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ON CONVEXITY OF PREIMAGES OF MONOTONE OPERATORS

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Abstract. In this paper we first study the relationship between local and global Minty-Browder monotone operators and then we show that these operators have generally convex preimages. Our results allow to show that positive semidefinitedness on the complement of a discrete set of the differential operator implies the Minty-Browder monotonicity of the operator itself. We also show that complex functions of one complex variable are Minty-Browder monotone under suitable conditions. Finally, we obtain some injectivity/univalency theorems that generalize some well-known results.

1. Introduction

Let us consider a function $A : \mathbb{R} \longrightarrow \mathbb{R}$. If either $(Ax - Ay)(x - y) \geq 0$ for all $x, y \in \mathbb{R}$ or $(Ax - Ay)(x - y) \leq 0$ for all $x, y \in \mathbb{R}$, then the preimage $A^{-1}z$ of any z in the range of A is an interval. The converse of the above statement is obviously false. However the converse works if we additionally assume that A is continuous. Note that A is *increasing* if and only if $(Ax - Ay)(x - y) \geq 0$, for all $x, y \in \mathbb{R}$ and A is *decreasing* if and only if $(Ax - Ay)(x - y) \leq 0$, for all $x, y \in \mathbb{R}$. Recall that A is said to be *monotone* if A is either increasing or decreasing. Consequently the assumption on the preimages $A^{-1}z$, $z \in R(A)$ to be intervals is a characterization of monotonicity, at least for continuous real valued functions of one real variable.

The concept of monotonicity has been extended to operators from a Banach space to its dual some fifty years ago by the celebrated works of Browder and Minty (see for example [2-4, 9, 10]). This extension (often called *Minty-Browder monotonicity*) has been the starting point for the development of nonlinear functional analysis, being also used in convex analysis. The monotone mappings appear in a rather wide variety of contexts. We only mention the equivalence between the convexity of a

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function and the monotonicity of its subdifferential (see for instance [6]). In what follows, for any set S denote by 2^S the collection of all subsets of S.

In order to define the concept of Minty-Browder monotonicity, let us recall that if X, Y are given sets and $T : X \longrightarrow 2^Y$ is a set-valued map, one defines the *domain, the range* and *the graph of* T as

$$D(T) := \{x \in X : Tx \neq \emptyset\}\,, R(T) := \{y \in Y : \exists x \in D(T)\ s.t.\ y \in Tx\}$$

and $G(T) := \{(x, y) \in X \times Y : x \in D(T), y \in Tx\}$ respectively. If $y \in Y$, then $T^{-1}y := \{x \in X : y \in Tx\}$ and a new operator

$$T^{-1} : Y \longrightarrow 2^X$$

is defined. Single-valued operators, denoted by $T : D \longrightarrow Y$, $D \subseteq X$, are understood as set-valued operators $T : X \longrightarrow 2^Y$ with $D(T) = D$ and $card\,(Tx) = 1$ for all $x \in D$. If X is a Banach space and X^* is its dual, then for $x \in X$ and $x^* \in X^*$ denote by $\langle x^*, x\rangle$ the scalar $x^*(x)$. Recall that an operator $T : X \to 2^{X^*}$is said to be *Minty-Browder monotone* if either $\langle u - v, x - y\rangle \geq 0$ for all $x, y \in D(T)$ and all $u \in Tx, v \in Ty$ or $\langle u - v, x - y\rangle \leq 0$ for all $x, y \in D(T)$ and all $u \in Tx, v \in Ty$. The inequality symbol "$\geq$" corresponds to Minty-Browder *increasing* operators, while the inequality symbol "$\leq$" corresponds to Minty-Browder *decreasing* operators.

The observation made at the beginning of this section on real-valued functions of one real variable, namely that the preimages $A^{-1}z,\ z \in R(A)$ are intervals, is a characterization of monotonicity in the case of continuous functions, leads to another type of monotonicity. More precisely the operator $T : X \longrightarrow 2^{X^*}$ is said to be *c-monotone* if $T^{-1}u$ is a convex set for every $u \in R(T)$.

