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SINGLE ELEMENTS IN SOME REFLEXIVE ALGEBRA MODULES

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Abstract. In this paper, we first introduce the concept of single elements in a module. A systematic study of single elements in Alg$\mathcal{L}$-module $\mathcal{U}_\phi$ is initiated, where $\mathcal{L}$ is a completely distributive subspace lattice on a Banach space $\mathcal{X}$ and ϕ is an order homomorphism from $\mathcal{L}$ into $\mathcal{L}$. For a reflexive Banach space $\mathcal{X}$ and a positive integer n (or $+\infty$), by virtue of the order homomorphism ϕ we give necessary and sufficient conditions for the existence of single elements of $\mathcal{U}_\phi$ of rank n (or $+\infty$).

1. Introduction and Preliminariesntroduction

An element S of an algebra $\mathcal{A}$ is called single if the condition $ASB = 0$ for A, B in $\mathcal{A}$ implies $AS = 0$ or $SB = 0$. It is easy to show that a rank one operator is a single element of any operator algebra containing it. It is primarily for this reason that the notion of 'single element' plays a role in the representation theory of C^*-algebras, or more generally, of semi-simple Banach algebras ([1], [2]). In yet another aspect, namely in the study of algebraic isomorphisms between reflexive operator algebras on a normed space, single elements have also proved a useful tool. This is mainly because single elements are carried to single elements under algebraic isomorphisms. So if, in a particular operator algebra, it is known that each single element is of rank one (the converse is always true), then the study of algebraic isomorphism is considerably simplified. Reflexive algebras that have been looked at from this point of view include nest algebras on a Hilbert space [9], algebras of operators leaving invariant the elements of a complete atomic Boolean lattice of subspaces on normed space [5]. On each of the above two mentioned operator algebras it is proved in [5] and [9] that single elements are of rank one.

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Using this, one shows that each algebraic isomorphism between a pair of such algebras is automatically continuous and spatial in the sense that it is of the form $\Phi(A) = T^{-1}AT$ for a suitable T ([5], [9]).

A systematic study of single elements of a reflexive operator algebra Alg$\mathcal{L}$, where $\mathcal{L}$ is completely distributive, was initiated by Lambrou in [6]. Amongst many other interesting results, he shows that single elements of any rank (including infinity) are possible. In [8], Longstaff and Penaia obtained a lattice-theoretic conditions for the existence of single element of rank n (or infinity). The present note owes much to the ideas contained in [6] and [8].

Definition 1.1. Let $\mathcal{R}$ be a ring and $\mathcal{M}$ a $\mathcal{R}$-bimodule. An element $s \in \mathcal{M}$ is called single if, whenever $asb = 0$ with $a, b \in \mathcal{R}$, then $as = 0$ or $sb = 0$.

Though this is a simple generalization of the notion of single elements in algebras, we hope that as single elements in algebras, single elements in modules will play a role in the representation theory of modules, and in the study of module isomorphisms between modules, especially in the operator modules theory. In this paper, we initiated the study of single elements in Alg$\mathcal{L}$-modules $\mathcal{U}_\phi$, where $\mathcal{L}$ is a completely distributive subspace lattice on a Banach space $\mathcal{X}$ and ϕ is an order homomorphism from $\mathcal{L}$ into itself. For a reflexive Banach space $\mathcal{X}$ and a positive integer n, we show that the existence of a single element of $\mathcal{U}_\phi$ of rank one n depends on ϕ and a lattice-theoretic condition on $\mathcal{L}$. Similar conditions also are obtained for the existence of a single element of infinite rank.

Let us introduce some notation and terminology. Throughout what follows $\mathcal{X}$ will denote a real or complex Banach space with topological dual $\mathcal{X}^*$. The terms operator on $\mathcal{X}$ and subspace of $\mathcal{X}$ shall mean bounded linear mapping of $\mathcal{X}$ into itself and closed linear manifold of $\mathcal{X}$ respectively. For non-zero vectors $e^* \in \mathcal{X}^*$ and $f \in \mathcal{X}$, the rank one operator defined by $x \to e^*(x)f$ is denoted by $e^* \otimes f$. Clearly $(e^* \otimes f)^* = \hat{f} \otimes e^*$, where $\hat{f}$ is the image of f under the canonical map of $\mathcal{X}$ into $\mathcal{X}^{**}$. The linear span of a vector g is denoted by $\langle g \rangle$.

