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INVARIANT MEANS AND FIXED POINT PROPERTIES OF SEMIGROUP OF NONEXPANSIVE MAPPINGS

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. This paper outlines some of my recent joint works with Q. Takahashi on fixed point properties or ergodic properties for semigroup of nonexpansive mappings on closed convex subsets of a Banach space and their relationship with existence of left invariant mean on certain subspaces of bounded real-valued functions on the semigroup.

1. Introduction

Let S be a semigroup, $\ell^\infty(S)$ be the Banach space of bounded real valued functions on S with the supremum norm. There is a strong connection between the existence of an invariant mean (or submean) on an invariant subspace of $\ell^\infty(S)$, fixed point properties or ergodic properties of S when S is represented as a semigroup of nonexpansive mappings on a closed convex subset of a Banach space. The first such relation was established by W. Takahashi [67] where he proved:

Theorem 1.1. (W. Takahashi) *Let S be a semigroup. If $\ell^\infty(S)$ has a left invariant mean, then S has the following fixed point property:*

(F) *Whenever $\mathcal{S} = \{T_s; s \in S\}$ is a representation of S as non-expansive mapping from a non-empty compact convex subset C of a Banach space into C, then C contains a common fixed point for $\mathcal{S}$.*

Theorem 1.1 was proved for commutative semigroup by R. DeMarr [9] for commutative semigroup. Later in [25], we show that fixed point property (F) is

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equivalent to the existence of a left invariant mean on $AP(S)$, the space of almost periodic functions on the semigroup S.

It is the purpose of this paper to outline some of my joint works with W. Takahashi and some recent results on such relations. It is our hope that this paper will generate further research in the connection between nonlinear analysis and amenability of semigroups.

This paper is organized as follows: In Section 3, we shall study the algebra of (nonlinear) submean on subspaces of $\ell^\infty(S)$. In Section 4, we shall outline the relationship between invariant means (or submeans) and fixed point properties of semigroups of nonexpansive mappings. Finally in Section 5 we shall discuss the relation between amenability and ergodic type theorems and approximation of fixed points.

2. Some Preliminaries

All topologies in this paper are assumed to be Hausdorff. If E is a Banach space and $A \subseteq E$, then $\overline{A}$ and $\overline{\text{co}}\ A$ will denote the closure of A and the closed convex hull of A in E, respectively.

Throughout this paper, all vector spaces are real. Let E be a Banach space and let C be a subset of E. A mapping T from C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. A Banach space E is said to be *uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(x+y)/2\| \leq 1 - \delta$ for each $x, y \in E$ satisfying $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$.

Let E^* be the topological dual of a Banach space E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$ or $x^*(x)$. With each $x \in E$, we associate the set

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $Jx \neq \emptyset$ for each $x \in E$. The multivalued operator J from E into E^* is called the *duality mapping* of E. Let $S(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit $\lim_{\lambda \to 0}(\|x + \lambda y\| - \|x\|)/\lambda$ exists uniformly for $y \in S(E)$. In this case, J is single-valued and

$$\lim_{\lambda \to 0} \sup_{\|y\|=1} \left| \frac{\frac{1}{2}\|x + \lambda y\|^2 - \frac{1}{2}\|x\|^2}{\lambda} - \langle y, Jx \rangle \right| = 0$$

for each $x \in E$; see [11] for details.

The Banach space E is called *smooth* if the duality mapping J is single-valued. E is said to be *strictly convex* if

$$\|x\| \leq 1,\ \|y\| \leq 1 \quad \text{and} \quad x \neq y \quad \text{imply} \quad \left\| \frac{x+y}{2} \right\| < 1.$$

Let D be a subset of B where B is a subset of a Banach space E and let P be a retraction of B onto D, that is, $Px = x$ for each $x \in D$. Then P is said to be *sunny* [64] if for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$,

$$P\big(Px + t(x - Px)\big) = Px.$$

A subset D of B is said to be a *sunny nonexpansive retract* of B if there exists a sunny nonexpansive retraction P of B onto D. We know that if E is smooth and P is a retraction of B onto D, then P is sunny and nonexpansive if and only if for each $x \in B$ and $z \in D$,

$$\langle x - Px, J(z - Px)\rangle \leq 0.$$

For more details, see [69].

Let S be a semigroup. Then a subspace X of $\ell^\infty(S)$ is *left (resp. right) translation invariant* if $\ell_a(X) \subseteq X$ (resp. $r_a(X) \subseteq X$) for all $a \in S$, where $(\ell_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa),\ s \in S$.