The following properties are immediate.

Remark 1.1. Let X be a topological vector space and $A : X \longrightarrow 2^{X^*}$ be an operator.

(1) A is increasing if and only if $-A$ is decreasing.

(2) $(-A)^{-1}(-v^*) = A^{-1}v^*$ for all $v^* \in X^*$. Consequently $A^{-1}v^*$ is convex for all $v^* \in R(A)$ if and only if $(-A)^{-1}u^*$ is convex for all $u^* \in R(-A)$. In other words A is c-monotone if and only if $-A$ is c-monotone.

The paper is organized as follows. In section two we first prove that a kind of local Minty-Browder monotonicity implies the global Minty-Browder monotonicity and, as a consequence, we prove, in a Hilbert space context, that the positive semidefinitedness of the differentials of a continuous function on a convex open set which is of class C^1 except on a discrete set, implies the Minty-Browder monotonicity of the operator itself. The latter is reworded at the end of the section in the context of complex-valued functions of one complex variable. More precisely the Minty-Browder monotonicity of such kind of functions can be shown under suitable assumptions as Corollary 2.8 below shows.

The relationship between Minty-Browder monotonicity and c-monotonicity is studied within section 3. It is well-known that Minty-Browder maximal monotone operators have closed and convex primages, being, in particular, c-monotone. We show that c-monotonicity of Minty-Browder monotone operators still holds, without assuming maximality, if we strengthen the hypothesis on their domain (see Theorem 3.5 below). On the other hand c-monotonicity does not imply, in general, Minty-Browder monotonicity as Example 3.3 below shows.

The last section is devoted to applications. Using our previous results we show some global injectivity theorems which contain, in particular, some well-known injectivity/univalency results such as Alexander, Noshiro, Warschawschi and Wolff's univalence criterion (see for instance [1]), Gale-Nikaido's [7] and Mocanu's [11] injectivity theorems.

2. Local Minty-browder Increasing/Decreasing Operators

In this section we first prove that a kind of local increasing/decreasing monotonicity concept implies global Minty-Browder increasing/decreasing monotonicity.

Lemma 2.1. *If $I \subseteq \mathbb{R}$ is an interval and $f : I \longrightarrow \mathbb{R}$ has the property that for each $t \in I$ there exists an open interval J_t with $t \in J_t$ such that $(f(x) - f(t))(x - t) \geq 0$ for all $x \in J_t \cap I$, then f is an increasing function.*

Proof. Assume that f is not increasing. This means that there exists $a, b \in I$, $a < b$ such that $f(a) > f(b)$. Consider the set $A := \{x \in [a, b] : f(x) > f(b)\}$, and observe that $A \neq \emptyset$, as well as, $a < s := \sup A$. In fact the inequality $s < b$ also holds, since $s \in J_s$ and $f(x) \leq f(s)$ for all $x \in J_s, x < s$. If $f(s) > f(b)$, then $s \in A$, which combined with the fact that $f(x) \geq f(s)$ for all $x \in J_s$, $x > s$, shows that $[s, b] \cap J_s \subseteq A$, a contradiction with $s = \sup A$. Thus $f(s) \leq f(b)$. On the other hand $f(b) < f(x)$ for all $x \in A$, as well as, $f(s) \geq f(x)$ for all $x \in J_s \cap [a, s]$. Consequently $f(s) \leq f(b) < f(x) \leq f(s)$ for all $x \in A \cap J_s \cap [a, s] \neq \emptyset$, which is absurd. ∎

Theorem 2.2. *Let X be a Banach space and $D \subseteq X$ be a convex open subset of X. If $A : X \longrightarrow X^*$ is an operator with the property that each $x \in X$ has an open neighbourhood U_x such that $\langle Ay - Ax, y - x\rangle \geq 0$ for all $y \in U_x$, then A is an increasing Minty-Browder operator.*

Proof. For $u, v \in X$, $u \neq v$ consider an open interval I containing $[0, 1]$ such that $u + t(v - u) \in D$ for all $t \in I$ and the maps

$$\gamma : I \longrightarrow X^*,\ \gamma(t) := A(u + t(v - u)) \text{ and } p : X^* \longrightarrow \mathbb{R},\ p(x^*) = \frac{\langle x^*, v - u\rangle}{||v - u||^2}$$

