

STABILITY OF IMPLICIT MULTIFUNCTIONS IN ASPLUND SPACES

N. Q. Huy and J.-C. Yao*

Abstract. The purpose of this paper is to present new sufficient conditions for both the metric regularity and the Lipschitzian stability of implicit multifunctions in Asplund spaces. The basic tools of our analysis involve the Fréchet normal coderivative and the Mordukhovich normal coderivative of set-valued mappings, the basic subgradient estimate for marginal functions and the Ekeland variational principle. Applications to the pointbased characterizations for the metric regularity and the Lipschitzian stability of solution mapping of parametric generalized equations are given.

1. INTRODUCTION

Consider $F: X \times Y \rightrightarrows Z$ a set-valued mapping between Banach spaces and the *generalized inequality systems*

$$(1.1) \quad 0 \in F(x, y).$$

The solution map $G: Y \rightrightarrows X$ associated with (1.1) which has the form

$$(1.2) \quad G(y) = \{x \in X : 0 \in F(x, y)\}$$

is said to be the *implicit multifunction* defined by the inclusion (1.1). In this paper we will focus on the study of the metric regularity in the Robinson's sense [24] and the Lipschitzian stability in the Aubin's sense [1] for the implicit multifunction (1.2).

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*Corresponding author.

The study of implicit multifunction has a long history. Stable properties of implicit multifunction such as the metric regularity and Lipschitzian stability which have attracted much attention of many researchers; see, e.g., [2–19, 22–27] and the references therein for more details and discussions.

In a special case of F in (1.1) which has the form

$$(1.3) \quad F(x, y) = f(x, y) + Q(x, y)$$

where $f: X \times Y \rightarrow Z$ is a single-valued mapping and $Q: X \times Y \rightrightarrows Z$ is a set-valued mapping between Asplund spaces, Mordukhovich [12, 14, 15] gave the characterizations of necessary as well as sufficient conditions for the Lipschitzian stability of (1.2) under the strict differentiability of f with the surjective property of partial gradient of f and either Q does not depend x or Q is graphically regular at a given point. Together with the Lipschitzian stability, the metric regularity of (1.2) was intensively investigated in *implicit* and *inverse multifunctions* (see, e.g., [2–11, 13, 16–19, 22–27]). Recently, Ledyev and Zhu [9], Ngai and Théra [22] established sufficient conditions for the metric regularity of (1.2) in terms of the Fréchet normal coderivatives in Banach space with the Fréchet-smooth bump function. Another set of sufficient conditions for the same property in Asplund spaces was given by Lee, Tam and Yen [10] in terms of the normal coderivatives. More recently, Yen and Yao [27] established some *pointbased* sufficient conditions for the metric regularity property of implicit multifunctions in finite-dimensional setting, and the same property in the WCG space setting was also presented in [7]. It is well known that for inverse multifunctions the metric regularity is equivalent to the Lipschitz-like property (see [3, 11, 23]). However, as well shown in [8, Example 3.6 and 3.7] that the metric regularity of implicit multifunctions is not equivalent to the Lipschitz-like property in general.

Our main objective of this paper is to establish new sufficient conditions for both the (local) metric regularity and Lipschitzian stability of the solution map (1.2) in Asplund space setting. The basic tools of our analysis involve the Fréchet normal coderivative and the Mordukhovich normal coderivative of set-valued mappings, the basic subgradient estimate for marginal functions and the Ekeland variational principle. Applications to the pointbased characterizations for both the metric regularity and the Lipschitzian stability of solution mapping of parametric generalized equations (1.1)–(1.3) are also given without any requirements of either the surjective property of partial gradient of f , Q does not depend x or Q is graphically regular at a given point. Some our results generalized the corresponding results presented in [7, 10, 27].

The paper is organized as follows. In Section 2 we recall some basic definitions and preliminaries from variational analysis and the generalized differentiation. In Section 3 we derive pointbased sufficient conditions for both the (local) metric

regularity and Lipschitzian stability of the implicit multifunction (1.2), and the relationships with the corresponding known results. Applications to stability analysis for solution mapping of parametric generalized equations are given in Section 4.

2. PRELIMINARIES

Throughout the paper we use standard notation of variational analysis and generalized differentiation. We refer the reader to the books by Mordukhovich [16, 17] for more details and discussions. Unless otherwise stated, all spaces under consideration are Banach spaces whose norms are always denoted by $\|\cdot\|$. For any X we consider its dual space X^* equipped with the weak* topology w^* where $\langle \cdot, \cdot \rangle$ means the canonical pairing. As usual, B_X and B_{X^*} stand for the closed unit balls of the Banach space X and its dual, respectively. The symbol A^* is the adjoint operator of a linear continuous operator A . The closed ball with center x and radius ρ is denoted by $B_\rho(x)$.

Given a set-valued mapping $F: X \rightrightarrows X^*$ between a Banach space X and its topological dual X^* , we denote by

$$\text{Lim sup}_{x \rightarrow \bar{x}} F(x) := \left\{ x^* \in X^* \mid \begin{array}{l} \exists \text{ sequences } x_k \rightarrow \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \\ \text{with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N} \end{array} \right\}$$

the *sequential Painlevé-Kuratowski upper/outer limit* with respect to the norm topology of X and the weak* topology of X^* , where $\mathbb{N} := \{1, 2, \dots\}$.

Given $\Omega \subset X$ and $\varepsilon \geq 0$, define the collection of ε -normals to Ω at $\bar{x} \in \Omega$ by

$$(2.1) \quad \widehat{N}_\varepsilon(\bar{x}; \Omega) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \rightarrow \bar{x}$ with $x \in \Omega$. When $\varepsilon = 0$, the set $\widehat{N}(\bar{x}; \Omega) := \widehat{N}_0(\bar{x}; \Omega)$ in (2.1) is a cone called the *prenormal cone* or the *Fréchet normal cone* to Ω at \bar{x} .

The *Mordukhovich normal cone* $N(\bar{x}; \Omega)$ is obtained from $\widehat{N}_\varepsilon(x; \Omega)$ by taking the sequential Painlevé-Kuratowski upper limit in the weak* topology of X^* as

$$(2.2) \quad N(\bar{x}; \Omega) := \text{Lim sup}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x; \Omega),$$

where one can put $\varepsilon = 0$ when Ω is closed around \bar{x} and the space X is *Asplund*, i.e., a Banach space whose separable subspaces have separable duals. The subset

$\Omega \subset X$ is said to be (locally) closed around \bar{x} if there is a neighborhood U of \bar{x} such that $\Omega \cap \text{cl}U$ is closed.

