will be established by means of continuation theorems. In general, a priori bounds are needed for establishing existence. Although such bounds are not difficult to establish, good ones are not. For this purpose, we will need to build novel inequalities for periodic sequences which are also sharp. Such inequalities can then be used to find sharp a priori bounds for periodic solutions of (1). Then by using Mawhin's continuation theorem we will be able to find periodic solutions of (1).

Let X and Y be two Banach spaces and $L : \mathrm{Dom}L \subset X \to Y$ is a linear mapping and $N : X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero $\dim \mathrm{Ker}L = \mathrm{codimIm}\, L < +\infty$, and $\mathrm{Im}\, L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}\, P = \mathrm{Ker}L$ and $\mathrm{Im}\, L = \mathrm{Ker}Q = \mathrm{Im}\,(I-Q)$. It follows that $L_{|\mathrm{Dom}L \cap \mathrm{Ker}P} : (I-P)\, X \to \mathrm{Im}\, L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN\left(\bar{\Omega}\right)$ is bounded and $K_P\,(I-Q)\, N : \bar{\Omega} \to X$ is compact. Since $\mathrm{Im}\, Q$ is isomorphic to $\mathrm{Ker}L$ there exist an isomorphism $J : \mathrm{Im}\, Q \to \mathrm{Ker}L$.

Theorem A. (Mawhin's continuation theorem [9]). Let L be a Fredholm mapping of index zero, and let N be L-compact on $\bar{\Omega}$. Suppose

(i) for each $\lambda \in (0,1)$, $x \in \partial\Omega, Lx \neq \lambda Nx$; and
(ii) for each $x \in \partial\Omega \cap \mathrm{Ker}L, QNx \neq 0$ and $\deg\,(JQN, \Omega \cap \mathrm{Ker}\ L, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \mathrm{dom}L$.

2. Existence Criteria

Our main result is the following.

Theorem 1. *Suppose there exist constants* $K > 0, D > 0, r_1 > 0, r_2 > 0$ *and* $r_3 > 0$ *such that*

(a) $|F(n,x)| \le r_1|x| + K$ *for*$(n,x) \in Z \times R$,

(b) $xG(n,x) > 0$ *and* $|G(n,x)| \geqslant r_2|x|$ *for* $n \in Z$ *and* $|x| > D$, *and*

(c) $\lim_{x\to-\infty} \max_{0\le n\le\omega-1} \frac{G(n,x)}{x} \le r_3$, *(or* $\lim_{x\to\infty} \max_{0\le n\le\omega-1} \frac{G(n,x)}{x} \le r_3$*).*

If

$$\omega \cdot \left(r_1 + r_3 \left(\frac{r_1}{r_2} + \frac{\omega}{2} \right) \right) < 1,$$

then (1) *has an* ω*-periodic solution.*

To prove our results, we proceed in steps. First of all, for any real sequence $\{u_n\}_{n\in Z}$,we recall the non-standard "summation" operation introduced in [10],

$$\bigoplus_{n=\alpha}^{\beta} u_n = \begin{cases} \sum_{n=\alpha}^{\beta} u_n & \alpha \le \beta \\ 0 & \beta = \alpha - 1 \\ -\sum_{n=\beta+1}^{\alpha-1} u_n & \beta \le \alpha - 1 \end{cases} . \tag{3}$$

It is then easy to see that $\{x_n\}_{n\in Z}$ is an ω-periodic solution of the following equation

$$\Delta x_n = \Delta x_0 - \bigoplus_{i=1}^{n} \left(F(i, \Delta x_{i-\sigma_i}) + G(i, x_{i-\tau_i}) - p_i \right), \tag{4}$$

if, and only if, $\{x_n\}_{n\in Z}$ is an ω-periodic solution of (1).

Let l_ω, where $\omega \geqslant 2$ is positive integer, be the set of all real ω-periodic sequences of the form $u = \{u_n\}_{n\in Z}$. Let X_ω be the Banach space of all real ω-periodic sequences in l_ω endowed with the usual linear structure as well as the norm $\|x\|_1 = \max_{0\le i\le\omega-1} |x_i|$. Let Y_ω be the Banach space of all real ω-periodic sequences of the form $y = \{y_n\}_{n\in Z} = \{n\alpha + h_n\}_{n\in Z}$, where $y_0 = 0$, $\alpha \in R$ and $\{h_n\}_{n\in Z} \in X_\omega$, and endowed with the usual linear structure as well as the norm $\|y\|_2 = |\alpha| + \|h\|_1$. Let the zero elements of X_ω and Y_ω be denoted by θ_1 and θ_2 respectively.

