TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 1, pp. 231-237, March 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

LOGARITHMIC CONVEXITY OF THE ONE-PARAMETER MEAN VALUES

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Abstract. In this article, the logarithmic convexity of the one-parameter mean values $J(r)$ and the monotonicity of the product $J(r)J(-r)$ with $r \in \mathbb{R}$ are presented. Some more general results are established.

1. Introduction

The one-parameter mean values $J(r; x, y)$ for two positive numbers x and y with $x \neq y$ are defined by

$$(1) \qquad J(r) \triangleq J(r; x, y) = \begin{cases} r(x^{r+1} - y^{r+1})/(r+1)(x^r - y^r), & r \neq 0, -1; \\ (x - y)/(\ln x - \ln y), & r = 0; \\ xy(\ln x - \ln y)/(x - y), & r = -1. \end{cases}$$

There has been some literature on the one-parameter mean values $J(r; x, y)$, see [1-4, 7].

The main purpose of this paper is to prove the logarithmic convexity of the one-parameter mean values $J(r; x, y)$ and the monotonicity of $J(-r)J(r)$ for $r \in \mathbb{R}$.

Our main results are as follows.

Theorem 1. *Let x and y be positive numbers with $x \neq y$. Then*

(i) *The one-parameter mean values $J(r)$ are strictly increasing in $r \in \mathbb{R}$;*

Received January 24, 2005, accepted January 23, 2006.
Communicated by H. M. Srivastava.
2000 *Mathematics Subject Classification*: Primary 26A48, 26A51; Secondary 26B25, 26D07.
Key words and phrases: Logarithmic convexity, Monotonicity, One-parameter mean values.
*Supported by a seed grant for basic research of the University of Hong Kong, Hong Kong SAR, People's Republic of China.
**Supported by the Science Foundation of Project for Fostering Innovation Talents at Universities of Henan Province of the People's Republic of China

(ii) *The one-parameter mean values $J(r)$ are strictly logarithmically convex in $(-\infty, -1/2)$ and strictly logarithmically concave in $(-1/2, \infty)$.*

Theorem 2. *Let $\mathcal{J}(r) = J(r)J(-r)$ with $r \in \mathbb{R}$ for fixed positive numbers x and y with $x \neq y$. Then the function $\mathcal{J}(r)$ is strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$.*

2. Proofs of Theorems

2.1. Proof of Theorem 1.

Let

$$g(t) \triangleq g(t; x, y) = \begin{cases} (y^t - x^t)/t, & t \neq 0 \\ \ln y - \ln x, & t = 0 \end{cases} \tag{2}$$

for positive numbers x and y with $x \neq y$.

In [5], Corollary 3 states that, for $y > x > 0$, if $t > 0$, then

$$g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3 < 0; \tag{3}$$

if $t < 0$, inequality (3) reverses.

2.1.1. Formula (3) implies that, for $y > x > 0$,

$$[g'(t)/g(t)]'' = \operatorname{sgn}(-t). \tag{4}$$

From this, we obtain that the function $[g'(t)/g(t)]'$ is strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$.

By using Cauchy-Schwartz integral inequality or Tchebycheff integral inequality, it is obtained [6-8] that $[g'(t)/g(t)]' > 0$ for $t \in \mathbb{R}$. Then the function $g'(t)/g(t)$ is strictly increasing in $(-\infty, \infty)$.

The one-parameter mean values $J(r)$ can be rewritten in terms of g as $J(r) = g(r+1)/g(r)$ with $r \in \mathbb{R}$ for $y > x > 0$. Taking the logarithm of $J(r)$ yields

$$\ln J(r) = \ln g(r+1) - \ln g(r) = \int_r^{r+1} \frac{g'(u)}{g(u)}\,\mathrm{d}u = \int_0^1 \frac{g'(u+r)}{g(u+r)}\,\mathrm{d}u \tag{5}$$

and $[\ln J(r)]' = g'(r+1)/g(r+1) - g'(r)/g(r) > 0$. Hence the functions $\ln J(r)$ and $J(r)$ are strictly increasing in $r \in (-\infty, \infty)$. This proves (i).

2.1.2. If $r < -1$, then $r < r+1 < 0$ and $[\ln J(r)]'' = [g'(r+1)/g(r+1)]' - [g'(r)/g(r)]' > 0$ which follows from the strictly increasing property of $[g'(r)/g(r)]'$ in $(-\infty, 0)$.

