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GENERALIZED CONVEXITY IN NONLINEAR ELASTICITY WITH APPLICATIONS TO UNILATERAL CONTACT

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Dedicated to the memory of Professor Dr. Tamas Rapcsák

Abstract. This research/survey paper firstly gives an overview of generalized convexity in calculus of variations and nonlinear elasticity, centered at the notions of quasiconvexity, polyconvexity, and rank-one-convexity. Then $\mathcal{A}$-convexity based on Young measures and relaxation are discussed. In this context a general version of the Jensen's inequality for $\mathcal{A}$-convex functions is given that extends the classical Jensen's inequality for convex functions.

Secondly new results for the unilateral contact problem in nonlinear elasticity are presented. In particular existence results are derived for the pure contact-traction problem under an appropriate recession condition for quasiconvex as well as for nonquasiconvex energy densities, using in the latter case the Young measure approach.

0. Introduction

This research/survey paper aims at two different purposes. Firstly we give an overview of the various notions of generalized convexity in nonconvex vectorial calculus of variations that are applicable in nonlinear elasticity. Our exposition centers at the notion of quasiconvexity in the sense of Morrey, polyconvexity due to Ball, and rank-one-convexity. Let us mention in passing that this notion of quasiconvexity should not be confused with quasiconvexity as convexity of lower level sets in optimization theory (see e.g. [37] for a recent survey) and also used in quasilinear elliptic equations [32].

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Thus to some part our paper is a follow up to the survey of Brechtken-Manderscheid and Heil [7] of 1993 and to the paper of Hartwig [25] on quasi-convexity that appeared in the Proceedings of the IVth International Workshop on Generalized Convexity at Pécs in 1994. On the other hand we do not strive to cover the whole field of nonlinear elasticity theory; therefore there is only a small overlap with the survey of Ball [4] of 2002 on outstanding open problems in nonlinear elasticity.

In our overwiew we include more recent results due to N. Chaudhuri and S. Müller [8], S. Conti [11], B. Dacorogna and J.-P. Haeberly [13], D. Faraco and X. Zhong [15] J. Kristensen [28], and M. Kruz´k [29]. These results provide a deeper understanding of generalized convexity in vectorial calculus of variations and show in particular how additional constraints may lead to new relations between quasiconvexity and rank-one-convexity and thus to a more refined existence analysis.

We focus our overview to first order problems of the calculus of variations; higher order problems as e.g. in [17, 18] are outside of the scope. Instead we elaborate on the general framework of Pedregal [36, Section 1.3] to parametrized measures in nonconvex calculus of variations. Since this exposition is directed to a larger audience, we provide some examples to motivate and illustrate the abstract theory. We discuss the recent notion of $\mathcal{A}$ - convexity, its relation to Young measure and its application to relaxation in nonlinear elasticity. In this context we make precise the Jensen's inequality for $\mathcal{A}$ - convex functions as sketched in [36], which is shown to extend the classical Jensen's inequality for convex functions.

Secondly we present new results for the unilateral contact problem in nonlinear elasticity. Here we do not enter in the derivation of a Euler Lagrange equation or a more general Euler Lagrange inclusion using Clarke's generalized differential calculus, what had recently been established by Schuricht [38] and by Habeck and Schuricht [23]. Also we do not discuss the formulation of penalty methods and study their convergence properties what is well known at least in convex programming to provide a constructive approach to the existence of Lagrange multipliers; for an investigation of penalty convergence in nonlinear elasticity in a polyconvex setting we refer to [22]. Instead we stick to the variational problem of minimizing the strain energy subject to the unilateral constraint of rigid friction-free contact with a given foundation.

In particular we study the pure contact-traction problem. Under the assumption of quasiconvexity and an appropriate recession condition we derive an existence result that parallels the existence result of Ciarlet and Nečas in their classical paper [10] (see also book[9] of Ciarlet) under the more stringent assumption of polyconvexity. Here for the reader's convenience we give a self-contained proof which uses an recession argument that in elasticity theory goes back at least to Fichera [16] and has raised to a higher abstract level by Baiocchi, Buttazzo, Gastaldi and

Tomarelli in their seminal paper [5]. Since then recession analysis for semicoercive or noncoercive variational problems has experienced many extensions and ramifications, we only note the book of Goeleven [20]; further [21] for an application to nonlinear von Kármán elasticity and [30] for the convergence study of asymptotic directions of unbounded sets in general normed spaces. Finally we deal with non-quasiconvex energy densities f and present an analogous existence result for the pure contact-traction problem using the Young measure approach.

1. Generalized Convexity in Calculus of Variations and Nonlinear Elasticity

In this section we formulate a general version of the Weierstrass principle, shortly recall the scalar case of calculus of variations, then focus to the vectorial case and give an overview of the various notions of generalized convexity in calculus of variations quasiconvexity, rank-one-convexity and polyconvexity. We conclude this section with a discussion of polyconvexity in nonlinear elasticity.

Let us consider the functional

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, \mathrm{d}x\,,$$

where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain,
$f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^* = \mathbb{R} \cup \{+\infty\}$ is a Lebesgue measurable function,
and $u : \Omega \to \mathbb{R}^m$ belongs to a reflexive Banach space

$$X \subset W^{1,1}(\Omega, \mathbb{R}^m) = \{u \in L^1(\Omega, \mathbb{R}^m) \ : \ \nabla u \in L^1(\Omega, \mathbb{R}^{m\times d})\}$$

such that the integrand is integrable. More specific assumptions will be discussed later when needed.

In nonlinear elasticity we have $d = m$ and the unknown u stands for the deformation field , i.e. the coordinates of an elastically deformed body under applied forces with respect to the reference configuration Ω. I describes the total elastic energy of the body under deformation including the work done by the volume forces and boundary tractions. To obtain an equilibrium state of the elastic body we use the fundamental principle of energy minimization. Thus these equilibrium states are exactly the minimizers of I.

The back bone of these so called direct methods in calculus of variations is

1.1. The Weierstrass principle

Let M be a weakly closed subset of a reflexive Banach space X, further let $I : X \to \mathbb{R}^* = \mathbb{R} \cup \{+\infty\}$, $I \not\equiv +\infty$ be weakly lower semicontinuous and

$$I(u) \geq \varphi(\|u\|) \quad \text{for} \quad \forall\, u \in M \quad \text{where} \ \varphi \in C(\mathbb{R}^+)\,, \ \varphi(\|u\|) \underset{\|u\| \to \infty}{\longrightarrow} +\infty\,.$$

Then there exists at least one $\tilde{u} \in M$ with $I(\tilde{u}) = \inf_{v \in M} I(v)$.

Proof. I is bounded from below in M. Hence there exists an infimizing sequence $(u_n)_{n\in\mathbb{N}} \subset M$ with $I(u_n) \underset{n\to\infty}{\longrightarrow} \inf_{v\in M} I(v)$. Using the coercivity of $I \not\equiv +\infty$ we obtain for some $C < \infty$

$$\limsup_{n\to\infty} \varphi\left(\|u_n\|\right) \leq \limsup_{n\to\infty} I(u_n) < C\,,$$

hence $\|u_n\| \leq \tilde{C} < \infty$ for sufficiently large n. By the theorem of Eberlein and Šmulyan we can extract a subsequence of (u_n) (not renamed) that weakly converges to some $\tilde{u}$ in the weakly closed subset M of X. Finally, the weak lower semicontinuity of I yields

$$I(\tilde{u}) \leq \liminf_{n\to\infty} I(u_n) = \inf_{v\in M} I(v)$$

which implies the assertion. ■

Remark. Clearly as a consequence of the Hahn - Banach theorem in the geometric version we obtain that $M \subset X$ is weakly closed, if M is closed and convex. As an example of a weakly closed, but nonconvex subset of a Banach space we note

$$D = \{(u,\delta) \,:\, \delta = \det \nabla u\} \subset W^{1,q}(\Omega,\mathbb{R}^d) \times L^{\frac{q}{d}}(\Omega)$$

provided $d < q < \infty$, since D is the graph of the nonlinear, however weakly continuous $\det$ function, see e.g. [14, Lemma 3, 3E].

According to the Weierstrass principle, an essential condition for the existence of minimizers of energy functionals $I : X \to \mathbb{R}^*$ is the weak lower semicontinuity. This condition is hard to verify in general and so one is led to examine the properties of f which ensure the weak lower semicontiuity of I. At first we shortly recall.

