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SOME ASYMPTOTES RELATED TO k-th-POWER FREE NUMBERS

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Abstract. In this note we find some relations between $k-$free numbers. We obtain some asymptotes of the error term in the summation of $k-$free integers $R_k(x)$. Further, we determine some constants which happen in the summation of $k-$free integers.

1. Introduction and Results

Let $g_k(n)$ be the index function of k-free number, i.e.

$$g_k(n) = \begin{cases} 1 & \text{if } n \text{ is k-free ,} \\ 0 & \text{otherwise.} \end{cases}$$

By the property of Möbius function, it is not difficult to show that the number of k-free natural numbers not exceeding x is

$$Q_k(x) = \sum_{n\le x} g_k(n) = \sum_{n\le x}\sum_{d^k|n} \mu(d) = \frac{x}{\zeta(k)} + R_k(x),$$

where $R_k(x)$ is the error term and

$$R_k(x) = O(x^{\frac{1}{k}}).$$

Since the generating function $\frac{\zeta(s)}{\zeta(ks)}$ has poles on the line $\Re(s) = \frac{1}{2k}$, it follows that

$$R_k(x) = \Omega(x^{\frac{1}{2k}}).$$

(See [2, 3]). And it is thus conjectured

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$$R_k(x) = O(x^{\frac{1}{2k}+\epsilon}),$$

where ϵ denote any small positive real number. The modern estimate belongs to Walfisz [4]. He showed that

$$R_k(x) = O(x^{\frac{1}{k}} \exp(-ck^{\frac{-8}{5}} (\log x)^{\frac{3}{5}} (\log\log x)^{\frac{-1}{5}})).$$

Clearly, any substantial sharpening of $R_k(x)$ results in a wider zero-free region for the Riemann Zeta function $\zeta(s)$.

In this note we study the average behaviour of the error terms $R_k(x)$ and provide a fundamental viewpoint to them . Some relations between k-free numbers are established (lemma 2.1, corollary 2.2). Applying these relations, we acquire some asymptotes of $R_k(x)$. The obtained results are stated as follows.

Theorem 1.1. *Let k,$h > 1$ be integers, x be a large number. The following asymptotes hold.*

(i)

$$\sum_{n \le x^{\frac{1}{k}}} g_h(n) R_k(\frac{x}{n^k}) = \frac{x^{\frac{1}{k}}}{(k-1)\zeta(k)\zeta(h)} + O(x^{\frac{1}{kh}}) \tag{1.1}$$

(ii)

$$\sum_{n \le x^{\frac{1}{k}}} R_k(\frac{x}{n^k}) = \frac{x^{\frac{1}{k}}}{(k-1)\zeta(k)} + O(1) \tag{1.2}$$

(iii)

$$\sum_{n \le x} \frac{g_k(n)}{n} = \frac{\log x}{\zeta(k)} + c_k + O(x^{\frac{1}{k}-1} \log x), \tag{1.3}$$

where $c_k = -\frac{k\zeta'(k)}{\zeta^2(k)} + \frac{\gamma}{\zeta(k)}$ are constants which depend on k. Here γ denotes the Euler's constant.

(iv)

$$\sum_{n \le x} g_k(n) R_k(\frac{x}{n}) = (\frac{\gamma - 1}{\zeta^2(k)} - \frac{k\zeta'(k)}{\zeta^3(k)})x + O(x^{\frac{1}{2}+\frac{1}{2k}} \log x). \tag{1.4}$$

The equations (1.1),(1.2) and (1.4) give us patterns of elimination of the summands $R_k(\frac{x}{n^k})$ and $R_k(\frac{x}{n})$. Equation (1.3) is necessary to deduce (1.4). It is obtained by elementary method. It is also a better result than what the partial summation method can give.

2. LEMMAS

The following lemmas will be applied in the proof of the theorem.

Lemma 2.1. *For any complex number s, we have*

$$\sum_{n\le x^{\frac{1}{k}}}\frac{g_h(n)}{n^{ks}}\sum_{m\le\frac{x}{n^k}}\frac{g_k(m)}{m^s}=\sum_{N\le x}\frac{g_{kh}(N)}{N^s}.$$

Proof. Clearly, if p is a prime and $a < kh$ is a natural number, then the $kh-$free number $N = p^a$ can be expressed uniquely as $N = n^k m g_h(n) g_k(m)$, where $n = p^b$ and $m = p^r$ are h- free and k- free integers respectively by the fact that $a = kQ + R$ has unique integer solution $(Q, R) = (b, r)$ under the restriction $0 \le r < k$.

