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ON A THEORY BY SCHECHTER AND TINTAREV

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. In this paper, we show that the beautiful theory developed by M. Schechter and K. Tintarev in [9] can be applied to the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

when

$$\limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2} < +\infty$$

and, for each λ in a suitable interval, the problem has a unique positive solution.

Here and in the sequel, X is an infinite-dimensional real Hilbert space and $J : X \to \mathbf{R}$ is a sequentially weakly continuous C^1 functional, with $J(0) = 0$.

For each $r > 0$, set

$$S_r = \{x \in X : \|x\|^2 = r\}$$

as well as

$$\gamma(r) = \sup_{x\in S_r} J(x) \ .$$

Also, set

$$r^* = \inf\{r > 0 : \gamma(r) > 0\} \ .$$

In Section 2 of [9], M. Schechter and K. Tintarev developed a very elegant, transparent and precise theory whose aspects which are relevant for the present paper can be summarized as follows:

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Theorem A. *Assume that J has no local maximum in $X \setminus \{0\}$. Moreover, let $I \subseteq]r^*, +\infty[$ be an open interval such that, for each $r \in I$, there exists a unique $\hat{x}_r \in S_r$ satisfying $J(\hat{x}_r) = \gamma(r)$.*

Then, the following conclusions hold:

(i) *the function $r \to \hat{x}_r$ is continuous in I ;*

(ii) *the function γ is C^1 and increasing in I ;*

(iii) *one has*

$$J'(\hat{x}_r) = 2\gamma'(r)\hat{x}_r$$

for all $r \in I$.

At page 895 of [9], the authors say: "It is not yet clear, if there are general conditions providing uniqueness of the point of spherical maximum $\hat{x}_r$." Then, in the subsequent Lemma 2.14, they note that such an uniqueness does occur if J is concave.

One year after [9], Schechter and Tintarev reconsidered this question assuming it as the starting point for [10]. Actually, at page 454 of [10], after declaring that the uniqueness assumption made in Theorem A is difficult to verify in applications, they recall, as just observed in [11], that it implied by the concavity of J, but, as they say, "this condition is rather restrictive". Finally, they declare that the purpose of [10] is to give applications of Theorem A in which the uniqueness hypothesis can be verified without assuming the concavity of J.

In [7], in spite of the above recalled "pessimistic" assertions of Schecther and Tintarev, we proved that if J' is Lipschitzian in a neighbourhood of 0 and $J'(0) \neq 0$, then there exists an explicitly determined $\delta > 0$ such that, for each $r \in]0, \delta[$, the restriction of J to S_r has a unique global maximum. Therefore, the result of [7] shows that Theorem A can actually be applied to a very large class of functionals.

We then applied the method of [7] (that we first introduced in [4] and adopted in [5, 6] too) to prove, in [8], a very general result that we now state in a (partial) form which is enough for our purposes (with the conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$):

Theorem B. *Let Y be a Hausdorff topological space and let $\Phi, \Psi : Y \to \mathbf{R}$ be such that the function $\Phi + \lambda\Psi$ has sequentially compact sub-level sets and admits a unique global minimum, say $\hat{v}_\lambda$, for all $\lambda \in]a, b[$, where $-\infty \leq a < b \leq +\infty$. Set*

$$\eta = \max\left\{\inf_Y \Psi, \sup_{V_b} \Psi\right\} ,$$

$$\theta = \min\left\{\sup_Y \Psi, \inf_{V_a} \Psi\right\} ,$$

where V_a (resp. V_b) denotes either the set of all global minima of the function $\Phi + a\Psi$ (resp. $\Phi + b\Psi$) or the empty set according to whether a (resp. b) is finite or not. Assume that $\eta < \theta$.

Then, for every $r \in]\eta, \theta[$, there exists $\lambda_r \in]a, b[$ such that $\hat{v}_{\lambda_r} \in \Psi^{-1}(r)$.

The aim of the present paper is to establish Theorem 1 below which can be regarded as the most complete fruit of a joint application of Theorems A and B.