A collection $\mathcal{L}$ of subspaces of $\mathcal{X}$ is called a subspace lattice on $\mathcal{X}$ if it contains (0) and $\mathcal{X}$, and is closed under the formation of arbitrary closed linear spans (denoted '$\bigvee$') and intersection (denoted '$\bigcap$'). If in a subspace lattice $\mathcal{L}$, the following infinite distributive identity

$$\bigcap_{a \in A} \bigvee_{b \in B} L_{a,b} = \bigvee_{f \in B^A} \bigcap_{a \in A} L_{a,f(a)}$$

and its dual holds, $\mathcal{L}$ is called completely distributive. The formal definition of complete distributivity just given is in practice, difficult to use. Alternative characterizations of completely distributivity have been proven to be more useful.

Lemma 1.1. ([7]) *For a subspace lattice $\mathcal{L}$, the followings are equivalent:*

(1) $\mathcal{L}$ *is completely distributive;*

(2) $L = \bigcap\{K_- : K \in \mathcal{L}, K \nsubseteq L\}$*, for every* $L \in \mathcal{L}$*;*

(3) $L = \bigvee\{K \in \mathcal{L} : K_- \nsupseteq L\}$*, for every* $L \in \mathcal{L}$*;*

(4) $L = \bigvee\{K_+ : K \in \mathcal{L}, K \nsupseteq L\}$*, for every* $L \in \mathcal{L}$*. here* $K_- = \bigvee\{G \in \mathcal{L} : G \nsupseteq K\}$ *and* $K_+ = \bigcap\{G \in \mathcal{L} : G \nsubseteq K\}$ *.*

If $\mathcal{L}$ is a subspace lattice, Alg$\mathcal{L}$ denote the set of operators on $\mathcal{X}$ leaving every member of $\mathcal{L}$ invariant. Obviously, Alg$\mathcal{L}$ is a unital Banach algebra, operator algebras of the type Alg$\mathcal{L}$ are called reflexive operator algebras. A nest $\mathcal{L}$ is a totally ordered subspace lattice, and Alg$\mathcal{L}$ is called the nest algebra associated with the nest $\mathcal{L}$. Suppose that ϕ is an order homomorphism of $\mathcal{L}$ into itself (that is, $L \leq L^{'}$ implies $\phi(L) \leq \phi(L'))$. Then the set

$$\mathcal{U}_\phi = \{X \in \mathcal{B}(\mathcal{X}) : XL \subseteq \phi(L), \forall L \in \mathcal{L}\}$$

is clearly a weakly closed (two sided) Alg$\mathcal{L}$-module. In [3], Erdos and Power show that every weakly closed Alg$\mathcal{L}$-module is of the above form.

If L is a subspace of $\mathcal{X}$, its annihilator is denoted by $L^\perp$. Thus $L^\perp = \{e^* \in \mathcal{X}^* : e^*(f) = 0, \text{ for every } f \in L\}$. Dually, if M is a subspace of $\mathcal{X}^*$, its pre-annihilator is denoted by ${}^\perp M$. Thus ${}^\perp M = \{f \in \mathcal{X} : e^*(f) = 0, \text{ for every } e^* \in M\}$. We have $\mathcal{X}^\perp = (0), (0)^\perp = \mathcal{X}^*$ and ${}^\perp(L^\perp) = L$. Also, if $\mathcal{X}$ is reflexive, we have $({}^\perp M)^\perp = M$. If $\mathcal{X}$ is a reflexive Banach space, we can easily show that $\mathcal{L}^\perp = \{L^\perp : L \in \mathcal{L}\}$ is a subspace lattice on $\mathcal{X}^*$ and Alg$\mathcal{L}^\perp = \{A^* : A \in$ Alg$\mathcal{L}\}$. If $\mathcal{L}$ is a completely distributive subspace lattice on $\mathcal{X}$, it follows from Lemma 1.1 that $\mathcal{L}^\perp$ is also completely distributive on $\mathcal{X}^*$.