If $r > 0$, then from the strictly decreasing property of $[g'(r)/g(r)]'$ in $(0,\infty)$, we have $[\ln J(r)]'' < 0$.

If $-1 < r < 0$, then $r < 0 < r+1$, and we have

$$
\begin{aligned}
\big[\ln J(r)\big]'' &= \left(\frac{g'(r+1)}{g(r+1)}\right)' - \left(\frac{g'(r)}{g(r)}\right)' \\
&= \frac{g''(r+1)g(r+1) - [g'(r+1)]^2}{g^2(r+1)} - \frac{g''(r)g(r) - [g'(r)]^2}{g^2(r)} \\
&= \frac{g''(u)g(u) - [g'(u)]^2}{g^2(u)} - \frac{g''(-r)g(-r) - [g'(-r)]^2}{g^2(-r)} \\
&= \frac{g''(u)g(u) - [g'(u)]^2}{g^2(u)} - \frac{g''(v)g(v) - [g'(v)]^2}{g^2(v)} \\
&= \left(\frac{g'(u)}{g(u)}\right)' - \left(\frac{g'(v)}{g(v)}\right)',
\end{aligned} \tag{6}
$$

where $u = r+1 > 0$ and $v = -r > 0$. Thus, $[\ln J(r)]'' < 0$ for $-1 < r < 0$ and $r+1 > -r$. This means that $[\ln J(r)]'' < 0$ for $r \in \big(-1/2, 0\big)$.

Similar as above, $[\ln J(r)]'' > 0$ for $-1 < r < 0$ and $-r > r+1$. This means that $[\ln J(r)]'' > 0$ for $r \in (-1, -1/2)$. This proves (ii).

Remark. From (4), (5) and by direct calculation, we have

$$
[\ln J(r)]'' = \int_0^1 \frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{g'(u+r)}{g(u+r)}\right)\mathrm{d}u < 0 \tag{7}
$$

for $r \in (0,\infty)$. This means that $J(r;x,y)$ is strictly logarithmically concave in $r \in (0,\infty)$, whether $x > y$ or $x < y$, since $J(r;x,y) = J(r;y,x)$ holds.

By straightforward computation, we have

$$
J(r) = \frac{xy}{J(-r-1)} \tag{8}
$$

for $r \in \mathbb{R}$. Hence, if $r \in (-\infty, -1)$, from (3), (4) and (7), it follows that $[\ln J(r)]'' = -[\ln J(-r-1)]'' = -\int_0^1 \{\,\mathrm{d}^2[g'(u-r-1)/g(u-r-1)]/\,\mathrm{d}r^2\}\,\mathrm{d}u > 0$. This tells us that the one-parameter mean values $J(r;x,y)$ are strictly logarithmically convex in $r \in (-\infty,-1)$, whether $x > y$ or $x < y$, since $J(r;x,y) = J(r;y,x)$.

2.2. Proof of Theorem 2.

It is easy to obtain that $\mathcal{J}(r) = xyJ(r)/J(r-1)$ for $r \in \mathbb{R}$. Then $\ln \mathcal{J}(r) = \ln(xy) + \ln J(r) - \ln J(r-1)$ and

$$[\ln \mathcal{J}(r)]' = \frac{J'(r)}{J(r)} - \frac{J'(r-1)}{J(r-1)}. \tag{9}$$

Theorem 1 states that the function $J(r)$ is strictly logarithmically convex in $(-\infty, -1/2)$. Thus, being the derivative of $\ln J(r)$, $J'(r)/J(r)$ is strictly increasing in $(-\infty, -1/2)$, that is, $J'(r)/J(r) > J'(r-1)/J(r-1)$, or, equivalently, $[\ln \mathcal{J}(r)]' > 0$ for $r \in (-\infty, -1/2)$, thus $\ln \mathcal{J}(r)$ and $\mathcal{J}(r)$ are strictly increasing in $(-\infty, -1/2)$.

From (8), it follows that $\ln J(r) = \ln(xy) - \ln J(-r-1)$ and $J'(r)/J(r) = J'(-r-1)/J(-r-1)$. Then (9) results in $[\ln \mathcal{J}(r)]' = J'(-r-1)/J(-r-1) - J'(r-1)/J(r-1)$.

For $r \in (-1/2, 0)$, we have $-3/2 < r-1 < -1$ and $-1 < -r-1 < -1/2$. Since $J'(r)/J(r)$ is strictly increasing in $(-\infty, -1/2)$, $[\ln \mathcal{J}(r)]' > 0$ for $r \in (-1/2, 0)$, therefore $\ln \mathcal{J}(r)$ and $\mathcal{J}(r)$ are also strictly increasing in $(-1/2, 0)$.