Theorem 1. *Set*

$$\rho = \limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2}$$

and

$$\sigma = \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2} \ .$$

Let a, b satisfy

$$\max\{0, \rho\} \leq a < b \leq \sigma \ .$$

Assume that J has no local maximum in $X \setminus \{0\}$, and that, for each $\lambda \in]a, b[$, the functional $x \to \lambda\|x\|^2 - J(x)$ has a unique global minimum, say $\hat{y}_\lambda$. Let M_a (resp. M_b if $b < +\infty$ or $M_b = \emptyset$ if $b = +\infty$) be the set of all global minima of the functional $x \to a\|x\|^2 - J(x)$ (resp. $x \to b\|x\|^2 - J(x)$ if $b < +\infty$). Set

$$\alpha = \max\left\{0, \sup_{x \in M_b} \|x\|^2\right\}$$

and

$$\beta = \inf_{x \in M_a} \|x\|^2 \ .$$

Then, the following assertions hold:

(a_1) *one has $r^* \leq \alpha < \beta$;*

(a_2) *the function $\lambda \to g(\lambda) := \|\hat{y}_\lambda\|^2$ is decreasing in $]a, b[$ and its range is $]\alpha, \beta[$;*

(a_3) *for each $r \in]\alpha, \beta[$, the point $\hat{x}_r := \hat{y}_{g^{-1}(r)}$ is the unique global maximum of $J_{|S_r}$ towards which every maximizing sequence in S_r converges ;*

(a_4) *the function $r \to \hat{x}_r$ is continuous in $]\alpha, \beta[$;*

(a_5) *the function γ is C^1, increasing and strictly concave in $]\alpha, \beta[$;*

(a_6) *one has*

$$J'(\hat{x}_r) = 2\gamma'(r)\hat{x}_r$$

for all $r \in]\alpha, \beta[$;

(a_7) *one has*

$$\gamma'(r) = g^{-1}(r)$$

for all $r \in]\alpha, \beta[$.

Before giving the proof of Theorem 1, let us recall the following proposition:

Proposition 1. ([4], Proposition 1). *Let* Y *be a non-empty set,* $\Phi, \Psi : Y \to \mathbf{R}$ *two functions, and* λ, μ *two real numbers, with* $\lambda < \mu$. *Let* $\hat{v}_\lambda$ *be a global minimum of the function* $\Phi + \lambda\Psi$ *and let* $\hat{v}_\mu$ *be a global minimum of the function* $\Phi + \mu\Psi$. *Then, one has*

$$\Psi(\hat{v}_\mu) \leq \Psi(\hat{v}_\lambda) \ .$$

If either $\hat{v}_\lambda$ *or* $\hat{v}_\mu$ *is strict and* $\hat{v}_\lambda \neq \hat{v}_\mu$, *then*

$$\Psi(\hat{v}_\mu) < \Psi(\hat{v}_\lambda) \ .$$

Now, we prove Theorem 1.

Proof of Theorem 1. First of all, observe that, by Proposition 1, the function g is non-increasing in $]a, b[$ and $g(]a, b[) \subseteq [\alpha, \beta]$. Now, let $I \subset]a, b[$ be a non-degenerate interval. If g was constant in I, then, by Proposition 1 again, the function $\lambda \to \hat{y}_\lambda$ would be constant in I. Let y^* be its unique value. Then, y^* would be a critical point of the functional $x \to \lambda\|x\|^2 - J(x)$ for all $\lambda \in I$. That is to say

$$2\lambda y^* = J'(y^*)$$

for all $\lambda \in I$. This would imply that $y^* = 0$, and so (since $J(0) = 0$) we would have $\inf_{x \in X}(\lambda\|x\|^2 - J(x)) = 0$ for all $\lambda \in I$, against the fact that $\inf_{x \in X}(\lambda\|x\|^2 - J(x)) < 0$ for all $\lambda < \sigma$. Consequently, g is decreasing in $]a, b[$, and so, in particular, $\alpha < \beta$. Next, observe that

$$\lim_{\|x\| \to +\infty} (\lambda\|x\|^2 - J(x)) = +\infty$$

for each $\lambda > \max\{0, \rho\}$. From this, recalling that J is sequentially weakly continuous, it clearly follows that we can apply Theorem B, taking $Y = X$ with the weak topology, $\Phi = -J$, $\Psi(\cdot) = \| \cdot \|^2$. Consequently, for every $r \in]\alpha, \beta[$, there exists $\lambda_r \in]a, b[$ such that $\|\hat{y}_{\lambda_r}\|^2 = r$. Therefore, by the strict monotonicity of g, we have $g(]a, b[) =]\alpha, \beta[$. Now, let us prove (a_3). Fix $r \in]\alpha, \beta[$. Clearly, we have