For $s, t \in \mathbb{R}$, we have successively:

$$\begin{aligned}
&[(p \circ \gamma)(s) - (p \circ \gamma)(t)](s - t) \\
&= [p(A(u + s(v - u))) - p(A(u + t(v - u)))](s - t) \\
&= \left(\frac{\langle A(u + s(v - u)), v - u\rangle}{||v - u||^2} - \frac{\langle A(u + t(v - u)), v - u\rangle}{||v - u||^2}\right)(s - t) \\
&= \langle A(u + s(v - u)) - A(u + t(v - u)), v - u\rangle \frac{s - t}{||v - u||^2} \\
&= \langle A(u + s(v - u)) - A(u + t(v - u)), (s - t)(v - u)\rangle \cdot \frac{1}{||v - u||^2}.
\end{aligned}$$

By using the hypothesis one gets that

$$\langle A(u + s(v - u)) - A(u + t(v - u)), (s - t)(v - u)\rangle \geq 0$$

for s sufficiently closed to t. In other words, for each $t \in \mathbb{R}$, there exists an open interval, say J_t, containing t such that $[(p \circ \gamma)(s) - (p \circ \gamma)(t)](s - t) \geq 0$, for all $s \in J_t$. By using Lemma 2.1 one gets that the function $p \circ \gamma$ is increasing, that is

$$\langle A(u + s(v - u)) - A(u + t(v - u)), (s - t)(v - u)\rangle \geq 0,$$

for all $s, t \in \mathbb{R}$, which particularly shows that $\langle Av - Au, v - u\rangle \geq 0$, and the proof is now completely done. ∎

In the sequel $\mathcal{D}'$ denotes the set of all cluster points of $\mathcal{D}$. For a differentiable mapping f, denote by $(df)_x$ the Fréchet differential of f at x.

The next result shows that, in a Hilbert space context, the positive semidefinitedness of the differentials of a continuous function on a convex open set which is of class C^1 except on a discrete set, implies the Minty-Browder monotonicity of the operator itself.

Proposition 2.3. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $D \subseteq H$ be a convex open subset of H. If $\mathcal{D}$ is a subset of D such that $\mathcal{D}' \cap D = \emptyset$ and $A : D \to H$ is a continuous operator which is of class C^1 on $D \setminus \mathcal{D}$ and has the property $\langle (dA)_x(y), y\rangle \geq 0$ for all $x \in D \setminus \mathcal{D}$ and all $y \in H$, then A is a Minty-Browder increasing operator.*

Proof. According to Theorem 2.2, it is enough to prove that each $x \in D$ has an open neighbourhood U_x such that $\langle Ay - Ax, y - x\rangle \geq 0$ for all $y \in U_x$. Let us consider $x \in D$ and $r_x > 0$ such that the open ball $B(x, r_x)$ is contained in D and

$\big(B(x, r_x) \setminus \{x\}\big) \cap \mathcal{D} = \emptyset$. For $y \in B(x, r_x)$, $y \neq x$ consider an open interval I containing $[0, 1]$ such that $x + t(y - x) \in D$ for all $t \in I$ and the maps

$$\gamma : I \longrightarrow H,\ \gamma(t) := A(x + t(y - x)) \text{ and } p : H \longrightarrow \mathbb{R},\ p(u) = \frac{\langle u, y - x\rangle}{||y - x||^2}.$$