1.2. The scalar case ($m = 1$ or $d = 1$)

Here classical convexity of f is the central condition for weak lower semicontinuity of I. The following results clarify the connection between the notions of convexity and weak lower semicontinuity in the scalar case. We summarize the results in the following equivalence.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $q \in [1,\infty]$ and let further $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{\max\{d,m\}} \to \mathbb{R}$ be a continuous function $(m = 1$ or $d = 1)$ with

$$\langle\, g_1(x)\,,\, v\,\rangle_{\mathbb{R}^{\max\{d,m\}}} + g_2(x) \leq |f(x,u,v)| \leq g(x,|u|\,,|v|)$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^m$, $v \in \mathbb{R}^{\max\{d,m\}}$, where $g_1 \in L^{q'}(\Omega, \mathbb{R}^{\max\{d,m\}})$, $g_2 \in L^1(\Omega)$ and $g \in L^1(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+)$. Then

$$f(x,u,\cdot) \quad \text{is convex for a.e. } x \in \Omega \quad \text{and all } u \in \mathbb{R}^m \quad \Longleftrightarrow$$
$$I(u) = \int_\Omega f(x,u,\nabla u)\,\mathrm{d}x \quad \text{is weakly lower semicontinuous in } W^{1,q}(\Omega, \mathbb{R}^m)\,.$$

1.3. The vectorial case ($m > 1$ and $d > 1$)

The situation becomes more complicated in this case. The convexity of f (with appropriate growth conditions) is still a sufficient condition for weak lower semicontinuity of I but it is far from being necessary. Therefore we adopt the notion of

1.3.1. Quasiconvexity

Definition. (Morrey 1952).
An integrable function $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(A) \le \frac{1}{|\Omega|}\int_\Omega f(A + \nabla\varphi(x))\,\mathrm{d}x \;, \;\forall\, A \in \mathbb{R}^{m\times d} \;, \;\forall\, \varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)\,.$$

Remark.

(1) In nonlinear elasticity quasiconvexity can be interpreted as follows. Among all deformations having affine boundary values given by A, energy minimization using the energy density f gives precisely the homogeneous, affine deformation determined by A.

(2) Convexity implies quasiconvexity:
Let $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ be convex. Hence Jensen‘s inequality

$$f\left(\frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x\right) \le \frac{1}{|\Omega|}\int_\Omega f\left(u(x)\right)\,\mathrm{d}x$$

is immediate for all step functions u. By density and continuity of f, Jensen‘s inequality extends to all $L^1\left(\Omega, \mathbb{R}^{m\times d}\right)$.
Set $u = A + \nabla\varphi$ for $A \in \mathbb{R}^{m\times d}$ and $\varphi \in C_0^\infty(\Omega, \mathbb{R}^m)$.

$$\Longrightarrow \qquad f\left(\frac{1}{|\Omega|}\int_\Omega (A + \nabla\varphi(x))\,\mathrm{d}x\right) \le \frac{1}{|\Omega|}\int_\Omega f\left(A + \nabla\varphi(x)\right)\,\mathrm{d}x$$

By Gauss' divergence theorem for scalar fields applied componentwise, the left hand side equals to $f(A)$. And since $C_0^\infty(\Omega, \mathbb{R}^m)$ is dense in $W_0^{1,\infty}(\Omega, \mathbb{R}^m)$, the implication is shown.

(3) Pedregal [36, Theorem 1.2] gives another proof of the basic Jensen's inequality used above that is based on the existence of a supporting hyperplane to the epigraph of a convex continuous function (subgradient).

(4) Later on we will see in subsection 2.3, that a generalized kind of convexity - $\mathcal{A}$ - convexity - is equivalent to the validity of a generalized form of Jensen's inequality that employs a special class of measures (Young measures).

Analogous to the scalar case we get similar weak lower semicontinuity results where convexity of f is replaced by quasiconvexity. From necessity and sufficiency results in [1], [12] and [31] on quasiconvexity we can extract the following equivalence.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $q \in [1, \infty]$ and $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ a continuous function bounded from below and above with

$$
\begin{aligned}
&0 \le f(x,u,A) \le C\,(1 + |u|^p + |A|^p) \quad \text{for some } C > 0\,, \text{ when } q < \infty \\
&|f(x,u,A)| \le g_1\,(x, |u|\,, |A|) \quad \text{for some } g_1 \in L^1(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+)\,, \\
&\text{when } q = \infty
\end{aligned}
$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^m$, $A = (a_{ij}) \in \mathbb{R}^{m\times d}$ with $|A|^2 = \sum_{i,j} a_{ij}^2$. Then the following equivalence holds

$$
\begin{aligned}
&f(x,u,\cdot) \quad \text{is quasiconvex for a.e. } x \in \Omega \quad \text{and all } u \in \mathbb{R}^m \quad \Longleftrightarrow \\
I(u) = &\int\limits_\Omega f(x,u,\nabla u)\,\mathrm{d}x \quad \text{is weakly lower semicontinuous in } W^{1,q}(\Omega, \mathbb{R}^m)\,.
\end{aligned}
$$

However, it is difficult to check the quasiconvexity of a function f by the definition above. Therefore one is interested in weakening resp. strengthening this notion to get at least necessary resp. sufficient conditions for weak lower semicontinuity of I which are easier to verify. With this motivation we have the weaker notion of

1.3.2. Rank-one-convexity

Definition.

$f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is said to be *rank-one-convex* if for all $\lambda \in [0,1]$ and for all $A, B \in \mathbb{R}^{m\times d}$ with $\operatorname{rank}(A - B) \le 1$ there holds

$$f(\lambda\, A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda)\, f(B)$$

and the stronger notion of

1.3.3. Polyconvexity

Definition. (Ball 1977).

For $A \in \mathbb{R}^{m\times d}$ let $T(A)$ denote the vector composed by A and all its quadratic minors. Then

$$T(A) \in \mathbb{R}^{\sigma(m,d)} \quad \text{with} \quad \sigma(m,d) = \sum_{l=1}^{\min(m,d)} \frac{m!\,d!}{(l!)^2\,(m-l)!\,(d-l)!}\,.$$

A function f is then called *polyconvex* if there exists a convex function $g\,:\,\mathbb{R}^{\sigma(m,d)} \to \mathbb{R}$ such that

$$f(A) \,=\, g(T(A))\,,\ \forall\, A \in \mathbb{R}^{m\times d}\,.$$

We see immediately: f is convex $\Rightarrow$ f is polyconvex.
(Set e.g. $g(T(A)) = f(A)\,,\ \forall\, A \in \mathbb{R}^{m\times d}$.)
As indicated above there is the following chain of implications for real-valued functions f between these notions.

$$f \text{ is convex} \implies f \text{ is polyconvex} \implies f \text{ is quasiconvex}$$
$$f \text{ is quasiconvex} \implies f \text{ is rank-one-convex}$$

These implications are not invertible in the vectorial case; see e.g. [3], [12], [39] for the second and third one. In the scalar case, the rank-one-condition in the definition of rank-one-convexity makes no restriction to classical convexity. Hence these notions coincide and we obtain an equivalence between the convexity-classes if $m = 1$ or $d = 1$.

1.4. Recent results in generalized convexity

The central notion of quasiconvexity is hard to handle and hence one of the main sources for recent investigations in the calculus of variations. J. Kristensen proves in [28], that for $m \geq 3\,,\ d \geq 2$ there is no local condition which is equivalent to quasiconvexity. This explains why quasiconvexity is difficult to check. To overcome this drawback to some extent, B. Dacorogna and J.-P. Haeberly [13] show how analytically and by some numerical computation bounds on parameters of a matrix function can be obtained, that insure quasi-convexity or convexity. D. Faraco and X. Zhong [15] provide a family of examples of quasiconvex functions, which contains the classic example due to V. Šverák. Another drawback of quasiconvexity is that following M. Kružík [29] quasiconvexity is not stable with transposition.

It is still an unsolved problem whether rank-one-convexity implies quasiconvexity for $m = 2\,,\ d \geq 2$. Thus it is resonable to deal with constraints of the space of matrices $\mathbb{R}^{2\times 2}$ which ensure an equivalence between rank-one-convexity and quasiconvexity. Such considerations can be found in [33] where an equivalence between these notions on diagonal matrices is shown. Another treatment on this subject is presented by N. Chaudhuri and S. Müller [8]. On the other hand S. Conti [11] deals with the convexity behaviour of extended real valued functions incorporating volumetric constraints; he proves that a quasiconvex function $W : \mathbb{M}^{d\times d} \to [0,\infty]$ which is finite on the set $\Sigma = \{F : \det F = 1\}$ is rank-one convex, and hence continuous, on Σ; and the same for constraints on minors.

