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A PROXIMAL METHOD FOR PSEUDOMONOTONE TYPE VARIATIONAL-LIKE INEQUALITIES

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Abstract. The purpose of this paper is to investigate the convergence of a proximal method for solving pseudomonotone type variational-like inequalities. The main result is given for the finite-dimensional case. However convergence can still be obtained in an infinite-dimensional Hilbert space under a strong pseudomonotone type assumption or a pseudo-Dunn type one on the operator involved.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let K be a nonempty closed convex subset of H. Let $T : K \to H$ and $\eta : K \times K \to H$ be two mappings. We consider the variational-like inequality problem which is to find $x^* \in K$ such that

$$\langle T(x^*), \eta(y, x^*) \rangle \geq 0, \ \forall y \in K. \tag{1}$$

Problem (1) was studied previously by many authors; see, e.g., [1,2 5, 22, 25, 27, 30]. A randomized version of this problem was considered by Ding [6] in 1997. Furthermore, if $\eta(x, y) = x - y, \forall x, y \in K$ and T is pseudomonotone, then problem (1) reduces to the following pseudomonotone variational inequality problem: find $x^* \in K$ such that

$$\langle T(x^*), y - x^* \rangle \geq 0, \ \forall y \in K. \tag{2}$$

This problem was studied by Yao [28, 29] and El Farouq [9, 10] for example. In [28, 29] some results on the existence of solutions were obtained. Utilizing

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a technique developed by Cohen [3], El Farouq [10] studied the convergence of the method based on the auxiliary problem principle under a pseudomonotonicity assumption on the operator T. In addition, El Farouq [9] analyzed the convergence of proximal methods for solving pseudomonotone variational inequalities (2).

The notion of pseudomonotonicity employed here was introduced by Karamardian [16]. It is closely related to that of pseudoconvexity of functions; see [20]. Other concepts of generalized monotone operators are presented in [12, 17-19, 24, 32]. Some results on the existence of solutions of pseudomonotone variational inequalities are given in [4, 13-16, 28-29].

In order to study the proposed proximal method for solving problem (1), we extend the notion of pseudomonotonicity of operators, the notion of pseudoconvexity of functions and other concepts of generalized monotone operators. At first the result on the existence of solutions for pseudomonotone type variational-like inequalities is established. Then we discuss two cases. In the finite-dimensional case the proximal method considered in this paper converges under a η-pseudomonotonicity assumption on the operator. In the infinite-dimensional case our convergence results are limited to η-pseudo-Dunn and η-strong pseudomonotonicity assumptions.

The paper is organized as follows. In Section 2 we give some new definitions and basic results for generalized η-monotonicity. In Section 3 we present the proximal method that is studied in this paper. Sections 4 and 5 are devoted to its convergence in the finite-dimensional case and the infinite-dimensional case, respectively.

2. Preliminaries

In this section we give various definitions and basic results on generalized η-monotonicity.

Definition 2.1. Let K be a nonempty subset of H and let $T : K \to H$, $\eta : K \times K \to H$ be two mappings. Then

(i) T is η -monotone on K if

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq 0,\ \forall x, y \in K;$$

(ii) T is η -strongly monotone on K if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq \alpha\|x - y\|^2,\ \forall x, y \in K;$$

(iii) T has the η-Dunn property on K if there exists a constant $A > 0$ such that

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq (1/A)\|T(x) - T(y)\|^2,\ \forall x, y \in K;$$

(iv) T is η-pseudomonotone on K if for each $x, y \in K$,

$$\langle T(x), \eta(y, x)\rangle \geq 0 \Longrightarrow \langle T(y), \eta(y, x)\rangle \geq 0;$$

(v) T is η-strongly pseudomonotone on K if there exists a constant $c > 0$ such that for each $x, y \in K$,

$$\langle T(x), \eta(y,x)\rangle \geq 0 \Longrightarrow \langle T(y), \eta(y,x)\rangle \geq c\|y-x\|^2;$$

(vi) T has the η-pseudo-Dunn property on K if there exists a constant $E > 0$ such that for each $x, y \in K$,

$$\langle T(x), \eta(y,x)\rangle \geq 0 \Longrightarrow \langle T(y), \eta(y,x)\rangle \geq (1/E)\|T(y)-T(x)\|^2;$$

(vii) T is η-quasimonotone on K if for each $x, y \in K$,

$$\langle T(x), \eta(y,x)\rangle > 0 \Longrightarrow \langle T(y), \eta(y,x)\rangle \geq 0;$$

(viii) T is η-weakly monotone on K if there exists $L > 0$ such that for each $x, y \in K$,

$$\langle T(x)-T(y), \eta(x,y)\rangle \geq -L\|x-y\|^2.$$

Remark 2.1. If $\eta(x,y) = x - y,\ \forall x, y \in K$, then Definition 2.1 reduces to Definition 2.1 in El Farouq [9] for example.

