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CENTRAL SEQUENCE ALGEBRAS OF VON NEUMANN ALGEBRAS

Junsheng Fang, Liming Ge and Weihua Li

Abstract. We prove that the ultrapower and the central sequence algebra of the hyperfinite II_1 factor are prime factors which implies that the central sequence algebra of the tensor product of two factors may not be the tensor product of the central sequence algebras of the factors respectively. This answers negatively a question of D. McDuff. Isomorphisms of ultrapowers of factors are also studied.

1. Introduction

F. J. Murray and J. von Neumann [9-13] introduced and studied certain algebras of Hilbert space operators. Those algebras are now called "Von Neumann Algebras." They are strong-operator closed self-adjoint subalgebras of the algebra of all bounded linear transformations on a Hilbert space. *Factors* are von Neumann algebras whose centers consist of scalar multiples of the identity. Every von Neumann algebra is a direct sum (or "direct integral") of factors. Thus factors are the building blocks for all von Neumann algebras.

Murray and von Neumann [9] classified factors by means of a relative dimension function. *Finite factors* are those for which this dimension function has a finite range. For finite factors, this dimension function gives rise to a (unique, when normalized) tracial state. In general, a von Neumann algebra admitting a faithful normal trace is said to be *finite*. Finite-dimensional "finite factors" are full matrix algebras $M_n(\mathbb{C})$, $n = 1, 2, \ldots$. Infinite-dimensional "finite factors" are called factors of type II_1. They are "continuous" matrix algebras. In [10], Murray and von Neumann showed that there are two non-isomorphic factors of type II_1 using ideas of central sequences. Central sequences in a factor form an algebra. Many people (see [2, 5, 8, 16]) used the algebraic properties of central sequences in factors of type II_1

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to study factors themselves. Dixmier [4] showed that the central sequence algebra of a type II_1 factor is either trivial (i.e., one-dimensional) or non-atomic (and thus, infinite-dimensional). In this paper, we show that a similar result holds for an irreducible inclusion of type II_1 factors. Using the language of ultrapowers, our result states that the relative commutant of any irreducible subfactor of a type II_1 factor in the ultrapower of the factor is either trivial or non-atomic.

The central sequence algebra for a finite von Neumann algebra can be viewed as the relative commutant of the algebra in the ultrapower of the algebra constructed from a given free ultrafilter on natural numbers $\mathbb{N}$. Ultrapowers for finite von Neumann algebras were first introduced and studied by D. McDuff [8]. She proved a remarkable result which says that if the central sequence algebra of a type II_1 factor is noncommutative, then the original factor is isomorphic to the tensor product of the hyperfinite II_1 factor with itself. Thus the structure of the ultrapower of a factor encodes rich structural properties of the factor itself. McDuff asked three questions in [8]. The first one, answered positively by Ge and Hadwin [6], asks whether the ultrapowers of a factor on two different free ultrafilters are isomorphic to each other. The second question asks if the central sequence algebra of the tensor product of two factors splits as the tensor product of respective central sequence algebras of the factors. The third question asks whether the central sequence algebra of the hyperfinite factor of type II_1 is again hyperfinite. In this paper, we shall answer both the second and the third negatively. Moreover, we shall show that both the ultrapower and the central sequence algebra of the hyperfinite factor of type II_1 are (non-separable) prime factors and also the ultrapower can be embedded into the central sequence algebra as a subfactor.

Recently, especially after the introduction of free entropy by Voiculescu in his probability theory (see [17]), there is a growing interest in Connes's approximate embedding problem [3], which asks if every factor of type II_1 with a separable predual can be embedded into the ultrapower of the hyperfinite II_1 factor. Connes's problem is equivalent to the non-emptiness of a set used to define free entropy. We shall also study some embedding problems of factors into the ultrapower of the hyperfinite one. For example, we show that any embedding of a factor with a separable predual into the ultrapower of the hyperfinite II_1 has a big relative commutant.

The isomorphism problem of two ultrapowers from two non-isomorphic factors seems to be an interesting one. In this direction, we show that the ultrapower of a property Γ factor will have the same property. Also, if the ultrapower of a factor is isomorphic to the ultrapower of the hyperfinite II_1 factor, then the factor must be a McDuff factor.