Let $F: X \rightrightarrows Y$ be a set-valued mapping between Banach spaces with the graph

$$\text{gph } F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

The *Mordukhovich normal coderivative* $D^*F(\bar{x}, \bar{y}): Y^* \rightrightarrows X^*$ of F at $(\bar{x}, \bar{y}) \in \text{gph } F$ is defined by

$$(2.3) \quad D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph } F)\}.$$

The *Fréchet normal coderivative* at $(\bar{x}, \bar{y}) \in \text{gph } F$ is defined by

$$(2.4) \quad \widehat{D}^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}((\bar{x}, \bar{y}); \text{gph } F)\} \quad \forall y^* \in Y^*.$$

A single-valued mapping $f: X \rightarrow Y$ is said to be *strictly differentiable* at \bar{x} if there is a linear continuous operator $\nabla f(\bar{x}): X \rightarrow Y$ such that

$$\lim_{x, u \rightarrow \bar{x}} \frac{f(x) - f(u) - \langle \nabla f(\bar{x}), x - u \rangle}{\|x - u\|} = 0.$$

We know that for such mappings one has

$$D^*f(\bar{x})(y^*) = \widehat{D}^*f(\bar{x})(y^*) = \{(\nabla f(\bar{x}))^*y^*\} \quad \forall y^* \in Y^*,$$

i.e., the Mordukhovich normal coderivative (resp., Fréchet normal coderivative) is a generalization of the adjoint operator to the classical Jacobian/strict derivative. For more details, we refer the reader to [16].

For an extended real-valued function $\varphi: X \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$, we define

$$\text{dom } \varphi = \{x \in X \mid |\varphi(x)| < \infty\}, \quad \text{epi } \varphi = \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\},$$

and say that φ is *lower semicontinuous* at $\bar{x} \in X$ if $\liminf_{x \rightarrow \bar{x}} \varphi(x) \geq \varphi(\bar{x})$. Here $\liminf_{x \rightarrow \bar{x}}$ denotes the lower limit of scalar functions in the classical sense.

Each extended real-valued function $\varphi: X \rightarrow \overline{\mathbb{R}}$ corresponds to a multifunction $\Phi: X \rightrightarrows \mathbb{R}$ defined by

$$\Phi(x) = E_\varphi(x) := \{\mu \in \mathbb{R} \mid \mu \geq \varphi(x)\} \quad \forall x \in X,$$

which is called the *associated epigraphical multifunction*. The *limiting or basic/Mordukhovich subdifferential* $\partial\varphi(\bar{x})$ of φ at $\bar{x} \in \text{dom } \varphi$ is defined via the coderivative $D^*\Phi(\bar{x}, \bar{y})$ as follows

$$\partial\varphi(\bar{x}) := D^*E_\varphi(\bar{x}, \varphi(\bar{x}))(1) = \{x^* \in X \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi)\}.$$

If $\bar{x} \notin \text{dom}\varphi$ then one puts $\partial\varphi(\bar{x}) = \emptyset$. If \bar{x} is a *local minimum* of φ , then

$$(0, -1) \in \widehat{N}((\bar{x}, \varphi(\bar{x})); \text{epi}\varphi) \subset N((\bar{x}, \varphi(\bar{x})); \text{epi}\varphi).$$

The *presubdifferential* or *Fréchet subdifferential* of φ at $\bar{x} \in \text{dom}\varphi$ is denoted by

$$\widehat{\partial}\varphi(\bar{x}) := \{x^* \in X \mid (x^*, -1) \in \widehat{N}((\bar{x}, \varphi(\bar{x})); \text{epi}\varphi)\}.$$

We recall that a set Ω is *sequentially normally compact* (SNC) at \bar{x} if for any sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$ one has

$$[x_k^* \xrightarrow{w^*} 0] \implies [\|x_k^*\| \rightarrow 0] \text{ as } k \rightarrow \infty,$$

where ε_k can be omitted if X is Asplund and if Ω is locally closed around \bar{x} . A set-valued mapping $F: X \rightrightarrows Y$ is *SNC* at $(\bar{x}, \bar{y}) \in \text{gph}F$ if its graph enjoys this property. F is *partially sequentially normally compact* (PSNC) at (\bar{x}, \bar{y}) if for any sequence $(\varepsilon_k, x_k, y_k, x_k^*, y_k^*) \in [0, \infty) \times (\text{gph}F) \times X^* \times Y^*$ satisfying

$$\varepsilon_k \downarrow 0, (x_k, y_k) \rightarrow (\bar{x}, \bar{y}), x_k^* \in \widehat{D}_{\varepsilon_k}^* F(x_k, y_k)(y_k^*), x_k^* \xrightarrow{w^*} 0 \text{ and } \|y_k^*\| \rightarrow 0$$

one has $\|x_k^*\| \rightarrow 0$ as $k \rightarrow \infty$.

Let $\varphi: X \rightarrow \overline{\mathbb{R}}$ be finite at \bar{x} . We say that φ is *sequentially normally epi-compact* (SNEC) at \bar{x} if its epigraph is (SNC) at $(\bar{x}, \varphi(\bar{x}))$.

Let us recall the definitions of the (local) metric regularity and Lipschitz-like multifunctions introduced by Robinson [24] and Aubin [1], respectively.

Definition 2.1. [16]. Let $\Phi: X \rightrightarrows Y$ be a multifunction between Asplund spaces. Let $(\bar{x}, \bar{y}) \in \text{gph}\Phi$.

(a) Φ is said to be *local-metrically regular* around (\bar{x}, \bar{y}) with modulus $c > 0$ if there exist a neighborhood U of \bar{x} , a neighborhood V of \bar{y} , and a number $\mu > 0$ such that

$$\text{dist}(x, \Phi^{-1}(y)) \leq c \text{dist}(y, \Phi(x))$$

for any $x \in U$ and $y \in V$ satisfying $\text{dist}(y, \Phi(x)) \leq \mu$.

(b) Φ is said to be *pseudo-Lipschitzian* or *Lipschitz-like* around (\bar{x}, \bar{y}) with modulus $\ell > 0$ if there exist a neighborhood U of \bar{x} and a neighborhood V of \bar{y} such that

$$\Phi(x_1) \cap V \subset \Phi(x_2) + \ell\|x_1 - x_2\|B_X \quad \forall x_1, x_2 \in U.$$

3. CHARACTERIZATIONS OF METRIC REGULARITY AND LIPSCHITZIAN STABILITY

In this section we establish the sufficient conditions for the metric regularity and the Lipschitzian stability of the implicit multifunction (1.2).