Lemma 2.1. *Let $\eta : K \times K \to H$ satisfy the condition*

$$\eta(x,y) + \eta(y,x) = 0,\ \forall x, y \in K.$$

(i) *If $T : K \to H$ is η-pseudomonotone, then for all solutions x_1^*, x_2^* of problem* (1),

$$\langle T(x_2^*), \eta(x_2^*, x_1^*)\rangle = 0. \tag{3}$$

(ii) *If T has the η-pseudo-Dunn property, then the set*

$$S = \{T(x^*) : \langle T(x^*), \eta(x, x^*)\rangle \geq 0,\ \forall x \in K\}$$

is a singleton.

(iii) *If T is η -strongly pseudomonotone and problem (1) has a solution, then it is unique.*

Proof. Let x_1^* and x_2^* in K be two solutions of problem (1). Then

$$\langle T(x_1^*), \eta(x_2^*, x_1^*)\rangle \geq 0 \text{ and } \langle T(x_2^*), \eta(x_1^*, x_2^*)\rangle \geq 0. \tag{4}$$

(i) If T is η-pseudomonotone, then (4) implies $\langle T(x_2^*), \eta(x_2^*, x_1^*)\rangle \geq 0$. Note that $\eta(x_1^*, x_2^*) = -\eta(x_2^*, x_1^*)$. Thus from (4) we get (3).

(ii) If T has the η-pseudo-Dunn property, then

$$0 = \langle T(x_2^*), \eta(x_2^*, x_1^*)\rangle \geq (1/E)\|T(x_2^*) - T(x_1^*)\|^2.$$

Therefore, $T(x_2^*) = T(x_1^*)$. Hence the set S is a singleton.

(iii) By using the same reasoning as in (ii) in case T is η-strongly pseudomonotone, we get $x_1^* = x_2^*$. ■

We illustrate below the relationships between the η-monotonicity assumption and some generalized η-monotonicity assumptions:

$$\begin{array}{ccccc}
\eta-\text{strong monotonicity} & \Rightarrow & \eta-\text{monotonicity} & \Leftarrow & \eta-\text{Dunn property} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\eta\text{–strong pseudomonotonicity} & \Rightarrow & \eta\text{–pseudomonotonicity} & \Leftarrow & \eta\text{–pseudo–Dunn property} \\
 & & \Downarrow & & \\
 & & \eta-\text{quasimonotonicity} & &
\end{array}$$

We easily obtain the following result from the definitions above.

Lemma 2.2. *If $T : K \to H$ is η-strongly pseudomonotone with constant c and Lipschitz continuous with constant L, then it has the η-pseudo-Dunn property with constant L^2/c.*

Definition 2.2 [1]. A mapping $\eta : K \times K \to H$ is called Lipschitz continuous if there is a constant $\lambda > 0$ such that $\|\eta(x, y)\| \leq \lambda\|x - y\|\ \forall x, y \in K$.

Definition 2.3 [1]. A differentiable function $h : K \to R$ on a convex subset K is called

(i) η-convex if

$$h(y) - h(x) \geq \langle h'(x), \eta(y, x)\rangle,\ \forall x, y \in K$$

where $h'(x)$ is the Frechet derivative of h at x;

(ii) η-strongly convex if there exists a constant $\mu > 0$ such that

$$h(y) - h(x) - \langle h'(x), \eta(y, x)\rangle \geq (\mu/2)\|x - y\|^2,\ \forall x, y \in K.$$

Proposition 2.1 [1]. *Let h be a differentiable η-strongly convex function on a convex subset K of H and let $\eta : K \times K \to H$ be a mapping such that $\eta(x,y) + \eta(y,x) = 0,\ \forall x, y \in K$. Then, h' is η-strongly monotone.*

Lemma 2.3 [1]. *Let $\eta(y,\cdot) : K \to H$ and h' be sequentially continuous from the weak topology to the weak topology and from the weak topology to the strong topology, respectively, where y is any fixed point in K. Then the function $g : K \to R$, defined as $g(x) = \langle h'(x), \eta(y,x)\rangle$ for each fixed $y \in K$, is weakly sequentially continuous.*

For each $D \subseteq H$, we denote by $\operatorname{co}(D)$ the convex hull of D. A point-to-set mapping $G : H \to 2^H$ is called a KKM mapping if for every finite subset $\{x_1, x_2, \cdots, x_k\}$ of H, $\quad \operatorname{co}(\{x_1, x_2, \cdots, x_k\}) \subseteq \cup_{i=1}^k G(x_i)$.