A *semitoplogical semigroup* S is a semigroup with Hausdorff topology such that for each $a \in S$, the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S into S are continuous. Examples of semitopological semigroups include all topological groups, the set $M(n, \mathbb{C})$ of all $n \times n$ matrices with complex entries, matrix multiplication and the usual topology, the unit ball of ℓ^∞ with weak*-topology and pointwise multiplication, or $\mathcal{B}(H)$ $(=$ the space of bounded linear operators on a Hilbert space $H)$ with the weak*-topology and composition.

If S is a semitopological semigroup, we denote by $CB(S)$ the closed subalgebra of $\ell^\infty(S)$ consisting of continuous functions. Let $LUC(S)$ (resp. $RUC(S)$) be the space of *left* (resp. *right*) *uniformly continuous functions* on S, i.e. all $f \in CB(S)$ such that the mapping from S into $CB(S)$ defined by $s \to \ell_s f$ (resp. $s \to r_s f$) is continuous when $CB(S)$ has the sup norm topology. Then as is known (see [5]), $LUC(S)$ and $RUC(S)$ are left and right translation invariant closed subalgebras of $CB(S)$ containing constants. Also let $AP(S)$ (resp. $WAP(S)$) denote the space of almost periodic (resp. weakly almost periodic) functions f in $CB(S)$, i.e. all $f \in CB(S)$ such that $\{\ell_a f; a \in S\}$ is relatively compact in the norm (resp. weak) topology of $CB(S)$, or equivalently $\{r_a f; a \in S\}$ is relatively compact in the norm (resp. weak) topology of $CB(S)$. Then as is known [5, p. 164], $AP(S) \subseteq LUC(S) \cap RUC(S)$, and $AP(S) \subseteq WAP(S)$. When S is a group, then $WAP(S) \subseteq LUC(S) \cap RUC(S)$ (see [5, p. 167]).

3. Submeans on Semigroups

Let S be a non-empty set and X be a subspace of $\ell^\infty(S)$ containing constants. Then $\mu \in X^*$ is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. As is well known, μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$$

for each $f \in X$.

By a (nonlinear) *submean* on X, we shall mean a real-valued function μ on X satisfying the following properties:

(1) $\mu(f+g) \leq \mu(f)+\mu(g)$ for every $f, g \in X$;

(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geq 0$;

(3) for $f, g \in X$, $f \leq q$ implies $\mu(f) \leq \mu(g)$;

(4) $\mu(c) = c$ for every constant function c.

Clearly every mean is a submean. The notion of submean was first introduced by Mizoguchi and Takahashi [59].

Let SM_X denote the set of submeans on X. For each $\phi \in SM_X$, $-\|f\| \leq \phi(f) \leq \|f\|$ by (3) and (4). Hence SM_X may be identified as a subset of the product space $\Pi_{f\in X}[-\|f\|, \|f\|\,]$, which is compact by Tychonoff's Theorem. Hence SM_X is a compact convex subset of the product topological vector space $\Pi_{f\in X}\mathbb{R}_f$, where each $\mathbb{R}_f = \mathbb{R}$.

If S is a semigroup, and $X \subseteq \ell^\infty(S)$ is a left translation invariant subspace of $\ell^\infty(S)$ containing constants, a mean (submean) μ on X is *left [right] invariant* if $\mu(\ell_a f) = \mu(f)[\mu(r_a f) = \mu(f)]$ for each $a \in S$, $f \in X$. (See [7], [61] and [62].)

We abbreviate left invariant submean to $LISM$ and left invariant mean to LIM.

Depending on time and circumstances, the value of a submean (or mean) μ at $f \in X$ will also be denoted by $\mu(f)$, $\langle \mu, f\rangle$ or $\mu_t f(t)$.

A semitopological semigroup S is *left reversible* if any two closed right ideals of S have non-void intersection.

Lemma 3.1. *Let S be a semitopological semigroup and X be a left translation invariant subspace of $CB(S)$ containing constants and which separates closed subsets of S. If X has a $LISM$, then S is left reversible.*

Proof. Let μ be a $LISM$ of X, I_1 and I_2 be disjoint nonempty closed right ideals of S. By assumption, there exists $f \in X$ such that $f \equiv 1$ on I_1 and $f \equiv 0$ on I_2. Now if $a_1 \in I_1$, then $\ell_{a_1} f = 1$. So,

$$\mu(f) = \mu(\ell_{a_1} f) = 1.$$

But if $a_2 \in I_2$, then $\ell_{a_2} f \equiv 0$. So $\mu(f) = \mu(\ell_{a_2} f) = 0$, which is impossible.