For a subspace lattice $\mathcal{L}$ we denote $\mathcal{J}$ the set of elements $\mathcal{J} = \{L \in \mathcal{L} : L \neq (0)$ and $L_- \neq \mathcal{X}\}$. It follows from Lemma 1.1 (3) that the linear manifold $\mathcal{X}_0 = span\{K : K \in \mathcal{J}\} = span\{K \in \mathcal{L} : K \neq (0)$ and $K_- \neq \mathcal{X}\}$ is norm dense in $\mathcal{X}$. Similarly, Lemma 1.1 (2) shows that $\mathcal{X}_1 = span\{K_-^\perp : K \in \mathcal{J}\}$ is weak* dense in $\mathcal{X}^*$.

Lemma 1.2. ([7]) *If* $\mathcal{L}$ *is a subspace lattice on a real or complex Banach space, then the rank one operator* $e^* \otimes f$ *belongs to Alg*$\mathcal{L}$ *if and only if there is an element* $L \in \mathcal{J}$ *such that* $f \in L$ *and* $e^* \in L_-^\perp$*.*

2. Elementary Properties of Single elements

Lemma 2.1. *Let* $\mathcal{L}$ *be a completely distributive subspace lattice on* $\mathcal{X}$*, and let* $A \in \mathcal{B}(\mathcal{X})$*. (1) If* $RA = 0$ *for all rank one operators* $R \in$ *Alg*$\mathcal{L}$*, then* $A = 0$*; (2) If* $AR = 0$ *for all rank one operators* $R \in$ *Alg*$\mathcal{L}$*, then* $A = 0$*.*

Proof. (1) If $R \in \mathrm{Alg}\mathcal{L}$ is of rank one, then $R = e^* \otimes f$, where $e^* \in K_-^\perp$ for some $K \in \mathcal{L}$. The condition $0 = (e^* \otimes f)A = A^*e^* \otimes f$ for all rank ones of $\mathrm{Alg}\mathcal{L}$ implies $\ker A^* \supseteq K_-^\perp$, so taking the linear span over all such K, we have $\ker A^* \supseteq \mathcal{X}_1$, where $\mathcal{X}_1$ is as defined just above the statement of Lemma 1.2. The weak* continuity of A^* and the weak* density of $\mathcal{X}_1$ implies that A^*, and hence A, is zero.

(2) The condition $0 = A(e^* \otimes f) = e^* \otimes Af$ implies $\ker A \supseteq K$ for each K with $K_- \neq \mathcal{X}$. So $\ker A \supseteq \mathcal{X}_0$, which is norm dense in $\mathcal{X}$, and so A is zero. ∎

Lemma 2.2. *Let $\mathcal{L}$ be a complete distributive subspace lattice on $\mathcal{X}$ and ϕ an order homomorphism from $\mathcal{L}$ into $\mathcal{L}$. Then an element S of $\mathcal{U}_\phi$ is single if and only if for each rank one operators R_1, R_2 of Alg$\mathcal{L}$ the condition $R_1SR_2 = 0$ implies R_1S or SR_2 is zero.*

Proof. If S is single, then the above condition is only a special case of the definition. Suppose that $ASB = 0$ for $A, B \in \mathrm{Alg}\mathcal{L}$. If $SB \neq 0$, then by Lemma 2.1 there exists a rank one operator R_2 of Alg$\mathcal{L}$ such that $SBR_2 \neq 0$. For any rank one R_1 of Alg$\mathcal{L}$ we have $R_1ASBR_2 = 0$, and clearly R_1A and BR_2 are are of rank one or zero. In either case the condition in the lemma implies R_1AS or SBR_2 is zero. But as $SBR_2 \neq 0$, we have for all rank one operators R_1 of Alg$\mathcal{L}$ that R_1AS is zero. Applying Lemma 2.1 once again, it follows that $AS = 0$, and this shows that S is a single element of $\mathcal{U}_\phi$. ∎