It is clear that the function $\mathcal{J}(r)$ is even in $(-\infty, \infty)$. So, it is easy to see that $\mathcal{J}(r)$ is strictly decreasing in $(0, \infty)$. The proof of Theorem 2 is completed.

3. Some Related Results

For $x \neq y$ and $\alpha > 0$, define for $r \in \mathbb{R}$

$$J_\alpha(r) \triangleq J_\alpha(r; x, y) = \begin{cases} [r(x^{r+\alpha} - y^{r+\alpha})/(r+\alpha)(x^r - y^r)]^{1/\alpha}, & r \neq 0, -\alpha; \\ [(x^\alpha - y^\alpha)/\alpha(\ln x - \ln y)]^{1/\alpha}, & r = 0; \\ [\alpha x^\alpha y^\alpha (\ln x - \ln y)/(x^\alpha - y^\alpha)]^{1/\alpha}, & r = -\alpha. \end{cases} \tag{10}$$

We call $J_\alpha(r; x, y)$ the generalized one-parameter mean values for two positive numbers x and y in the interval $(-\infty, \infty)$.

It is clear that $J_1(r; x, y) = J(r; x, y)$ and $J_\alpha(r; x, y) = [g(r+\alpha)/g(r)]^{1/\alpha}$. By the same arguments as in the proofs of Theorem 1 and Theorem 2, we can obtain the following

Theorem 3. *Let x and y be positive numbers with $x \neq y$. Then*

(1) *The generalized one-parameter mean values $J_\alpha(r)$ are strictly increasing in $r \in \mathbb{R}$;*

(2) *The mean values $J_\alpha(r)$ are strictly logarithmically convex in $(-\infty, -\alpha/2)$ and strictly logarithmically concave in $(-\alpha/2, \infty)$;*

(3) *Let* $\mathcal{J}_\alpha(r) = J_\alpha(r)J_\alpha(-r)$ *with* $r \in \mathbb{R}$ *for positive numbers* x *and* y *with* $x \neq y$. *Then the function* $\mathcal{J}_\alpha(r)$ *is strictly increasing in* $(-\infty, 0)$ *and strictly decreasing in* $(0, \infty)$.

Proof. These follow from combining the identities $[J_\alpha(r; x, y)]^\alpha = J(r/\alpha; x^\alpha, y^\alpha)$ and $[\mathcal{J}_\alpha(r)]^\alpha = \mathcal{J}(r/\alpha)$ with Theorem 1 and Theorem 2. ∎

Theorem 4. *The function* $(r+\alpha)[J_\alpha(r)]^\alpha$ *is strictly increasing and strictly convex in* $(-\infty, \infty)$, *and is strictly logarithmically concave for* $r > -\alpha/2$.

Proof. Direct computation gives

$$(r+\alpha)[J_\alpha(r; x, y)]^\alpha = \alpha\Big(\frac{r}{\alpha}+1\Big)J\Big(\frac{r}{\alpha}; x^\alpha, y^\alpha\Big), \tag{11}$$

$$\frac{\mathrm{d}^2 \ln\big\{(r+\alpha)[J_\alpha(r)]^\alpha\big\}}{\mathrm{d}r^2} = -\frac{1}{(r+\alpha)^2} + \alpha[\ln J_\alpha(r)]''. \tag{12}$$

From the result by Alzer in [3] that the function $(r+1)J(r; x, y)$ is strictly convex in $(-\infty, \infty)$, it is not difficult to obtain that the function $(r+\alpha)[J_\alpha(r; x, y)]^\alpha$ is also strictly convex in $(-\infty, \infty)$ by using (11).