Corollary 3.2. *If S is normal and $CB(S)$ has a $LISM$, then S is left reversible.*

Corollary 3.3. *If S is normal and $CB(S)$ has a $LISM$, then $AP(S)$ has a LIM.*

Proof. This follows from Corollary 3.2 and [25, Corollary 3.3].

Remark 3.4. The class $\mathbb{S}$ of all left reversible semitopological semigroups includes trivially all topological semigroups which are algebraically groups, and all commuting topological semigroups.

The class $\mathbb{S}$ is closed under the following operations.

(a) If $S \in \mathbb{S}$ and S' is a continuous homomorphic image of S, then $S' \in \mathbb{S}$.

(b) Let $S_\alpha \in \mathbb{S},\ \alpha \in I$ and S be the topological semigroup consisting of the set of all functions f on I such that $f(\alpha) \in S_\alpha,\ \alpha \in I$, the binary operation defined by $fg(\alpha) = f(\alpha)g(\alpha)$ for all $\alpha \in I$ and $f, g \in S$, and the product topology. Then $S \in \mathbb{S}$.

(c) Let S be a topological semigroup and $S_\alpha,\ \alpha \in I$, topological sub-semigroups of S with the property that $S = \cup S_\alpha$ and, if $\alpha_1, \alpha_2 \in I$, then there exists $\alpha_3 \in I$ such that $S_{\alpha_3} \supseteq S_{\alpha_1} \cup S_{\alpha_2}$. If $S_\alpha \in \mathbb{S}$ for each $\alpha \in I,\ S \in \mathbb{S}$.

Let SM be the set of submeans on $\ell^\infty(S)$. For $\mu \in SM$ and $f \in \ell^\infty(S)$, define

$$\mu_\ell(f)(s) = \mu(\ell_s f)$$

for each $s \in S$. Then

$$\|f\| \le \inf\,(\ell_s f)(t) \le \mu(\ell_s f)(t) \le \sup\,(\ell_s f) \le \|f\|$$

for each $s \in S$. So $\mu_\ell f \in \ell^\infty(S)$. Hence if $\psi, \mu \in SM$, we may define

$$\langle \psi \odot \mu, f\rangle = \langle \psi, \mu_\ell(f)\rangle.$$

If $\psi, \mu \in SM$, then $\psi \odot \mu \in SM$.

A semigroup S is called a *left zero semigroup* if all of its elements are left zeros which means that $xy = x$ for all $x, y \in S$. Similarly S is called a *right zero semigroup* if $xy = y$ for all $x, y \in S$. The (possibly empty) set of idempotents of a semigroup S is denoted by $E(S)$.

Let X, Y be nonempty sets and G be a group. Let $K = X \times G \times Y$. Given a map $\delta : X \times Y \to G$, we define a sandwich product on K by

$$(x, g, y) \circ (x', g', y') = \big(x, g\delta(y, x')g', y'\big).$$

Then $(K, \circ)$ is a simple group (i.e. no proper two-sided ideals) and any semigroup isomorphic to a simple group of this kind is called a *paragroup*.

Let S be a compact semigroup. It is called a *right topological semigroup* if the translations $x \mapsto xs \ \ (s \in S)$ are continuous.

Theorem 3.5. $\Pi = (SM, \odot)$ *is a compact right topological semigroup. Further, the following conditions hold:*

(a) Π *has a minimal ideal* K *and*

$$K \simeq E(p\Pi) \times p\Pi p \times E(\Pi p)$$

where p *is any idempotent of* K *and* $p\Pi = \{p \odot s : s \in \Pi\}$ *with similar definition for* $p\Pi p$ *and* Πp. *Also,* $E(p\Pi)$ *is a right zero semigroup,* $E(\Pi p)$ *is a left zero semigroup and* $p\Pi p = p\Pi \cap \Pi p$ *is a group.*

(b) *The minimal ideal* K *need not be a direct product, but is a paragroup with respect to the natural map*

$$\delta : E(p\Pi) \times E(\Pi p) \to p\Pi p : (x, y) \mapsto x \odot y.$$

(c) *For any idempotent* $p \in K,$ $\ p\Pi$ *is a minimal right ideal and* Πp *is a minimal left ideal.*

(d) *The minimal left ideals in* Π *are closed and homeomorphic to each other.*

Remark 3.6.

(a) Theorem 3.5 remain valid if SM is replaced by SM_X when X is a left translation invariant and left introverted subspace of $\ell^\infty(S)$ containing constants, i.e. for each $\mu \in SM_X$ and $f \in X$, the function $\mu_\ell(f) \in X$.