Lemma 2.4 [11]. *Let K be an arbitrary nonempty subset in a Hausdorff topological vector space E and let $G : K \to 2^E$ be a KKM mapping. If $G(x)$ is closed for all $x \in K$ and is compact for at least one $x \in K$, then $\cap_{x\in K} G(x) \neq \emptyset$.*

3. A Proximal Method

In this section, we present a proximal method [7, 8, 21, 23, 26, 31]. The proximal method was initially introduced by Martinet [21] as a regularization method in the context of convex optimization in Hilbert spaces. In the early seventies it has been extended widely to the general framework of maximal monotone inclusions by Rockafellar. Recently, it has also been extended to develop the proximal method for solving problem (2); see, e.g., [9].

We consider an auxiliary function $h : K \to R$ which is chosen differentiable and η-strongly convex, and a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of positive numbers. For some $x \in H$, we introduce the problem

$$(5) \qquad \langle h'(\widetilde{y}(x)) + \varepsilon_n T(\widetilde{y}(x)) - h'(x), \eta(y, \widetilde{y}(x))\rangle \geq 0,\ \forall y \in K.$$

If $\widetilde{y}(x)$ exists and is equal to x, then it is a solution of the original problem (1).

Algorithm 3.1. Proximal Algorithm.

(i) Start from some initial point x_0 in K.

(ii) At stage n, knowing $x_n \in K$, compute x_{n+1} as a solution of the variational-like inequality

$$(6) \qquad \langle \varepsilon_n T(x_{n+1}) + h'(x_{n+1}) - h'(x_n), \eta(y, x_{n+1})\rangle \geq 0\ \forall y \in K$$

where h' is the Frechet derivative of a function $h : K \to R$ at x.

Lemma 3.1. *Let $x \in H$. Suppose that*

(*i*) $T : K \to H$ *is η-weakly monotone with constant L;*

(*ii*) $h : K \to R$ *is η-strongly convex with constant μ;*

(*iii*) $\eta(x, y) + \eta(y, x) = 0,\ \forall x, y \in K$;

(*iv*) $\varepsilon_n < \mu/L$.

Then the operator $F_n(y) = \varepsilon_n T(y) + h'(y) - h'(x_n)$ is η-strongly monotone on K with constant $\mu - \varepsilon_n L$.

Proof. Since h is η-strongly convex with constant μ, we obtain

$$h(y_2) - h(y_1) - \langle h'(y_1), \eta(y_2, y_1)\rangle \geq \frac{\mu}{2}\|y_1 - y_2\|^2,$$

and hence

$$\langle h'(y_1), \eta(y_1, y_2)\rangle \geq \frac{\mu}{2}\|y_1 - y_2\|^2 + h(y_1) - h(y_2).$$

Similarly, we obtain

$$-\langle h'(y_2), \eta(y_1, y_2)\rangle \geq \frac{\mu}{2}\|y_2 - y_1\|^2 + h(y_2) - h(y_1).$$

Thus, we derive

$$\langle h'(y_1) - h'(y_2), \eta(y_1, y_2)\rangle \geq \mu\|y_1 - y_2\|^2;$$

that is, h' is η-strongly monotone with constant μ. Since T is η-weakly monotone with constant L, we have for each $y_1, y_2 \in K$,

$$\begin{aligned}
&\langle F_n(y_1) - F_n(y_2), \eta(y_1, y_2)\rangle \\
= \ & \varepsilon_n\langle T(y_1) - T(y_2), \eta(y_1, y_2)\rangle + \langle h'(y_1) - h'(y_2), \eta(y_1, y_2)\rangle \\
\geq \ & -\varepsilon_n L\|y_1 - y_2\|^2 + \mu\|y_1 - y_2\|^2 \\
= \ & (\mu - \varepsilon_n L)\|y_1 - y_2\|^2.
\end{aligned}$$

■

Lemma 3.2. *Suppose that problem (1) has a solution. Let $T : K \to H$ be η-weakly monotone with constant L. Assume that*

(i) $\eta : K \times K \to H$ *satisfies the following conditions:*

  (a) $\eta(x, y) + \eta(y, x) = 0,\ \forall x, y \in K$,

  (b) $\eta(x, y) = \eta(x, z) + \eta(z, y),\ \forall x, y, z \in K$,

  (c) *for each fixed $y \in K, \eta(\cdot, y) : K \to H$ is affine,*

  (d) *for each fixed $x \in K, \eta(x, \cdot) : K \to H$ is sequentially continuous from the weak topology to the weak topology;*