Note that p is linear and $p \circ \gamma$ is continuous on [0,1] and differentiable on $(0, 1)$. In fact $p \circ \gamma$ is differentiable on the whole interval $[0, 1]$ if $x \notin \mathcal{D}$, while it might not be differentiable at 0 if $x \in \mathcal{D}$. On the other hand,

$$(p \circ \gamma)(1) - (p \circ \gamma)(0) = \langle Ay - Ax, y - x\rangle \frac{1}{||y - x||^2}$$

and, by using the mean value theorem, there exists $c \in (0, 1)$ such that

$$(p \circ \gamma)(1) - (p \circ \gamma)(0) = (p \circ \gamma)'(c).$$

Consequently one gets successively:

$$\begin{aligned}&\langle Ay - Ax, y - x\rangle \frac{1}{||y - x||^2} = (p \circ \gamma)(1) - (p \circ \gamma)(0) = (p \circ \gamma)'(c)\\
&= (dp)_{\gamma(c)}(\gamma'(c)) = (dp)_{\gamma(c)}\Big((dA)_{x+c(y-x)}(y - x)\Big) = p\Big((dA)_{x+c(y-x)}(y - x)\Big)\\
&= \Big\langle (dA)_{x+c(y-x)}(y - x), y - x\Big\rangle \frac{1}{||y - x||^2}.\end{aligned}$$

This shows that $\langle Ay - Ax, y - x\rangle = \Big\langle (dA)_{x+c(y-x)}(y - x), y - x\Big\rangle \geq 0$. ∎

Remark 2.4.

(1) Observe that within the hypotheses of Proposition 2.3 the assumption $\mathcal{D}' \cap D = \emptyset$ implies that $\mathcal{D}$ is a discrete subset of D, i.e. every $x \in \mathcal{D}$ admits a neighbourhood U_x such that $U_x \cap \mathcal{D} = \{x\}$.

(2) Let $D \subseteq H$ be a connected open set and $\mathcal{D} \subseteq D$ be a set such that $\mathcal{D}' \cap D = \emptyset$. If $A : D \longrightarrow H$ is a continuous operator which is of class C^1 on $D \setminus \mathcal{D}$ and has the property $\langle (dA)_x(y), y\rangle \geq 0$ for all $x \in D \setminus \mathcal{D}$ and all $y \in H$, then A is a local Minty-Browder increasing operator in the sense that each point has a neighbourhood on which the operator A is Minty-Browder increasing. In order to justify this fact we consider around an arbitrary point $x \in D$ an open ball, say $B(x, r_x)$, such that $B(x, r_x) \subseteq D$ and observe that obviously $B(x, r_x) \cap \big(B(x, r_x) \cap \mathcal{D}\big)' = \emptyset$. By using Proposition 2.3 one can deduce, taking into account that $B(x, r_x)$ is a convex open set, that $A\big|_{B(x,r_x)}$ is a Minty-Browder increasing operator. We wonder whether the convexity assumption on D, in Theorem 2.2, might be relaxed by replacing it with its connectedness.

Next we study the Minty-Browder monotonicity of complex functions of one complex variable.

Corollary 2.5. *If $D \subseteq \mathbb{C}$ is a convex open set and $f : D \longrightarrow \mathbb{C}$ is a holomorphic mapping such that $Re(f'(z)) \geq 0$, for all $z \in D$, then f is a Minty-Browder increasing operator.*

Proof. According to Proposition 2.3, it is enough to prove that $\langle (df)_z(w), w\rangle \geq 0$ for all $z \in D$ and all $w \in \mathbb{C}$. Indeed, by using the obvious equality $(df)_z(w) = wf'(z)$ for all $z \in D$ and all $w = a + ib \in \mathbb{C}$ one gets:

$$\langle (df)_z(w), w\rangle = Re\big(\bar{w}wf'(z)\big) = Re\big(|w|^2 f'(z)\big) = |w|^2 Re(f'(z)) \geq 0. \qquad \blacksquare$$

Lemma 2.6. *Let $D \subseteq \mathbb{C}$ be an open set and $f : D \longrightarrow \mathbb{C}$, $f = u + iv$ be a function of class C^1. If*

$$Re\frac{\partial f}{\partial z}(z) > \Big|\frac{\partial f}{\partial \bar{z}}(z)\Big|,$$

for some $z \in D$, then the quadratic form $Q_z : \mathbb{C} \longrightarrow \mathbb{R}$, $Q_z(w) = \langle (df)_z(w), w\rangle$ is definite (positive or negative).