(b) If $X \subseteq \ell^\infty(S)$ is left translation invariant and left introverted and contains constants, then

(i) X is right translation invariant,

(ii) for each $f \in X,\ K_f =$ the w^*-closed convex hull of $\{r_a f : a \in S\} \subseteq X$.

The following is an analogue of Lemma 1 in [13] and the Localization Theorem (Theorem 5.2) in [72] (see also [13] and [27]):

Theorem 3.7. *Let* X *be a left invariant and left introverted subspace of* $\ell^\infty(S)$ *containing constants. Then* X *has a left invariant submean if and only if for each* $f \in X$, *there exists a submean* μ *(depending on* f*) such that* $\mu(f) = \mu(\ell_s f)$ *for all* $s \in S$.

Example 3.8. If μ is a left invariant mean on $\ell^\infty(S)$, then $\mu(h) = 0$ for any $h = (f_1 - \ell_{a_1} f_1) + \cdots + (f_n - \ell_{a_n} f_n), f_1, \ldots, f_n \in \ell^\infty(S), a_1, \ldots, a_n \in S$. But this is no longer true for left invariant submean.

Let $S =$ free group on two generators a, b. Define $\mu(f) = \sup\ f(s)$. Then $\mu(\ell_a f) = \mu(f)$ for all $a \in S$ (this is the case when $aS = S$ for all $a \in S$, i.e. μ is a left invariant submean on $\ell^\infty(S)$. But if $A =$ all elements in S that begin with a or a^{-1} (reduced word), and $f = 1_A$, and

$$h = \big(\ell_{ba^{-1}} f - \ell_{ab^{-1}a}(\ell_{ba^{-1}} f)\big) + \big((-f) - \ell_{b^{-1}a^{-1}}(-f)\big),$$

then $\mu(h) < 0$ (see Theorem 3.8).

Theorem 3.9. *Let X be a left translation subspace of $\ell^\infty(S)$ containing constants. The following are equivalent:*

(a) *X has a left invariant mean.*

(b) *For $s_1, \ldots, s_n \in S$ and $f_1, \ldots, f_n \in X^+ = \{f \in X : f \geq 0\}$, there exists a submean μ on X such that*

$$\mu\Big(\sum_{i=1}^{n} f_i\Big) \leq \mu\Big(\sum_{i=1}^{n} \ell_{s_i} f_i\Big).$$

Notes and remarks.

(i) Results in this section are contained in [45].

(ii) Using implicitly the notion of the submean $\mu(f) = \sup\{f(t); t \in G\}$ of a group G, $f \in \ell^\infty(G)$, Despic and Ghahramami gave in [10] a simple proof of a result of B.E. Johnson on weak amenability of the group algebra of a locally compact group.

4. Submeans and Fixed Point Property of Semigroups

A closed convex subset K of a Banach space E has *normal structure* [6, p. 39] if for each bounded closed convex subset H of K which contains more than one point, there is a point $x \in H$ which is not a diametral point of H, i.e., $\sup\{\|x - \nu\| : \nu \in H\} < \delta(H)$, where $\delta(H) =$ the diameter of H.

Belluce and Kirk [4] first proved that if K is a nonempty weakly compact convex subset of a Banach space and if K has complete normal structure, then every family of commuting nonexpansive self-maps on K has a common fixed point. Later Lim [52, Theorem 3] extended this theorem to a continuous representation of a left reversible semitopological semigroup S as nonexpansive mappings on a weakly compact convex set K with normal structure.

If S is a semigroup and X is left translation invariant, a submean μ on X is left *subinvariant* if $\mu(\ell_a f) \geq \mu(f)$ for each $f \in X$ and $a \in S$. A representation

$\mathcal{S} = \{T_s; s \in S\}$ as mappings from a subset C of a Banach space into C is called X-*admissible* if for each $x, y \in C$, the function $t \to \|T_t x - y\|$ belongs to X.

Theorem 4.1. *Let C be a non-empty weakly compact convex subset of a Banach space E. If C has more than one point and normal structure, then C satisfies:*
(P) *Whenever S is a semigroup, X is a closed left translation invariant subspace of $\ell^\infty(S)$ containing constants with a left subinvariant submean μ, $\mathcal{S} = \{T_s; s \in S\}$ is an X-admissible representation of S as nonexpansive mappings from C into C, then the set $A_x = \{y \in C; \mu_t\|T_t x - y\| = \rho_x\}$ is a* proper *subset of C for some $x \in C$, where $\rho_x = \inf\{\mu_t\|T_t x - y\|; y \in C\}$. Furthermore, for each $x \in C$ the set A_x is non-empty, closed, convex, and T_s-invariant.*

This theorem is the key to prove the following generalization of Lim's fixed point theorem [52] for left reversible semigroups of nonexpansive mappings.