(ii) $h : K \to R$ *is* η*-strongly convex with constant* μ*, and its derivative* h' *is sequentially continuous from the weak topology to the strong topology;*

(iii) *for each fixed* $n \geq 0$ *and* $z \in K, \{x \in K : \langle \varepsilon_n T(x) + h'(x) - h'(z), \eta(y, x)\rangle \geq 0\}$ *is bounded for at least one* $y \in K$*;*

(iv) *for each fixed* $y \in K, \langle T(\cdot), \eta(y, \cdot)\rangle : K \to R$ *is weakly upper semicontinuous.*

If $\{\varepsilon_n\}$ *satisfies* $\varepsilon_n < \mu/L$*, for each iterate* x_n *there exists a unique solution* $x = x_{n+1} \in K$ *of*

$$\langle \epsilon_n T(x) + h'(x) - h'(x_n), \eta(y, x)\rangle \geq 0, \ \forall y \in K. \tag{7}$$

Proof. Existence of a solution of problem (7).

We write (7) as follows: find $\overline{x} \in K$ such that

$$\langle \varepsilon_n T(\overline{x}) + h'(\overline{x}) - h'(x_n), \eta(y, \overline{x})\rangle \geq 0, \ \forall y \in K.$$

For each fixed $n \geq 0$ and each $y \in K$, we define

$$G(y) = \{x \in K : \langle \varepsilon_n T(x) + h'(x) - h'(x_n), \eta(y, x)\rangle \geq 0\}.$$

Since $y \in G(y), G(y)$ is nonempty for each $y \in K$. Now, we claim that G is a KKM mapping. Suppose to the contrary that there exists a finite subset $\{y_1, y_2, \cdots, y_k\}$ of K and $\alpha_i \geq 0, \ \forall i = 1, 2, \cdots, k$ with $\sum_{i=1}^k \alpha_i = 1$ such that $\hat{x} = \sum_{i=1}^k \alpha_i y_i \notin G(y_i), \ \forall i = 1, 2, \cdots, k$. Then by virtue of assumptions (a), (c) in (i), we have

$$\begin{aligned} 0 &= \langle \varepsilon_n T(\hat{x}) + h'(\hat{x}) - h'(x_n), \eta(\hat{x}, \hat{x})\rangle \\ &\leq \sum_{i=1}^k \alpha_i \langle \varepsilon_n T(\hat{x}) + h'(\hat{x}) - h'(x_n), \eta(y_i, \hat{x})\rangle < 0, \end{aligned}$$

a contradiction. Hence, G is a KKM mapping.

In view of conditions (i) (d), (ii), (iv) and Lemma 2.3, we can readily see that $G(y)$ is a weakly closed subset of K for each $y \in K$. Moreover, from condition (iii) we know that $G(y)$is weakly compact for at least one point $y \in K$. Hence, by Lemma 2.4, we have $\cap_{y\in K} G(y) \neq \emptyset$ which implies that there exists at least one solution of (7).

Uniqueness of solutions of problem (7).

Let x_1, x_2 be two solutions of (7). Then

$$\langle \varepsilon_n T(x_1) + h'(x_1) - h'(x_n), \eta(y, x_1)\rangle \geq 0, \tag{8}$$

$$(9) \qquad \langle \varepsilon_n T(x_2) + h'(x_2) - h'(x_n), \eta(y, x_2)\rangle \geq 0$$

for all $y \in K$. Taking $y = x_2$ in (8) and $y = x_1$ in (9) and adding these inequalities, we obtain

$$(10) \qquad \langle F_n(x_1) - F_n(x_2), \eta(x_1, x_2)\rangle \leq 0,$$

where $F_n(y) = \varepsilon_n T(y) + h'(y) - h'(x_n)$. According to Lemma 3.1, the operator F_n is η-strongly monotone on K with constant $\mu - \varepsilon_n L$. Thus from (10) we get

$$(\mu - \varepsilon_n L)\|x_1 - x_2\|^2 \leq \langle F_n(x_1) - F_n(x_2), \eta(x_1, x_2)\rangle \leq 0.$$

and therefore $x_1 = x_2$ since $\mu - \varepsilon_n L > 0$. Hence the solution of (7) is unique. ■

4. Convergence Results in the Finite-dimensional Case

In this section we assume that H is a finite-dimensional space. We present at first a general convergence result based on η-pseudomonotonicity. Then we give stronger results under a η-pseudo-Dunn assumption or a η-strong pseudomonotonicity assumption. We finally study the special case of the proximal algorithm. We will make the following assumptions throughout the remainder of this section.