$$\|\hat{x}_r\|^2 = r \ .$$

Since

$$g^{-1}(r)\|\hat{x}_r\|^2 - J(\hat{x}_r) \leq g^{-1}(r)\|x\|^2 - J(x)$$

for all $x \in X$, we then have

$$J(x) \leq J(\hat{x}_r)$$

for all $x \in S_r$. Hence, $\hat{x}_r$ is a global maximum of $J_{|S_r}$. On the other hand, if v is a global maximum of $J_{|S_r}$, then

$$g^{-1}(r)\|v\|^2 - J(v) = g^{-1}(r)\|\hat{x}_r\|^2 - J(\hat{x}_r)$$

and hence, since

$$\inf_{x \in X}(g^{-1}(r)\|x\|^2 - J(x)) = g^{-1}(r)\|\hat{x}_r\|^2 - J(\hat{x}_r) \ ,$$

we have $v = \hat{x}_r$. In other words, $\hat{x}_r$ is the unique global maximum of $J_{|S_r}$. Since the sub-level sets of the functional $x \to g^{-1}(r)\|x\|^2 - J(x)$ are sequentially weakly compact, it is a classical remark ([2], p. 3) that any minimizing sequence of this functional in X converges weakly to $\hat{x}_r$. Now, let $\{w_n\}$ be any sequence in S_r such that $\lim_{n\to\infty} J(w_n) = \gamma(r)$. Then, we have

$$\lim_{n\to\infty}(g^{-1}(r)\|w_n\|^2 - J(w_n)) = \inf_{x \in X}(g^{-1}(r)\|x\|^2 - J(x))$$

and so $\{w_n\}$ converges weakly to $\hat{x}_r$. But then, since $\lim_{n\to\infty}\|w_n\| = \|\hat{x}_r\|$ and X is a Hilbert space, we have $\lim_{n\to\infty}\|w_n - \hat{x}_r\| = 0$ by a classical result. Let us prove that $r^* \leq \alpha$. Arguing by contradiction, assume that $\alpha < r^*$. Choose $r \in]\alpha, \min\{r^*, \beta\}[$. Then, since γ is non-decreasing in $]0, +\infty[$ (see Lemma 2.1 of [9]) and J is continuous, we would have $\gamma(r) = 0$, and so $J(\hat{x}_r) = 0$, and this would contradict the fact that $\inf_{x\in X}(g^{-1}(r)\|x\|^2 - J(x)) < 0$ since $g^{-1}(r) < \sigma$. At this point, we are allowed to apply Theorem A taking $I =]\alpha, \beta[$. Consequently, the function γ is C^1 and increasing in $]\alpha, \beta[$, and (a_4), (a_6) come directly from (i), (iii) respectively. Fix $r \in]\alpha, \beta[$ again. Since $\hat{x}_r$ is a critical point of the functional $x \to g^{-1}(r)\|x\|^2 - J(x)$, we

$$2g^{-1}(r)\hat{x}_r = J'(\hat{x}_r)$$

and then (a_7) follows from a comparison with (a_6). Finally, from (a_7), since g^{-1} is decreasing in $]\alpha, \beta[$, it follows that γ is strictly concave there, and the proof is complete. ■

The following two remarks show two very broad classes of functionals to which Theorem 1 can be applied.

Remark 1. If J' is Lipschitzian in X, with Lipschitz constant L, then, for each $\lambda > \frac{L}{2}$, the functional $\lambda\|x\|^2 - J(x)$ is coercive and has a unique global minimum in X ([4]).

Remark 2. If the derivative of J is compact and if, for some $\lambda > \rho$, the functional $x \to \lambda\|x\|^2 - J(x)$ has at most two critical points in X, then the same functional has a unique global minimum in X. Indeed, if this functional had at least two global minima, taken into account that it satisfies the classical Palais-Smale condition ([11], Example 38.25), it would have at least three critical points by Corollary 1 of [3].

Remark 2, in particular, allows a systematic application of Theorem 1 to boundary value problems admitting a unique non-zero solution, provided that the involved non-linearity has, for instance, a linear growth. Note that the specific case treated in the already recalled [10] falls in this setting.

The remainder of the paper is just devoted to this point.