Lemma 2.3. *Let $\mathcal{L}$ be a completely distributive subspace lattice on the Banach space $\mathcal{X}$, ϕ an order homomorphism from $\mathcal{L}$ into itself and S a nonzero single element of $\mathcal{U}_\phi$. Then there exists an M in $\mathcal{L}$ with $M_- \neq \mathcal{X}$ such that $S\mid_M$ is nonzero. Moreover, for any $L \in \mathcal{L}$ with $L_- \neq \mathcal{X}$ and $S\mid_L$ nonzero, the operator $S\mid_L$ is of rank one.*

Proof. By Lemma 2.1 there is a rank one operator $R \in \mathrm{Alg}\mathcal{L}$ such that $SR \neq 0$. By Lemma 1.2, R is of the form $e^* \otimes f$ where $f \in M, e^* \in M_-^\perp$ for some $M \in \mathcal{L}$ and $M_- \neq \mathcal{X}$. But then $0 \neq S(e^* \otimes f) = e^* \otimes Sf$ shows that $Sf \neq 0$, and the first part of the lemma is proved.

By Lemma 2.1 there is a rank one $T \in \mathrm{Alg}\mathcal{L}$ such that $TS \neq 0$. Let now $L \in \mathcal{L}$ satisfy the condition in the statement of the lemma. We are to prove that if $x, y \in L$ then Sx and Sy are linearly dependent, so there is no loss in assuming that Sx, Sy are nonzero.

The operator TS is of rank one, so there exist scalars λ, μ not both zero, such that $TS(\lambda x + \mu y) = \lambda(TSx) + \mu(TSy) = 0$. For any nonzero $x^* \in L_-^\perp$, we have $x^* \otimes (\lambda x + \mu y) \in \mathrm{Alg}\mathcal{L}$ and

$$TS[x^* \otimes (\lambda x + \mu y)] = x^* \otimes TS(\lambda x + \mu y) = 0.$$

However, S is single and $TS \neq 0$, so

$$x^* \otimes S(\lambda x + \mu y) = S[x^* \otimes (\lambda x + \mu y)] = 0,$$

which in turn implies $\lambda(Sx) + \mu(Sy) = 0$, and so Sx, Sy are linearly dependent. Thus the operator $S\mid_L$ is of rank one. ∎

Lemma 2.4. *Let $\mathcal{L}$ be a completely distributive subspace lattice, ϕ an order homomorphism from $\mathcal{L}$ into itself and S a nonzero single element of $\mathcal{U}_\phi$. Then there exists an M in $\mathcal{L}$ with $M_- \neq \mathcal{X}$ such that $S^*\mid_{M_-^\perp}$ is nonzero. Moreover, for any $L \in \mathcal{L}$ with $L \neq (0), L_- \neq \mathcal{X}$ and $S^*\mid_{L_-^\perp}$ nonzero, the operator $S^*\mid_{L_-^\perp}$ is of rank one.*

Proof. The proof is similar to Lemma 3.3 in [6] and is omitted. ∎

Proposition 2.1. *Let $\mathcal{L}$ be a completely distributive subspace lattice, ϕ an order homomorphism from $\mathcal{L}$ into itself and S a nonzero single element of $\mathcal{U}_\phi$. If $S\mid_M$ is nonzero for some $M \in \mathcal{L}$ with $M_- \neq \mathcal{H}$, then $S(\mathcal{X}) \subseteq \phi(M)$.*