Assumption 4.1.

(i) K is bounded, closed and convex;

(ii) T is continuous and η-weakly monotone with constant L on K;

(iii) h is η-strongly convex with constant μ on K and its derivative h' is continuous;

(iv) ε_n satisfies $\alpha < \varepsilon_n < \mu/L$ for some $\alpha > 0$.

The following theorem is based on η-pseudomonotonicity.

Theorem 4.1. *Suppose that problem (1) has a solution x^*. Assume that $\eta : K \times K \to H$ is Lipschitz continuous with constant λ such that*

(a) $\eta(x, y) + \eta(y, x) = 0, \ \forall x, y \in K,$

(b) $\eta(x, y) = \eta(x, z) + \eta(z, y), \ \forall x, y, z \in K,$

(c) *for each fixed $y \in K, \eta(\cdot, y) : K \to H$ is affine.*

Under Assumption 4.1, starting from any $x_0 \in K$, *Algorithm 3.1 generates a well-defined sequence* $\{x_n\}$. *Moreover, if* T *is* η*-pseudomonotone on* K, *then* $\{\|x_{n+1} - x_n\|\}$ *converges to zero. In addition, if* h' *is Lipschitz continuous, then the sequence* $\{x_n\}$ *converges to a solution of problem* (1).

Proof. We consider the function Φ which is the Bregman distance between x and x^* induced by the η-strongly convex function $h(\cdot)$ and defined by

$$\Phi(x) = h(x^*) - h(x) - \langle h'(x), \eta(x^*, x)\rangle. \tag{11}$$

From the η-strong convexity of h on K, we get

$$\Phi(x_n) \geq (\mu/2)\|x_n - x^*\|^2 \geq 0. \tag{12}$$

We now study the variation of Φ for one step of Algorithm 3.1,

$$\triangle_n^{n+1} = \Phi(x_{n+1}) - \Phi(x_n).$$

We see that

$$\triangle_n^{n+1} = s_1 + s_2$$

with

$$\begin{aligned} s_1 &= h(x_n) - h(x_{n+1}) - \langle h'(x_n), \eta(x_n, x_{n+1})\rangle, \\ s_2 &= \langle h'(x_n) - h'(x_{n+1}), \eta(x^*, x_{n+1})\rangle. \end{aligned}$$

The η-strong convexity of h on K yields

$$\begin{aligned} s_1 &= -[h(x_{n+1}) - h(x_n) - \langle h'(x_n), \eta(x_{n+1}, x_n)\rangle] \\ &\leq -(\mu/2)\|x_n - x_{n+1}\|^2. \end{aligned}$$

By using (7) with $x = x_{n+1}$ and $y = x^\star$, we obtain

$$s_2 \leq \varepsilon_n \langle T(x_{n+1}), \eta(x^*, x_{n+1})\rangle.$$

By using (1) with $y = x_{n+1}$ and the η-pseudomonotonicity of T, we get

$$s_2 \leq 0.$$

Thus

$$\triangle_n^{n+1} \leq -(\mu/2)\|x_{n+1} - x_n\|^2 \tag{13}$$

and hence $\triangle_n^{n+1}$ is negative unless $x_{n+1} = x_n$ in which case (7) shows that x_n is a solution of (1). It follows that the sequence $\{\Phi(x_n)\}$ is strictly decreasing.

Since it is positive, it must converge, and the difference between two consecutive terms tends to zero. Therefore $\|x_{n+1} - x_n\|$ converges to zero. Moreover since the sequence $\{\Phi(x_n)\}$ is convergent, it is bounded. Thus from (13) we deduce

$$\|x_n - x^*\|^2 \leq (2/\mu) \cdot \sup_{n \geq 0} \Phi(x_n).$$

Let $\bar{x}$ be a cluster point of the sequence $\{x_n\}$ and let $\{x_{n_i}\}$ be a subsequence converging to $\bar{x}$. Since η is Lipschitz continuous with constant λ, h' is Lipschitz continuous with constant B and $\varepsilon_n > \alpha$, we have in view of (7) for any $y \in K$

$$\begin{aligned}\langle T(x_{n+1}), \eta(y, x_{n+1})\rangle &\geq -(1/\varepsilon_n)\langle h'(x_{n+1}) - h'(x_n), \eta(y, x_{n+1})\rangle \\ &\geq -(1/\alpha)B\|x_{n+1} - x_n\| \cdot \lambda\|y - x_{n+1}\| \\ &= (-\lambda B/\alpha)\|x_{n+1} - x_n\|\|y - x_{n+1}\|.\end{aligned}$$

Since T is continuous and $\|x_{n_i+1} - x_{n_i}\|$ converges to zero, taking the limit for the subsequence $\{n_i\}$ in the last inequality yields

$$\langle T(\bar{x}), \eta(y, \bar{x})\rangle \geq 0, \ \forall y \in K.$$

Hence $\bar{x}$ is a solution of problem (1).