The paper is organized as follows. In Section 2, we shall review the construction of ultrapowers of finite von Neumann algebras as well as some technical results. Section 3 contains results on relative property Γ for irreducible inclusions. We show that the relative commutant of any irreducible subfactor of a type II_1 factor in the ultrapower of the factor is either trivial or non-atomic. In Section 4, we show that the ultrapower and the central sequence algebra of the hyperfinite II_1 factor are prime. We also consider the central sequence algebras of certain tensor products. The isomorphism problem of ultrapowers is studied in Section 5. In Section 6, we shall prove that the central sequence algebra of the hyperfinite II_1 factor does not contain a Cartan subalgebra and also it contains the ultrapower of the factor as a subfactor.

2. Preliminaries

There are two main classes of examples of von Neumann algebras constructed by Murray and von Neumann [10,11]. One is obtained from the "group-measure space construction;" the other is based on regular representations of a (discrete) group G (with unit e). The second basic construction is more related to the topics discussed in this paper. We shall give more details. The Hilbert space $\mathcal{H}$ is $l^2(G)$ (with its usual inner product). We assume that G is countable so that $\mathcal{H}$ is separable. For each g in G, let L_g denote the left translation of functions in $l^2(G)$ by g^{-1}. Then $g \to L_g$ is a faithful unitary representation of G on $\mathcal{H}$. Let $\mathcal{L}_G$ be the von Neumann algebra generated by $\{L_g : g \in G\}$ (or the strong-operator closure of the linear span). Similarly, let R_g be the right translation by g on $l^2(G)$ and $\mathcal{R}_G$ be the von Neumann algebra generated by $\{R_g : g \in G\}$. Then the commutant $\mathcal{L}_G'$ of $\mathcal{L}_G$ is equal to $\mathcal{R}_G$ and $\mathcal{R}_G' = \mathcal{L}_G$. The function u_g that is 1 at the group element g and 0 elsewhere is a cyclic trace vector for $\mathcal{L}_G$ (and for $\mathcal{R}_G$). In general, $\mathcal{L}_G$ and $\mathcal{R}_G$ are finite von Neumann algebras. The vector state given by any u_g is a trace. They are factors (of type II_1) precisely when each conjugacy class in G (other than that of e) is infinite. In this case we say that G is an *infinite conjugacy class* (i.c.c.) group.

Specific examples of such II_1 factors result from choosing for G any of the free groups F_n on n generators ($n \geq 2$), or the permutation group Π of integers $\mathbb{Z}$ (consisting of those permutations that leave fixed all but a finite subset of $\mathbb{Z}$). Murray and von Neumann [11] proved that $\mathcal{L}_{F_n}$ and $\mathcal{L}_\Pi$ are not * isomorphic to each other using ideas of central sequences.

Suppose $\mathcal{M}$ is a factor of type II_1 with the unique (normalized) trace τ. We shall use $\|X\|_2 = \tau(X^*X)^{1/2}$, $X \in \mathcal{M}$, to denote the trace norm induced by τ. A uniform bounded sequence $\{A_n\}$ in a factor $\mathcal{M}$ is called *central* if $\|A_nX - XA_n\|_2 \to 0$ for any X in $\mathcal{M}$ as n tends to infinity. A central sequence $\{A_n\}$ in a type II_1 factor $\mathcal{M}$ is non-trivial when it is central and A_n's are away from the scalars, i.e., $\|A_n - \tau(A_n)I\|_2$ has no limit or a non-zero limit. Murray and von Neumann referred to factors of type II_1 with a non-trivial sequence as property Γ

factors. A factor is *hyperfinite* if it is the ultraweak closure of the ascending union of a family of finite-dimensional self-adjoint subalgebras. In fact, $\mathcal{L}_{\Pi}$ is the *unique* hyperfinite factor of type II_1 and it has non-trivial central sequences (see [11]). While Murray and von Neumann showed that the free group factor $\mathcal{L}_{F_n}$, $n \geq 2$, has no non-trivial sequences-a deep result.