Define the mappings $L : X_\omega \to Y_\omega$ and $N : X_\omega \to Y_\omega$ respectively by

$$(Lx)_n = \Delta x_n - \Delta x_0,\ n \in Z, \tag{5}$$

and

$$(Nx)_n = -\bigoplus_{i=1}^{n} \left(F(i, \Delta x_{i-\sigma_i}) + G(i, x_{i-\tau_i}) - p_i \right),\ n \in Z. \tag{6}$$

Let

$$\overline{h}_n = -\bigoplus_{i=1}^{n} \left(F(i, \Delta x_{i-\sigma_i}) + G(i, x_{i-\tau_i}) - p_i\right) \tag{7}$$
$$+\frac{n}{\omega}\bigoplus_{i=1}^{\omega} \left(F(i, \Delta x_{i-\sigma_i}) + G(i, x_{i-\tau_i}) - p_i\right), \ n \in Z.$$

Since $\left\{\overline{h}_n\right\}_{n\in Z} \in X_\omega$ and $\overline{h}_0 = 0$, N is a well-defined operator from X_ω into Y_ω.

Let us define $P : X_\omega \to X_\omega$ and $Q : Y_\omega \to Y_\omega$ respectively by

$$(Px)_n = x_0, n \in Z, \text{ for } x = \{x_n\}_{n\in Z} \in X_\omega, \tag{8}$$

and

$$(Qy)_n = n\alpha, n \in Z, \text{ for } y = \{n\alpha + h_n\}_{n\in Z} \in Y_\omega. \tag{9}$$

It is easy to see from (5) that the following Lemma 1 is true.

Lemma 1. *The mapping L defined by (5) satisfies*

$$\mathrm{Ker}L = \{x \in X_\omega;\ x_n = x_0, n \in Z, x_0 \in R\}. \tag{10}$$

Lemma 2. *The mapping L defined by (5) satisfies*

$$\mathrm{Im}\, L = \{y \in X_\omega : y_0 = 0\} \subset Y_\omega. \tag{11}$$

Proof. It suffices to show that for each $y = \{y_n\}_{n\in Z} \in X_\omega$ that satisfies $y_0 = 0$, there is $x = \{x_n\}_{n\in Z} \in X_\omega$ such that

$$y_n = \Delta x_n - \Delta x_0, n \in Z. \tag{12}$$

Indeed, if we let

$$x_n = \bigoplus_{i=0}^{n-1} y_i - \frac{n}{\omega}\bigoplus_{i=0}^{\omega-1} y_i,\ n \in Z, \tag{13}$$

then it is not difficult to check that $x = \{x_n\}_{n\in Z} \in X_\omega$ as required . This completes the proof. ∎

In view of (8), (9), Lemma 1 and Lemma 2, we know that the operators P and Q are projections, $X_\omega = \mathrm{Ker}P \oplus \mathrm{Ker}L$ and $Y_\omega = \mathrm{Im}\, L \oplus \mathrm{Im}\, Q$. Furthermore, It is easy to see that $\dim \mathrm{Ker}L = 1 = \dim \mathrm{Im}Q = \mathrm{codimIm}\, L$, and that $\mathrm{Im}\, L$ is closed in Y_ω. Thus the following lemma is true.

Lemma 3. *The mapping L defined by (5) is a Fredholm mapping of index zero.*

Next we recall that a subset S of a Banach space X is relatively compact if, and only if, for each $\varepsilon > 0$, it has a finite ε-net.

Lemma 4. *A subset S of X_ω is relatively compact if, and only if S is bounded.*

Proof. It is easy to see that if S is relatively compact in X_ω, then S is bounded. Conversely, if the subset S of $X_\omega$is bounded, then there is a subset

$$\Gamma := \{x \in X_\omega : \|x\|_1 \leq H\}, \tag{14}$$

where H is a positive constant, such that $S \subset \Gamma$. It suffices to show that Γ is relatively compact in X_ω. Note that for each $\varepsilon > 0$, we may choose numbers $y_0 < y_1 < \cdots < y_l$ such that $y_0 = -H$, $y_l = H$ and $y_{i+1} - y_i < \varepsilon$ for $i = 0, 1, ..., l-1$. Then

$$\left\{v = \{v_n\}_{n\in Z} \in X_\omega : v_j \in \{y_0, y_1, ..., y_{l-1}\}, j = 0, ..., \omega - 1\right\} \tag{15}$$

is a finite ε-net of Γ. This completes the proof. ■

Lemma 5. *Let L and N be defined by (5) and (6) respectively. Suppose Ω is an open and bounded subset of X_ω. Then N is L-compact on $\overline{\Omega}$.*

Proof. Since $\overline{\Omega}$ is bounded, there is a positive constant H such that for any $x = \{x_n\}_{n\in Z} \in \overline{\Omega}$,

$$-\frac{H}{2} \leq x_n \leq \frac{H}{2}, \; n \in Z. \tag{16}$$

¿From this we have

$$-H \leq \Delta x_n \leq H, \; n \in Z. \tag{17}$$

Let

$$A = \max_{1\leq i\leq\omega, --H\leq u\leq H} |F(i,u)|, \tag{18}$$

$$B = \max_{1\leq i\leq\omega, --H\leq u\leq H} |G(i,u)|, \tag{19}$$

and

$$C = \max_{1\leq i\leq\omega} |p_i|.$$

It is easy to see from (6), (7) and (9) that

$$(20)\qquad \begin{aligned}(QNx)_n &= -\frac{n}{\omega}\bigoplus_{i=1}^{\omega}\left(F(i,\Delta x_{i-\sigma_i})+G(i,x_{i-\tau_i})-p_i\right)\\ &= -\frac{n}{\omega}\bigoplus_{i=1}^{\omega}\left(F(i,\Delta x_{i-\sigma_i})+G(i,x_{i-\tau_i})\right)\end{aligned}$$