Now we prove that the sequence $\{x_n\}$ has a unique cluster point. Assume that it has two cluster points $\bar{x}$ and $\hat{x}$. Then both cluster points can be used as x^* in (11) to define the Lyapunov function Φ. This yields two possible Lyapunov functions denoted by $\bar{\Phi}$ and $\hat{\Phi}$, respectively. It was proved that $\Phi(x_n)$ has a limit. It may depend on the solution x^* used to define Φ. The corresponding limits will be denoted by $\bar{l}$ and $\hat{l}$, respectively. Consider subsequences $\{n_i\}$ and $\{m_j\}$ such that x_{n_i} and x_{m_j} converge to $\bar{x}$ and $\hat{x}$, respectively. One has

$$\hat{\Phi}(x_{n_i}) = \bar{\Phi}(x_{n_i}) + R(x_{n_i})$$

where

$$R(x_{n_i}) = h(\hat{x}) - h(\bar{x}) - \langle h'(x_{n_i}), \eta(\hat{x}, \bar{x})\rangle$$

hereby using assumption (b). In the limit, $\bar{\Phi}(x_{n_i})$ and $\hat{\Phi}(x_{n_i})$ tend to $\bar{l}$ and $\hat{l}$, respectively. Since h' is continuous and the subsequence $\{x_{n_i}\}$ converges to $\bar{x}, R(x_{n_i})$ converges to a limit l such that

$$l \geq (\mu/2)\|\hat{x} - \bar{x}\|^2.$$

The latter inequality stems from the η-strong convexity of h on K. Therefore

$$\hat{l} \geq \bar{l} + (\mu/2)\|\hat{x} - \bar{x}\|^2.$$

By interchanging the roles of $\bar{x}$ and $\hat{x}$ and of the subsequences $\{n_j\}$ and $\{m_j\}$, the same calculations yield

$$\bar{l} \geq \hat{l} + (\mu/2)\|\bar{x} - \hat{x}\|^2.$$

Then, the following inequalities hold simultaneously:

$$0 \leq (\mu/2)\|\hat{x} - \bar{x}\|^2 \leq \hat{l} - \bar{l},$$

$$0 \leq (\mu/2)\|\hat{x} - \bar{x}\|^2 \leq \bar{l} - \hat{l}.$$

This implies that $\bar{x} = \hat{x}$ which completes the proof. ■

Corollary 4.1. *With the assumptions of Theorem 4.1, if T has the η-pseudo-Dunn property with constant E on K, then the following statements hold:*

(i) $\|x_{n+1} - x_n\|$ *converges to zero;*

(ii) $\{T(x_n)\}$ *converges to* $T(x^*)$.

Proof. We proceed as in the proof of Theorem 4.1. By using the fact that T has the η-pseudo-Dunn property with constant E and since $\varepsilon_n > \alpha$, with calculations analogous to those leading to (13), we get

$$\begin{aligned}\triangle_n^{n+1} &= s_1 + s_2 \\ &\leq -(\mu/2)\|x_{n+1} - x_n\|^2 + \varepsilon_n\langle T(x_{n+1}, \eta(x^*, x_{n+1})\rangle \\ &\leq -(\mu/2)\|x_{n+1} - x_n\|^2 - (\alpha/E)\|T(x_{n+1}) - T(x^*)\|^2.\end{aligned}$$

Hence $\triangle_n^{n+1}$ is negative unless $T(x_{n+1}) = T(x^*)$ and $x_{n+1} = x_n$ in which case (7) shows that x_n is a solution of (1). The sequence $\{\Phi(x_n)\}$ is strictly decreasing. Since it is positive, it must converge. The difference between two consecutive terms tends to zero. Therefore $\|T(x_{n+1}) - T(x^*)\|$ converges to zero. ■

Corollary 4.2. *Suppose that, in addition to the assumptions of Theorem 4.1, except for the Lipschitz continuity assumption on h', T is η-strongly pseudomonotone with constant c on K. Then the sequence $\{x_n\}$ converges to the unique solution x^* of problem* (1).