Proof. Let $L \in \mathcal{L}$ with $L \not\subseteq \phi(M)$. We shall first show that $S(\mathcal{X}) \subseteq L_-$. If $L_- = \mathcal{X}$ we have nothing to prove, so assume $L_- \neq \mathcal{X}$. The condition $L \not\subseteq \phi(M)$ implies $\phi(M) \subseteq L_-$, so if $m \in M$, it follows from $S \in \mathcal{U}_\phi$ that

$$Sm \in S(M) \subseteq \phi(M) \subseteq L_-.$$

Let now $l^* \in L_-^\perp$ be arbitrary. Choose nonzero $l \in L$ and $m \in M$ with $Sm \neq 0$ and nonzero $m^* \in M_-^\perp$. Then by Lemma 1.2 the rank ones $l^* \otimes l$ and $m^* \otimes m$ belong to Alg$\mathcal{L}$, and $S(m^* \otimes m) \neq 0$. However, $Sm \in L_-$ and $l^* \in L_-^\perp$, so

$$(l^* \otimes l)S(m^* \otimes m) = (l^* \otimes l)(m^* \otimes Sm) = l^*(Sm)m^* \otimes l = 0.$$

The assumption that S is single implies $(l^* \otimes l)S = 0$. Thus for any $x \in \mathcal{X}$, we have

$$l^*(Sx)l = (l^* \otimes l)Sx = 0$$

and so $Sx \in {}^\perp(L_-^\perp) = L_-$ and $S(\mathcal{X}) \subseteq L_-$, as required.

Since $\mathcal{L}$ is a completely distributive, it follows from Lemma 1.1 (2) that $\phi(M) = \wedge\{L_- : L \in \mathcal{L}, L \not\subseteq \phi(M)\}$. Hence $S(\mathcal{X}) \subseteq \wedge\{L_- : L \in \mathcal{L}, L \not\subseteq \phi(M)\} = \phi(M)$. ∎

Let ϕ be an order homomorphism from $\mathcal{L}$ into $\mathcal{L}$, we define

$$\overline{\mathcal{U}}_\phi = \{T \in \mathcal{B}(\mathcal{X}) : T\phi(L) \subseteq L, \forall L \in \mathcal{L}\}.$$

Theorem 2.1. *Let $\mathcal{L}$ be a completely distributive subspace lattice and ϕ an order homomorphism from $\mathcal{L}$ into $\mathcal{L}$. If S is a single element of $\mathcal{U}_\phi$ with $S\overline{\mathcal{U}}_\phi S \neq 0$, then S is of rank one.*

Proof. Let $A \in \overline{\mathcal{U}}_\phi$ be such that $SAS \neq 0$, and let l_2 be such that $SASl_2 \neq 0$. Put $l_1 = ASl_2$ and $l = Sl_1$. We will show that $S(\mathcal{X}) \subseteq \mathbf{C}l$. First we show that if $K \in \mathcal{L}$ with $K_- \neq \mathcal{X}$, then $S(K) \subseteq \mathbf{C}l$. Indeed, if $S(K) = (0)$ we have nothing to prove. If instead $S(K) \neq (0)$, Proposition 2.1 implies that $S(\mathcal{X}) \subseteq \phi(K)$, and so $Sl_2 \in \phi(K)$. But then $l_1 = ASl_2 \in A(\phi(K)) \subseteq K$, since $A \in \overline{\mathcal{U}}_\phi$. Note that $S\mid_K$ is nonzero, since $l_1 \in K$ and $Sl_1 = SASl_2 \neq 0$. Thus Lemma 2.3 implies $S\mid_K$ is of rank one. Hence $S(K) \subseteq \mathbf{C}Sl_1 = \mathbf{C}l$, as claimed. Denoting now by $\mathcal{X}_0$ the linear span of $\{K \in \mathcal{L} : K \neq \mathcal{X}\}$, the above shows that also $S(\mathcal{X}_0) \subseteq \mathbf{C}l$. But it follows from Lemma 1.1 that $\mathcal{X}_0$ is dense in $\mathcal{X}$, so

$$S(\mathcal{X}) = S(\overline{\mathcal{X}_0}) \subseteq \overline{S(\mathcal{X}_0)} \subseteq \overline{\mathbf{C}l} = \mathbf{C}l,$$

this completes the proof. ∎

3. Single Elements of Rank n and ∞

Throughout this section $\mathcal{L}$ will denote a completely distributive subspace lattice on a reflexive Banach space $\mathcal{X}$, and ϕ an order homomorphism from $\mathcal{L}$ into $\mathcal{L}$. Recall that $\mathcal{J}$ denote the set of elements $\mathcal{J} = \{L \in \mathcal{L} : L \neq (0) \text{ and } L_- \neq \mathcal{X}\}$.