Let Z be an Asplund space and let $\Omega \subset Z$. We say that Ω has the *robustness* property at \bar{z} belonging to closure of Ω if for any sequences $z_k \xrightarrow{\Omega} \bar{z}$ and $z_k^* \xrightarrow{w^*} z^*$ with $z_k^* \in N(z_k; \Omega)$, $k = 1, 2, \dots$, there exists a subsequence $\{z_{k_j}^*\}$ of $\{z_k^*\}$ satisfying $z_{k_j}^* \rightarrow z^*$ in the normal topology of Z^* and $z^* \in N(\bar{z}; \Omega)$. Obviously, the robustness property of Ω at \bar{z} automatically holds if Z is a finite-dimensional space.

Let $\Xi: X \rightrightarrows Y$ be a multifunction and let $(x_0, y_0) \in \text{gph } \Xi$. The Mordukhovich normal coderivative of Ξ is said to be robust at (x_0, y_0) if its graph has the robustness property at (x_0, y_0) .

One says that Ξ is *inner semicompact* around $x_0 \in X$ if there exists a neighborhood U of x_0 such that for any $x \in U$ and any sequence $x_k \rightarrow x$, there is a sequence $y_k \in \Xi(x_k)$, $k = 1, 2, \dots$, which contains a subsequence convergent in the norm topology of Y .

Definition 3.1. [24]. For the variational system defined as in (1.1) and (1.2), let $\omega_0 := (x_0, y_0, 0) \in \text{gph } F$. The implicit multifunction G is said to be *locally metrically regular* around ω_0 with modulus $c > 0$ if there exist a neighborhood U of x_0 , a neighborhood V of y_0 and a number $\mu > 0$ such that

$$(3.1) \quad \text{dist}(x, G(y)) \leq c \text{dist}(0, F(x, y))$$

for any $x \in U$ and $y \in V$ satisfying $\text{dist}(0, F(x, y)) \leq \mu$.

Given a subset $\Omega \subset X$ and a point $u \in X$, we denote the set of the *metric projections* of u on the closure of Ω by $\mathcal{M}(u, \Omega)$, that is

$$\mathcal{M}(u, \Omega) = \{x \in \text{cl } \Omega \mid \|x - u\| = \text{dist}(u, \Omega)\}$$

where $\text{dist}(u, \Omega) := \inf_{z \in \Omega} \|z - u\|$ is the *distance* from u to Ω and $\text{cl } \Omega$ denotes the *closure* of Ω .

For the variational system defined as in (1.1) and (1.2), let $\omega_0 := (x_0, y_0, 0) \in \text{gph } F$ and let $F_y(\cdot) := F(\cdot, y)$. Consider the following assumption (SC):

There are a neighborhood U_0 of x_0 , a neighborhood V_0 of y_0 such that for any $y \in V_0$ and for any $x \in U_0$, the multifunction $\mathcal{M}(0, F_y(\cdot))$ is inner semicompact around x .

We first give the pointbased sufficient conditions for the metric regularity and the Lipschitzian stability of implicit multifunction (1.2).

Theorem 3.2. . Let X, Y, Z be Asplund spaces, a multifunction $F : X \times Y \rightrightarrows Z$ and a multifunction $G : Y \rightrightarrows X$ be defined as in (1.1) and (1.2). Let $\omega_0 := (x_0, y_0, 0) \in \text{gph } F$. Suppose that $\text{gph } F$ is locally closed around ω_0 and (SC) is valid. If the Mordukhovich normal coderivative of F has the robustness property at ω_0 and the following condition

$$(PB) \quad \forall (y^*, z^*) \in Y^* \times Z^*, (0, y^*) \in D^*F(\omega_0)(z^*) \implies (y^*, z^*) = (0, 0)$$

holds, then G is local-metrically regular around ω_0 with modulus $\frac{1}{\sigma} > 0$, i.e., there exist a neighborhoods U of x_0 , a neighborhood V of y_0 , a number $\sigma > 0$ and a number $\mu > 0$ such that

$$(3.2) \quad \text{dist}(x, G(y)) \leq \frac{1}{\sigma} \text{dist}(0, F(x, y))$$

for any $x \in U$ and $y \in V$ satisfying $\text{dist}(0, F(x, y)) \leq \mu$. Moreover, for any $x^* \in X^*$,

$$(3.3) \quad D^*G(y_0, x_0)(x^*) \subset \bigcup_{z^* \in Z^*} \{y^* : (-x^*, y^*) \in D^*F(\omega_0)(z^*)\}$$

and G is Lipschitz-like around (y_0, x_0) .

For proving Theorem 3.2, we need the following auxiliary results.

We first consider the parametric minimization problem

$$(3.4) \quad \min\{\varphi(x, y) \mid y \in \Phi(x)\}$$

depending on the parameter x and the corresponding *marginal function*

$$(3.5) \quad m(x) := \inf\{\varphi(x, y) : y \in \Phi(x)\},$$

where $\varphi : X \times Y \rightarrow \bar{\mathbb{R}}$ is an extended real-valued function and $\Phi : X \rightrightarrows Y$ is a multifunction between Banach spaces. Let

$$(3.6) \quad M(x) := \{y \in \Phi(x) \mid \varphi(x, y) = m(x)\}$$

denote the parametric *solution set* of (3.4).

Lemma 3.3. (Basic subgradient of marginal function) [16, Theorem 3.38]. Let $\Phi : X \rightrightarrows Y$ be a closed-graph multifunction between Asplund spaces X and Y , let $\varphi : X \times Y \rightarrow \bar{\mathbb{R}}$ be lower semicontinuous on $\text{gph } \Phi$. Suppose that for any $\bar{y} \in M(\bar{x})$, the multifunction M in (3.6) be inner semicompact at (\bar{x}, \bar{y}) and either φ is SNEC at (\bar{x}, \bar{y}) or $\text{gph } \Phi$ is SNC at (\bar{x}, \bar{y}) and the qualification constraint

$$(3.7) \quad \partial^\infty \varphi(\bar{x}, \bar{y}) \cap (-N((\bar{x}, \bar{y}); \text{gph } \Phi)) = \{0\}$$

is satisfied. Then one has the inclusion

$$(3.8) \quad \partial m(\bar{x}) \subset \bigcup \{x^* + D^* \Phi(\bar{x}, \bar{y})(y^*) : (x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y}), \bar{y} \in M(\bar{x})\}.$$

Given a single-valued mapping $f: X \rightarrow Y$ between Banach spaces. Let $\bar{x} \in X$. f is said to be *locally Lipschitzian* around \bar{x} if there exist a neighborhood U of \bar{x} and a number $\ell \geq 0$ such that

$$\|f(x_1) - f(x_2)\| \leq \ell \|x_1 - x_2\| \text{ for all } x_1, x_2 \in U.$$

We recall from [16, Corollary 1.81] that if X is a Banach space and $\varphi: X \rightarrow \bar{\mathbb{R}}$ is locally Lipschitzian around \bar{x} , then $\partial^\infty \varphi(\bar{x}) = \{0\}$.