Proof. The technique of the proof remains the same as the one developed for Theorem 4.1. By using similar calculations and the fact that T is η-strongly pseudomonotone, we get

$$\begin{aligned}\triangle_n^{n+1} &= s_1 + s_2 \\ &\leq -(\mu/2)\|x_{n+1} - x_n\|^2 + \varepsilon_n\langle T(x_{n+1}), \eta(x^*, x_{n+1})\rangle \\ &\leq -(\mu/2)\|x_{n+1} - x_n\|^2 - c\alpha\|x_{n+1} - x^*\|^2.\end{aligned}$$

$\triangle_n^{n+1}$ is negative unless $x_{n+1} = x_n = x^*$. The sequence $\{T(x_n)\}$ is strictly decreasing. Since it is positive, it must converge, and the difference between two consecutive terms tends to zero. Therefore x_n converges to x^*. ■

5. Convergence Results in the Infinite-dimensional Case

In this section we assume that H is an infinite-dimensional space.

Definition 5.1. T is called Hölder continuous on K if there exist constants $\beta > 0$ and $D \geq 0$ such that for all $x_1, x_2 \in K$,

$$\|T(x_2) - T(x_1)\| \leq D\|x_2 - x_1\|^{\beta}.$$

Assumption 5.1.

(i) K is a bounded, closed and convex subset in H;

(ii) T is Hölder continuous and η-weakly monotone with constant L on K;

(iii) h is η-strongly convex with constant μ on K and its derivative h' is Lipschitz continuous with constant B;

(iv) ε_n satisfies $\alpha < \varepsilon_n < \mu/L$ for some $\alpha > 0$.

Theorem 5.1. *Suppose that problem (1) has a solution x^*. Assume that*

(i) $\eta : K \times K \to H$ *is Lipschitz continuous with constant* λ *such that*

  (a) $\eta(x,y) + \eta(y,x) = 0,\ \forall x, y \in K,$

  (b) $\eta(x,y) = \eta(x,z) + \eta(z,y),\ \forall x, y, z \in K,$

  (c) *for each fixed* $y \in K, \eta(\cdot, y) : K \to H$ *is affine,*

  (d) *for each fixed* $x \in K, \eta(x, \cdot) : K \to H$ *is sequentially continuous from the weak topology to the weak topology;*

(ii) h' *is sequentially continuous from the weak topology to the strong topology;*

(iii) *for each fixed* $y \in K, \langle T(\cdot), \eta(y, \cdot)\rangle : K \to R$ *is weakly upper semicontinuous.*

Under Assumption 5.1, starting from any $x_0 \in K$, Algorithm 3.1 generates a well-defined sequence$\{x_n\}$. Moreover if T has the η-pseudo-Dunn property with constant E on K, then the following statements hold:

(i) $\|x_{n+1} - x_n\|$ converges to zero;

(ii) $T(x_n)$ strongly converges to $T(x^*)$;

(iii) $\{x_n\}$ weakly converges to a solution of problem (1).

Proof. According to Lemma 3.2, starting from any $x_0 \in K$, Algorithm 3.1 generates a well-defined sequence $\{x_n\}$. We proceed as in the proof of Theorem 4.1. By using the fact that T has the η-pseudo-Dunn property with constant E and since $\varepsilon_n > \alpha$, with calculations analogous to those leading to (13), we obtain

$$\begin{aligned}\triangle_n^{n+1} &= s_1 + s_2\\ &\leq -(\mu/2)\|x_{n+1} - x_n\|^2 + \varepsilon_n\langle T(x_{n+1}), \eta(x^*, x_{n+1})\rangle\\ &\leq -(\mu/2)\|x_{n+1} - x_n\|^2 - (\alpha/E)\|T(x_{n+1}) - T(x^*)\|^2.\end{aligned}$$

Hence $\triangle_n^{n+1}$ is negative unless $T(x_{n+1}) = T(x^*)$ and $x_{n+1} = x_n$ in which case (7) shows that x_n is a solution of (1). The sequence $\{\Phi(x_n)\}$ is strictly decreasing. Since it is positive, it must converge, and the difference between two consecutive terms tends to zero. Therefore $\|x_{n+1} - x_n\|$ converges to 0 and $\|T(x_{n+1}) - T(x^*)\|$ converges to zero by the Hölder continuity of T.