Now, suppose that each $\mathcal{M}_n$ is a finite von Neumann algebra with a faithful normal tracial state τ_n. Let $\prod_{n\in\mathbb{N}} \mathcal{M}_n$ be the l^∞-product of the $\mathcal{M}_n$'s. Then $\prod_n \mathcal{M}_n$ is a von Neumann algebra (with pointwise multiplication). Let ω be a free ultrafilter on $\mathbb{N}$ (ω may be viewed as an element in $\beta\mathbb{N} \setminus \mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Céch compactification of $\mathbb{N}$). If $\{X_n\}$ and $\{Y_n\}$ are two elements in $\prod_n \mathcal{M}_n$, then we define $\{X_n\} \sim \{Y_n\}$ when $\lim_{n\to\omega} \|X_n - Y_n\|_2 = 0$. Then the *ultraproduct*, denoted by $\prod^\omega \mathcal{M}_n$, of $\mathcal{M}_n$ (with respect to the free ultrafilter ω) is the quotient von Neumann algebra of $\prod_n \mathcal{M}_n$ modulo the equivalence relation $\sim$ and the limit of τ_n at ω gives rise to a tracial state on $\prod^\omega \mathcal{M}_n$. We shall use τ again to denote the tracial state on $\prod^\omega \mathcal{M}_n$. When $\mathcal{M}_n = \mathcal{M}$ for all n, then $\prod^\omega \mathcal{M}_n$ is called the *ultrapower* of $\mathcal{M}$, denoted by $\mathcal{M}^\omega$. The initial algebra $\mathcal{M}$ is embedded in $\mathcal{M}^\omega$ as constant sequences given by elements in $\mathcal{M}$. Sakai [15] showed that an ultrapower of a finite von Neumann algebra with respect to a faithful normal trace is again a von Neumann algebra. Central sequences in $\mathcal{M}$ give rise to elements in $\mathcal{M}^\omega$ which commute with all constant sequences (along ω). Therefore the relative commutant of $\mathcal{M}$ in $\mathcal{M}^\omega$ is called the central sequence algebra of $\mathcal{M}$ and denote by $\mathcal{M}_\omega$ $(= \mathcal{M}' \cap \mathcal{M}^\omega)$. Dixmier [4] proved that if $\mathcal{M}_\omega$ is non-trivial, then it is non-atomic. In the following section we show that if $\mathcal{N} \subset \mathcal{M}$ is an irreducible inclusion, then $\mathcal{N}' \cap \mathcal{M}^\omega$ is either trivial or non-atomic. Throughout this paper, ω will be an arbitrarily given free ultrafilter on $\mathbb{N}$. Our initial factors are always assumed to have separable preduals although their resulting ultrapowers have non-separable preduals [6]. For basics on operator algebras we refer to [7]. For basic results and techniques on ultrapowers and ultraproducts we refer to [6].

3. Relative Central Sequence Algebras

In [1], D. Bisch studied central sequences of inclusions $\mathcal{N} \subset \mathcal{M}$. He considered sequences from $\mathcal{N}$ being central with respect to $\mathcal{M}$. In this section, we shall discuss a different relative central sequence algebra of an inclusion $\mathcal{N} \subset \mathcal{M}$, i.e., the sequences in the bigger factor $\mathcal{M}$ which are central with respect to the subfactor. For simplicity, we shall consider only irreducible inclusions since elements in the relative commutant give rise to "trivial" central sequences. Suppose $\mathcal{N} \subset \mathcal{M}$ is an irreducible inclusion of factors of type II_1. Sequences in $\mathcal{M}$ that are central with respect to $\mathcal{N}$ give rise to elements in $\mathcal{N}' \cap \mathcal{M}^\omega$. Thus we shall call the von Neumann algebra $\mathcal{N}' \cap \mathcal{M}^\omega$ the central sequence algebra of $\mathcal{N}$ in $\mathcal{M}$. In the following we shall show that $\mathcal{N}' \cap \mathcal{M}^\omega$ is either one-dimensional or non-atomic. We need few lemmas before proving the result.