1.5. Polyconvexity in nonlinear elasticity

The rather abstract definition of polyconvexity reduces to a more concrete one in nonlinear elasticity with $m = d = 3$. Hence $\sigma(3,3) = 19$ and

$$T(A) \,=\, (A, \text{adj}A, \det A) \in \mathbb{R}^{19} \quad \text{for} \quad A \in \mathbb{R}^{3\times 3}\,.$$

Thus $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is polyconvex if there exists a convex $g : \mathbb{R}^{19} \to \mathbb{R}$ with

$$f(A) \,=\, g\,(A, \text{adj}A, \det A)\ ,\ \forall\, A \in \mathbb{R}^{3\times 3}\,.$$

Some simple examples for polyconvex functions are

(a) $f(A) \,=\, \det A$, which is an example for a polyconvex function that is not convex,

(b) $f(A) \,=\, \langle\, A\,,\, B\,\rangle^2_{\mathbb{R}^{3\times 3}} := \big(\text{tr}(A\,B^T)\big)^2$ for a fixed $B \in \mathbb{R}^{3\times 3}$,

(c) $f(A) \,=\, h(\|A\|_{\mathbb{R}^{3\times 3}})$ for convex functions h on $\mathbb{R}^+$.

There is a wide range of material models which posses a polyconvex stored energy function. Well known and often used models especially with rubber and rubber-like materials are

$$\begin{aligned} \text{Neo-Hookean materials:} \quad f(A) &= a_1\,\|A\|^2_{\mathbb{R}^{3\times 3}} + h(\det A) \\ \text{Mooney-Rivlin materials:} \quad f(A) &= a_1\,\|A\|^2_{\mathbb{R}^{3\times 3}} + a_2\,\|\text{adj}\,A\|^2_{\mathbb{R}^{3\times 3}} + h(\det A) \end{aligned}$$

for some $a_1, a_2 > 0$ and a convex function $h : \mathbb{R} \to \mathbb{R}$. More polyconvex material models can be found in e.g. [35], [24].

On the other hand there are many material models which do not have a polyconvex energy such as the Simo/Ortiz - energy:

$$f(A) \,=\, \frac{1}{a_1}\,\|A\|^{a_1}_{\mathbb{R}^{3\times 3}} + \ln(\det\,A)^{a_2} - \ln(\det\,A) \quad \text{for}\ a_1, a_2 \geq 2\,.$$

The lack of polyconvexity is shown in e.g. [34]. So we have lost the pointwise sufficient condition for quasiconvexity. In fact it can be shown that the Simo/Ortiz-energy is not quasiconvex. This one and many other examples for material models in nonlinear elasticity motivate a further generalization of convexity.

2. Minimizing Non(quasi)convex Integrands

In many applications we have to deal with the minimization of energies with non(quasi)convex integrands. As the quasiconvexity of f is a necessary condition for the weak lower semicontinuity of the functional I, we need alternative methods which characterize the minimum state of I. An advantageous method in this direction, as we shall see below, is the Young measure approach.

In what follows, let $(u_j)_{j\in\mathbb{N}} \subset L^\infty(\Omega,\mathbb{R}^m)$ and $u_j \overset{*}{\rightharpoonup} u$. We consider the sequence $\left(f\,(u_j)\right)_{j\in\mathbb{N}}$ with $f \in C_0(\mathbb{R}^m)$, where

$$C_0(\mathbb{R}^m) := \left\{ f \in C(\mathbb{R}^m,\mathbb{R}) \, : \, \lim_{|z|\to\infty} f(z) \to 0 \right\}$$

denotes the space of continuous functions vanishing at infinity, which is endowed with the supremum norm $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{R}^m\}$.

Motivating Example. Examine the oscillating sequence $u_j : \mathbb{R} \to \mathbb{R}$ given by

$$u_j(x) \;=\; \mathbb{I}_{\{(k,k+1/3)\,,\,k\in\mathbb{Z}\}}(j\,x) = \begin{cases} 1 \;, & 0 < j\,x - \lfloor j\,x \rfloor < 1/3 \\ 0 \;, & 1/3 < j\,x - \lfloor j\,x \rfloor < 1 \end{cases} .$$

The weak-$*$-limit of (u_j) in $L^\infty(\mathbb{R})$ is the mean value $1/3$, since

$$\int_{\mathbb{R}} u_j(x)\,\alpha(x)\,\mathrm{d}x \;=\; \sum_{k\in\frac{1}{j}\mathbb{Z}} \int_{k/j}^{(k+1/3)/j} \alpha(x)\,\mathrm{d}x \;\underset{j\to\infty}{\longrightarrow}\; \int_{\mathbb{R}} \frac{1}{3}\,\alpha(x)\,\mathrm{d}x \;, \quad \forall\,\alpha \in L^1(\mathbb{R})\,.$$

Now look at the asymptotic behaviour of $f(u_j)$ for $f \in C_0(\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}} f(u_j(x))\,\alpha(x)\,\mathrm{d}x \;&=\; \sum_{k\in\frac{1}{j}\mathbb{Z}} \left(\int_{k/j}^{(k+1/3)/j} f(1)\,\alpha(x)\,\mathrm{d}x + \int_{(k+1/3)/j}^{(k+1)/j} f(0)\,\alpha(x)\,\mathrm{d}x \right) \\ &\underset{j\to\infty}{\longrightarrow}\; \int_{\mathbb{R}} \left(\frac{1}{3}\,f(1) + \frac{2}{3}\,f(0) \right) \alpha(x)\,\mathrm{d}x \;, \quad \forall\,\alpha \in L^1(\mathbb{R})\,. \end{aligned}$$

Consequently

$$f(u_j) \overset{*}{\rightharpoonup} \frac{1}{3}\,f(1) + \frac{2}{3}\,f(0)\,,$$

which is not equal to $f(1/3)$ for nonlinear f in general. ∎

2.1. Young measures generated by function sequences

Let us first recall some functional analysis and measure theory. With respect to the supremum norm, the space $C_0(\mathbb{R}^m)$ is a separable Banach space. The representation theorem of Riesz - Radon provides a unique correspondence between positive linear forms $F : C_0(\mathbb{R}^m) \to \mathbb{R}^+$ and positive Radon measures $\mu \in \mathcal{M}^+(\mathbb{R}^m)$ with finite mass ($\mu(\mathbb{R}^m) < \infty$):

$$F(f) \;=\; \int\limits_{\mathbb{R}^m} f(z)\,\mathrm{d}\mu(z) \;=\; \langle\, \mu\,,\, f\,\rangle\;\;, \quad \forall f \in C_0(\mathbb{R}^m)\;.$$

This relation extends to arbitrary signed measures $\mu \in \mathcal{M}(\mathbb{R}^m)$ [6]. So the dual of $C_0(\mathbb{R}^m)$ identifies with the space of Radon measures with finite mass:

$$(C_0(\mathbb{R}^m))^* \;\cong\; \mathcal{M}(\mathbb{R}^m)\,.$$

Fundamental property and existence theorem.

Let $(f\,(u_j))_{j\in\mathbb{N}} \subset L^1(\Omega)$. Then *Young measures* are parametrized measures $\nu_x \in (C_0(\mathbb{R}^m))^*$, $x \in \Omega$ depending measurably on x that satisfy the *fundamental property*

$$\lim_{j\to\infty} \int_\Omega f(u_j(x))\,\alpha(x)\,\mathrm{d}x \;=\; \int_\Omega \left(\int_{\mathbb{R}^m} f(z)\,\mathrm{d}\nu_x(z) \right)\,\alpha(x)\,\mathrm{d}x \tag{1}$$

for any $\alpha \in L^\infty(\Omega)$.

The subsequent theorem confirms the existence of Young measures as families of Radon measures. The Young measure is a representation of the weak limit in the case '$(f(u_j))_{j\in\mathbb{N}}$ converges weakly in $L^1(\Omega)$'. In applications to minimization problems, the convergence itself has to be shown separately, depending on the specific structure of the variational problem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be a measurable and bounded set and $(u_j)_{j\in\mathbb{N}} \subset L^q(\Omega,\mathbb{R}^m)$ for some $q \in [1,\infty]$. Moreover, let $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ be a Lebesgue measurable function, such that the sequence $(f(u_j))_{j\in\mathbb{N}}$ is weakly convergent in $L^1(\Omega)$. Then there exists a measurable family $(\nu_x)_{x\in\Omega} \subset (C_0(\mathbb{R}^m))^*$ (Young measure) which represents the weak limit of $(f(u_j))_{j\in\mathbb{N}}$ according to (1).*

Sketch of Proof. We treat the case $f \in C_0(\mathbb{R}^m)$ and detect the weak limit of $(f(u_j))_{j\in\mathbb{N}}$ as a family of Radon measures. Set

$$(\nu_x^j)_{j\in\mathbb{N}} = \Big(\delta_{u_j(x)}\Big)_{j\in\mathbb{N}} \subset \mathcal{M}(\mathbb{R}^m) \quad \text{for a.e.} \quad x \in \Omega$$

and obtain

$$\Big\langle\, \tilde{f}\,,\, \nu_x^j \,\Big\rangle = \int_{\mathbb{R}^m} \tilde{f}(z)\,\mathrm{d}\delta_{u_j(x)}(z) = \tilde{f}(u_j(x))\,, \ \forall\, \tilde{f} \in C_0(\mathbb{R}^m)\,. \tag{2}$$