Likewise, if $p_1, p_2, \cdots p_t$ are prime numbers and $\alpha_1, \alpha_2, \cdots \alpha_t$ are natural numbers less than kh, then by the fundamental theorem of arithmetic, the kh- free number $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ can be expressed uniquely as $N = n^k m g_h(n) g_k(m)$, where $n = p_1^{\beta_1} p_2^{\beta_2} \dots p_t^{\beta_t}$ and $m = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_t^{\gamma_t}$ are h- free and k- free integers respectively by the fact that the equation system

$$\begin{cases}\alpha_1 &= Q_1k+R_1\\ \alpha_2 &= Q_2k+R_2\\ \quad . & . \quad .\\ \alpha_t &= Q_tk+R_t\end{cases}$$

has unique integer solution $(Q_1, Q_2, \cdots, R_1, R_2, \cdots) = (\beta_1, \beta_2, \dots, \gamma_1, \gamma_2, \dots)$ under the restriction $0 \le \gamma_i < k$ for $1 \le i \le t$.

Therefore the summation items of the left-hand side and the right-hand side are exactly the same and the identity is thus valid.

Choosing h large enough such that $2^h > x^{\frac{1}{k}}$, we obtain:

Corollary 2.2. *Under the same hypothesis of Lemma 2.1, the following equality holds.*

$$\sum_{n\le x^{\frac{1}{k}}}\frac{1}{n^{ks}}\sum_{m\le\frac{x}{n^k}}\frac{g_k(m)}{m^s}=\sum_{n\le x}\frac{1}{n^s}$$

Lemma 2.3. *For any complex number $s \ne 1$ such that the real part $\Re(s) = \sigma > \frac{1}{k}$, we have*

$$\sum_{n\le x}\frac{g_k(n)}{n^s}=\frac{x^{1-s}}{(1-s)\zeta(k)}+\frac{\zeta(s)}{\zeta(ks)}+O(x^{\frac{1}{k}-\sigma}).$$

Proof. Let $x < M$. Denote $Q_k(x, M) = \sum_{x<n\leq M} g_k(n)$ and $S_{x,M} = \sum_{x<n\leq M} \frac{g_k(n)}{n^s}$. By the partial summation method, we have

$$\begin{aligned} S_{x,M} &= \frac{Q_k(x,M)}{M^s} - \int_x^M Q_k(x,t)dt^{-s} \\ &= \frac{M - x + O(M^{\frac{1}{k}})}{\zeta(k)M^s} + \int_x^M \frac{t - x + O(t^{\frac{1}{k}})}{\zeta(k)} st^{-s-1}dt \\ &= \frac{M^{1-s} - x^{1-s}}{\zeta(k)(1-s)} + O(M^{\frac{1}{k}-\sigma}) + O(x^{\frac{1}{k}-\sigma}). \end{aligned}$$

Note that $\sum_{n<\infty} \frac{g_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)}$ for $\sigma > 1$. Then as $\sigma > 1$ and M approaches infinity, we have

$$\sum_{n\leq x} \frac{g_k(n)}{n^s} = \lim_{M\to\infty} \sum_{n\leq M} \frac{g_k(n)}{n^s} - S_{x,M} = \frac{\zeta(s)}{\zeta(ks)} + \frac{x^{1-s}}{(1-s)\zeta(k)} + O(x^{\frac{1}{k}-\sigma}).$$

Thus we obtain this lemma for $\sigma > 1$.

Denote $f_x(s) = \sum_{n\leq x} \frac{g_k(n)}{n^s} - \frac{x^{1-s}}{(1-s)\zeta(k)}$. Clearly, $f_x(s)$ converges to $\frac{\zeta(s)}{\zeta(ks)}$ for $\sigma > 1$. Besides, we have

$$|f_M(s) - f_x(s)| = |S_{x,M} - \frac{M^{1-s} - x^{1-s}}{(1-s)\zeta(k)}| = O(M^{\frac{1}{k}-\sigma}) + O(x^{\frac{1}{k}-\sigma}).$$

By Cauchy condition for uniform convergence, $f_x(s)$ converges for $\sigma > \frac{1}{k}$. Furthermore, by the principle of analytic continuation of functions on the complex plane, $f_x(s)$ converges to the same function $\frac{\zeta(s)}{\zeta(ks)}$ not only for $\sigma > 1$ but also for $\sigma > \frac{1}{k}$.

Now this lemma can be obtained for $\sigma > \frac{1}{k}$. We have

$$\begin{aligned} \sum_{n\leq x} \frac{g_k(n)}{n^s} &= \lim_{M\to\infty} f_M(s) + \frac{M^{1-s}}{(1-s)\zeta(k)} - S_{x,M} \\ &= \frac{\zeta(s)}{\zeta(ks)} + \frac{x^{1-s}}{(1-s)\zeta(k)} + O(x^{\frac{1}{k}-\sigma}). \end{aligned}$$

Lemma 2.4. *If γ is Euler's constant, then the following statement holds.*

$$\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x}).$$

Proof. See [1] theorem 3.2.