Theorem 4.2. *Let S be a semitopological semigroup, let C be a nonempty weakly compact convex subset of a Banach space E which has normal structure and let $\mathcal{S} = \{T_s; s \in S\}$ be a continuous representation of S as nonexpansive self mappings on C such that the map $S \times C \to C$ defined by $(s, x) \to T_s x$, $s \in S$, $x \in X$ is continuous when $S \times C$ has the product topology. Suppose $RUC(S)$ has a left subinvariant submean. Then $\mathcal{S}$ has a common fixed point in C.*

Corollary 4.3. ([52]). *Let S be a left reversible semitopological semigroup. Let D be a nonempty weakly compact convex subset of a Banach space E which has normal structure and let $\mathcal{S} = \{T_s; s \in S\}$ be a continuous representation of S as nonexpansive self mappings on D. Then $\mathcal{S}$ has a fixed point in D.*

Proof. If S is left reversible, define $\mu(f) = \inf_s \sup_{t \in sS} f(t)$. Then the proof of Lemma 3.6 in [42] shows that μ is a submean on $CB(S)$ such that $\mu(\ell_a f) \geq \mu(f)$ for all $f \in CB(S)$ and $a \in S$, i.e., μ is left subinvariant.

Example 4.4. E. Hewitt [15] has constructed a regular Hausdorff space T such that the only real continuous functions on T are the constant functions. Let $(U, \circ)$ be any discrete left amenable semigroup (e.g., $U =$ a commutative semigroup). Let $S = T \times U$ with multiplication defined by

$$(t_1, u_1) \cdot (t_2, u_2) = (t_1, u_1 \circ u_2),$$

with $t_1, t_2 \in T, u_1, u_2 \in U$ and product topology. Then S is a semitopological semigroup which is not left reversible since $(t, u) \cdot S = \{(t, u \circ u'); u' \in U\}$ is a closed right ideal of S, and $(t_1, u) \cdot S \cap (t_2, u) \cdot S = \phi$ if $t_1 \neq t_2$. However, if ϕ

is a left invariant mean on $\ell^\infty(U)$ and $t_0 \in T$, then the positive functional m on $CB(S)$ defined by

$$m(t) = \phi(f_{t_0}) \quad \text{for every} \quad f \in CB(S)$$

is a left invariant mean, where $f_{t_0}(u) = f(t_0, u)$ for every $u \in U$. In particular, for any such semitopological semigroup S, our fixed point theorem Theorem 4.2 applies but the fixed point theorem of Lim [52] for left reversible semigroup (Corollary 4.3) does not (see also [17]).

A discrete semigroup is called *left amenable* if $\ell^\infty(S)$ has a left invariant mean.

When S is a discrete semigroup, the following implication diagram is known:

$$\begin{array}{lll}
S \quad \text{left amenable} & & \\
\quad\Downarrow \;\; \not\Uparrow & & \\
S \quad \text{left reversible} & & \\
& \searrow \not\nwarrow & \\
\quad\Downarrow \;\; \not\Uparrow & & WAP(S) \quad \text{has} \quad LIM \\
& \swarrow \nearrow & \\
AP(S) \quad \text{has} \quad LIM & &
\end{array}$$

The implication "S is left reversible $\Longrightarrow$ $AP(S)$ has a LIM" for any semitopological semigroup was established in [25]. During the 1984 Richmond, Virginia conference on analysis on semigroups, T. Mitchell [58] gave two examples to show that for discrete semigroups "$AP(S)$ has LIM" $\not\Longrightarrow$ "S is left reversible" (see [29] or [49]). The implication "S is left reversible $\Longrightarrow$ $WAP(S)$ has LIM" for discrete semigroups was proved by Hsu [20]. Recently, it is shown in [49] that if S_1 is the bicyclic semigroup generated by $\{e, a, b, c\}$ such that e is the unit of S_1 and $ab = e$ and $ac = e$, then $WAP(S)$ has a LIM, but S_1 is not left reversible. Also if S_2 is the bicyclic semigroup generated by $\{e, a, b, c, d\}$, where e is the unit element and $ac = bd = e$, then $AP(S_2)$ has a LIM, but $WAP(S_2)$ does not have a LIM.

Theorem 4.5. *Let S be a left reversible discrete semigroup. Then S has the following fixed point property:*

Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as norm nonexpansive weak*-weak* continuous mappings of a norm-separable weak*-compact convex subset C of a dual Banach space E into C, then C contains a common fixed point for S.