Proof. We have successively:

$$\begin{aligned} Re\frac{\partial f}{\partial z}(z) > \Big|\frac{\partial f}{\partial \bar{z}}(z)\Big| &\iff Re\Big(\frac{\partial f}{\partial x}(z) - i\frac{\partial f}{\partial y}(z)\Big) > \Big|\frac{\partial f}{\partial x}(z) + i\frac{\partial f}{\partial y}(z)\Big| \\ &\iff u_x(z)+v_y(z) > \sqrt{\big(u_x(z)-v_y(z)\big)^2+\big(u_y(z)+v_x(z)\big)^2} \\ &\implies 4u_x(z)v_y(z) > (u_y(z)+v_x(z))^2. \end{aligned}$$

This shows that $u_x(z)$, $v_y(z) > 0$ or $u_x(z)$, $v_y(z) < 0$. In the first case we actually have

$$u_x(z) > 0 \text{ and } \begin{vmatrix} u_x(z) & \dfrac{u_y(z)+v_x(z)}{2} \\ \dfrac{u_y(z)+v_x(z)}{2} & v_y(z) \end{vmatrix} > 0,$$

which shows that the quadratic form

$$\begin{aligned} Q_z(a+ib) &= u_x(z)a^2 + \big(u_y(z)+v_x(z)\big)ab + v_y(z)b^2 \\ &= [a\ b] \begin{bmatrix} u_x(z) & \dfrac{u_y(z)+v_x(z)}{2} \\ \dfrac{u_y(z)+v_x(z)}{2} & v_y(z) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}$$

is positive definite and in the second case one gets

$$u_x(z) < 0 \text{ and } \begin{vmatrix} u_x(z) & \dfrac{u_y(z)+v_x(z)}{2} \\ \dfrac{u_y(z)+v_x(z)}{2} & v_y(z) \end{vmatrix} > 0,$$

which shows that the quadratic form Q_z is negative definite. ∎

Corollary 2.7. *If $D \subseteq \mathbb{C}$ is an open connected set and $f : D \longrightarrow \mathbb{C}$ is a function of class C^1 such that*

$$Re\frac{\partial f}{\partial z}(z) > \left|\frac{\partial f}{\partial \bar{z}}(z)\right|,$$

for all $z \in D$, then either all quadratic forms Q_z, $z \in D$ are positive definite or they are all negative definite.

Proof. As we have already seen in the proof of Lemma 2.6 the inequality $4u_x(z)v_y(z) > (u_y(z)+v_x(z))^2$ shows that Q_z is either positive or negative definite. Assume that Q_z is positive definite and Q_w is negative definite, for some $z, w \in D, z \neq w$, namely $u_x(z) > 0$ and $u_x(w) < 0$. Since D is connected, there exists some $\xi \in D$ such that $u_x(\xi) = 0$, which shows that $0 = 4u_x(\xi)v_y(\xi) > (u_y(\xi) + v_x(\xi))^2 \geq 0$, which is absurd. ∎

Corollary 2.8. *Let $D \subseteq \mathbb{C}$ be a convex open set and $\mathcal{D} \subseteq D$ be a set such that $\mathcal{D}' \cap D = \emptyset$. If $f : D \longrightarrow \mathbb{C}$ is a continuous function which is also of class C^1 on $D \setminus \mathcal{D}$ and satisfies the inequality*

$$Re\frac{\partial f}{\partial z}(z) > \left|\frac{\partial f}{\partial \bar{z}}(z)\right|,$$

for all $z \in D \setminus \mathcal{D}$, then f is a Minty-Browder monotone operator.

Proof. We only use the connectedness of $D \setminus \mathcal{D}$, Corollary 2.7 and Proposition 2.3. ∎

3. On the Preimages of Minty-browder Monotone Operators

Recall that the operator $T : X \to 2^{X^*}$ is said to be (*Minty-Browder*) *maximal monotone* if for every Minty-Browder monotone operator $T' : X \to 2^{X^*}$ such that $G(T) \subseteq G(T')$, one has $T = T'$. The next result is well-known from the theory of Minty-Browder monotone operators.