for $n \in Z$. Thus

$$\|QNx\|_2 = \left|\frac{1}{\omega}\bigoplus_{i=1}^{\omega}\left(F(i,\Delta x_{i-\sigma_i})+G(i,x_{i-\tau_i})\right)\right| \le A+B,$$

so that $QN\left(\overline{\Omega}\right)$ is bounded. We denote the inverse of the mapping $L_{|\mathrm{Dom}L\cap\mathrm{Ker}P}:(I-P)\,X \to \mathrm{Im}\,L$ by K_P. Then

$$\begin{aligned}((I-Q)\,Nx)_n = &-\bigoplus_{i=1}^{n}\left(F(i,\Delta x_{i-\sigma_i})+G(i,x_{i-\tau_i})-p_i\right)\\ &+\frac{n}{\omega}\bigoplus_{i=1}^{\omega}\left(F(i,\Delta x_{i-\sigma_i})+G(i,x_{i-\tau_i})\right), n\in Z.\end{aligned}$$

By direct calculations, we may obtain

$$(21)\qquad \begin{aligned}(K_P\,(I-Q)\,Nx)_n =\; &-\bigoplus_{i=0}^{n-1}\bigoplus_{k=1}^{i}\left(F(k,\Delta x_{k-\sigma_k})+G(k,x_{k-\tau_k})-p_k\right)\\ &+\bigoplus_{i=0}^{n-1}\frac{i}{\omega}\bigoplus_{k=1}^{\omega}\left(F(k,\Delta x_{k-\sigma_k})+G(k,x_{k-\tau_k})\right)\\ &+\frac{n}{\omega}\bigoplus_{i=0}^{\omega-1}\left\{\bigoplus_{k=1}^{i}\left(F(k,\Delta x_{k-\sigma_k})+G(k,x_{k-\tau_k})-p_k\right)\right.\\ &\left.-\frac{i}{\omega}\bigoplus_{k=1}^{\omega}\left(F(k,\Delta x_{k-\sigma_k})+G(k,x_{k-\tau_k})\right)\right\}.\end{aligned}$$

It follows that

$$(22)\qquad \begin{aligned}\|K_P\,(I-Q)\,Nx\|_1 \le\; &2\bigoplus_{i=0}^{\omega-1}\bigoplus_{k=1}^{i}\left|\left(F(k,\Delta x_{k-\sigma_k})+G(k,x_{k-\tau_k})-p_k\right)\right|\\ &+2\bigoplus_{i=0}^{\omega-1}\bigoplus_{k=1}^{\omega}\left|\left(F(k,\Delta x_{k-\sigma_k})+G(k,x_{k-\tau_k})\right)\right|\\ \le\; &4\bigoplus_{i=0}^{\omega-1}\bigoplus_{k=1}^{\omega}\left(|F(k,\Delta x_{k-\sigma_k})|+|G(k,x_{k-\tau_k})|+|p_k|\right)\\ \le\; &4\omega^2\,(A+B+C)\,.\end{aligned}$$

Thus $K_P(I-Q)N(\overline{\Omega})$ is bounded in X_ω. In view of Lemma 4, $K_P(I-Q)N(\overline{\Omega})$ is relatively compact in X_ω and hence N is L-compact on $\overline{\Omega}$. This completes the proof .

Lemma 6. ([10]). *If $u=\{u_n\}_{n\in Z}\in l_\omega$, then*

$$\max_{0\le i,j\le\omega-1}|u_i-u_j|\le\frac{1}{2}\sum_{k=0}^{\omega-1}|\Delta u_k|, \tag{23}$$

where the constant factor $\frac{1}{2}$ is the best possible.

Lemma 7. *For any $u=\{u_n\}_{n\in Z}\in l_\omega$, we have*

$$\max_{0\le k\le\omega-1}|\Delta u_k|\le\frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta^2 u_k\right|. \tag{24}$$

Furthermore, if $\omega\geqslant 3$, then the constant factor $\frac{1}{2}$ in (24) is the best possible.