Lemma 3.1. *Operators of the type $\phi(L)TL_-^\perp$ belong to $\mathcal{U}_\phi$, where $L \in \mathcal{L}$ and $T \in \mathcal{B}(\mathcal{X})$.*

Proof. For any $M \in \mathcal{L}$, we deal with two cases separately. Case 1. $L \subseteq M$. We have $\phi(L)TL_-^\perp M \subseteq \phi(L) \subseteq \phi(M)$; Case 2. $L \not\subseteq M$. From the definition of L_-, $L_- \supseteq M$. So $\phi(L)TL_-^\perp M = (0) \subseteq \phi(M)$.

It follows from Case 1 and Case 2 that for any $M \in \mathcal{L}$, we have $\phi(L)TL_-^\perp M \subseteq \phi(M)$ and so $\phi(L)TL_-^\perp \in \mathcal{U}_\phi$. ∎

Proposition 3.1. *If n is a positive integer and there exist sets $\{M_i \in \mathcal{J} : 1 \leq i \leq n\}$ and $\{K_j \in \mathcal{J} : 1 \leq j \leq n\}$ satisfying (1) $M_i \bigcap M_j = (0),$ if $\; i \neq j$; (2) $K_{i-} \bigvee K_{j-} = \mathcal{X},$ if $\; i \neq j$; (3) $\bigvee_{i=1}^{n} M_i \subseteq \bigcap_{j=1}^{n} \phi(K_j)$,*
then, for any choice of non-zero vectors $f_i \in M_i$ and $e_j^* \in K_{j-}^\perp, 1 \leq i, j \leq n$, the operator $S = \sum_{j=1}^{n} e_j^* \otimes g_j$ is a single element of $\mathcal{U}_\phi$ of rank n, where $g_1 = \sum_{k=1}^{n} f_k$ and $g_j = (\sum_{k\neq j} f_k) - f_j$ for $2 \leq j \leq n$.

Proof. Note that $g_j \in \phi(K_j)$, for every $1 \leq j \leq n$, so $e_j^* \otimes g_j = \phi(K_j)(e_j^* \otimes g_j)K_{j-}^\perp \in \mathcal{U}_\phi$ by Lemma 3.1. It follows that $S \in \mathcal{U}_\phi$. Since $M_i \bigcap (\bigvee_{j\neq i} M_j) =$

$\bigvee_{j\neq i}(M_i\bigcap M_j)=(0)$, for every $1\leq i\leq n$, the set of vectors $\{f_i:1\leq i\leq n\}$ is linearly independent. It readily follows that $\{g_j:1\leq j\leq n\}$ is linearly independent. Similarly $\{e_j^*:1\leq j\leq n\}$ is linearly independent since $K_{j-}^{\perp}\bigcap(\bigvee_{i\neq j}K_{i-}^{\perp})=\bigvee_{i\neq j}(K_{i-}^{\perp}\bigcap K_{j-}^{\perp})=(0)$, for every $1\leq j\leq n$. Thus S has rank n.

Let $A,B\in \mathrm{Alg}\mathcal{L}$ satisfy $ASB=\sum_{k=1}^{n}B^*e_k^*\otimes Ag_k=0$, and suppose that $AS=\sum_{k=1}^{n}e_k^*\otimes Ag_k\neq 0$. Then $Ag_j\neq 0$, for some $1\leq j\leq n$. In fact, $Ag_k\neq 0$, for every $1\leq k\leq n$. For, if $Ag_k=0$, then for every $1\leq i\leq n$, $Af_i\in M_i\bigcap(\bigvee_{l\neq i}M_l)=(0)$, so $Af_i=0$ and contradicts $Ag_j\neq 0$ for some $1\leq j\leq n$. Now, for every $1\leq j\leq n$, there exists $h_j^*\in\mathcal{X}^*$ such that $h_j^*(Ag_j)\neq 0$. So