Proposition 3.1. *If $T : X \to 2^{X^*}$ is a maximal monotone operator, then $T^{-1}y$ is a closed convex subset of X, for every $y \in X^*$. In particular T is c-monotone.*

Proof. Maximal monotonicity of T is equivalent to maximal monotonicity of T^{-1} (see for instance [12, pp. 105]). On the other hand, the maximal monotonicity of T^{-1}implies that $T^{-1}y$ is a closed convex subset of X, for every $y \in X^*$ (see for instance [12, pp. 105]). ■

Corollary 3.2. *The continuous Minty-Browder monotone operators $A : X \to X^*$ are c-monotone.*

Proof. By Corollary 2.3 of [12, pp. 106]), A is maximal monotone. Thus, the result follows by Proposition 3.1. ■

Example 3.3. The converse of Corollary 3.2 is not true. For example, the operator $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $A(x_1, x_2) = (x_1, -x_2)$ has convex preimages, but A is not a Minty-Browder monotone operator. Indeed each preimage $A^{-1}y$ is a one-point set, but $\langle A(1,0) - A(0,0), (1,0) - (0,0)\rangle = ||(1,0)||^2 = 1 > 0$ and $\langle A(0,1) - A(0,0), (0,1) - (0,0)\rangle = \langle (0,-1), (0,1)\rangle = -1 < 0$.

Remark 3.4. Let X be a topological vector space and $A : X \longrightarrow X^*$ be an operator.

(1) If $A : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then the converse of Corollary 3.2 also works in this case.

(2) In Remark 3.4(1) the dimension one of the space is essential: the equivalence does not hold if the involved space has dimension two, as we have already seen in Example 3.3. In fact, Example 3.3 can be included in a larger class of operators which are not Minty-Browder monotone but have convex preimages. Indeed, a linear involution $s : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ $(s \circ s = id)$ with nontrivial set of fixed points which is not the identity, has convex preimages, being one-to-one, but it is not Minty-Bowder monotone. In this respect, we just need to consider a nonzero fixed point $x \in \mathbb{R}^{n+1}$ of s and a vector $y \in \mathbb{R}^{n+1}$ which is not a fixed point of s. Observe that $\langle s(x), x\rangle = \langle x, x\rangle = ||x||^2 > 0$, while $\langle s(y - s(y)), y - s(y)\rangle = -||y - s(y)||^2 < 0$. Consequently, the only linear Minty-Browder monotone involutions of $\mathbb{R}^{n+1}$ are the identity of $\mathbb{R}^{n+1}$and the antipodal map $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, Ax = -x$.

The next result shows that the preimages of single-valued Minty-Browder monotone operators are still convex if one removes the maximality assumption, but under the continuity and some other additional assumptions on the domain of the involved operator.

Theorem 3.5. *If $(H\, \langle\cdot,\cdot\rangle)$ is a Hilbert space, $D \subseteq H$ is a convex open set and $A : D \to H$ is a continuous Minty-Browder monotone operator, then A is c-monotone.*

Proof. Assume, without loss of generality, that A is increasing (see Remark 1.1(1)). If $A^{-1}y$ is one point there is nothing to be proved. Otherwise consider $x_1, x_2 \in A^{-1}y$, $x_1 \neq x_2$ and assume that the segment $[x_1x_2] := \{tx_1 + (1-t)x_2 : t \in [0,1]\}$ is not completely contained in $A^{-1}y$, which is equivalent to the existence of a $t_0 \in [0,1]$ such that $x_0 := t_0x_1 + (1-t_0)x_2 \notin A^{-1}y \Leftrightarrow Ax_0 =: y_0 \neq y := Ax_1 = Ax_2$. Since $\langle y_0 - y, y - y_0\rangle = -\langle y - y_0, y - y_0\rangle = -||y - y_0||^2 < 0$, it follows by the continuity of A that $\langle y_\varepsilon - y, y - y_0\rangle < 0$ for $\varepsilon > 0$ sufficiently small, where $y_\varepsilon := Ax_\varepsilon$ and $x_\varepsilon := x_0 + \varepsilon(y - y_0)$. Because of the Minty-Browder monotonicity of A we get successively:

$$\begin{cases} \langle Ax_\varepsilon - Ax_1, x_\varepsilon - x_1\rangle \geq 0 \\ \langle Ax_\varepsilon - Ax_2, x_\varepsilon - x_2\rangle \geq 0 \end{cases} \Leftrightarrow \begin{cases} \langle y_\varepsilon - y, x_0 - x_1\rangle + \varepsilon\langle y_\varepsilon - y, y - y_0\rangle \geq 0 \\ \langle y_\varepsilon - y, x_0 - x_2\rangle + \varepsilon\langle y_\varepsilon - y, y - y_0\rangle \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \langle y_\varepsilon - y, (1-t_0)(x_2 - x_1)\rangle + \varepsilon\langle y_\varepsilon - y, y - y_0\rangle \geq 0 \\ \langle y_\varepsilon - y, t_0(x_1 - x_2)\rangle + \varepsilon\langle y_\varepsilon - y, y - y_0\rangle \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (1-t_0)\langle y_\varepsilon - y, x_2 - x_1\rangle + \varepsilon\langle y_\varepsilon - y, y - y_0\rangle \geq 0 \\ -t_0\langle y_\varepsilon - y, x_2 - x_1\rangle + \varepsilon\langle y_\varepsilon - y, y - y_0\rangle \geq 0. \end{cases}$$

But the above relations cannot be simultaneously satisfied since $\varepsilon\langle y_\varepsilon - y, y - y_0\rangle < 0$ and at least one of the expressions $(1-t_0)\langle y_\varepsilon - y, x_2 - x_1\rangle$ and $-t_0\langle y_\varepsilon - y, x_2 - x_1\rangle$ is nonpositive. ∎

Note that the proof of Theorem 3.5 is elementary in the sense that no fundamental results are used. Despite of this, the proof of Corollary 3.2 uses implicitly the strong separation theorem in Banach spaces [12, pp. 106]).

Example 3.6. The assumption on D to be open in Theorem 3.5 is essential. Indeed, the operator $A : P \longrightarrow \mathbb{C}$, $Az = z^2$, where $P := \{z \in \mathbb{C} : Re(z) \geq 0\}$, is Minty-Browder monotone, but $A^{-1}(-1) = \{-i, i\}$ is not even connected. In order to prove the Minty-Browder monotonicity of A, observe that

$$\langle Az - Aw, z - w\rangle = [Re(z) + Re(w)]\,|z - w|^2 \geq 0$$

for every $z, w \in H$.

4. Applications. Global Injectivity Theorems

In this section we prove some global injectivity results. The statement below is an easy consequence of our previous statements.

Proposition 4.1. *Let H be a Hilbert space and let $A : D \to H$ be a local homeomorphism, where $D \subseteq H$ is a convex open set. If A is additionally a Minty-Browder monotone operator, then A is injective.*

Proof. Since A is a local homeomorphism, its preimages are discrete subsets of D. On the other hand, according to Theorem 3.5, its preimages are also convex subsets of D. Consequently the preimages of A are actually one point sets, which shows the injectivity of A. ■

Another consequence of our results is the next univalency theorem.

Corollary 4.2. (Alexander, Noshiro, Warschawski and Wolff [1], [8, p. 63]). *If $D \subseteq \mathbb{C}$ is a convex open set and $f : D \longrightarrow \mathbb{C}$ is a holomorphic mapping such that $Re(f'(z)) > 0$, for all $z \in D$, then f is univalent on D.*

Proof. Combining Corollary 2.5 with Theorem 3.5 one gets that $f^{-1}(w)$, $w \in R(f)$ is convex. Since the preimages of holomorphic functions with $Re(f'(z)) > 0$, for all $z \in D$, are also discrete, it follows that the preimages $f^{-1}(w)$, $w \in R(f)$ are actually one point sets. ■