Proof. Since $u=\{u_n\}_{n\in Z}\in l_\omega$, we see that $\{\Delta u_n\}_{n\in Z}\in l_\omega$ and $\left\{\Delta^2 u_n\right\}_{n\in Z}\in l_\omega$. For any $s,i\in\{0,1,,...,\omega-1\}$, we have

$$\Delta u_s=\Delta u_i+\bigoplus_{k=i+1}^{s}\Delta^2 u_{k-1}, \tag{25}$$

and

$$\Delta u_i=\Delta u_{i+\omega}=\Delta u_s+\bigoplus_{k=s+1}^{i+\omega}\Delta^2 u_{k-1}. \tag{26}$$

(25) and (26) lead us to

$$\Delta u_s=\Delta u_i+\frac{1}{2}\left\{\bigoplus_{k=i+1}^{s}\Delta^2 u_{k-1}-\bigoplus_{k=s+1}^{i+\omega}\Delta^2 u_{k-1}\right\}. \tag{27}$$

Let $u_a=\max_{0\le k\le\omega-1}u_k,\ u_b=\min_{0\le k\le\omega-1}u_k,\ |\Delta u_c|=\max_{0\le k\le\omega-1}|\Delta u_k|,$ where $a,b,c\in\{0,1,,...,\omega-1\}$. Then $\Delta u_a\le 0$ and $\Delta u_b\geqslant 0$. If $\Delta u_c=\max_{0\le k\le\omega-1}|\Delta u_k|$, then from (27), we see that

$$\begin{aligned}\max_{0\le k\le\omega-1}|\Delta u_k| &= \Delta u_c=\Delta u_a+\frac{1}{2}\left\{\bigoplus_{k=a+1}^{c}\Delta^2 u_{k-1}-\bigoplus_{k=c+1}^{a+\omega}\Delta^2 u_{k-1}\right\}\\ &\le \frac{1}{2}\left\{\bigoplus_{k=a+1}^{s}\Delta^2 x_{k-1}-\bigoplus_{k=s+1}^{a+\omega}\Delta^2 x_{k-1}\right\}\\ &\le \frac{1}{2}\sum_{k=a+1}^{a+\omega}\left|\Delta^2 x_k\right|=\frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta^2 x_k\right|.\end{aligned} \tag{28}$$

If $-\Delta u_c = \max_{0\le k\le \omega-1} |\Delta u_k|$, from (27) we have

$$\begin{aligned} \max_{0\le k\le \omega-1} |\Delta u_k| &= -\Delta u_c = -\Delta u_b - \frac{1}{2}\left\{ \bigoplus_{k=b+1}^{c} \Delta^2 u_{k-1} - \bigoplus_{k=c+1}^{b+\omega} \Delta^2 u_{k-1} \right\} \\ &\le -\frac{1}{2}\left\{ \bigoplus_{k=b+1}^{c} \Delta^2 u_{k-1} - \bigoplus_{k=c+1}^{b+\omega} \Delta^2 u_{k-1} \right\} \\ &\le \frac{1}{2}\sum_{k=b+1}^{b+\omega} \left|\Delta^2 u_k\right| = \frac{1}{2}\sum_{k=0}^{\omega-1} \left|\Delta^2 u_k\right| . \end{aligned} \tag{29}$$

According to (28) and (28), we see that (24) holds. When $\omega \geqslant 3$, we assert that if β is a constant and $\beta < \frac{1}{2}$, then there is $u = \{u_n\}_{n\in Z} \in l_\omega$ such that

$$\max_{0\le k\le \omega-1} |\Delta u_k| > \beta \sum_{k=0}^{\omega-1} \left|\Delta^2 u_k\right| . \tag{30}$$

Indeed, in case $\omega = 3$, if we let

$$u_n = \begin{cases} 0 & n \equiv 0,1 \bmod 3 \\ \omega - 1 & n \equiv 2 \bmod 3 \end{cases} \tag{31}$$

then we have

$$\Delta u_n = \begin{cases} 0 & n \equiv 0 \bmod 3 \\ \omega - 1 & n \equiv 1 \bmod 3 \\ 1-\omega & n \equiv 2 \bmod 3 \end{cases} \tag{32}$$

and

$$\Delta^2 u_n = \begin{cases} \omega - 1 & n \equiv 0 \bmod 3 \\ 2-2\omega & n \equiv 1 \bmod 3 \\ \omega - 1 & n \equiv 2 \bmod 3 \end{cases} . \tag{33}$$

In case $\omega \geqslant 4$, if we let

$$u_n = \begin{cases} 0 & n \equiv 0,1,...,\omega-2 \bmod \omega, \\ \omega - 1 & n \equiv \omega - 1 \bmod \omega \end{cases} , \tag{34}$$

then we have

$$\Delta u_n = \begin{cases} 0 & n \equiv 0,1,...,\omega-3 \bmod \omega \\ \omega - 1 & n \equiv \omega-2 \bmod \omega \\ 1-\omega & n \equiv \omega-1 \bmod \omega \end{cases} \tag{35}$$

and

$$\Delta^2 u_n = \begin{cases} 0 & n \equiv 0, 1, ..., \omega - 4 \bmod \omega \\ \omega - 1 & n \equiv \omega - 3 \bmod \omega \\ 2 - 2\omega & n \equiv \omega - 2 \bmod \omega \\ \omega - 1 & n \equiv \omega - 1 \bmod \omega \end{cases} . \tag{36}$$

Thus when $\omega \geqslant 3$, from (32) and (33) (or (35) and (36)), we see that

$$\beta \sum_{k=0}^{\omega-1} \left|\Delta^2 u_k\right| = 4\beta\left(\omega - 1\right) < 2\left(\omega - 1\right) = \max_{0 \le k \le \omega - 1} \left|\Delta u_k\right|$$

as required. This shows that the constant $1/2$ in (24) is the best possible. The proof is complete. ∎

Now, we consider the following auxiliary equation

$$\Delta x_n = \Delta x_0 - \lambda \bigoplus_{i=1}^{n} \left(F(i, \Delta x_{i-\sigma_i}) + G(i, x_{i-\tau_i}) - p_i\right), \, n \in Z, \tag{37}$$

where $\lambda \in (0, 1)$.