$$0=(ASB)^*(h_j^*)=\sum_{k=1}^{n}h_j^*(Ag_k)B^*e_k^*$$

gives

$$B^*e_j^*=-\sum_{k\neq j}\frac{h_j^*(Ag_k)}{h_j^*(Ag_j)}B^*e_k^*.$$

Thus $B^*e_j^*\in K_{j-}^{\perp}\bigcap(\bigvee_{k\neq j}K_{k-}^{\perp})=(0)$, so $B^*e_j^*=0$, for every $1\leq j\leq n$. Hence $SB=\sum_{k=1}^{n}B^*e_k^*\otimes g_k=0$. This shows that S is a single element of $\mathcal{U}_\phi$ and the proof is completed. ∎

Theorem 3.1. *Let $\mathcal{L}$ be a completely distributive subspace lattice on a reflexive Banach space $\mathcal{X}$ and ϕ an order homomorphism from $\mathcal{L}$ into $\mathcal{L}$. If n is a positive integer, then $\mathcal{U}_\phi$ contains a single element of rank n if and only if there exist sets $\{M_i\in\mathcal{J}:1\leq i\leq n\}$ and $\{K_j\in\mathcal{J}:1\leq j\leq n\}$ satisfying* (1) $M_i\bigcap M_j=(0)$, *if* $i\neq j$; (2) $K_{i-}\bigvee K_{j-}=\mathcal{X}$, *if* $i\neq j$; (3) $\bigvee_{i=1}^{n}M_i\subseteq\bigcap_{j=1}^{n}\phi(K_j)$.

Proof. The sufficiency of the condition follows from Proposition 3.1. We only need to prove the necessity.

Suppose that $\mathcal{U}_\phi$ contains a single element S of rank n. Since $\mathcal{J}\neq\emptyset$, we may suppose that $n>1$. By complete distributivity $\mathcal{X}=\bigvee\{L:L\in\mathcal{J}\}$, thus $R(S)=S(\mathcal{X})=\bigvee\{S(L):L\in\mathcal{J}\}$. It follows from Lemma 2.3 that $S(L)$ is at most one dimensional and so there is a basis $\{g_j:1\leq j\leq n\}$ of $R(S)$ satisfying $S(K_j)=\langle g_j\rangle, 1\leq j\leq n$, for some set $\{K_j\in\mathcal{J}:1\leq j\leq n\}$. Again by complete distributivity, $(0) = \bigcap\{L_- : L \in \mathcal{J}\}$. Thus $\mathcal{X}^* = \bigvee\{L_-^\perp : L \in \mathcal{J}\}$, and $R(\mathcal{X}^*) = S^*(\mathcal{X}^*) = \bigvee\{S^*(L_-^\perp) : L \in \mathcal{J}\}$. It follows from Lemma 2.4 that $S^*(L_-^\perp)$ is at most one dimensional for every $L \in \mathcal{J}$, so there is a basis $\{e_i^* : 1 \le i \le n\}$ for $R(S^*)$ satisfying $S^*(M_{i-}^\perp) = \langle e_i^* \rangle$, for $1 \le i \le n$, for some set $\{M_i \in \mathcal{J} : 1 \le i \le n\}$.

Let $1 \le i \ne j \le n$. Since $\bigcap\{L_- : L \in \mathcal{J}\} = (0)$, to show $M_i \bigcap M_j = (0)$, it is enough to show that $M_i \bigcap M_j \subseteq L_-$ for every $L \in \mathcal{J}$. If this were not the case it would follow that for some $L \in \mathcal{J}$, $L \subseteq M_i \bigcap M_j$, so $L_- \subseteq M_{i-} \bigcap M_{j-}$. Then $L_-^\perp$ would contain both $M_{i-}^\perp$ and $M_{j-}^\perp$, and it follows from Lemma 2.4 that $S^*(M_{i-}^\perp) = S^*(M_{j-}^\perp) = S^*(L_-^\perp)$. This would contradicts the linear independence of $\{e_i^*, e_j^*\}$. Thus $M_i \bigcap M_j = (0)$. Since $\bigvee\{K : K \in \mathcal{J}\} = \mathcal{X}$, to show that $K_{i-} \bigvee K_{j-} = \mathcal{X}$, it is enough to show that $K \subseteq K_{i-} \bigvee K_{j-}$, for every $K \in \mathcal{J}$. Now K cannot contain both K_i and K_j since otherwise it follows from Lemma 2.3 that $S(K_i) = S(K_j) = S(K)$ and this would contradicts the linear independence of $\{g_i, g_j\}$. Thus $K \subseteq K_{i-} \bigvee K_{j-}$ since either $K_i \not\subseteq K$ (so $K \subseteq K_{i-}$) or $K_j \not\subseteq K$ (so $K \subseteq K_{j-}$). Thus $K_{i-} \bigvee K_{j-} = \mathcal{X}$.