For $\mu \in \mathcal{M}(\mathbb{R}^m)$ we use the dual norm $\|\mu\|_{\mathcal{M}(\mathbb{R}^m)} = \sup\limits_{\|\tilde{f}\|_\infty \leq 1} \Big|\Big\langle\, \tilde{f}\,,\, \mu \,\Big\rangle\Big|$ and conclude for a.e. $x \in \Omega$.

$$\begin{aligned}\sup_{j\in\mathbb{N}}\Big(\big\|\nu_x^j\big\|_{\mathcal{M}(\mathbb{R}^m)}\Big) &= \sup_{j\in\mathbb{N}}\left(\sup_{\|\tilde{f}\|_\infty \leq 1}\Big|\Big\langle\, \tilde{f}\,,\, \nu_x^j \,\Big\rangle\Big|\right)\\ &= \sup_{j\in\mathbb{N}}\left(\sup_{\|\tilde{f}\|_\infty \leq 1}\Big|\tilde{f}\,(u_j(x))\Big|\right) \leq 1\,.\end{aligned}$$

Hence $(\nu_x^j)_{j\in\mathbb{N}}$ is uniformly bounded in $\mathcal{M}(\mathbb{R}^m)$. The predual space $C_0(\mathbb{R}^m)$ of $\mathcal{M}(\mathbb{R}^m)$ is a separable Banach space. Thus by the Banach - Alaoglu theorem, there exists a subsequence of $(\nu_x^j)_{j\in\mathbb{N}}$ with the pointwise limit

$$\nu_x^j \overset{*}{\rightharpoonup} \nu_x \quad \text{in} \quad \mathcal{M}(\mathbb{R}^m) \quad \text{for a.e.} \quad x \in \Omega\,.$$

This means that

$$\Big\langle\, \tilde{f}\,,\, \nu_x^j \,\Big\rangle \underset{j\to\infty}{\longrightarrow} \Big\langle\, \tilde{f}\,,\, \nu_x \,\Big\rangle\,,\ \forall\, \tilde{f} \in C_0(\mathbb{R}^m) \quad \text{and a.e.} \quad x \in \Omega\,. \tag{3}$$

This pointwise convergence is understood in the following way. We consider $(\nu_x^j)_{j\in\mathbb{N}}$ as a sequence of weakly - $*$ - measurable mappings $x \mapsto \nu_x^j$ i.e. as a sequence in $L^\infty(\Omega, \mathcal{M}(\mathbb{R}^m))$, that is uniformly bounded with respect to the norm $\operatorname*{ess\,sup}\limits_{x\in\Omega}\Big\|\nu_x^j\Big\|_{\mathcal{M}(\mathbb{R}^m)}$. The separability of the predual space $L^1(\Omega, C_0(\mathbb{R}^m))$ yields the weak - $*$ - convergence of $(\nu_x^j)_{j\in\mathbb{N}}$ (see [19],[36]).

Note since Ω is bounded, there is an integrable majorant $g_{\tilde{f}} \in L^1(\Omega)$ given by $g_{\tilde{f}}(x) := \sup\limits_{j\in\mathbb{N}}\Big|\Big\langle\, \tilde{f}\,,\, \nu_x^j \,\Big\rangle\Big| \leq \|\tilde{f}\|$. By Lebesgue's convergence theorem we can integrate the pointwise limit in (3) and obtain

$$\lim_{j\to\infty}\int\limits_\Omega \Big\langle\, \tilde{f}\,,\, \nu_x^j \,\Big\rangle\,\mathrm{d}x = \int\limits_\Omega \lim_{j\to\infty}\Big\langle\, \tilde{f}\,,\, \nu_x^j \,\Big\rangle\,\mathrm{d}x = \int\limits_\Omega \Big\langle\, \tilde{f}\,,\, \nu_x \,\Big\rangle\,\mathrm{d}x$$

where $\left\langle \tilde{f}\,,\,\nu_x \right\rangle$ is Lebesgue measurable in x.

Finally we choose $\tilde{f} = f$ of the theorem and multiply relation (3) with an arbitrary $\alpha \in L^\infty(\Omega)$. The product $\left(\left|\left\langle f\,,\,\nu_x^j \right\rangle \alpha(x)\right|\right)$ is still dominated by $(g_f\;\|\alpha\|_\infty) \in L^1(\Omega)$. So we can integrate over Ω and arrive at

$$\lim_{j\to\infty} \int_\Omega f(u_j(x))\,\alpha(x)\,\mathrm{d}x \;=\; \int_\Omega (\langle\, f\,,\,\nu_x\,\rangle)\;\alpha(x)\,\mathrm{d}x\,, \quad \forall\,\alpha \in L^\infty(\Omega)\,,$$

the desired fundamental property for $(f\,(u_j))_{j\in\mathbb{N}}\,,\; f \in C_0(\mathbb{R}^m)$.

Now, the extension to a Lebesgue measurable $f\;:\;\mathbb{R}^m \;\to\; \bar{\mathbb{R}}$ can be done with an approximation by an auxiliary sequence $(f_k)_{k\in\mathbb{N}} \subset C_0(\mathbb{R}^m)$ which has the same asymptotic properties as the weak convergent $(f(u_j))_{j\in\mathbb{N}}$. For a detailed construction we refer to [36], [19]. ■

Remark. In view of the imbedding $L^p(\Omega) \subset L^1(\Omega)$ for $p \in [1,\infty]$ with Ω bounded, we can replace the convergence assumption in Theorem 1 by '$(f(u_j))_{j\in\mathbb{N}}$ converges weakly in $L^p(\Omega)$'.

2.2. $\mathcal{A}$-convexity and $\mathcal{A}$-convex relaxations

Functions competing in variational problems belong to appropriate Banach spaces and -in addition- they have to satisfy special constraints. To absorb such constraints for infimizing sequences, we consider a special class $\mathcal{A}$ of sequences in $L^q(\Omega,\mathbb{R}^m)$ for some fixed $q \in [1,\infty]$. This gives rise to the $\mathcal{A}$-convex relaxation of an arbitrary Lebesgue measurable function $f\,:\,\mathbb{R}^m \;\to\; \bar{\mathbb{R}}$ defined by

$$f^{\mathcal{A}}(z) \;:=\; \inf_{(u_j)\in\mathcal{A}} \left\{ \liminf_{j\to\infty} \frac{1}{|\Omega|} \int_\Omega f(u_j(x))\,\mathrm{d}x\,;\; u_j \rightharpoonup z \text{ in } \mathcal{A} \right\};\; z \in \mathbb{R}^m\,. \tag{4}$$

Here $u_j \rightharpoonup z$ means weak convergence to the constant function $z(x) = z$ and the convergence "$u_j \rightharpoonup u \text{in} \mathcal{A}$" depends on the choice of $\mathcal{A}$ (e.g. $u_j \rightharpoonup u$ in $W^{1,q}(\Omega,\mathbb{R}^m)$). As we require $\mathcal{A} \subset (L^q(\Omega,\mathbb{R}^m))^{\mathbb{N}}$, the "$\mathcal{A}$-convergence" should at least imply $u_j \rightharpoonup u$ in $L^q(\Omega, R^m)$.

By using the Young measure notation we can simplify the definition of $f^{\mathcal{A}}$. To this end we introduce for some $p \in [1,\infty]$ the set

$$C_p^{\mathcal{A}} \;=\; \left\{ f\;:\;\mathbb{R}^m \;\to\; \bar{\mathbb{R}}\,;\;\; (u_j) \in \mathcal{A} \;\;\Rightarrow\;\; (f(u_j))_{j\in\mathbb{N}} \text{ converges weakly in } L^p(\Omega) \right\}\,.$$

By Theorem 1 the forthcoming notions are well defined. We adopt the set $\mathcal{A}^*$ of homogeneus Young measures ($\nu_x \;=\; \nu$ for almost every $x \in \Omega$) generated by sequences in $\mathcal{A}$, i.e.

$$\mathcal{A}^* = \left\{ \nu \in (C_0(\mathbb{R}^m))^*\;;\; \langle\, f\,,\,\nu\,\rangle = \lim_{j\to\infty} \frac{1}{|\Omega|} \int_\Omega f(u_j(x))\,\mathrm{d}x\,, (u_j) \in \mathcal{A}\,, f \in C_p^{\mathcal{A}} \right\}$$

Thus we obtain

$$f^{\mathcal{A}}(z) = \inf_{\nu \in \mathcal{A}^*} \{ \langle f , \nu \rangle \; ; \; \langle id , \nu \rangle = z \} \, . \tag{5}$$

Finally we call a Lebesgue measurable function $f : \mathbb{R}^m \to \mathbb{R}$ $\mathcal{A}$ - *convex*, if it is equal to its $\mathcal{A}$ - convex relaxation $f^{\mathcal{A}}$.