Remark 4.3. If $D \subseteq \mathbb{C}$ is an open connected set and $f : D \longrightarrow \mathbb{C}$, $f = u + iv$ is a holomorphic function, by using the Cauchy-Riemann relations, we get successively:

$$(Jf)_z = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix},$$

where $(Jf)_z$ is the Jacobi matrix of f at $z = x + iy \in D$. The characteristic polynomial of $(df)_z$ is $\lambda^2 - 2u_x\lambda + u_x^2 + v_x^2$ and shows that

$$Spec\big((df)_z\big) = \{u_x \pm iv_x\} = \{f'(z), \overline{f'(z)}\}.$$

Consequently, the condition $Re(f'(z)) > 0$, for all $z \in D$ in Corollary 4.2 is equivalent to $Re\big(Spec(df)_z\big) > 0$ for all $z \in D$, where the relation $Re(Z) > 0$ for a subset Z of $\mathbb{C}$ is understood in the sense that $Re(z) > 0$ for all $z \in Z$. We wonder whether the new version of Alexander, Noshiro, Warschawski and Wolff univalence criterion obtained by replacing the holomorphy assumption on f with its C^1 differentiability and condition $Re(f'(z)) > 0$, for all $z \in D$, with $Re\big(Spec(df)_z\big) > 0$ for all $z \in D$, remains true.

Proposition 4.4. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $D \subseteq H$ be a convex open subset of H. If $\mathcal{D}$ is a subset of D such that $\mathcal{D}' \cap D = \emptyset$ and $A : D \longrightarrow H$ is a continuous operator which is C^1 on $D \setminus \mathcal{D}$ and has the property $\langle (dA)_x(y), y\rangle > 0$ for all $x \in D \setminus \mathcal{D}$ and all $y \in H \setminus \{0\}$, then A is injective.*

Proof. We first observe that, according to Proposition 2.3, A is a Minty-Browder increasing operator. In fact the proof of Proposition 2.3 shows that each $x \in D$ has a neighborhood U_x such that $\langle Ay - Ax, y - x\rangle > 0$ for all $y \in U_x \setminus \{x\}$. This implies that $A^{-1}(Ax)$ is discrete for all $x \in D$. Indeed, otherwise there would exist $x \in D$ and $y \in U_x$, $y \neq x$ such that $Ay = Ax$, which shows that

$\langle Ay - Ax, y - x\rangle = 0$. Consequently $A^{-1}y$ is discrete for all $y \in R(A)$, which combined with its convexity, ensured by Theorem 3.5, implies that $A^{-1}y$ is a one point set for every $y \in R(A)$. ■

A proof of the injectivity of local Minty-Browder strictly monotone operators was done by M. Cristea in [5].

A direct consequence of Proposition 4.4 is the following classical result.

Corollary 4.5. (Gale, Nikaido [7]). *Let D be a convex open subset of $\mathbb{R}^n$ and let $A : D \longrightarrow \mathbb{R}^n$ be a C^1 map. If $\langle (dA)_x(y), y\rangle > 0$ for all $x \in D$ and all $y \in \mathbb{R}^n$, then A is injective.*

We close our paper by showing that our results also allow a generalization of one of Mocanu's injectivity theorem.

Corollary 4.6. *Let $D \subseteq \mathbb{C}$ be a convex open set and $\mathcal{D} \subseteq D$ be a set such that $\mathcal{D}' \cap D = \emptyset$. If $f : D \longrightarrow \mathbb{C}$ is a continuous function which is also of class C^1 on $D \setminus \mathcal{D}$ and satisfies the inequality*

$$Re\frac{\partial f}{\partial z}(z) > \left|\frac{\partial f}{\partial \bar{z}}(z)\right|,$$

for all $z \in D \setminus \mathcal{D}$, then f is injective.

Proof. The statement follows by combining Corollary 2.7 (applied to the open connected set $D \setminus \mathcal{D}$) with Proposition 4.4. ■

Corollary 4.7. (Mocanu [11]). *If $D \subseteq \mathbb{C}$ is a convex open set and $f : D \longrightarrow \mathbb{C}$ is a function of class C^1 such that*

$$Re\frac{\partial f}{\partial z}(z) > \left|\frac{\partial f}{\partial \bar{z}}(z)\right|,$$

for all $z \in D$, then f is injective.

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