Lemma 3.1. *Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible inclusion of factors of type* II_1. *Suppose P is a projection of trace r, $0 < r < 1$, in $\mathcal{M}$. Then, for any $\varepsilon > 0$, there is a unitary operator U in $\mathcal{N}$ such that $\|PU - UP\|_2 \geq \sqrt{r - r^2} - \varepsilon$.*

Proof. Assume on the contrary that there is a positive ε_0 such that, for any unitary U in $\mathcal{N}$, $\|PU - UP\|_2 < \sqrt{r - r^2} - \varepsilon_0$, i.e.,

$$\|U^*PU - P\|_2 < \sqrt{r - r^2} - \varepsilon_0.$$

Let $\mathrm{co}\{U^*PU : U \in \mathcal{N}\}$ be the minimal convex set containing all U^*PU with U a unitary element in $\mathcal{N}$. For any Y in $\mathrm{co}\{U^*PU : U \in \mathcal{N}\}$, write $Y = \sum_{j=1}^n \lambda_j U_j^* P U_j$ for unitary U_j in $\mathcal{N}$ and positive constant λ_j with $\sum_j \lambda_j = 1$. Then we have

$$\|Y - P\|_2 = \|\sum_{j=1}^n \lambda_j U_j^* P U_j - P\|_2 \leq \sum_{j=1}^n \lambda_j \|U_j^* P U_j - P\|_2 \leq \sqrt{r - r^2} - \varepsilon_0.$$

Since $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}I$, $\tau(P)I$ lies in the weak-operator closure of $\mathrm{co}\{U^*PU : U \in \mathcal{N}\}$. Then

$$\sqrt{r - r^2} = \|rI - P\|_2 \leq \sqrt{r - r^2} - \varepsilon_0.$$

This shows that our initial assumption is false. Thus our lemma follows. ∎

The following corollary follows immediately from the above lemma and the definition of ultrapowers.

Corollary 3.2. *Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible inclusion of factors of type* II_1. *Suppose P is a projection of trace r, $0 < r < 1$, in $\mathcal{M}^\omega$ for some free ultrafilter ω. Then there is a unitary operator U in $\mathcal{N}^\omega$ such that $\|PU - UP\|_2 \geq \sqrt{r - r^2}$.*

Before we state and prove our main theorem (Theorem 3.5) in this section, we need two more lemmas.

Lemma 3.3. *Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible inclusion of factors of type* II_1. *Suppose $P = \{P_n\}$ is a projection of trace r, $0 < r < 1$, in $\mathcal{N}' \cap \mathcal{M}^\omega$ for some free ultrafilter ω. Then, for any projection Q in $\mathcal{M}$ with trace s and any positive ε, there is a P_{n_0} in $\{P_n\}$ such that $\|P_{n_0}Q\|_2^2 \geq rs - \varepsilon$.*

Proof. Without loss of generality, we may assume that P_n are projections in $\mathcal{M}$ with $\tau(P_n) = r$. From the proof of Lemma 3.1, we know that for any $\varepsilon > 0$, there are unitary elements $U_1, \ldots, U_n$ in $\mathcal{N}$ and positive numbers $\lambda_1, \ldots, \lambda_n$ with $\sum_{j=1}^n \lambda_j = 1$ so that

$$\|\sum_{j=1}^{n} \lambda_j U_j^* Q U_j - \tau(Q)I\|_2 < \frac{\varepsilon}{2}.$$

From our assumption that P commutes with U_j's, there is an n_0 such that $\|P_{n_0}U_j - U_jP_{n_0}\|_2 < \frac{\varepsilon}{2}$, for $j = 1, \ldots, n$. Thus we have

$$\begin{aligned}
&\| \textstyle\sum_{j=1}^n \lambda_j U_j^* P_{n_0} Q U_j - \tau(Q) P_{n_0}\|_2 \\
&\le \| \textstyle\sum_{j=1}^n \lambda_j U_j^* P_{n_0} Q U_j - \sum_{j=1}^n \lambda_j P_{n_0} U_j^* Q U_j\|_2 + \|P_{n_0}(\sum_{j=1}^n \lambda_j U_j^* Q U_j - \tau(Q)I)\|_2 \\
&\le \textstyle\sum_{j=1}^n \lambda_j \|(U_j^* P_{n_0} - P_{n_0} U_j^*)\|_2 \|QU_j\| + \| \sum_{j=1}^n \lambda_j U_j^* Q U_j - \tau(Q) I\|_2 \\
&< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Since $\tau(U_j^* P_{n_0} Q U_j) = \tau(P_{n_0} Q)$, we have $|\tau(P_{n_0}Q) - rs| < \varepsilon$. This shows that $\|P_{n_0}Q\|_2^2 = \tau(P_{n_0}QP_{n_0}) > rs - \varepsilon$. ∎

The following lemma is an easy exercise. We omit its proof.