Example 1. Consider

$$\mathcal{A} = \left\{ (\nabla u_j)_{j\in\mathbb{N}} : \; (u_j)_{j\in\mathbb{N}} \subset W^{1,q}(\Omega, \mathbb{R}^m) \, , \; \sup_{j\in\mathbb{N}} \|u_j\|_{W^{1,q}(\Omega,\mathbb{R}^m)} < \infty \right\} .$$

Then by (4) and since weak lower semicontinuity implies quasiconvexity, we have that any $\mathcal{A}$ - convex function is simply quasiconvex.

Example 2. We compute a $\mathcal{A}$ - convex relaxation of a non - $\mathcal{A}$ - convex function φ explicitly. We set

$$\Omega = (0,1) \quad \text{and} \quad \mathcal{A} = \left\{ (u_j)_{j\in\mathbb{N}} \subset L^2((0,1)) \, ; \; \sup_{j\in\mathbb{N}} \|u_j\|_{L^2((0,1))} < \infty \right\} .$$

Consider $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(z) = z^4 - z^2 + \frac{1}{4}$.

(1) This function is not $\mathcal{A}$ - convex since $\varphi(0) = \frac{1}{4}$ and on the other hand we have

$$\varphi^{\mathcal{A}}(0) \le \lim_{j\to\infty} \int_0^1 (u_j^4 - u_j^2 + \frac{1}{4}) \, \mathrm{d}x \quad \text{whenever} \quad u_j \rightharpoonup 0 \text{ in } L^2((0,1)) \, .$$

Choose $u_j(x) = \sin(j\,x)$ which converges weakly to zero in $L^2((0,1))$ and

$$\lim_{j\to\infty} \int_0^1 \left((\sin(j\,x))^4 - (\sin(j\,x))^2 + \frac{1}{4} \right) \mathrm{d}x = \lim_{j\to\infty} \frac{4\,j + \sin(4\,j)}{32\,j} = \frac{1}{8} \, ,$$

hence indeed $\varphi^{\mathcal{A}}(0) < \varphi(0)$.

(2) Now we compute the $\mathcal{A}$- convex relaxation $\varphi^{\mathcal{A}}$. At first we observe $\varphi^{\mathcal{A}}(z) \ge 0$ $\forall\, z \in \mathbb{R}$ because of

$$\int_0^1 \varphi(u_j(x)) \, \mathrm{d}x = \int_0^1 \left(u_j^2(x) - \frac{1}{2} \right)^2 \mathrm{d}x \ge 0 \;, \forall\, (u_j) \in \mathcal{A} \, .$$

Next we let $z \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right]$ and set for $\lambda \in [0,1]$,

$$u_j^{(\lambda)}(x) = \frac{\sqrt{2}}{2} \, \mathbb{I}_{\{(k,k+\lambda)\,,\, k\in\mathbb{Z}\}}(j\,x) - \frac{\sqrt{2}}{2} \, \mathbb{I}_{\{(k+\lambda,k+1)\,,\, k\in\mathbb{Z}\}}(j\,x) \, .$$

In the fashion of the motivating example we see

$$u_j^{(\lambda)} \underset{j\to\infty}{\rightharpoonup} \lambda \frac{\sqrt{2}}{2} - (1-\lambda)\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(2\lambda - 1) =: z$$

in $L^2((0,1))$. In this way we obtain an oscillating sequence weakly converging to $z \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ for $\lambda \in [0,1]$. Plugging this sequence in $\varphi(z) = z^4 - z^2 + \frac{1}{4}$ we get

$$\varphi\left(u_j^{(\lambda)}(x)\right) = 0 \;\; \forall\, \lambda \in [0,1] \text{ and a.e. } x \in [0,1]$$

and hence $\varphi^{\mathcal{A}} = 0$ on $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$. Now define

$$\tilde{\varphi}(z) = \begin{cases} 0\ , & z \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \\ \varphi(z)\ , & z \in \mathbb{R} \setminus \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \end{cases},$$

which is convex in the classical sense. We claim $\varphi^{\mathcal{A}} \equiv \tilde{\varphi}$. The inequality $\varphi^{\mathcal{A}} \leq \tilde{\varphi}$ is clear by definition. The reverse one follows from Jensen's inequality for $\tilde{\varphi}$,

$$\int_0^1 \varphi(u_j(x))\,\mathrm{d}x \geq \int_0^1 \tilde{\varphi}(u_j(x))\,\mathrm{d}x \geq \tilde{\varphi}\left(\int_0^1 u_j(x)\,\mathrm{d}x\right).$$

Since this inequality holds for arbitrary sequences $(u_j) \in \mathcal{A}$ that converge weakly to $z \in \mathbb{R}$, by definition of $\varphi^{\mathcal{A}}$ we arrive at $\varphi^{\mathcal{A}} \geq \tilde{\varphi}$ and the claim is proved. ∎

Remark. The example above illustrates the general fact that with

$$\mathcal{A} = \left\{ (u_j)_{j\in\mathbb{N}} \subset L^p(\Omega)\ ;\ \sup_{j\in\mathbb{N}} \|u_j\|_{L^p(\Omega)} < \infty \right\},$$

the $\mathcal{A}$-convex relaxation $\varphi^{\mathcal{A}}$ and the standard convexification φ^{**} by means of the Legendre-Fenchel transform $\varphi \mapsto \varphi^*$ coincide, see e.g. [36].

2.3. Jensen's inequality for Young measures

The representation formula (5) of the $\mathcal{A}$-convex relaxation provides the following equivalent characterization of $\mathcal{A}$-convex functions:

$$f^{\mathcal{A}} = f \quad \Longleftrightarrow \quad f(z) \leq \langle\, f\,,\,\nu\,\rangle \;\; \forall\,\nu \in \mathcal{A}^* \text{ with } \int_{\mathbb{R}^m} \tilde{z}\,\mathrm{d}\nu(\tilde{z}) = z \tag{6}$$

Here the right hand side is an abstract version of Jensen's inequality for general $f \in C_p^{\mathcal{A}}$ using the Young measure device. As shown below, this abstract version contains the classical Jensen's inequality as a special case.

In this connection let us discuss the function class $C_p^{\mathcal{A}}$. The formulas (5) and (6) make sense only with the requirement $f \in C_p^{\mathcal{A}}$ such that Theorem 1 applies to guarantee the existence of a representing Young measure. For

$$\mathcal{A} = \left\{ (u_j)_{j\in\mathbb{N}} \subset L^q(\Omega, \mathbb{R}^m) \; ; \; \sup_{j\in\mathbb{N}} \|u_j\|_{L^q(\Omega,\mathbb{R}^m)} < \infty \right\}$$

a growth condition on f determines an explicit subset of $C_{q/p}^{\mathcal{A}}$. Indeed with $0 < p < q$, any Lebesgue measurable function $f : \mathbb{R}^m \to \mathbb{R}$ that satisfies $|f(z)| \leq C\,(1+|z|^p)$ for some $C > 0$ and all $z \in \mathbb{R}^m$ admits a weakly convergent sequence $(f(u_j))_{j\in\mathbb{N}}$ in $L^{q/p}(\Omega)$. Other useful subsets of $C_p^{\mathcal{A}}$ may be determined with an explicitly given variational functional I and with associated infimizing sequences $(u_j) \in \mathcal{A}$.

To derive from (6) the classical Jensen's inequality, we take

$$\mathcal{A} = \left\{ (u_j)_{j\in\mathbb{N}} \subset L^1(\Omega, \mathbb{R}^m) \; ; \; \sup_{j\in\mathbb{N}} \|u_j\|_{L^1(\Omega,\mathbb{R}^m)} < \infty \right\} .$$

Then as already remarked above, $\mathcal{A}$-convexity coincides with classical convexity. Thus (6) reads

$$f \text{ is convex} \quad \Longleftrightarrow \quad f\left(\lim_{j\to\infty} \frac{1}{|\Omega|} \int_\Omega u_j(x)\,\mathrm{d}x\right) \leq \liminf_{j\to\infty} \frac{1}{|\Omega|} \int_\Omega f(u_j(x))\,\mathrm{d}x$$

for $u_j \rightharpoonup z \in \mathbb{R}^m$ in $L^1(\Omega, \mathbb{R}^m)$. Now we choose constant sequences $u_j = u \in L^1(\Omega, \mathbb{R}^m)$ with $\frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x = z$ and obtain the classical Jensen's inequality,

$$f\left(\frac{1}{|\Omega|} \int_\Omega u(x)\,\mathrm{d}x\right) \leq \frac{1}{|\Omega|} \int_\Omega f(u(x))\,\mathrm{d}x \,, \;\; \forall\, u \in L^1(\Omega, \mathbb{R}^m)\,.$$