Lemma 8. *Suppose the conditions (a), (b) and (c) of Theorem 1 hold and*

$$\omega\left(r_1 + r_3\left(\frac{r_1}{r_2} + \frac{\omega}{2}\right)\right) < 1,$$

then (1) *has an* ω*-periodic solution. Then there is a positive number* D_0 *such that for any* ω*-periodic solution* $x = \{x_n\}_{n\in Z}$ *of (37), we have*

$$\|x\|_1 \le D_0. \tag{38}$$

Proof. We only give the proof in case

$$\lim_{x \to -\infty} \max_{0 \le n \le \omega - 1} \frac{G(n, x)}{x} \le r_3, \tag{39}$$

since the other case can be treated in similar manners. Let $x = \{x_n\}_{n\in Z}$ be an ω-periodic solution of (37). It is easy to see that $\{\Delta x_n\}_{n\in Z} \in X_\omega$. By (37) and the assumption that $\sum_{n=0}^{\omega-1} p_n = 0$, we have

$$\bigoplus_{i=1}^{\omega} \left(F(i, \Delta x_{i-\sigma_i}) + G(i, x_{i-\tau_i})\right) = 0. \tag{40}$$

It is then easy to see that there are $\xi, \eta \in \{1, 2, ..., \omega\}$ such that

$$F(\xi, \Delta x_{\xi-\sigma_\xi}) + G(\xi, x_{\xi-\tau_\xi}) \leq 0 \tag{41}$$

and

$$F(\eta, \Delta x_{\eta-\sigma_\eta}) + G(\eta, x_{\eta-\tau_\eta}) \geq 0. \tag{42}$$

Next, we will prove that

$$\|x\|_1 \leq \left(\frac{r_1}{r_2} + \frac{\omega}{2}\right) \max_{0\leq k\leq \omega-1} |\Delta x_k| + D + \frac{K}{r_2}. \tag{43}$$

Indeed, either there is $\delta \in \{0, 1, ..., \omega - 1\}$ such that

$$|x_\delta| \leq D, \tag{44}$$

or $|x_k| > D$ for all $k \in \{0, 1, ..., \omega - 1\}$. In the former case, from Lemma 6,

$$\|x\|_1 = \max_{0\leq k\leq \omega-1} |x_k| - |x_\delta| \leq \max_{0\leq i,j\leq \omega-1} |x_i - x_j| \leq \frac{1}{2}\sum_{k=0}^{\omega-1} |\Delta x_k| , \tag{45}$$

so that

$$\|x\|_1 \leq |x_\delta| + \frac{1}{2}\sum_{k=0}^{\omega-1} |\Delta x_k| \leq D + \frac{\omega}{2} \max_{0\leq k\leq \omega-1} |\Delta x_k| . \tag{46}$$

This shows that (43) holds. In the latter case, there are three subcases:

Case I. There are $i_0, j_0 \in \{0, 1, ..., \omega - 1\}$ such that $x_{i_0} > D$ and $x_{j_0} < -D$. Let $x_a = \max_{0\leq k\leq \omega-1} x_k$ and $x_b = \min_{0\leq k\leq \omega-1} x_k$. We know that $x_a > D$ and $x_b < -D$. By Lemma 6, we have

$$x_a - x_b \leq \max_{0\leq i,j\leq \omega-1} |x_i - x_j| \leq \frac{1}{2}\sum_{k=0}^{\omega-1} |\Delta x_k| \leq \frac{\omega}{2} \max_{0\leq k\leq \omega-1} |\Delta x_k| . \tag{47}$$

It follows that

$$x_a \leq x_b + \frac{\omega}{2} \max_{0\leq k\leq \omega-1} |\Delta x_k| \leq -D + \frac{\omega}{2} \max_{0\leq k\leq \omega-1} |\Delta x_k| \tag{48}$$

and

$$x_b \geqslant x_a - \frac{\omega}{2} \max_{0\leq k\leq \omega-1} |\Delta x_k| \geqslant D - \frac{\omega}{2} \max_{0\leq k\leq \omega-1} |\Delta x_k| . \tag{49}$$

From (48) and (49), we have for any $n \in \{0, 1, ..., \omega - 1\}$,

$$D - \frac{\omega}{2} \max_{0\leq k\leq \omega-1} |\Delta x_k| \leq x_b \leq x_n \leq x_a \leq -D + \frac{\omega}{2} \max_{0\leq k\leq \omega-1} |\Delta x_k| . \tag{50}$$

It follows that

$$\|x\|_1 \le \left| -D + \frac{\omega}{2} \max_{0\le k\le \omega-1} |\Delta x_k| \right| \le D + \frac{\omega}{2} \max_{0\le k\le \omega-1} |\Delta x_k| \,. \tag{51}$$

This shows that (43) also holds.