Let $\bar{x}$ be a weak cluster point of the sequence $\{x_n\}$, and let $\{x_{n_i}\}$ be a subsequence weakly converging to $\bar{x}$. Since η is Lipschitz continuous with constant λ, h' is Lipschitz continuous with constant B and $\varepsilon_n > \alpha$, we have in view of (7) that for each $y \in K$,

$$\begin{aligned}\langle T(x_{n+1}), \eta(y, x_{n+1})\rangle &\geq (-1/\varepsilon_n)\langle h'(x_{n+1}) - h'(x_n), \eta(y, x_{n+1})\rangle\\ &\geq (-\lambda B/\alpha)\|x_{n+1} - x_n\|\|y - x_{n+1}\|.\end{aligned}$$

Since $\|x_{n_i+1} - x_{n_i}\|$converges to 0, using assumption (iii) and taking the superior limit for the subsequence $\{n_i\}$ in the last inequality, we conclude that for each $y \in K$

$$\begin{aligned}\langle T(\bar{x}), \eta(y, \bar{x})\rangle &\geq \text{limsup}_{n_i\to\infty}\langle T(x_{n_i+1}), \eta(y, x_{n_i+1})\rangle\\ &\geq \text{limsup}_{n_i\to\infty}(-\lambda B/\alpha)\|x_{n_i+1} - x_{n_i}\|\|y - x_{n_i+1}\| = 0.\end{aligned} \tag{14}$$

This implies that $\bar{x}$ is a solution of problem (1).

Now we prove that the sequence $\{x_n\}$ has a unique weak cluster point. Assume that it has two weak cluster points $\bar{x}$ and $\hat{x}$. Then both weak cluster points can be used as $x^*$to define the Lyapunov function Φ. This yields two possible Lyapunov functions, denoted by $\bar{\Phi}$ and $\hat{\Phi}$, respectively. It was proved that $\Phi(x_n)$ has a limit. It may depend on the solution x^* used to define Φ. The corresponding limits will be denoted by $\bar{l}$ and $\hat{l}$, respectively. Consider subsequences $\{n_i\}$ and $\{m_j\}$ such that x_{n_i} and x_{m_j} weakly converge to $\bar{x}$ and $\hat{x}$, respectively. One has that

$$\hat{\Phi}(x_{n_i}) = \bar{\Phi}(x_{n_i}) + R(x_{n_i})$$

where

$$R(x_{n_i}) = h(\hat{x}) - h(\bar{x}) - \langle h'(x_{n_i}), \eta(\hat{x}, \bar{x})\rangle.$$

Note that $\bar{\Phi}(x_{n_i})$ and $\hat{\Phi}(x_{n_i})$ tend to $\bar{l}$ and $\hat{l}$, respectively. Since the subsequence $\{x_{n_i}\}$ weakly converges to $\bar{x}$, by using assumption (ii) we infer that $R(x_{n_i})$ converges to a limit l such that

$$l \geq (\mu/2)\|\hat{x} - \bar{x}\|^2.$$

The latter inequality stems from the η-strong convexity of h on K. Therefore,

$$\hat{l} \geq \bar{l} + (\mu/2)\|\hat{x} - \bar{x}\|^2.$$

By interchanging the roles of $\bar{x}$ and $\hat{x}$ and of the subsequences $\{n_i\}$ and $\{m_j\}$, the same calculations yield

$$\bar{l} \geq \hat{l} + (\mu/2)\|\hat{x} - \bar{x}\|.$$

Then the following inequalities hold simultaneously:

$$0 \leq (\mu/2)\|\hat{x} - \bar{x}\|^2 \leq \hat{l} - \bar{l},$$

$$0 \leq (\mu/2)\|\hat{x} - \bar{x}\|^2 \leq \bar{l} - \hat{l}.$$

This shows that $\bar{x} = \hat{x}$. The proof is now complete. ∎

The proof of the following result is the same as the one of Corollary 4.2 and hence will be omitted.

Corollary 5.1. *Suppose that problem* (1) *has a solution* x^*. *Under the previous assumptions, starting from any* $x_0 \in K$, *Algorithm 3.1 generates a well-defined sequence* $\{x_n\}$. *Moreover, if* T *is* η*-strongly pseudomonotone with constant* c *on* K, *then the sequence* $\{x_n\}$ *strongly converges to the unique solution* x^* *of problem* (1).

6. Conclusion

In this paper, we studied the convergence of the proximal method for solving certain non-monotone variational-like inequalities. We showed that the sequence generated by the proximal algorithm is still well-defined under a η-weak monotonicity assumption on the operator involved in the variational-like inequality problem and proved the convergence in the finite-dimensional case under a η-pseudomonotonicity assumption. In the infinite-dimensional case, the convergence can still be established under a η-pseudo-Dunn assumption or a η-strong pseudomonotonicity assumption on the operator.

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