3. Contact Problem in Nonlinear Elasticity

To set up the contact problem we need additional notation. For the given open and bounded reference configuration $\Omega \subset \mathbb{R}^d$ of the elastic body, the Lipschitz boundary $\Gamma = \partial\Omega$ decomposes as $\Gamma = \overline{\Gamma}_U \cup \overline{\Gamma}_\tau \cup \overline{\Gamma}_C$. Γ_U is the Dirichlet boundary part with prescribed deformations $U \in L^q(\Gamma, \mathbb{R}^d)$, $q \in (1, \infty)$, Γ_τ is the boundary part with prescribed tractions $\tau \in L^{q'}(\Gamma, \mathbb{R}^d)$ with the dual exponent $q' = \frac{q}{q-1}$, and Γ_C is the contact part of the boundary. The set of admissible deformation fields is described by a weakly closed subset K of $V = \{v \in W^{1,q}(\Omega, \mathbb{R}^d) : v|_{\Gamma_U} = U\}$. K can be given by a box constraint

$$u(x) \in Q \text{ for } x \in E$$

for given closed Q containing $cl(\Omega)$ with $E = \Gamma_C$ as with [10] or with [5] more generally with a set E of positive capacity such that $E \subset cl(\Omega)$. A more concrete instance is Signorini's boundary condition. For that let $\tilde{g} \in L^q(\Gamma_C)$ be the initial gap function between the elastic body and the rigid support as a unilateral constraint for the displacement field $\tilde{u} = u - id$. Then with the normal component $v_N = \langle v, N\rangle_{\mathbb{R}^d}$, we are led to $K = \{v \in V : v_N|_{\Gamma_C} \leq g\}$, where $g = \tilde{g} + \langle id, N\rangle_{\mathbb{R}^d}$.

Finally we have volume forces $h \in L^{q'}(\Omega, \mathbb{R}^d)$. Thus we obtain the external work

$$F(\tilde{v}) = \int_\Omega h\, \tilde{v}\, \mathrm{d}x + \int_{\Gamma_\tau} \tau\, \tilde{v}\, \mathrm{d}S$$

done by the loads h and τ. As before a nonnegative stored energy (density) function $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ defines the stored energy

$$\tilde{I}(v) = \int_\Omega f(\nabla v)\, \mathrm{d}x$$

such that the total elastic energy

$$I(v) = \tilde{I}(v) - F(v)\,.$$

In these terms the contact problem reads:

$$\text{Find a minimizer } u \in K \text{ of } I \text{ on } K! \tag{7}$$

Since we have to verify the closedness of K in $W^{1,q}(\Omega, \mathbb{R}^d)$ from Signorini's boundary condition, it is necessary to understand the relation $v_N|_{\Gamma_C} \leq g$ in $L^q(\Gamma)$. To this end we introduce a refined outer measure on $\mathbb{R}^d$ - the capacity

$$C_d^{\alpha,q}(B) = \inf\left\{\|\varphi\|^q_{W^{\alpha,q}(\mathbb{R}^d)} : \varphi \in C_0^\infty(\mathbb{R}^d)\,,\ \varphi \geq 1 \text{ on } B\right\}$$

where $B \subset \overline{\Omega}$ is compact. Capacities are useful for distinguishing functions in Sobolev spaces $W^{\alpha,q}$. They are studied in e.g. [2] in great detail.

3.4. Existence results for contact problems

3.4.1. The case $|\Gamma_U| > 0$

With the notations above we formulate the following

Theorem 2. *Let $f : \mathbb{R}^{d\times d} \to \bar{\mathbb{R}}$ in (7) be quasiconvex and satisfy the growth condition*

$$c_1 \, \|A\|^q_{\mathbb{R}^{d\times d}} \le f(A) \le c_2 + c_3 \, \|A\|^q_{\mathbb{R}^{d\times d}}$$

for some constants $c_1, c_2, c_3 > 0$. Moreover, let $C^{0,q}_{d-1}(\overline{\Gamma}_U) > 0$. Then there exists a solution $u \in K$ to problem (7).

Proof. In order to obtain the existence of a minimizer of (7), we use the Weierstrass principle.

It is clear that $W^{1,q}(\Omega, \mathbb{R}^d)$, $q \in (1,\infty)$ is a reflexive Banach space and $K = \left\{v \in W^{1,q}(\Omega, \mathbb{R}^d) \,:\, v|_{\Gamma_U} = U \,,\, v_N|_{\Gamma_C} \le g\right\}$ a convex subset.

(a) *Closedness of K in $W^{1,q}(\Omega, \mathbb{R}^d)$*

Let $u_n \to u$ in $W^{1,q}(\Omega, \mathbb{R}^d)$ and $(u_n)_{n\in\mathbb{N}} \subset K$. By continuity of the trace mapping we have $u|_{\Gamma_U} = U$ in $L^q(\Gamma_U, \mathbb{R}^d)$. It remains to show $u_N|_{\Gamma_C} \le g$ in $L^q(\Gamma_C)$.

We have $(u_n)_{n\in\mathbb{N}} \subset K$ and hence $(u_n)_N|_{\Gamma_C} - g \le 0$.

$$\begin{aligned} &\Longrightarrow \quad (u_n)_N|_{\Gamma_C} - u_N|_{\Gamma_C} + u_N|_{\Gamma_C} - g \,\le\, 0 \\ &\Longrightarrow \quad u_N|_{\Gamma_C} - g \,\le\, |(u_n)_N|_{\Gamma_C} - u_N|_{\Gamma_C}| \end{aligned} \tag{8}$$

By the continuity of the trace mapping we have for some $c > 0$ that

$$\|v|_\Gamma\|_{L^q(\Gamma)} \,\le\, c\, \|v\|_{W^{1,q}(\Omega)} \tag{9}$$

holds for $v \in W^{1,q}(\Omega)$. With

$$C^{0,q}_{d-1}(D) \,=\, \inf\left\{\|\varphi|_\Gamma\|^q_{L^q(\Gamma)} \,:\, \varphi \in C^\infty_0(\mathbb{R}^d)\,,\, \varphi \ge 1 \text{ on } D\right\},$$

for compact $D \subset \overline{\Gamma}_C$, we obtain with (9)

$$C^{0,q}_{d-1}(\overline{\Gamma}_C) \,\le\, c\, C^{1,q}_d(\overline{\Omega})\,. \tag{10}$$

Further since $u_n \longrightarrow u$ in $W^{1,q}(\Omega, \mathbb{R}^d)$,

$$\begin{aligned} & |(u_n)_N - u_N| \underset{n\to\infty}{\longrightarrow} 0 \quad C^{1,q}_d - \text{a.e.} \\ \overset{(10)}{\Longrightarrow} \quad & |(u_n)_N|_{\Gamma_C} - u_N|_{\Gamma_C}| \underset{n\to\infty}{\longrightarrow} 0 \quad C^{0,q}_{d-1} - \text{a.e.} \\ \overset{(8)}{\Longrightarrow} \quad & u_N|_{\Gamma_C} - g \,\le\, 0 \quad C^{0,q}_{d-1} - \text{a.e.} \end{aligned}$$

which is the meaning of $u_N|_{\Gamma_C} \le g$ in $L^q(\Gamma_C)$.

(b) *Coercivity of I in $K \subset W^{1,q}(\Omega, \mathbb{R}^d)$*

We show $I(u) \geq \varphi\left(\|u\|_{W^{1,q}(\Omega,\mathbb{R}^d)}\right)$, $\forall\, u \in K$ for some appropriate continuous $\varphi : \mathbb{R}^+ \to \mathbb{R}$ with $\lim\limits_{s\to\infty} \varphi(s) = +\infty$.

By the growth condition on f we obtain

$$\begin{aligned}
&I(u) \\
&= \int\limits_\Omega f(\nabla u)\,\mathrm{d}x - F(u) \\
&\geq c_1\, \|\nabla u\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} - \int\limits_\Omega h\,u\,\mathrm{d}x - \int\limits_{\Gamma_\tau} \tau\,u\,\mathrm{d}S \\
&\geq c_1\, \|\nabla u\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} - \|h\|_{L^{q'}(\Omega,\mathbb{R}^d)}\, \|u\|_{L^q(\Omega,\mathbb{R}^d)} - \|\tau\|_{L^{q'}(\Gamma,\mathbb{R}^d)}\, \|u\|_{L^q(\Gamma,\mathbb{R}^d)} \\
&\geq c_1\, \|\nabla u\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} - \underbrace{c\left(\|h\|_{L^{q'}(\Omega,\mathbb{R}^d)} + \|\tau\|_{L^{q'}(\Gamma,\mathbb{R}^d)}\right)}_{\tilde{c}} \|u\|_{W^{1,q}(\Omega,\mathbb{R}^d)}
\end{aligned}$$

Since $C^{0,q}_{d-1}(\overline{\Gamma}_U) > 0$, we have an equivalence between the norms $\|\nabla u\|_{L^q(\Omega,\mathbb{R}^{d\times d})}$ and $\|u\|_{W^{1,q}(\Omega,\mathbb{R}^d)}$ on K. Therefore

$$I(u) \geq c_1\, \|u\|^q_{W^{1,q}(\Omega,\mathbb{R}^d)} - \tilde{c}\, \|u\|_{W^{1,q}(\Omega,\mathbb{R}^d)} .$$

With $\varphi(s) = c_1\, s^q - \tilde{c}\, s$ we obtain the coercivity of I.