Notes and remarks

(i) Both Theorems 4.1 and 4.2 are contained in [43]. Theorem 4.5 was proved in [42, Theorem 5.3] (see also [46] for a more general result).

For other related results, we refer the readers to [6, 9, 12, 18, 19, 26, 30, 32, 37, 41, 44, 53, 54, 56, 57, 66] and [67].

(ii) It can be shown that the following fixed point property on a discrete semigroup S implies S is left amenable:

(G) Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as norm non-expansive weak*-weak* continuous mappings of a weak*-compact convex subset C of a dual Banach space E into C, then C contains a common fixed point for S.

Problem 1. Does left amenability of S imply (G)?

(iii) Quite recently the author and Y. Zhang [49] are able to establish the following related fixed point property.

Theorem 4.6. *Let S be a separable semitopological semigroup. If $WAP(S)$ has a left invariant mean, then S has the following fixed point property:*

Whenever $\mathcal{S} = \{T_s; s \in S\}$ is a continuous representation of S as non-expansive self mappings on a weakly compact convex subset C of a Banach space E such that the closure of $\mathcal{S}$ in C^C with the product of weak topology consists entirely of continuous functions, then C contains a common fixed point of C.

(iv) It has been proved by Hsu [20] (see also [6]) that if S is discrete and left reversible, and $\mathcal{S} = \{T_s; s \in S\}$ is a representation of S as weakly continuous nonexpansive mappings on a weakly compact convex subset C of a Banach space, then C has a common fixed point for $\mathcal{S}$. Note that it follows from Alspach's example [1] that there exists a commutative semigroup of nonexpansive mappings on a weakly compact convex subset of $L_1[0,1]$ with no common fixed point. But as is well known, $\ell^\infty(S)$ always has an invariant mean when S is commutative. Also Schechtman [66] has shown that there exists a weakly compact convex subset W of $L_1[0,1]$ and a sequence $T_1, T_2, \ldots$ of commuting nonexpansive operators of W into itself such that any finite number of them have a common fixed point but there is no common fixed point for the entire sequence.

Note that $L^1[0,1]$ is isometrically isomorphic to $L^1(\mathbb{T})$, the group algebra of the circle group $\mathbb{T}$, and $A(\mathbb{Z})$, the Fourier algebra of the integer group $(\mathbb{Z}, +)$. For related works (motivated by Alspach's example [1]) concerning the weak fixed point property on the group or Fourier algebra of a locally compact group and other related geometric properties, we refer the reader to: [33, 34, 47, 31], and [50].

5. Ergodic Theorems and Approximation of Fixed Points

Let S be a semigroup and X be a translation invariant subspace of $\ell^\infty(S)$ containing 1. A mean $\mu \in X^*$ is called a *finite mean* if there exists $\lambda_1, \ldots, \lambda_n \geq 0$,

$t_1 \ldots, t_n \in S$ such that $\sum\limits_{i\equiv 1}^{n} \lambda_i = 1$, and $\mu = \sum\limits_{i\equiv 1}^{n} \lambda_i \delta_{t_i}$, where δ_t is the point evaluation at t, $t \in S$.

A net $\{\mu_\alpha\}$ of means on X is said to be *asymptotically invariant* if

$$\lim_\alpha \big(\mu_\alpha(\ell_s f) - \mu_\alpha(f)\big) = 0 \quad \text{and} \quad \lim_\alpha \big(\mu_\alpha(r_s f) - \mu_\alpha(f)\big) = 0$$

for each $f \in X$ and $s \in S$, and it is said to be *strongly regular* if

$$\lim_\alpha \|\ell_s^* \mu_\alpha - \mu_\alpha\| = 0 \quad \text{and} \quad \lim_\alpha |r_s^* \mu_\alpha - \mu_\alpha\| = 0$$

for each $s \in S$, where ℓ_s^* and r_s^* are the adjoint operators of ℓ_s and r_s, respectively. Such nets were first studied by Day in [7], where they were called *weak* invariant nets* and *norm invariant nets*, respectively.

It is well known [7] that a semigroup S is left amenable if and only if there is a net $\{\mu_\alpha\}$ of finite means on $\ell^\infty(S)$ such that $\{\mu_\alpha\}$ is asymptotically left invariant, i.e. $\lim\limits_\alpha \big(\mu_\alpha(\ell_s f) - \mu_\alpha(f)\big) = 0$ for each $f \in \ell^\infty(S)$, $s \in S$. A remarkable result of Day [7] shows that if S is left amenable, then there is a net $\{\mu_\alpha\}$ of finite means which is left strongly regular (i.e. $\|\ell_s \mu_\alpha - \mu_\alpha\| \to 0$ for each $s \in S$ see also [60] for an elegant proof of this).