Finally, we show that $\bigvee\limits_{i=1}^{n} M_i \subseteq \bigcap\limits_{j=1}^{n} \phi(K_j)$. Let $1 \le i, j \le n$ and suppose that $M_i \not\subseteq \phi(K_j)$. Choose $0 \ne f \in K_j$ such that $Sf \ne 0$ and choose $0 \ne e^* \in K_{j-}^\perp$. Further, choose $0 \ne g^* \in M_{i-}^\perp$ such that $S^*g^* \ne 0$ and choose $0 \ne h \in M_i$. Then by Lemma 1.2, $e^* \otimes f$ and $g^* \otimes h$ are rank one operators of Alg$\mathcal{L}$. Now $(g^* \otimes h)S = S^*g^* \otimes h \ne 0$ and $S(e^* \otimes f) = e^* \otimes Sf \ne 0$. However, since $\phi(K_j) \subseteq M_{i-}$ and $S \in \mathcal{U}_\phi$, $Sf \in \phi(K_j) \subseteq M_{i-}$. Hence

$$(g^* \otimes h)S(e^* \otimes f) = g^*(Sf)(e^* \otimes f) = 0,$$

and this contradicts the fact that S is a single element of $\mathcal{U}_\phi$. This completes the proof. ■

Next we present the ‘infinite’ version of Theorem 3.1. A part from modifications necessitated by the requirements of convergence, the proof is similar. For this reason we omit the proof (see Theorem 2 in [8]).

Theorem 3.2. *$\mathcal{U}_\phi$ contains a single element of infinite rank if and only if there exist sets $\{M_i \in \mathcal{J} : i \ge 1\}$ and $\{K_j \in \mathcal{J} : j \ge 1\}$ satisfying (1) $M_i \bigcap M_j = (0)$, if $\ i \ne j$; (2) $K_{i-} \bigvee K_{j-} = \mathcal{X}$, if $\ i \ne j$; (3) $\bigvee\limits_{i=1}^{+\infty} M_i \subseteq \bigcap\limits_{j=1}^{+\infty} \phi(K_j)$.*

Combining Theorem 3.1 and Theorem 3.2 leads immediately to the following corollaries.

Corollary 3.1. *If $m \in \mathrm{Z}^+$ and $\mathcal{U}_\phi$ contain a single element of rank m, then it contains a single element of rank n, for every $1 \le n < m$.*

Corollary 3.2. *The followings are equivalent.*

(1) *Every non-zero single element in* $\mathcal{U}_\phi$ *has rank one;*

(2) $\mathcal{U}_\phi$ *does not contain a single element of rank two;*

(3) *There are not* $M_1, M_2, K_1, K_2 \in \mathcal{J}$ *satisfying*

(*a*) $M_1 \bigcap M_2 = (0)$*;*

(*b*) $K_{1-} \bigvee K_{2-} = \mathcal{X}$*;*

(*c*) $M_1 \bigvee M_2 \subseteq \phi(K_1) \bigcap \phi(K_2)$.

Corollary 3.3. *If* $\mathcal{L}$ *is a nest, then every non-zero single element of any weakly closed Alg*$\mathcal{L}$*-module has rank one.*

Proof. It follows from [3] Theorem 1.5 and Corollary 3.2. ■

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