Case II. For all $k \in \{0, 1, ..., \omega-1\}$, we have $x_k > D$. In view of conditions (a), (b) and (41), we have

$$\begin{aligned} r_2 \left| x_{\xi-\tau_\xi} \right| &\le \left| G(\xi, x_{\xi-\tau_\xi}) \right| = G(\xi, x_{\xi-\tau_\xi}) \le -F(\xi, \Delta x_{\xi-\sigma_\xi}) \\ &= \left| F(\xi, \Delta x_{\xi-\sigma_\xi}) \right| \le r_1 |\Delta x_{\xi-\sigma_\xi}| + K \le r_1 \max_{1\le k\le\omega} |\Delta x_k| + K. \end{aligned} \tag{52}$$

Thus,

$$\left| x_{\xi-\tau_\xi} \right| \le \frac{r_1}{r_2} \max_{1\le k\le\omega} |\Delta x_k| + \frac{K}{r_2}. \tag{53}$$

Since $\xi - \tau_\xi \in Z$ and $x = \{x_n\}_{n\in Z}$ is ω-periodic, there is a $c \in \{0, 1, ..., \omega-1\}$ such that $x_c = x_{\xi-\tau_\xi}$. From (53), we have

$$|x_c| \le \frac{r_1}{r_2} \max_{0\le k\le\omega-1} |\Delta x_k| + \frac{K}{r_2}. \tag{54}$$

Furthermore, by Lemma 6 we have

$$\|x\|_1 \le |x_c| + \frac{1}{2} \sum_{k=0}^{\omega-1} |\Delta x_k| \le \left(\frac{r_1}{r_2} + \frac{\omega}{2} \right) \max_{0\le k\le\omega-1} |\Delta x_k| + \frac{K}{r_2}. \tag{55}$$

This shows that (43) also holds.

Case III. For all $k \in \{0, 1, ..., \omega-1\}$, we have $x_k < -D$. In view of conditions (a), (b) and (42), we have

$$\begin{aligned} r_2 \left| x_{\eta-\tau_\eta} \right| &\le \left| G(\eta, x_{\eta-\tau_\eta}) \right| = -G(\eta, x_{\eta-\tau_\eta}) \le F(\eta, \Delta x_{\eta-\sigma_\eta}) \\ &= \left| F(\eta, \Delta x_{\eta-\sigma_\eta}) \right| \le r_1 |\Delta x_{\eta-\sigma_\eta}| + K \le r_1 \max_{1\le k\le\omega} |\Delta x_k| + K. \end{aligned} \tag{56}$$

Thus,

$$\left| x_{\eta-\tau_\eta} \right| \le \frac{r_1}{r_2} \max_{1\le k\le\omega} |\Delta x_k| + \frac{K}{r_2}. \tag{57}$$

Since $\eta - \tau_\eta \in Z$ and $x = \{x_n\}_{n\in Z}$ is ω-periodic, there is a $d \in \{0, 1, ..., \omega-1\}$ such that $x_d = x_{\eta-\tau_\eta}$. From (57), we have

$$|x_d| \le \frac{r_1}{r_2} \max_{0\le k\le\omega-1} |\Delta x_k| + \frac{K}{r_2}. \tag{58}$$

It easy to prove from Lemma 6 and (58) that (43) also holds.

Next, note that the condition $\omega\left(r_1+r_3\left(\frac{r_1}{r_2}+\frac{\omega}{2}\right)\right) < 1$ implies there is a positive number $\varepsilon > 0$ such that

$$\eta_1 = \omega\left(r_1 + (r_3+\varepsilon)\left(\frac{r_1}{r_2}+\frac{\omega}{2}\right)\right) < 1. \tag{59}$$

In view of condition (c), we have a $\rho > D$ such that for $n \in Z$ and $x < -\rho$,

$$G(n;x) \le (r_3+\varepsilon)\,|x|\,. \tag{60}$$

Let

$$E_1 = \{n \in \{0,1,...,\omega-1\} : x_n < -\rho\}\,, \tag{61}$$

$$E_2 = \{n \in \{0,1,...,\omega-1\} : |x_n| \le \rho\}\,, \tag{62}$$

$$E_3 = \{0,1,...,\omega-1\} \setminus (E_1 \cup E_2) \tag{63}$$

and

$$M_0 = \max_{0\le i\le \omega-1, |x|\le\rho} |G(n;x)|\,. \tag{64}$$

By (43), (60) and (61), we have

$$\begin{aligned}\sum_{n\in E_1} |G(n;x_{n-\tau_n})| &\le \sum_{n\in E_1} (r_3+\varepsilon)\,|x_{n-\tau_n}| \le \omega\,(r_3+\varepsilon)\,\|x\|_1 \\ &\le \omega\,(r_3{+}\varepsilon)\left\{\left(\frac{r_1}{r_2}{+}\frac{\omega}{2}\right)\max_{0\le k\le\omega-1}|\Delta x_k|{+}D{+}\frac{K}{r_2}\right\}.\end{aligned} \tag{65}$$