Lemma 3.4. (Sum rules for basic subgradient) [16, Theorem 3.36]. *Let X be an Asplund space, let $\varphi_i: X \rightarrow \bar{\mathbb{R}}$, $i = 1, 2$, be lower semicontinuous at \bar{x} , and one of these functions be SNEC at \bar{x} . Suppose that*

$$[x_i^* \in \partial^\infty \varphi_i(\bar{x}), i = 1, 2, x_1^* + x_2^* = 0] \implies x_1^* = x_2^* = 0.$$

Then one has the inclusion

$$\partial(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial \varphi_1(\bar{x}) + \partial \varphi_2(\bar{x}).$$

The following result derives a sufficient condition for the metric regularity of G via the partial Mordukhovich normal coderivative of F in x .

Theorem 3.5. *Let X, Z be Asplund spaces, Y a metric space, $F: X \times Y \rightrightarrows Z$ and $G: Y \rightrightarrows X$ multifunctions are defined as in (1.1) and (1.2), respectively. Let $\omega_0 := (x_0, y_0, 0) \in \text{gph } F$. Suppose that $\text{gph } F$ is locally closed around ω_0 and (SC) is valid. If there exist $\sigma > 0$ and neighborhood $U_0 \times V_0$ of (x_0, y_0) such that for any $(x, y) \in U_0 \times V_0$ with $0 \notin F(x, y)$ satisfying*

$$(3.9) \quad \sigma \leq \inf \{\|x^*\| : x^* \in D^* F_y(x, z)(z^*), z \in \mathcal{M}(0, F_y(x)), \|z^*\| = 1\},$$

then G is metrically regular around ω_0 with modulus $\frac{1}{\sigma} > 0$, i.e., there exist a neighborhood U of x_0 , a neighborhood V of y_0 and a number $\mu > 0$ such that

$$(3.10) \quad \text{dist}(x, G(y)) \leq \frac{1}{\sigma} \text{dist}(0, F(x, y))$$

for any $x \in U$ and $y \in V$ satisfying $\text{dist}(0, F(x, y)) \leq \mu$. Moreover, if $F: X \times Y \rightrightarrows Z$ is partially Lipschitz-like in y with rank ℓ around ω_0 , i.e., there exist

a neighborhood U_1 of x_0 , a neighborhood V_1 of y_0 and a neighborhood W_0 of 0 such that, for any $(x, y) \in U_1 \times V_1$ and $(x, y') \in U_1 \times V_1$,

$$F(x, y') \cap W_0 \subset F(x, y) + \ell \text{dist}(y', y) B_Y,$$

then G is Lipschitz-like with rank $\frac{\ell}{\sigma}$ around (y_0, x_0) , i.e., there exist a neighborhood V of y_0 and a neighborhood U of x_0 such that

$$G(y') \cap U \subset G(y) + \frac{\ell}{\sigma} \text{dist}(y', y) B_X \quad \forall y, y' \in V.$$

Proof. Let $\omega_0 := (x_0, y_0, 0) \in \text{gph } F$. By our assumptions, there are a neighborhood U_0 of x_0 and a neighborhood V_0 of y_0 such that (3.9) is valid. Choose a number $\mu > 0$ and a number $\rho > 0$ such that

$$(3.11) \quad \mu < \sigma \rho \quad \text{and} \quad B_\rho(x_0) \subset U_0.$$

We now examine the case $(x, y) \in \text{dom } F \cap (U_0 \times V_0)$ satisfying

$$(3.12) \quad \text{dist}(0, F(x, y)) \leq \mu.$$

For convenience we will ignore $(x, y) \in \text{dom } F$. Let $U := U_0$ and $V := V_0$. We want to show that U, V together with constants μ and σ satisfy the conclusion of the theorem. Fix any $x \in U$ and $y \in V$ and assume that (3.12) is satisfied. Put $\alpha := \text{dist}(0, F(x, y))$. By (3.11) and (3.12), $\alpha < \sigma \rho$. It remains to show that

$$(3.13) \quad \text{dist}(x, G(y)) \leq \frac{\alpha}{\sigma}.$$

Consider the function $v_y(u) := \text{dist}(0, F_y(u))$, $u \in U$. We claim that $v_y(\cdot)$ is lower semicontinuous on U . Indeed, if there is a sequence $x_k \rightarrow u$ and $\varepsilon > 0$ such that $v_y(x_k) \leq v_y(u) - \varepsilon$ for every $k \in \mathbb{N}$, then, by (SC), there exist a sequence $\{k_j\} \subset \{k\}$ and a sequence $z_{k_j} \in \mathcal{M}(0, F_y(x_{k_j}))$ such that $z_{k_j} \rightarrow z \in Z$. Since $z_{k_j} \in F_y(x_{k_j}) = F(x_{k_j}, y)$ for every $j \in \mathbb{N}$, the closedness of $\text{gph } F$ around ω_0 implies $z \in F_y(u)$. Hence, from the relation

$$\|z_{k_j}\| = v_y(x_{k_j}) \leq v_y(u) - \varepsilon \quad \forall j \in \mathbb{N},$$

it follows that $v_y(u) \leq \|z\| \leq v_y(u) - \varepsilon$, which is impossible and our claim is proved. Therefore, $v(\cdot)$ is lower semicontinuous on $B_\rho(x_0)$. Fixing any $\delta \in (\frac{\alpha}{\rho}, \sigma)$,

we have $v_y(x) = \alpha < \alpha \frac{\sigma}{\delta}$. Putting $t := \frac{\delta}{\alpha} v_y(x)$, we see that

$$v_y(x) = t \frac{\alpha}{\delta}, \quad \text{and } t \in (0, \sigma).$$

Clearly, $v_y(x) \leq \inf_{u \in B_\rho(x_0)} v_y(u) + t \frac{\alpha}{\delta}$. From the Ekeland variational principle [16, Theorem 2.26], it follows that there is $\bar{x} \in B_\rho(x_0)$ such that

$$(3.14) \quad v_y(\bar{x}) \leq v_y(x), \quad \|\bar{x} - x\| \leq \frac{\alpha}{\delta}$$

and

$$(3.15) \quad v_y(\bar{x}) \leq v_y(u) + t\|u - \bar{x}\| \quad \forall u \in B_\rho(x_0).$$