Theorem 5.1. *If S is an amenable semigroup, then there is a strong regular net $\{\mu_\alpha\}$ consisting of finite means on $\ell^\infty(S)$. Furthermore, if S is countable, then $\{\mu_\alpha\}$ may be taken to be a sequence.*

Let S be a semigroup and let C be a closed, convex subset of a reflexive Banach space F. Let $\mathcal{S} = \{T_t : t \in S\}$ be a representation of S as nonexpansive (or simply a nonexpansive semigroup on C) mappings from C into itself such that $\{T_t(x); x \in S\}$ is relatively weakly compact for each $x \in S$. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, i.e., $\cap_{t\in S}\{x \in C : T_t x = x\}$. For $x \in C$, we also denote by $Q(x)$ the set $\cap_{s\in S} \overline{\text{co}}\{T_{ts}x : t \in S\}$. A mapping A from C into itself is said to be a finite average of $\mathcal{S}$ if there exists $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $t_1, \ldots, t_n \in S$ such that $A = \sum_{i=1}^n \lambda_i T_{t_i}$. Let X be a subspace of $\ell^\infty(S)$ such that $1 \in X$ and $\mathcal{S}$ is X-admissible. Then $T_\mu x$ or $\int T_t x d\mu(t)$ defines an element in C using the bi-polar theorem (see [35]) for each mean μ on X and $x \in C$. We remark that A is a finite average of $\mathcal{S}$ if and only if there is a finite mean μ on X such that $A = T_\mu$.

The following result which we shall need is well known; for example, see [21].

Lemma 5.2. *Let S be a semigroup, let C be a closed, convex subset of a Banach space E, let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t : t \in S\}$ is relatively weakly compact for each $x \in C$, let X be a subspace of $B(S)$ such that $1 \in X$ and $\mathcal{S}$ is X-admissible. Let μ be a mean on X. Then*

(i) *T_μ is a nonexpansive mapping from C into itself;*

(ii) *$T_\mu x = x$ for each $x \in F(\mathcal{S})$;*

(iii) *$T_\mu x \in \overline{\text{co}}\{T_t x : t \in S\}$ for each $x \in C$;*

(iv) *if X is r_s-invariant for each $s \in S$ and μ is right invariant, then $T_\mu T_t = T_\mu$ for each $t \in S$.*

In 1975, Baillon [3] proved that: If *C is a closed, convex subset of a Hilbert space and T is a nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro mean*

$$S_n(x) = \frac{1}{n} \sum_{k=1}^{n} T^k x$$

converges weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction from C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{co}}\{T^n x : n = 1, 2, \dots\}$ for each $x \in C$. In [68], Takahashi proved the existence of such a retraction for an amenable semigroup of nonexpansive mappings on a Hilbert space: *If S is an amenable semigroup, C is a closed, convex subset of a Hilbert space H and $\mathcal{S} = \{T_t : t \in S\}$ is a nonexpansive semigroup on C such that the set $F(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is nonempty, then there exists a nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for each $t \in S$ and $Px \in \overline{\text{co}}\{T_t x : t \in S\}$ for each $x \in C$.* Rodé [65] also found a sequence of means on a semigroup, generalizing the Cesàro means, and extended Baillon's theorem as follows: *If S, C, H, and $\mathcal{S}$ are as above and $\{\mu_\alpha\}$ is an asymptotically invariant net of means, then for each $x \in C, \{T_{\mu_\alpha} x\}$ converges weakly to an element of $F(\mathcal{S})$. Further, for each $x \in C$, the limit point of $\{T_{\mu_\alpha} x\}$ is the same for all asymptotically invariant nets $\{\mu_\alpha\}$ of means.* From their results, we know that for each $x \in C, \{T_{\mu_\alpha} x\}$ converges weakly to Px for all asymptotically invariant nets $\{\mu_\alpha\}$ of means; see [70]. These results were extended to a uniformly convex Banach space whose norm is Fréchet differentiable in the case where S is *commutative* by Hirano, *et al.* [16]. However, it has been an open problem whether Takahashi's result and Rodé's result can be fully extended to such a Banach space for an *amenable* semigroup; see [71]. On the other hand, Day [7] proved the following ergodic theorem for an amenable semigroup of bounded linear operators on a Banach space: *If S is an amenable semigroup and $\mathcal{S} = \{T_t : t \in S\}$ is a bounded representation of S as bounded linear operators on a Banach space E, then there exists a net $\{A_\alpha\}$ of finite averages of $\mathcal{S}$ such that $\lim_\alpha \|A_\alpha(T_t - I)\| = 0$ and $\lim_\alpha \|T_t - I)A_\alpha\| = 0$ for each $t \in S$. In this case, there is also a projection P from E onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for each $t \in S$ and $Px \in \overline{\text{co}}\{T_t x : t \in S\}$ for each $x \in E$.*

The following ergodic theorem answered the open problem posed above.