From (62) and (64), we have

$$\sum_{n\in E_2} |G(n;x_{n-\tau_n})| \le \omega M_0. \tag{66}$$

It follows from condition (a) that

$$\bigoplus_{i=0}^{\omega-1} |F(i,\Delta x_{i-\sigma_i})| \le \bigoplus_{i=0}^{\omega-1} (r_1|\Delta x_{i-\sigma_i}|+K) \le \omega\left(r_1 \max_{0\le k\le\omega-1}|\Delta x_k|+K\right). \tag{67}$$

In view of (b), (40), (63), (65), (66) and (67), we get

$$
\begin{aligned}
&\sum_{n\in E_3}|G(n;x_{n-\tau_n})| = \sum_{n\in E_3} G(n;x_{n-\tau_n}) \\
&= \bigoplus_{i=0}^{\omega-1} G(i,x_{i-\tau_i}) - \sum_{n\in E_1} G(n;x_{n-\tau_n}) - \sum_{n\in E_2} G(n;x_{n-\tau_n}) \\
&\le -\bigoplus_{i=0}^{\omega-1} F(i,\Delta x_{i-\sigma_i}) - \sum_{n\in E_1} G(n;x_{n-\tau_n}) - \sum_{n\in E_2} G(n;x_{n-\tau_n}) \\
&\le \bigoplus_{i=0}^{\omega-1} |F(i,\Delta x_{i-\sigma_i})| + \sum_{n\in E_1} |G(n;x_{n-\tau_n})| + \sum_{n\in E_2} |G(n;x_{n-\tau_n})| \\
&\le \omega\,(r_3+\varepsilon)\left\{\left(\frac{r_1}{r_2}+\frac{\omega}{2}\right)\max_{0\le k\le\omega-1}|\Delta x_k| + D + \frac{K}{r_2}\right\} \\
&\quad +\omega\left(r_1 \max_{0\le k\le\omega-1}|\Delta x_k| + K\right) + \omega M_0. \\
&\le \omega\left(r_{1+}\,(r_3+\varepsilon)\left(\frac{r_1}{r_2}+\frac{\omega}{2}\right)\right)\max_{0\le k\le\omega-1}|\Delta x_k| + M_1.
\end{aligned}
\tag{68}
$$

for some positive number M_1. On the other hand, by (37), we see that

$$
\Delta^2 x_{n-1} = -\lambda\,(F(n,\Delta x_{n-\sigma_n}) + G(n,x_{n-\tau_n}) - p_n)\,,\; n\in Z. \tag{69}
$$

It follows from (b), (65), (66), (67), (68) and (69) that

$$
\begin{aligned}
&\bigoplus_{i=0}^{\omega-1}\left|\Delta^2 x_{i-1}\right| \le \bigoplus_{n=0}^{\omega-1}|F(i,\Delta x_{i-\sigma_i})| + \bigoplus_{i=0}^{\omega-1}|G(i,x_{i-\tau_i})| + \bigoplus_{i=0}^{\omega-1}|p_i| \\
&= \bigoplus_{n=0}^{\omega-1}|F(i,\Delta x_{i-\sigma_i})| + \sum_{n\in E_1}|G(n;x_{n-\tau_n}|) \\
&\quad + \sum_{n\in E_2}|G(n;x_{n-\tau_n}| + \sum_{n\in E_3}|G(n;x_{n-\tau_n}| + \bigoplus_{i=0}^{\omega-1}|p_i| \\
&\le \omega\left(r_1 \max_{0\le k\le\omega-1}|\Delta x_k| + K\right) \\
&\quad +\omega\,(r_3+\varepsilon)\left\{\left(\frac{r_1}{r_2}+\frac{\omega}{2}\right)\max_{0\le k\le\omega-1}|\Delta x_k| + D + \frac{K}{r_2}\right\} \\
&\quad +\omega M_0+\omega\left(r_{1+}\,(r_3+\varepsilon)\left(\frac{r_1}{r_2}+\frac{\omega}{2}\right)\right)\max_{0\le k\le\omega-1}|\Delta x_k| + M_1 + \bigoplus_{i=0}^{\omega-1}|p_i| \\
&\le 2\eta_1 \max_{0\le k\le\omega-1}|\Delta x_k| + M_2.
\end{aligned}
\tag{70}
$$

for some positive number M_2. On the other hand, by Lemma 7, we have

$$\max_{0\leq k\leq\omega-1}|\Delta x_k|\leq\frac{1}{2}\bigoplus_{i=0}^{\omega-1}\left|\Delta^2 x_k\right|. \tag{71}$$

¿From (70) and (71), we get

$$\max_{0\leq k\leq\omega-1}|\Delta x_k|\leq\eta_1\max_{0\leq k\leq\omega-1}|\Delta x_k|+\frac{M_2}{2}. \tag{72}$$