We next claim that

$$0 \in F_y(\bar{x}).$$

Conversely, suppose that $0 \notin F_y(\bar{x})$. It follows from (3.15) that \bar{x} is a local minimum of the function

$$\psi(u) := v_y(u) + \chi(u), \quad u \in B_\rho(x_0),$$

where $\chi(u) := t\|u - \bar{x}\|$ is a Lipschitz function. From the nonsmooth version of Fermat's rule [16, Proposition 1.114], it follows that

$$0 \in \partial\psi(\bar{x}).$$

By the Lipschitzian property of χ and Corollary 1.81 in [16], we have $\partial^\infty\chi(\bar{x}) = \{0\}$ and χ is sequentially normally epi-compact at \bar{x} . From Lemma 3.4, we have

$$(3.16) \quad 0 \in \partial v_y(\bar{x}) + tB_{X^*}.$$

Let us now compute $\partial v_y(\bar{x})$. Define

$$\begin{aligned} \Phi(u) &:= F_y(u), \quad \varphi(u, z) := \|z\|, \\ m(u) &:= v_y(u) = \inf\{\varphi(u, z) : z \in \Phi(u)\}. \end{aligned}$$

Take arbitrary $\bar{z} \in M(\bar{x}) := \mathcal{M}(0, F_y(\bar{x}))$. Since φ is locally Lipschitzian at (\bar{x}, \bar{z}) , it is sequentially normally epi-compact at this point and

$$\partial^\infty\varphi(\bar{x}, \bar{z}) = \{(0, 0)\}.$$

Define $\theta(z) := \|z\|$. It is easy to check that

$$\partial\varphi(\bar{x}, \bar{z}) = \{0\} \times \partial\theta(\bar{z}).$$

Applying Lemma 3.3, we obtain

$$(3.17) \quad \partial v_y(\bar{x}) \subset \bigcup [D^*F_y(\bar{x}, \bar{z})(z^*) \mid z^* \in \partial\theta(\bar{z}), \bar{z} \in \mathcal{M}(0, F_y(\bar{x}))].$$

The condition $0 \notin F_y(\bar{x})$ implies $\bar{z} \neq 0$ for every $\bar{z} \in \mathcal{M}(0, F_y(\bar{x}))$. It follows that

$$(3.18) \quad \partial\theta(\bar{z}) = \{z^* \in Z^* \mid \|z^*\| = 1, \langle z^*, \bar{z} \rangle = \|\bar{z}\|\}.$$

From (3.16)–(3.18), there exist vectors $\bar{z} \in \mathcal{M}(0, F_y(\bar{x}))$, $z^* \in Z^*$ with $\|z^*\| = 1$, and $x^* \in D^*F_y(\bar{x}, \bar{z})(z^*)$ such that $\|x^*\| \leq t < \sigma$. Besides, by (3.9), we can assert that $\sigma \leq \|x^*\|$, this contradicts the inequality $\|x^*\| < \sigma$. We have thus shown that $0 \in F_y(\bar{x})$, i.e., $\bar{x} \in G(y)$. Hence, by (3.14),

$$\text{dist}(x, G(y)) \leq \|x - \bar{x}\| \leq \frac{\alpha}{\delta}.$$

Letting $\delta \rightarrow \sigma$ we obtain $\text{dist}(x, G(y)) \leq \frac{1}{\sigma} \text{dist}(0, F(x, y))$ and G is local-metrically regular around ω_0 .

Suppose that F is partially Lipschitz-like in y with rank ℓ around ω_0 . Then there exist a neighborhood U_1 of x_0 , a neighborhood V_1 of y_0 and a neighborhood W_0 of 0 such that, for any $(x, y), (x, y') \in U_1 \times V_1$,

$$(3.19) \quad F(x, y') \cap W_0 \subset F(x, y) + \ell \text{dist}(y', y) B_Y.$$

Choose $\mu > 0$ such that

$$B_\mu(0) \subset W_0.$$

It follows from the regularity of G around ω_0 that there exist a neighborhood U_2 of x_0 , a neighborhood V_2 of y_0 and a number $\sigma > 0$ such that

$$(3.20) \quad \text{dist}(x, G(y)) \leq \frac{1}{\sigma} \text{dist}(0, F(x, y))$$

for any $x \in U_2$ and $y \in V_2$ satisfying $\text{dist}(0, F(x, y)) \leq \mu$. Let $U := U_1 \cap U_2$ and $V := V_1 \cap V_2$. Then

$$(3.21) \quad G(y') \cap U \subset G(y) + \frac{\ell}{\sigma} \text{dist}(y', y) B_X \quad \forall y, y' \in V.$$

Indeed, take arbitrary $x \in G(y') \cap U$. From (3.19) and (3.20), it follows that

$$\text{dist}(x, G(y)) \leq \frac{1}{\sigma} \text{dist}(0, F(x, y)) \leq \frac{\ell}{\sigma} \text{dist}(y', y) \quad \forall y, y' \in V.$$

Hence, $x \in G(y) + \frac{\ell}{\sigma} \text{dist}(y', y) B_X \quad \forall y, y' \in V$. So, (3.21) follows. The proof is complete. \blacksquare

Remark 3.6. Obviously, (A1) in [10, Theorem 3.1] implies (3.9). The corresponding results for the metric regularity of (1.2) in [10, Theorem 3.2] always require the lower continuity of $F(x_0, \cdot)$ at y_0 and $F(\cdot)$ is lower semicontinuous at (x_0, y_0) . Hence, Theorem 3.5 extends Theorem 3.1, Theorem 3.3 in [10] and Theorem 3.4 in [7]. The condition (3.9) is similar to the condition (iv') in [9, Theorem 3.6], but Ledyaev and Zhu [9] assumed that X and Z are Banach spaces with Fréchet-smooth Lipschitzian bump functions, $F(x_0, \cdot)$ is lower semicontinuous at y_0 and for any fixed $y \in V_0$, $F(\cdot, y)$ is upper semicontinuous.