Theorem 5.3. *If S is an amenable semigroup $\mathcal{S} = \{T_t : t \in S\}$ is a nonexpansive semigroup on a closed, convex subset C in a uniformly convex Banach space E such that for each $x \in C, \{T_t x;\ t \in S\}$ is bounded and $F(\mathcal{S}) \neq \emptyset$, then there exists a nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for each $t \in S$ and $Px \in \overline{\text{co}}\{T_t x : t \in S\}$ for each $x \in C$. In this case, there exists a net $\{A_\alpha\}$ of finite averages of $\mathcal{S}$ such that for each $t \in S$ and for each bounded subset B of C, $\lim_\alpha \|A_\alpha T_t x - A_\alpha x\| = 0$ and $\lim_\alpha \|T_t A_\alpha x - A_\alpha x\| = 0$ uniformly for $x \in B$. Also, if the norm of E is Fr échet differentiable then for each $x \in C$, Px is the unique common fixed point in $Q(x) = \cap_{s \in S} \overline{\text{co}}\{T_{ts} x : t \in S\}$. Furthermore, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means, then for each $x \in C, \{T_{\mu_\alpha} x\}$ converges weakly to Px. If S is countable, then $\{A_\alpha\}$ can be chosen to be a sequence.*

For the case of compact convex subsets of a strictly convex and smooth Banach space, we have the following strong convergence theorems for approximation of common fixed points of Halpern's types (see [14]):

Theorem 5.4. *Let S be a left amenable countable semigroup and $\mathcal{S} = \{T_t : t \in S\}$ be a representation of S as nonexpansive mappings from a compact convex subset C of a strictly convex and smooth Banach space E into C. Then there exists a $\{A_n\}$ of finite averages of $\mathcal{S}$ such that for each sequence $\{\alpha_n\}$ in $[0, 1]$ with properties $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, the sequence $\{x_n\}$ defined by $x_1 = x$, and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) A_n(x_n)$$

converges strongly to Px, where P denotes the unique sunny nonexpansive retraction of C onto $F(\mathcal{S})$.

Notes and remarks

(i) Theorem 5.1 is the consequence of a more general result on φ-amenability in [24, Proposition 5.6]. It was used implicitly in the proof of Theorem 4.4 in [39]. Furthermore if S is a countable left amenable semigroup, then there is a sequence $\{\mu_n\}$ which is strongly left regular i.e. $\|\ell_s^* \mu_n - \mu_n\| \to 0$ for each $s \in S$ (see [24, Proposition 3.6]). Such a sequence is important for approximation of fixed points.

(ii) Lemma 5.2 implies that if $\mathcal{S} = \{T_t; t \in S\}$ is a representation of a semigroup S as non-expansive mappings from a weakly compact convex subset C of a Banach space E into C, then $F(T_\mu)$ contains $F(\mathcal{S})$ for any mean μ on a subspace X of $\ell^\infty(S)$ such that $1 \in X$ and the functions $f(t) = \langle T_t x, x^* \rangle, x \in C$, $x^* \in E^*$. The following theorem was proved in [35].

Theorem 5.5. *Let S be a semigroup with an identity, and $\mathcal{S} = \{T_t : t \in S\}$ be a representaiton of S as nonexpansive mappings from a compact convex subset C of a Banach space E into C. Then for any left invariant mean μ on $AP(S)$, $F(\mathcal{S}) = F(T_\mu)$.*

The following problem is still open:

Problem. Let S be a left amenable semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as nonexpansive mappings from a weakly compact convex subset C of a Banach space into C and C has normal structure. Then $F(\mathcal{S}) \neq 0$ (see [52]). Let μ be a left invariant mean on $\ell^\infty(S)$. Does $F(\mathcal{S}) = F(T_\mu)$?

See also [22] and [23] for related results.

(iii) Theorem 5.3 is contained in the paper [39], and Theorem 5.4 follows also from Theorem 4.1 of [35] and Theorem 5.1 above. However, the existence of a (sequential) strongly regular net for a (countable) amenable semigroup is established only recently in [24, Proposition 3.6]. This partially answers Problem 8 in [44].

Approximations of common fixed point for semigroup of nonexpansive mappings have been of significant interest in recent years. Related works can also be found in [28, 36, 40, 51, 55, 63].

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