It follows that

$$\max_{0\leq k\leq\omega-1}|\Delta x_k|\leq D_1, \tag{73}$$

where $D_1=M_2/2(1-\eta_1)$. From (43) and (73), we get

$$\|x\|_1\leq\left(\frac{r_1}{r_2}+\frac{\omega}{2}\right)\max_{0\leq k\leq\omega-1}|\Delta x_k|+D+\frac{K}{r_2}\leq D_0, \tag{74}$$

where $D_0=\left(\frac{r_1}{r_2}+\frac{\omega}{2}\right)D_1+D+\frac{K}{r_2}$. This completes the proof. ■

We now turn to the proof of Theorem 1. Let L, N, P and Q be defined by (5), (6), (8) and (9) respectively. Take a positive number $\overline{D}$ which is greater than D_0+D and let

$$\Omega=\left\{x\in X_\omega:\|x\|_1<\overline{D}\right\}. \tag{75}$$

It is easy to see that Ω is an open and bounded subset of X_ω. Furthermore, in view of Lemma 3 and Lemma 5, we know that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. Noting that $\overline{D}>D_0$, by Lemma 8, for each $\lambda\in(0,1)$ and $x=\{x_n\}_{n\in Z}\in\partial\Omega$, we have $Lx\neq\lambda Nx$. Next note that a sequence $x=\{x_n\}_{n\in Z}\in\partial\Omega\cap\text{Ker } L$ must be a constant: $x_n=\overline{D}$ or $x_n=-\overline{D}$ for $n\in Z$. Hence, by (6), (9) and the assumption that $\sum_{n=0}^{\omega-1}p_n=0$, we see that

$$(QNx)_n=-\frac{n}{\omega}\bigoplus_{i=0}^{\omega-1}\left(F(i,\Delta x_{i-\sigma_i})+G(i,x_{i-\tau_i})\right), \tag{76}$$

so that

$$QNx\neq\theta_2. \tag{77}$$

The isomorphism $J:\text{Im}Q\to\text{Ker } L$ is defined by $(J(n\alpha))_n=\alpha$ for $\alpha\in R$ and $n\in Z$. Thus from the conditions (a) and (b),

$$(JQNx)_n=-\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1}\left(F(i,\Delta x_{i-\sigma_i})+G(i,x_{i-\tau_i})\right)\neq 0,\ n\in Z. \tag{78}$$

In particular, we see that if $x = \{\overline{D}\}_{n\in Z}$, then

$$
\begin{aligned}
&-\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1}(F(i,\Delta x_{i-\sigma_i})+G(i,x_{i-\tau_i})) \\
&= -\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1}\left(F(i,0)+G(i,\overline{D})\right) = -\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1}G(i,\overline{D}) < 0,
\end{aligned} \tag{79}
$$

and if $x = \{-\overline{D}\}_{n\in Z}$, then

$$
\begin{aligned}
&-\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1}(F(i,\Delta x_{i-\sigma_i})+G(i,x_{i-\tau_i})) \\
&= -\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1}\left(F(i,0)+G(i,-\overline{D})\right) = -\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1}G(i,-\overline{D}) > 0.
\end{aligned} \tag{80}
$$

This shows that

$$\deg\left(JQN, \Omega\cap \operatorname{Ker} L, \theta_1\right) \neq 0.$$

In view of Theorem A, there exists an ω-periodic solution of (1). The proof is complete. ∎

To see that our Theorem is not vacuous, we provide the following example.

Example. Consider a Rayleigh equation of the form (1) where $\sigma_n = \left[\sin\frac{2\pi n}{\omega}\right]$ and $\tau_n = \left[\cos\frac{2\pi n}{\omega}\right]$ (where $[\cdot]$ is a greatest-integer function and the integer $\omega \geqslant 3$),

$$
\begin{aligned}
G(n,x) &= \frac{4\exp\left(\sin\frac{2\pi n}{\omega}\right)^2 h(x)}{13\omega(\omega+2)}, \\
F(n,x) &= \left(\frac{1+\cos\frac{2\pi n}{\omega}}{6\omega(2+\omega)}\right)x + \exp\left(-x^2\right) - 1, \\
h(x) &= \begin{cases} x^3 & x \geqslant 0 \\ x & x < 0 \end{cases},
\end{aligned}
$$

and

$$
p_n = \begin{cases} 1 & n \equiv 0 \bmod \omega \\ -1 & n \equiv 1 \bmod \omega \\ 0 & n \equiv 2, ..., \omega-1 \bmod \omega \end{cases}.
$$

It is then easy to verify that all the assumptions in Theorem 1 are satisfied with

$$K = 2,\ D = 1,\ r_1 = \frac{1}{3\omega(2+\omega)},\ r_2 = \frac{4}{13\omega(2+\omega)},\ r_3 = \frac{4e}{13\omega(2+\omega)}.$$

Thus our equation has an ω-periodic solution. Furthermore, this solution is nontrivial since $y = \{0\}_{n \in Z}$ is not a solution of this equation.

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