Lemma 3.7. (Partial coderivative) [16, Corollary 3.17]. *Let $F : X \times Y \rightrightarrows Z$ be multifunction between Asplund spaces. Let $\bar{\omega} := (\bar{x}, \bar{y}, \bar{z}) \in \text{gph } F$ and let $F_y(\cdot) := F(\cdot, y)$. Suppose that $\text{gph } F$ is locally closed around $\bar{\omega}$ and PSNC at this point, and that*

$$(0, y^*) \in D_{MF}^*(\bar{\omega})(0) \implies y^* = 0;$$

these conditions automatically hold when F is Lipschitz-like around $\bar{\omega}$. Then one has the inclusion

$$(3.22) \quad D^*F_{\bar{y}}(\bar{x}, \bar{z})(z^*) \subset \left\{ x^* \in X^* : \exists y^* \in Y^* \text{ such that } (x^*, y^*) \in D^*F(\bar{\omega})(z^*) \right\}.$$

Next, we give the relationship between the pointbased criteria of F and the partial Mordukhovich normal coderivative of F around a given point.

Theorem 3.8. *Let X, Y, Z be Asplund spaces and a multifunction $F : X \times Y \rightrightarrows Z$. Let $\bar{\omega} := (\bar{x}, \bar{y}, 0) \in \text{gph } F$. Suppose that $\text{gph } F$ is locally closed around $\bar{\omega}$ and its Mordukhovich normal coderivative has the robustness property at this point. Consider the following statements:*

(i) *For any $(y^*, z^*) \in Y^* \times Z^*$, $(0, y^*) \in D^*F(\bar{\omega})(z^*) \implies (y^*, z^*) = (0, 0)$;*

(ii) *There exist a constant $c > 0$, a neighborhood U of \bar{x} , a neighborhood V of \bar{y} and a neighborhood W of 0 such that for any point $\omega = (x, y, z) \in \text{gph } F \cap (U \times V \times W)$, it holds*

$$(3.23) \quad \|z^*\| \leq c\|x^*\| \quad \forall z^* \in Z^*, \forall x^* \in D^*F_y(x, z)(z^*);$$

(iii) *There exist a number $\sigma > 0$, a neighborhood U of \bar{x} and a neighborhood V of \bar{y} such that for any $(x, y) \in U \times V$ with $0 \notin F(x, y)$*

$$(3.24) \quad \sigma \leq \inf\{\|x^*\| : x^* \in D^*F_y(x, z)(z^*), z \in \mathcal{M}(0, F_y(x)), \|z^*\| = 1\}.$$

Then (i) \implies (ii) \implies (iii).

Proof. We first prove that (i) \implies (ii). Suppose that (i) holds. We first claim that there exist a neighborhood U of \bar{x} , a neighborhood V of \bar{y} and a neighborhood W of 0 such that for any point $\omega = (x, y, z) \in \text{gph } F \cap (U \times V \times W)$ and for any $(y^*, z^*) \in Y^* \times Z^*$ satisfying

$$(3.25) \quad (0, y^*) \in D^*F(\omega)(z^*) \implies (y^*, z^*) = (0, 0).$$

Indeed, if our claim is false, then there exist sequences $\omega_k = (x_k, y_k, z_k) \in \text{gph } F$ and $(y_k^*, z_k^*) \in Y^* \times Z^* \setminus \{(0, 0)\}$ such that for every $k = 1, 2, \dots$

$$(3.26) \quad (0, y_k^*) \in D^*F(\omega_k)(z_k^*) \text{ and } \omega_k \rightarrow \bar{\omega} \text{ as } k \rightarrow \infty.$$

Without loss of generality we can assume that $\|z_k^*\| = 1$ for every $k \in \mathbb{N}$. Consider the following two cases:

Case 1. $\{y_k^*\}$ is bounded. Since Y is a Asplund space, the unit ball of the dual space Y^* is sequentially weak* compact. Taking into account the boundedness of $\{(y_k^*, z_k^*)\}$, one may assume that $(y_k^*, z_k^*) \xrightarrow{w^*} (y^*, z^*) \in Y^* \times Z^*$. Clearly,

$$(0, y_k^*, -z_k^*) \in N(\omega_k; \text{gph } F).$$

It follows from the robustness property of Mordukhovich normal coderivative of F at ω_0 that $(y_k^*, z_k^*) \rightarrow (y^*, z^*)$ as $k \rightarrow \infty$ in the norm topology of $Y^* \times Z^*$ and $(0, y^*, -z^*) \in N(\bar{\omega}; \text{gph } F)$. Then $\|z^*\| \neq 0$ and $(0, y^*) \in D^*F(\bar{\omega})(z^*)$ which is a contradiction to (i).

Case 2. $\{y_k^*\}$ is not bounded. Then there is a subsequence $\{y_{k_j}^*\}$ of $\{y_k^*\}$ such that $\|y_{k_j}^*\| \rightarrow \infty$ as $j \rightarrow \infty$. Without loss of generality, we may assume that

$$\left(\frac{y_{k_j}^*}{\|y_{k_j}^*\|}, \frac{z_{k_j}^*}{\|y_{k_j}^*\|} \right) \xrightarrow{w^*} (y^*, z^*) \in Y^* \times Z^*.$$

Analysis similar to that as in Case 1 shows that $\|y^*\| \neq 0$ and $(0, y^*) \in D^*F(\bar{\omega})(z^*)$ which contradicts (i). Hence, our claim is proved. Since the Mordukhovich normal coderivative of F is robust at $\bar{\omega}$, it follows that F is SNC at this point. Combining this with (i) and Theorem 4.10 in [16], we deduce that F is Lipschitz-like around $\bar{\omega}$. By (3.25) and Lemma 3.7, we have

$$(3.27) \quad D^*F_y(x, z)(z^*) \subset \left\{ x^* \in X^* : \exists y^* \in Y^* \text{ such that } (x^*, y^*) \in D^*F(\omega)(z^*) \right\}$$

for any $\omega = (x, y, z) \in \text{gph } F \cap (U \times V \times W)$ and $z^* \in Z^*$.