So, from now on, $\Omega \subset \mathbf{R}^n$ is an open, bounded and connected set, with sufficiently smooth boundary, and X denotes the space $H_0^1(\Omega)$, with the usual norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} .$$

Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying

$$\sup_{\xi \in \mathbf{R}} \frac{|f(\xi)|}{1 + |\xi|^p} < +\infty ,$$

where $0 < p < \frac{n+2}{n-2}$ if $n > 2$ and $0 < p < +\infty$ if $n = 2$, and with no growth condition if $n = 1$. For each $u \in X$, set

$$J_f(u) = \int_\Omega F(u(x))dx$$

where

$$F(\xi) = \int_0^\xi f(t)dt .$$

From classical results, the functional J_f is C^1 and J_f' is compact, and so J_f is sequentially weakly continuous.

For $\lambda > 0$, we consider the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

As usual, a weak solution of (P_λ) is any $u \in X$ such that

$$\int_\Omega \nabla u(x) \nabla v(x) dx = \lambda \int_\Omega f(u(x))v(x)dx$$

for all $v \in X$.

Hence, the weak solutions of (P_λ) are exactly the critical points in X of the functional $u \to \frac{1}{2}\|u\|^2 - \lambda J_f(u)$.

A classical solution of (P_λ) is any $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, zero on $\partial\Omega$, which satisfies the equation pointwise in Ω. If f is locally Hölder continuous, then the weak solutions of (P_λ) are classical.

Let us also recall that if f is zero in $]-\infty, 0[$ and non-negative in $[0, +\infty[$, then any non-zero classical solution of the problem is positive in Ω.

Consequently, if f is zero in $]-\infty, 0[$ and non-negative and locally Hölder continuous in $[0, +\infty[$, and if problem (P_λ) has a unique positive classical solution u, then u turns out to be the only non-zero weak solution of (P_λ).

The result about problem (P_λ) coming out from Theorem 1 reads as follows:

Theorem 2. *Set*

$$\rho_f = \liminf_{\|u\| \to +\infty} \frac{J_f(u)}{\|u\|^2} ,$$

$$\sigma_f = \sup_{u \in X \setminus \{0\}} \frac{J_f(u)}{\|u\|^2}$$

and

$$\gamma_f(r) = \sup_{\|u\|^2 = r} J_f(u)$$

for all $r > 0$. Let a, b satisfy

$$\max\{0, \rho_f\} \leq a < b \leq \sigma_f .$$

Assume that J_f has no local minima in $X \setminus \{0\}$ and that, for each $\lambda \in]a, b[$, the problem

$$\begin{cases} -\Delta u = \dfrac{1}{2\lambda} f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique non-zero weak solution, say $\hat{u}_\lambda$. Let M_a (resp. M_b if $b < +\infty$ or $M_b = \emptyset$ if $b = +\infty$) be the set of all global minima in X of the functional $u \to a\|u\|^2 - J_f(u)$ (resp. $u \to b\|u\|^2 - J_f(u)$ if $b < +\infty$). Set

$$\alpha_f = \max\left\{0, \sup_{u \in M_b} \|u\|^2\right\}$$

and

$$\beta_f = \inf_{u \in M_a} \|u\|^2 .$$

Then, the following assertions hold:

(b_1) *one has* $\alpha_f < \beta_f$ *;*

(b_2) *the function* $\lambda \to g_f(\lambda) := \|\hat{u}_\lambda\|^2$ *is decreasing in* $]a, b[$ *and its range is* $]\alpha_f, \beta_f[$ *;*

(b_3) *for each* $r \in]\alpha_f, \beta_f[$*, the function* $\hat{v}_r := \hat{u}_{g_f^{-1}(r)}$ *is the unique global maximum of* $(J_f)_{|S_r}$ *towards which every maximizing sequence in* S_r *converges ;*

(b_4) *the function* $r \to \hat{v}_r$ *is continuous in* $]\alpha_f, \beta_f[$ *;*

(b_5) *the function* γ_f *is* C^1*, increasing and strictly concave in* $]\alpha_f, \beta_f[$ *;*

(b_6) *for each* $r \in]\alpha_f, \beta_f[$*, the function* $\hat{v}_r$ *is the unique non-zero solution of the problem*

$$\begin{cases} -\Delta u = \dfrac{1}{2\gamma_f'(r)} f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ ; \end{cases}$$