Lemma 3.4. *Suppose $\mathcal{M}$ is a finite von Neumann algebra with trace τ and P, Q are projections in $\mathcal{M}$. If $\|P(I-Q)\|_2 \ge \varepsilon > 0$, then $\|P \vee Q - Q\|_2 \ge \varepsilon$.*

Lemma 3.5. *Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible inclusion of factors of type* II_1. *Suppose $\mathcal{N}' \cap \mathcal{M}^\omega$ is non-trivial. Then $\mathcal{N}' \cap \mathcal{M}^\omega$ is non-atomic.*

Proof. It is clear that $\mathcal{N}' \cap \mathcal{M}^\omega$ is a finite von Neumann algebra. Suppose on the contrary that there is a minimal projection P in $\mathcal{N}' \cap \mathcal{M}^\omega$ and $P \neq 0, I$. Let Q in $\mathcal{N}' \cap \mathcal{M}^\omega$ be the central carrier of P. Then there is a k such that $Q(\mathcal{N}' \cap \mathcal{M}^\omega)Q \cong M_k(\mathbb{C})$. Denote $\tau(P) = r$ for some $0 < r < 1$. Let $\mathcal{P}$ be the manifold of all minimal projections in $Q(\mathcal{N}' \cap \mathcal{M}^\omega)Q$ with respect to the metric given by $\| \, \|_2$. Then $\mathcal{P}$ is a separable space and let $E_1, E_2, \ldots$ be a countable dense sequence in $\mathcal{P}$. Let $P = \{P_n\}$ such that $\{P_n\}$ is central with respect to $\mathcal{N}$ and $\tau(P_n) = r$ for all n. Thus any subsequence of $\{P_n\}$ gives rise to an element in $\mathcal{N}' \cap \mathcal{M}^\omega$ with trace r. Similarly, we choose projections Q_n and $E_n^{(j)}$ such that $Q = \{Q_n\}$ and $E_j = \{E_n^{(j)}\}$ for $j = 1, 2, \ldots$. Here we may assume that $\tau(Q_n) = kr$ and $\tau(E_n^{(j)}) = r$, for all n and j.

From Corollary 3.2, for each j, there is a unitary element $U_j = \{U_n^{(j)}\}$ in $\mathcal{N}^\omega$ such that

$$\|E_jU_j - U_jE_j\|_2 \ge \sqrt{r - r^2}.$$

For any given n and with Q_n in $\mathcal{M}$, from Lemma 3.3 we have that there is an $m_n \in \mathbb{N}$ and P_{m_n} in $\{P_l\}_{l\in\mathbb{N}}$ such that

$$(*) \qquad \|P_{m_n}Q_n\|_2^2 \geq kr^2 - \frac{1}{n}.$$

Since P is central with respect to $\mathcal{N}$, with $U_k^{(l)}$ in $\mathcal{N}$, we can also assume that

$$(**) \qquad \|P_{m_n}U_k^{(l)} - U_k^{(l)}P_{m_n}\|_2 < \frac{1}{n},$$

for all $1 \leq k, l \leq n$. Now we define an element $F = \{P_{m_n}\}_n$ in $\mathcal{M}^\omega$. Clearly $F \in \mathcal{N}' \cap \mathcal{M}^\omega$ and $\tau(F) = r$. Moreover, from $(*)$ we have $\|FQ\|_2^2 \geq kr^2$ and thus $FQ \neq 0$. Since Q belongs to the center of $\mathcal{N}' \cap \mathcal{M}^\omega$, FQ must be a minimal projection and thus $FQ = F$. This shows that $F \in \mathcal{P}$. From $(**)$, we get $FU_j = U_jF$ for all j. By our choice of U_j, we have, for all j,

$$\begin{aligned}\sqrt{r-r^2} &\leq \|E_jU_j - U_jE_j\|_2 \\ &= \|(E_j - F)U_j - U_j(E_j - F)\|_2 \leq 2\|E_j - F\|_2.\end{aligned}$$

This contradicts to the assumption that $\{E_j\}_j$ is a dense subset of $\mathcal{P}$. And thus there is no minimal projection in $\mathcal{N}' \cap \mathcal{M}^\omega$. ■

4. Primeness of Ultrapowers and Central Sequence Algebras

Suppose $\mathcal{R}$ is the hyperfinite II_1 factor. In [14], Popa showed that $\mathcal{R}^\omega$ does not contain a Cartan subalgebra (i.e., a maximal abelian subalgebra whose normalizer generates the algebra). In this section, we show that both $\mathcal{R}^\omega$ and $\mathcal{R}_\omega$ are prime factors. By techniques similar to Popa's, one can also show that $\mathcal{R}_\omega$ does not contain any Cartan subalgebras.