Our next claim that (ii) holds. On the contrary, suppose that the conclusion of (ii) is not true. Then we can find sequences $\omega_k = (x_k, y_k, z_k) \in \text{gph } F$ and

$(x_k^*, z_k^*) \in X^* \times Z^*$ such that $\omega_k \rightarrow \bar{\omega}$, $x_k^* \in D^*F_{y_k}(x_k, z_k)(z_k^*)$, and $\|z_k^*\| > k\|x_k^*\|$ for all $k \in \mathbb{N}$. There is no loss of generality in assuming that $\|z_k^*\| = 1$ for all $k \in \mathbb{N}$. Then

$$(3.28) \quad x_k^* \in D^*F_{y_k}(x_k, z_k)(z_k^*) \text{ and } \|x_k^*\| \leq \frac{1}{k} \quad \forall k \in \mathbb{N}.$$

It follows from (3.27) that there exists a sequence $\{y_k^*\} \subset Y^*$ such that $(x_k^*, y_k^*) \in D^*F(\omega_k)(z_k^*)$, where $\omega_k := (x_k, y_k, z_k)$. Hence

$$(x_k^*, y_k^*, -z_k^*) \in N(\omega_k; \text{gph } F).$$

If the sequence $\{y_k^*\}$ is bounded then, by the same analysis as in Case 1, it follows that there exists a subsequence $\{(x_{k_j}^*, y_{k_j}^*, z_{k_j}^*)\}$ of $\{(x_k^*, y_k^*, z_k^*)\}$ converging to $(0, y^*, z^*)$ in the norm topology of $X^* \times Y^* \times Z^*$ such that $\|z^*\| \neq 0$ and $(0, y^*, -z^*) \in N(\bar{\omega}; \text{gph } F)$. Hence $\|z^*\| \neq 0$ and $(0, y^*) \in D^*F(\bar{\omega})(z^*)$ which contradicts (i). If the sequence $\{y_k^*\}$ is not bounded then, by the same method as in Case 2, it follows that there exists a subsequence $\left\{ \left(\frac{x_{k_j}^*}{\|y_{k_j}^*\|}, \frac{y_{k_j}^*}{\|y_{k_j}^*\|}, \frac{z_{k_j}^*}{\|y_{k_j}^*\|} \right) \right\}$ of $\left\{ \left(\frac{x_k^*}{\|y_k^*\|}, \frac{y_k^*}{\|y_k^*\|}, \frac{z_k^*}{\|y_k^*\|} \right) \right\}$ converging to $(0, y^*, z^*)$ in the norm topology of $X^* \times Y^* \times Z^*$ such that $\|y^*\| \neq 0$ and $(0, y^*) \in D^*F(\bar{\omega})(z^*)$ which is a contradiction to (i).

From the above, it follows that there must exist a constant $c > 0$, a neighborhood U of \bar{x} , a neighborhood V of \bar{y} and a neighborhood W of 0 such that for any point $\omega = (x, y, z) \in \text{gph } F \cap (U \times V \times W)$, (3.23) is fulfilled. The implication [(ii) \implies (iii)] is trivial. The proof is complete. \blacksquare

Proof of Theorem 3.2. Obviously, the local-metric regularity of G at ω_0 immediately follows from Theorem 3.5 and Theorem 3.8. Now let us examine the formula (3.3). We first observe that the graph of the mapping G under consideration can be represented as follows

$$\text{gph } G = \{(y, x) \in Y \times X \mid g(x, y) \in \Theta\} \text{ with } \Theta := \text{gph } F,$$

where $g(x, y) := (x, y, 0)$. Obviously, g is a strictly differentiable function and

$$(3.29) \quad (\nabla g(x_0, y_0))^*(x^*, y^*, z^*) = (x^*, y^*) \quad \forall (x^*, y^*, z^*) \in X^* \times Y^* \times Z^*.$$

We have

$$(3.30) \quad N(\omega_0; \Theta) \cap \ker (\nabla g(x_0, y_0))^* = \{0\}.$$

Indeed, let $(x^*, y^*, z^*) \in N(\omega_0; \Theta) \cap \ker(\nabla g(x_0, y_0))^*$. Then, by (3.29), $(x^*, y^*) = (0, 0)$ and $(0, 0) \in D^*F(\omega_0)(z^*)$. It follows from (PB) that $z^* = 0$ and (3.30) is fulfilled. Applying Corollary 3.42 in [16] we have

$$(3.31) \quad N(\omega_0; g^{-1}(\Theta)) \subset (\nabla g(\omega_0))^*N(\omega_0; \Theta).$$

For each $x^* \in X^*$, let $y^* \in D^*G(y_0, x_0)(x^*)$. Then $(y^*, -x^*) \in N((y_0, x_0); \text{gph } G)$. It is easy to check that

$$(3.32) \quad (y^*, -x^*) \in N((y_0, x_0); \text{gph } G) \iff (-x^*, y^*) \in N((x_0, y_0); g^{-1}(\Theta)).$$

From (3.29) and (3.31) it follows that

$$(3.33) \quad (-x^*, y^*) \in N((x_0, y_0); g^{-1}(\Theta)) \implies \exists z^* \in Z^*, (-x^*, y^*) \in D^*F(\omega_0)(z^*).$$

Therefore, (3.3) immediately follows from (3.32) and (3.33). From (3.3) it follows that $D^*G(y_0, x_0)(0) = \{0\}$. Since the Mordukhovich normal coderivative of F is robust at ω_0 , it follows that it is SNC at ω_0 . By Theorem 3.84 in [16], $g^{-1}(\Theta)$ is SNC at (x_0, y_0) (note that, by the special structure of g as above, we see check at once that Theorem 3.84 in [16] is still fulfilled with the local closedness of Θ around ω_0). Since $\text{gph } F$ is locally closed around ω_0 it follows that $\text{gph } G$ is locally closed around (y_0, x_0) . Applying Theorem 4.10 in [16] we conclude that G is Lipschitz-like around (y_0, x_0) . The proof is complete. ■

The following corollary extends the corresponding result in [27].

Corollary 3.9. *Let $F : \mathbb{R}^m \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a multifunction and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ a multifunction defined as in (1.1) and (1.2). Let $\omega_0 := (x_0, y_0, 0) \in \text{gph } F$. Suppose that $\text{gph } F$ is closed around ω_0 . If the following pointbased condition*

$$\forall (y^*, z^*) \in \mathbb{R}^n \times \mathbb{R}^p, (0, y^*) \in D^*F(\omega_0)(z^*) \implies (y^*, z^*) = (0, 0)$$

holds, then G is local-metrically regular around ω_0 and Lipschitz-like around (y_0, x_0) .