By standard argument, we have

$$\begin{aligned}\lim_{r\to-\infty}\{[J_\alpha(r)]^\alpha\}' &= \lim_{r\to-\infty}[\alpha(z^{r+\alpha}-1)/(r+\alpha)(z^r-1)]\\ &\quad - \lim_{r\to-\infty}[rz^r(z^\alpha-1)\ln z/(z^r-1)^2] = 0\end{aligned}$$

and $\lim_{r\to-\infty}[J_\alpha(r)]^\alpha = \min\{x^\alpha, y^\alpha\}$, where $z = y/x \neq 1$. This leads to

$$\begin{aligned}\lim_{r\to-\infty}\{(r+\alpha)[J_\alpha(r)]^\alpha\}' &= \lim_{r\to-\infty}[J_\alpha(r)]^\alpha + \lim_{r\to-\infty}(r+\alpha)\{[J_\alpha(r)]^\alpha\}'\\ &= \min\{x^\alpha, y^\alpha\} > 0.\end{aligned} \tag{13}$$

The convexity of $(r+\alpha)[J_\alpha(r)]^\alpha$ means that $\{(r+\alpha)[J_\alpha(r)]^\alpha\}'$ is strictly increasing, in view of (13), $\{(r+\alpha)[J_\alpha(r)]^\alpha\}' > 0$, and so $(r+\alpha)[J_\alpha(r)]^\alpha$ is strictly increasing in $(-\infty, \infty)$.

Since $J_\alpha(r)$ is strictly logarithmically concave in $(-\alpha/2, \infty)$, $[\ln J_\alpha(r)]'' < 0$, then $\mathrm{d}^2 \ln\{(r+\alpha)[J_\alpha(r)]^\alpha\}/\,\mathrm{d}r^2 < 0$ by (12). This means that the function $(r+\alpha)[J_\alpha(r)]^\alpha$ is strictly logarithmically concave in $(-\alpha/2, \infty)$. ∎

Corollary 1. *If* $r < -\alpha$, *then*

$$0 < \frac{\{[J_\alpha(r)]^\alpha\}'}{[J_\alpha(r)]^\alpha} = \frac{\{[J_\alpha(-r-\alpha)]^\alpha\}'}{[J_\alpha(-r-\alpha)]^\alpha} < -\frac{1}{r+\alpha}, \tag{14}$$

$$0 < \frac{\{[J_\alpha(r)]^\alpha\}''}{\{[J_\alpha(r)]^\alpha\}'} < -\frac{2}{r+\alpha}. \tag{15}$$

Proof. From the monotonicity and convexity of $(r+\alpha)J_\alpha(r)$, we have

$$\{(r+\alpha)[J_\alpha(r)]^\alpha\}' = [J_\alpha(r)]^\alpha + (r+\alpha)\{[J_\alpha(r)]^\alpha\}' > 0, \tag{16}$$

$$\{(r+\alpha)[J_\alpha(r)]^\alpha\}'' = 2\{[J_\alpha(r)]^\alpha\}' + (r+\alpha)\{[J_\alpha(r)]^\alpha\}'' > 0. \tag{17}$$

Inequality (14) follows from combining (16) with $[J_\alpha(r)]^\alpha = xy/[J_\alpha(-r-\alpha)]^\alpha$. Inequality (15) is a direct consequence of (17). ∎

Theorem 5. *The function* $r \ln J_\alpha(r)$ *is strictly convex in* $(-\alpha/2, 0)$.

Proof. Direct calculation yields $[r\ln J_\alpha(r)]'' = 2[\ln J_\alpha(r)]' + r[\ln J_\alpha(r)]''$. Since $J_\alpha(r)$ is strictly increasing in $(-\infty,\infty)$ and strictly logarithmically concave in $(-\alpha/2,\infty)$, it follows that $[\ln J_\alpha(r)]' > 0$ and $[\ln J_\alpha(r)]'' < 0$ in $(-\alpha/2,\infty)$. Therefore, $[r\ln J_\alpha(r)]'' > 0$ and $r\ln J_\alpha(r)$ is strictly convex in $(-\alpha/2, 0)$. ∎

Remark. If $\alpha = 1$, then $r\ln J(r)$ is strictly convex in $(-1/2, 0)$. This partially answers the question raised by Alzer in [3].

4. Open Problems

Finally, we pose the following

Open Problem 1. The generalized one-parameter mean values $J_\alpha(r)$ defined by (10) are strictly concave in $(-\alpha/2,\infty)$.

Open Problem 2. The function $\mathcal{J}_\alpha(t) = J_\alpha(t)J_\alpha(-t)$ is strictly logarithmically convex for $t \notin [-\frac{\alpha}{2}, \frac{\alpha}{2}]$ and strictly concave and strictly logarithmically concave for $t \in (-\alpha/2, \alpha/2)$.

Open Problem 3. Discuss the monotonic and (logarithmically) convex properties of the function $J_\alpha(r) + J_\alpha(-r)$.

Acknowledgment

The authors would like to express their many thanks to Professor H. M. Srivastava and Dr. A. Witkowski for their valuable comments on this paper.

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