(c) *Weak lower semicontinuity of I in $W^{1,q}(\Omega, \mathbb{R}^d)$*

With $I(u) = \tilde{I}(u) - F(u) = \int\limits_\Omega f(\nabla u)\,\mathrm{d}x - \int\limits_\Omega h\,u\,\mathrm{d}x - \int\limits_{\Gamma_\tau} \tau\,u\,\mathrm{d}S$ we have a decomposition into a stored energy and an external energy part. We observe that F is a continuous linear form with respect to the weak topology of $W^{1,q}(\Omega, \mathbb{R}^d)$. Consequently, it suffices to show weak lower semicontinuity of $\tilde{I}$. The latter one is guaranteed by the quasiconvexity and the growth condition of f. Hence all conditions of the Weierstrass principle are fulfilled and we obtain the existence of a minimizer of (7). ■

3.4.2. The case $\Gamma_U = \emptyset$

In this section we admit $\Gamma_U = \emptyset$. Thus the norms $\|\nabla u\|_{L^q(\Omega,\mathbb{R}^{d\times d})}$ and $\|u\|_{W^{1,q}(\Omega,\mathbb{R}^d)}$ are not equivalent on K anymore.

Necessary condition. Let us formulate a necessary condition for the existence of a minimizer $u \in K$ of (7). If $\Gamma_U = \emptyset$ the structure of K - as we shall see - allows to add a constant $b \in B \subset \mathbb{R}^d$ to $u \in K$ such that $u + b \in K$.

Firstly with a constant function b, clearly $u + b \in W^{1,q}(\Omega, \mathbb{R}^d)$. Secondly we can choose $B \subset \mathbb{R}^d$ with $\langle b , N \rangle_{\mathbb{R}^d} \leq 0$, $\forall b \in B$ where N denotes the outer normal on Γ_C. This implies $\langle u + b , N \rangle_{\mathbb{R}^d} |_{\Gamma_C} \leq g$. Hence, for

$$B = \Big\{ b \in \mathbb{R}^d \, : \, \langle b , N \rangle_{\mathbb{R}^d} \leq 0 \, , \, N \text{ is the outer normal on } \Gamma_C \Big\}$$

we have $u \in K \ \Rightarrow \ u + b \in K \quad \forall b \in B$. Suppose now that $u \in K$ minimizes I, i.e.

$$I(u) \leq I(u+b) = \int_\Omega f(\nabla u)\, \mathrm{d}x - F(u) - F(b) = I(u) - F(b)$$

$$\Longrightarrow \quad F(b) = \left\langle \int_\Omega h \, \mathrm{d}x + \int_{\Gamma_\tau} \tau \, \mathrm{d}S \, , \, b \right\rangle_{\mathbb{R}^d} \leq 0 \quad \forall b \in B \, . \tag{11}$$

Directions of escape and sufficiency We slightly strengthen the necessary condition (11) to $F(b) < 0 \ \forall b \in B$. With a more explicit indication of B as the set of ‘average directions of escape’ due to G. Fichera and Ciarlet / Nečas [10], [16] this will provide a sufficent condition for the existence of a minimizer.

Theorem 3. *Let $f : \mathbb{R}^{d\times d} \to \bar{\mathbb{R}}$ in (7) be quasiconvex and satisfy the growth condition*

$$c_1 \, \|A\|^q_{\mathbb{R}^{d\times d}} \leq f(A) \leq c_2 + c_3 \, \|A\|^q_{\mathbb{R}^{d\times d}}$$

for some constants $c_1, c_2, c_3 > 0$. Moreover, let $C^{0,q}_{d-1}(\overline{\Gamma}_C) > 0$ and suppose $F(b) < 0 \, , \ \forall b \in \tilde{B}$, where

$$\tilde{B} = \{ b \in \mathbb{R}^d \, : \quad b = \lim_{j\to\infty} b_j / \|b_j\|_{\mathbb{R}^d} \, , \, b_j := \tfrac{1}{|\Gamma_C|} \textstyle\int_{\Gamma_C} v_j \, dS \, ,$$
$$\|b_j\|_{\mathbb{R}^d} \underset{j\to\infty}{\longrightarrow} \infty \, , (v_j)_{j\in\mathbb{N}} \subset K \} \, .$$

Then there exists a solution $u \in K$ to problem (7).

Remark. The condition

$$F(b) = \left\langle \int_\Omega h \, \mathrm{d}x + \int_{\Gamma_\tau} \tau \, \mathrm{d}S \, , \, b \right\rangle_{\mathbb{R}^d} < 0 \quad \forall b \in \tilde{B}$$

is also mechanically reasonable. In order to get the existence of an equilibrium state, the applied forces h and τ have to form an obtuse angle with the possible directions of escape $b \in \tilde{B}$.

Proof. The proof is analogue to the one of theorem 2. The delicate point is the coercivity of I. Equivalently, we will show

$$(u_j)_{j\in\mathbb{N}} \subset K\,, \ \lim_{j\to\infty} \|u_j\|_{W^{1,q}(\Omega,\mathbb{R}^3)} = \infty \\ \Longrightarrow \ a = \lim_{j\to\infty} \frac{I(u_j)}{\|u_j\|_{W^{1,q}(\Omega,\mathbb{R}^3)}} > 0\,. \tag{12}$$

Just as the coercivity of I, this implication proves the weak convergence in $W^{1,q}(\Omega,\mathbb{R}^d)$ of infimizing sequences of I.

Indeed, assume $(u_j)_{j\in\mathbb{N}} \subset K$ is infimizing and $\lim\limits_{j\to\infty} \|u_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)} = \infty$. Then (12) implies $\lim\limits_{j\to\infty} I(u_j) = \infty$ and thus a contradiction to the infimizing property of $(u_j)_{j\in\mathbb{N}}$.

Thus it suffices to show (12) and to argue by contradiction. So let sequences $(v_j)_{j\in\mathbb{N}} \subset K$ and $(a_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ exist with

$$\lim_{j\to\infty} \|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)} = \infty\,, \quad I(v_j) = a_j\, \|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)} \quad \text{and} \quad \lim_{j\to\infty} a_j \le 0\,.$$

The growthness condition on f yields

$$I(u) \ \ge\ c_1\, \|\nabla u\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} - F(u)$$

and thus

$$a_j\, \|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)} \ \ge\ c_1\, \|\nabla v_j\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} - F(v_j)\,.$$

Now we introduce the normed sequence $\tilde v_j := v_j / \|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)}$ and obtain

$$a_j \ \ge\ c_1\, \|v_j\|^{q-1}_{W^{1,q}(\Omega,\mathbb{R}^d)}\, \|\nabla \tilde v_j\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} - F(\tilde v_j)\,. \tag{13}$$

The conditions $\lim\limits_{j\to\infty} a_j \le 0$, $\lim\limits_{j\to\infty} \|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)} = \infty$, $q > 1$ and $|F(\tilde v_j)| \le C\, \|\tilde v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)} = C < \infty$ imply

$$\lim_{j\to\infty} \|\nabla \tilde v_j\|^q_{L^q(\Omega,\mathbb{R}^{3\times 3})} = 0\,. \tag{14}$$

As $C^{0,q}_{d-1}(\overline{\Gamma}_C) > 0$, the generalized Poincaré inequality provides

$$\|w\|^q_{W^{1,q}(\Omega,\mathbb{R}^d)} \ \le\ C \left(\|\nabla w\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} + \left| \int_{\Gamma_C} w \,\mathrm{d}S \right|^q \right) \quad \forall\, w \in W^{1,q}(\Omega,\mathbb{R}^d)\,.$$

We apply this inequality to $w = (v_j - b_j) / \|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)}$ where $(b_j)_{j\in\mathbb{N}} \subset \mathbb{R}^d$ is defined with $b_j := \frac{1}{|\Gamma_C|} \int_{\Gamma_C} v_j \, \mathrm{d}S$ and get

$$\begin{aligned}\frac{\|v_j - b_j\|^q_{W^{1,q}(\Omega,\mathbb{R}^d)}}{\|v_j\|^q_{W^{1,q}(\Omega,\mathbb{R}^d)}} &\le C \left\| \left(\frac{\nabla(v_j - b_j)}{\|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^3)}} \right) \right\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} \\ &= C \, \|\nabla \tilde{v}_j\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} \, .\end{aligned}$$