3. Proof of Theorem

Taking $s = 0$ for lemma 2.1 and expanding it by lemma 2.3, we have

$$\begin{aligned}\sum_{n\le x^{\frac{1}{k}}} g_h(n) \sum_{m\le \frac{x}{n^k}} g_k(m) &= \sum_{n\le x^{\frac{1}{k}}} g_h(n)(\frac{x}{\zeta(k)n^k} + R_k(\frac{x}{n^k}))\\ &= \frac{x}{\zeta(k)}(\frac{(x^{\frac{1}{k}})^{1-k}}{\zeta(h)(1-k)} + \frac{\zeta(k)}{\zeta(kh)}\\ &\quad +O((x^{\frac{1}{k}})^{\frac{1}{h}-k})) + \sum_{n\le x^{\frac{1}{k}}} g_h(n) R_k(\frac{x}{n^k}).\end{aligned}$$

On the other hand,

$$\sum_{n\le x} g_{kh}(n) = \frac{x}{\zeta(kh)} + O(x^{\frac{1}{kh}}).$$

Equating the previous both equalities, we obtain (1.1).

Choosing h large enough such that $2^h > x^{\frac{1}{k}}$, we have on the left-hand side of (1.1)

$$\sum_{n\le x^{\frac{1}{k}}} g_h(n) R_k(\frac{x}{n^k}) = \sum_{n\le x^{\frac{1}{k}}} R_k(\frac{x}{n^k})$$

and on the right-hand side

$$\begin{aligned}\frac{x^{\frac{1}{k}}}{(k-1)\zeta(k)\zeta(h)} + O(x^{\frac{1}{kh}}) &= \frac{x^{\frac{1}{k}}}{(k-1)\zeta(k)}(1 + \frac{\mu(2)}{2^h} + \frac{\mu(3)}{3^h} + \cdots) + O((2^h)^{\frac{1}{h}})\\ &= \frac{x^{\frac{1}{k}}}{(k-1)\zeta(k)} + O(\frac{2^h}{2^h} + \frac{2^h}{3^h} + \frac{2^h}{5^h} + \cdots) + O(1)\\ &= \frac{x^{\frac{1}{k}}}{(k-1)\zeta(k)} + O(1).\end{aligned}$$

Thus we obtain (1.2) .

Furthermore, by lemma 2.4 and the fact

$$\sum_{d=1}^{\infty} \frac{\mu(d)\log d}{d^s} = \frac{\zeta'(s)}{\zeta^2(s)} \qquad \text{for } \Re(s) > 1,$$

(1.3) can be proved straightforwardly. We have

$$\begin{aligned}\sum_{n\le x}\frac{g_k(n)}{n} &= \sum_{n\le x}\frac{1}{n}\sum_{d^k|n}\mu(d)\\ &= \sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\sum_{m\le\frac{x}{d^k}}\frac{1}{m}\\ &= \sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\left(\log\frac{x}{d^k}+\gamma+O\left(\frac{d^k}{x}\right)\right)\\ &= \frac{\log x}{\zeta(k)}-\frac{k\zeta'(k)}{\zeta^2(k)}+\frac{\gamma}{\zeta(k)}+O\left(x^{\frac{1}{k}-1}\log x\right).\end{aligned}$$

Now we are ready to prove (1.4). Applying (1.3) to the following iterated sum, we get

$$\begin{aligned}S &= \sum_{n\le x}\sum_{m\le\frac{x}{n}}g_k(n)g_k(m)\\ &= \sum_{n\le x}g_k(n)\left(\frac{x}{\zeta(k)n}+R_k\left(\frac{x}{n}\right)\right)\\ &= \frac{x}{\zeta(k)}\left(\frac{\log x}{\zeta(k)}+c_k+O(x^{\frac{1}{k}-1}\log x)\right)+\sum_{n\le x}g_k(n)R_k\left(\frac{x}{n}\right).\end{aligned}$$

Note that the iterated sum S may be counted in another way by its symmetry of summation with respect to the line $y=f(x)=x$. Let $u=x^{\frac{1}{2}}$, we have

$$\begin{aligned}S &= 2\sum_{n\le u}\sum_{m\le\frac{x}{n}}g_k(n)g_k(m)-\left(\sum_{n\le u}g_k(n)\right)^2\\ &= 2\sum_{n\le u}g_k(n)\left(\frac{x}{\zeta(k)n}+O\left(\left(\frac{x}{n}\right)^{\frac{1}{k}}\right)\right)-\left(\frac{u}{\zeta(k)}+O(u^{\frac{1}{k}})\right)^2\\ &= \frac{2x}{\zeta(k)}\left(\frac{\log u}{\zeta(k)}+c_k+O(u^{\frac{1}{k}-1}\log x)\right)+O\left(x^{\frac{1}{k}}u^{1-\frac{1}{k}}\right)-\frac{u^2}{\zeta^2(k)}+O\left(u^{1+\frac{1}{k}}\right).\end{aligned}$$

Equating both of previous equations, we acquire (1.4).

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