(b_7) *one has*

$$\gamma_f'(r) = g_f^{-1}(r)$$

for all $r \in]\alpha_f, \beta_f[$ *;*

(b_8) *for each* $\lambda \in]a, b[$*, there exists a unique* $r \in]\alpha_f, \beta_f[$ *such that* $\lambda = \gamma_f'(r)$ *and* $\hat{u}_\lambda = \hat{v}_r$*.*

Proof. If $\lambda \in]a, b[$, the functional $u \to \lambda\|u\|^2 - J_f(u)$ is coercive and has negative infimum in X, and hence $\hat{u}_\lambda$ turns out to be the unique global minimum of it. At this point, we are allowed to apply Theorem 1 taking $J = J_f$. So, each (b_i), with $i < 8$, follows directly from the corresponding (a_i). Concerning (b_8), it is clear that, for each $\lambda \in]a, b[$, the unique $r \in]\alpha_f, \beta_f[$ with the claimed property is $g_f(\lambda)$. ∎.

Remark 3. The hypotheses of Theorem 2 are most general but do not deal directly with f. It is therefore useful to point out some conditions, involving directly f, which imply them. To this end, let λ_1 denote the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \ . \end{cases}$$

Recall that $\|u\|_{L^2(\Omega)} \leq \lambda_1^{-\frac{1}{2}}\|u\|$ for all $u \in X$. Now, set

$$\tilde{\rho}_f = \limsup_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^2}$$

and

$$\tilde{\sigma}_f = \liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} \ .$$

It is easily seen that

$$\rho_f \le \frac{\tilde{\rho}_f}{\lambda_1}$$

and that

$$\sigma_f \ge \frac{\tilde{\sigma}_f}{\lambda_1}\ .$$

Moreover, it is clear that $\sigma_f > 0$ when $\sup_{\xi\in\mathbf{R}} F(\xi) > 0$. Consequently, Theorem 2 is potentially applicable when $\max\{0, \tilde{\rho}_f\} < \tilde{\sigma}_f$ or when $\tilde{\rho}_f \le 0$ and $\sup_{\xi\in\mathbf{R}} F(\xi) > 0$. Further, note that J_f has no local maxima in X if either $f(0) \neq 0$ or f is zero in $]-\infty, 0]$ and positive in $]0, +\infty[$ (see [10], p. 456).

To conclude, we show a sample of application of Theorem 2.

Proposition 2. *Let $g \in C^1([0, +\infty[)$. Assume that $g(0) = 0$, $g'(0) > 0$, $g(\xi) > 0$ for all $\xi > 0$, $\lim_{\xi\to+\infty} \xi g'(\xi) = 0$, $\lim_{\xi\to+\infty} g(\xi)$ exists and is finite and positive. Let f defined by*

$$f(\xi) = \begin{cases} g(\xi) & \text{if } \xi \ge 0 \\ 0 & \text{if } \xi < 0\ . \end{cases}$$

Then, the conclusions of Theorem 2 hold with $a = 0$, with some $b \in \left]0, \frac{g'(0)}{2\lambda_1}\right]$ and with $\beta_f = +\infty$.

Proof. By [1], there exists $\lambda^* > 0$ such that, for every $\mu > \lambda^*$, the problem

$$\begin{cases} -\Delta u = \mu g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution. Consequently, if $\lambda \in \left]0, \frac{1}{2\lambda^*}\right[$, the problem

$$\begin{cases} -\Delta u = \dfrac{1}{2\lambda} f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique non-zero weak solution. Of course, we have $\rho_f \le 0$ and $\tilde{\sigma}_f = \frac{g'(0}{2}$. At this point, we are allowed to apply Theorem 2 taking $a = 0$ and $b = \min\left\{\frac{1}{2\lambda^*}, \frac{g'(0)}{2\lambda_1}\right\}$. Moreover, we have $\beta_f = +\infty$ since J_f has no global maxima. ■

For instance, Proposition 2 can be applied to the function

$$g(\xi) = \text{arctg}\xi + c\frac{\xi^q}{\xi^p + 1}$$

where $1 \leq q \leq p$ and $c \geq 0$.

Remark 4. Note that Theorem 2 applies also when f is zero in $]-\infty, 0]$ and $\xi \to \frac{f(\xi)}{\xi}$ is positive and decreasing in $]0, +\infty[$. This is just the case treated in [10]. Another case where Theorem 2 applies is when $f(0) \neq 0$ and f is Lipschitzian (see [4]).

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