Hence (14) implies

$$\lim_{j\to\infty} \frac{\|v_j - b_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)}}{\|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)}} = 0 \, . \tag{15}$$

We have

$$\lim_{j\to\infty} \|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)} = \infty \text{ and } \left| \frac{\|b_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)}}{\|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)}} - 1 \right| \le \frac{\|v_j - b_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)}}{\|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)}} \tag{16}$$

and so $\lim\limits_{j\to\infty} \|b_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)} = \infty$.
$(b_j)_{j\in\mathbb{N}} \subset \mathbb{R}^d$ is a constant sequence and so we have $\|b_j\|_{\mathbb{R}^d} \underset{j\to\infty}{\longrightarrow} \infty$. Hence this sequence defines a $b \in \tilde{B}$ by $b := \lim\limits_{j\to\infty} b_j / \|b_j\|_{\mathbb{R}^d}$.
Observe now that by (16) the sequence $(c_j)_{j\in\mathbb{N}} \subset \mathbb{R}^d$, $c_j := b_j / \|v_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)}$ is uniformly bounded in $W^{1,q}(\Omega,\mathbb{R}^d)$. So there exists a subsequence, again denoted with $(c_j)_{j\in\mathbb{N}}$, such that

$$\lim_{j\to\infty} c_j = b \quad \text{in} \quad W^{1,q}(\Omega,\mathbb{R}^d) \, . \tag{17}$$

From (15) we know $\|\tilde{v}_j - c_j\|_{W^{1,q}(\Omega,\mathbb{R}^d)} \underset{j\to\infty}{\longrightarrow} 0$. Together with (17) this provides

$$\lim_{j\to\infty} \tilde{v}_j = b \quad \text{in} \quad W^{1,q}(\Omega,\mathbb{R}^d) \, .$$

Testing now the assumption $F(b) < 0 \; (\forall \, b \in \tilde{B})$ we obtain by continuity of F and by (13),

$$\begin{aligned} F(b) &= \lim_{j\to\infty} F(\tilde{v}_j) \\ &\ge \lim_{j\to\infty} \left(c_1 \, \|v_j\|^{q-1}_{W^{1,q}(\Omega,\mathbb{R}^d)} \, \|\nabla \tilde{v}_j\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})} - a_j \right) \ge 0 \end{aligned}$$

a contradiction. ■

3.5. Lack of quasiconvexity and Young measure representation of the infimizing state

The coercivity of I is essential for weak convergence of an infimizing sequence $(u_j)_{j\in\mathbb{N}}$ in $W^{1,q}(\Omega,\mathbb{R}^d)$. Often it is also the decisive reason for the choice of a suitable function space. Weak lower semicontinuity is assured by the quasiconvexity assumption on f then.

Nevertheless there are material models - such as the mentioned Simo-Ortiz energy - which are not even quasiconvex. In this case we have no weak lower semicontinuity of I in $W^{1,q}(\Omega,\mathbb{R}^d)$. To describe the infimizing behaviour of I, we can use the Young measure device. Weak convergent infimizing sequences of I - which exist due to its coercivity - develop oscillations and generate Young measures. These describe the energy-infimizing shape of the elastic body Ω.

If one can prove weak convergence of the according energy density sequence $(f(\nabla u_j))_{j\in\mathbb{N}}$ in $L^p(\Omega)$ for some $p\in[1,\infty]$, the existence of such a measure is guaranteed by an application of Theorem 1. The following theorem provides an extension of Theorem 2 and 3 to nonquasiconvex energy densities f.

Theorem 4. *Let $f:\mathbb{R}^{d\times d}\to\bar{\mathbb{R}}$ and I in (7) fulfill the following conditions $\exists c_1>0$ such that $f(A)\geq c_1\,\|A\|^q_{\mathbb{R}^{d\times d}}\ \forall A\in\mathbb{R}^{d\times d}$, $f\in C^I_p$ for some $p\in[1,\infty]$ and $I\not\equiv+\infty$ in K where C^I_p denotes the set $\{f:\mathbb{R}^{d\times d}\to\bar{\mathbb{R}};\ (u_j) infimizes\ I\ \Rightarrow (f(u_j))_{j\in\mathbb{N}}$ converges weakly in $L^p(\Omega)\}$.*

Moreover - with the notations of the preceding theorems - let at least one of the following conditions hold:

(*i*) $C^{0,q}_{d-1}(\overline{\Gamma}_U)>0$

(*ii*) $C^{0,q}_{d-1}(\overline{\Gamma}_C)>0$ *and* $F(b)<0\ \ \forall\, b\in\tilde{B}$.

Then there exists an infimizing sequence $(u_j)_{j\in\mathbb{N}}\subset K$ of problem (7) with $u_j\rightharpoonup u$ in $W^{1,q}(\Omega,\mathbb{R}^d)$ and a Young measure $(\nu_x)_{x\in\Omega}\subset\big(C_0(\mathbb{R}^{d\times d})\big)^*$ such that the infimizing state of I is represented by

$$\inf_{v\in K} I(v) \;=\; \int_\Omega \langle\, f\,,\,\nu_x\,\rangle\;\mathrm{d}x - F(u)\,.$$

Proof. We have $I\not\equiv+\infty$ in K and thus an infimizing sequence $(u_j)_{j\in\mathbb{N}}\subset K$. With the assumptions above it is clear that infimizing sequences converge weakly (at least up to a subsequence):

$$u_j \rightharpoonup u \quad \text{in} \quad W^{1,q}(\Omega,\mathbb{R}^d)\,.$$

Now, since $(\nabla u_j)_{j\in\mathbb{N}}$ is uniformly bounded in $L^q(\Omega,\mathbb{R}^{d\times d})$ and $f\in C^I_p$, there exists a measurable family $(\nu_x)_{x\in\Omega}\subset\big(C_0(\mathbb{R}^{d\times d})\big)^*$ such that

$$\lim_{j\to\infty}\int_\Omega f(\nabla u_j(x))\,\alpha(x)\,\mathrm{d}x \;=\; \int_\Omega\left(\int_{\mathbb{R}^m} f(z)\,\mathrm{d}\nu_x(z)\right)\alpha(x)\,\mathrm{d}x\,, \quad \forall\,\alpha\in L^{p'}(\Omega)$$

by Theorem 1. With $\alpha = 1$ and the weak continuity of F in $W^{1,q}(\Omega,\mathbb{R}^d)$ we obtain

$$\inf_{v\in K} I(v) \;=\; \lim_{j\to\infty}\left(\int_\Omega f(\nabla u_j)\,\mathrm{d}x + F(u_j)\right) \;=\; \int_\Omega \langle\, f\,,\,\nu_x\,\rangle\,\mathrm{d}x + F(u)\,. \qquad \blacksquare$$

Remark. Whether a specific f belongs to C_p^I has to be verified separately. Nevertheless we can observe two general facts.

(1) Since '(u_j) infimizes $I \;\Rightarrow\; (\nabla u_j)$ converges weakly in $L^q(\Omega,\mathbb{R}^{d\times d})$'; we obtain with

$$\mathcal{A} = \left\{ (\nabla u_j)_{j\in\mathbb{N}} \subset L^q(\Omega,\mathbb{R}^{d\times d})\,,\; \sup_{j\in\mathbb{N}} \|\nabla u_j\|_{L^q(\Omega,\mathbb{R}^{d\times d})} < \infty \right\}$$

the relation $C_p^{\mathcal{A}} \subset C_p^I$.

(2) $p = 1$: By definition of infimizing sequences we immediately obtain '(u_j) infimizes $I \;\Rightarrow\; (f(\nabla u_j))$ is uniformly bounded in $L^1(\Omega)$'. Now, the additional condition of equiintegrability

$$\lim_{k\to\infty}\left(\sup_{j\in\mathbb{N}} \int_{\{|f(\nabla u_j)|\geq k\}} |f(\nabla u_j)|\,\mathrm{d}x\right) \;=\; 0$$

provides weak convergence of $(f(\nabla u_j))_{j\in\mathbb{N}}$ in $L^1(\Omega)$. Thus we have

$$C_1^I = \Big\{ f:\mathbb{R}^{d\times d}\to\bar{\mathbb{R}}; \\ (u_j)_{j\in\mathbb{N}} \text{ infimizes } I \;\;\Rightarrow\;\; (f(\nabla u_j))_{j\in\mathbb{N}} \text{ is equiintegrable over } \Omega \Big\}\,.$$

The implicit character of '$f \in C_p^I$' is compensated for the flexibility of this condition in applications. The kind of convergence and the choice of a suitable function space depend on the energy density f then.

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