Proof. The proof is immediate from Theorem 3.2, so can be omitted. ■

4. APPLICATIONS TO STABILITY ANALYSIS FOR SOLUTION MAPPING OF PARAMETRIC GENERALIZED EQUATIONS

In this section we consider a special case of F in (1.1) which has the form $F(x, y) = f(x, y) + Q(x, y)$ where $f : X \times Y \rightarrow Z$ is a single-valued mapping

and $Q: X \times Y \rightrightarrows Z$ is a set-valued mapping between Asplund spaces. Then (1.1) becomes

$$(4.1) \quad 0 \in f(x, y) + Q(x, y).$$

This generalized equation was introduced by Robinson [24]. It is well known that model (4.1) provides a convenient framework for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibrium, etc.; see, e.g., [16, 17, 25] and the references therein for more information and discussions. When $Q(x, y) = N(y; \Omega)$ the normal cone operator for a convex set Ω , the parametric generalized equations (4.1) is reduced to the *parametric variational inequality*:

$$(4.2) \quad \text{Find } y \in \Omega \text{ such that } \langle f(x, y), z - y \rangle \geq 0 \quad \forall z \in \Omega,$$

which is of particular interest for applications.

The solution map $G: Y \rightrightarrows X$ associated with (4.1) is defined by

$$(4.3) \quad G(y) = \{x \in X : 0 \in f(x, y) + Q(x, y)\}.$$

For the variational system defined as in (4.1) and (4.3), let $\omega_0 := (x_0, y_0, -f(x_0, y_0)) \in \text{gph } Q$ and let $Q_y(\cdot) := Q(\cdot, y)$. Consider the following assumption (SC[?]):

There are a neighborhood U_0 of x_0 , a neighborhood V_0 of y_0 such that for any $y \in V_0$ and for any $x \in U_0$, the multifunction $\mathcal{M}(-f(\cdot, y), Q_y(\cdot))$ is inner semicompact around x .

In what follows we establish the sufficient conditions ensuring the metric regularity and Lipschitzian stability for parametric generalized equation (4.3).

Theorem 4.1. *Let X, Y, Z be Asplund spaces. Let $f: X \times Y \rightarrow Z$ be a single-valued mapping, $Q: X \times Y \rightrightarrows Z$ and $G: Y \rightrightarrows X$ multifunctions defined as in (4.1) and (4.3). Let $\omega_0 := (x_0, y_0, -f(x_0, y_0)) \in \text{gph } Q$. Suppose that f is strictly differentiable at (x_0, y_0) , $\text{gph } Q$ is locally closed around ω_0 and (SC[?]) is valid. If the Mordukhovich normal coderivative of Q has the robustness property at ω_0 and for each $(y^*, z^*) \in Y^* \times Z^*$, the following condition holds*

$$(4.4) \quad (-\nabla f_x(x_0, y_0)^*(z^*), y^* - \nabla f_y(x_0, y_0)^*(z^*)) \in D^*Q(\omega_0)(z^*) \implies (y^*, z^*) = (0, 0)$$

then G is local-metrically regular around ω_0 with modulus $\frac{1}{\sigma} > 0$, i.e., there exist a neighborhoods U of x_0 , a neighborhood V of y_0 , a number $\sigma > 0$ and a number $\mu > 0$ such that

$$(4.5) \quad \text{dist}(x, G(y)) \leq \frac{1}{\sigma} \text{dist}(-f(x, y), Q(x, y))$$

for any $x \in U$ and $y \in V$ satisfying $\text{dist}(-f(x, y), Q(x, y)) \leq \mu$. Moreover, for any $x^* \in X^*$,

$$(4.6) \quad D^*G(y_0, x_0)(x^*) \subset \bigcup_{z^* \in Z^*} \{y^* : (-x^* - \nabla f_x(x_0, y_0)^*(z^*), y^* - \nabla f_y(x_0, y_0)^*(z^*)) \in D^*Q(\omega_0)(z^*)\}$$

and G is Lipschitz-like around (y_0, x_0) .

Proof. Let $F(x, y) = f(x, y) + Q(x, y)$, $(x, y) \in X \times Y$. Since f is strictly differentiable at (x_0, y_0) and $\text{gph } Q$ is locally closed around ω_0 , it follows from Theorem 3.5 in [20] that

$$(4.7) \quad D^*F(x_0, y_0, 0)(z^*) = \nabla f(x_0, y_0)^*(z^*) + D^*Q(\omega_0)(z^*) \quad \forall z^* \in Z^*.$$

Hence, the Mordukhovich normal coderivative of F has the robustness property at $(x_0, y_0, 0)$. Clearly, $\text{gph } F$ is locally closed around $(x_0, y_0, 0)$. From (4.4) and (4.7) it follows that

$$\forall (y^*, z^*) \in Y^* \times Z^*, \quad (0, y^*) \in D^*F(x_0, y_0, 0)(z^*) \implies (y^*, z^*) = (0, 0).$$

Applying Theorem 3.2 we can assert that G is both local-metrically regular around ω_0 and Lipschitz-like around (y_0, x_0) , and (4.6) is fulfilled. The proof is complete. ■

The following corollary follows immediately from Theorem 4.1.

Corollary 4.2. *Let $f: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a single-valued mapping, $Q: \mathbb{R}^m \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $G: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ multifunctions defined as in (4.1) and (4.3). Let $\omega_0 := (x_0, y_0, -f(x_0, y_0)) \in \text{gph } Q$. Suppose that f is strictly differentiable at (x_0, y_0) and $\text{gph } Q$ is closed around ω_0 . If for each $(y^*, z^*) \in \mathbb{R}^n \times \mathbb{R}^p$ the following condition holds*

$$(-\nabla f_x(x_0, y_0)^*(z^*), y^* - \nabla f_y(x_0, y_0)^*(z^*)) \in D^*Q(\omega_0)(z^*) \implies (y^*, z^*) = (0, 0),$$

then G is both local-metrically regular around ω_0 and Lipschitz-like around (y_0, x_0) .

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N. Q. Huy
Department of Applied Mathematics,
National Sun Yat-sen University,
Kaohsiung 804, Taiwan
and
Department of Mathematics,
Hanoi Pedagogical University,
No. 2, Vinh Phuc,
Vietnam
E-mail: huyngq@yahoo.com

J.-C. Yao
Department of Applied Mathematics,
National Sun Yat-Sen University,
Kaohsiung 804, Taiwan
E-mail: yaojc@math.nsysu.edu.tw