The following lemma is an easy consequence of the polar decomposition of operators in factors of type II_1.

Lemma 4.1. *Let $\mathcal{M}$ be a factor of type II_1 with a tracial state τ. Suppose $E_1, \ldots, E_m$ and $F_1, \ldots, F_m$ are projections in $\mathcal{M}$ such that $\sum_{j=1}^m E_j = I$, $\sum_{j=1}^m F_j = I$, $\tau(E_j) = \tau(F_j) = \frac{1}{m}$. Then there is a unitary element U in $\mathcal{M}$ such that $U^*E_jU = F_j$ for $1 \leq j \leq m$. Moreover if $\|E_j - F_j\|_2 < \delta$ for each $1 \leq j \leq m$, then we may choose U so that $\|I - U\|_2 < 4\sqrt{m}\delta$.*

Proof. For each j, let V_j be the partial isometry in $\mathcal{M}$ obtained from the polar decomposition of F_jE_j $(= V_j|F_jE_j|)$ so that $V_j^*V_j = E_j' \leq E_j$ and $V_jV_j^* = F_j' \leq$

F_j. From $\|E_j - F_j\|_2 < \delta$, one easily shows that

$$\begin{aligned}\|E_j - V_j\|_2 &\leq \|E_j - F_jE_j\|_2 + \|F_jE_j - V_j\|_2 \\ &= \|(E_j - F_j)E_j\|_2 + \|V_j|F_jE_j| - V_jE_j\|_2 \\ &\leq \delta + \|(|F_jE_j| + E_j)(|F_jE_j| - E_j)\|_2 \\ &= \delta + \|E_jF_jE_j - E_j\|_2 < 2\delta.\end{aligned}$$

Let V_j' be a partial isometry so that $V_j'^*V_j' = E_j - E_j'$ and $V_j'V_j'^* = F_j - F_j'$ and $U_j = V_j + V_j'$. Then $U_j^*U_j = E_j$ and $U_jU_j^* = F_j$. From $V_j(E_j - E_j') = 0$, we have $\|E_j - E_j'\|_2 = \|(E_j - V_j)(E_j - E_j')\|_2 \leq \|E_j - V_j\|_2 < 2\delta$. Therefore, $\|E_j - U_j\|_2 \leq \|E_j - V_j\|_2 + \|V_j'\|_2 < 4\delta$.

Let $U = \sum_{j=1}^m U_j$. Then U is a unitary element and $\|I - U\|_2^2 = \|\sum_{j=1}^m E_j - \sum_{j=1}^m U_j\|_2^2 = \tau(\sum_{j,k=1}^m (E_k - U_k^*)(E_j - U_j)) = \sum_{j=1}^m \|E_j - U_j\|_2^2 < 16m\delta^2$. Thus $\|I - U\|_2 < 4\sqrt{m}\delta$. ■

The following two lemmas are the key to our main results.

Lemma 4.2. *Suppose ω is a free ultrafilter on $\mathbb{N}$ and $\mathcal{M}$ a factor of type II_1. Let $\mathcal{M}^\omega$ be the ultrapower of $\mathcal{M}$ and $\mathcal{A}_1$, $\mathcal{A}_2$ be two non-atomic abelian von Neumann subalgebras of $\mathcal{M}^\omega$ with separable preduals. Then there is a unitary element U in $\mathcal{M}^\omega$ such that $U^*\mathcal{A}_1U = \mathcal{A}_2$.*

Proof. From our assumption that $\mathcal{A}_1, \mathcal{A}_2$ are non-atomic abelian von Neumann algebras with separable preduals, they are isomorphic to $L^\infty[0,1]$. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are generated by Haar unitary elements U_1 and U_2 respectively. We write $U_1 = \{U_1^{(n)}\}_n$ and $U_2 = \{U_2^{(n)}\}_n$ for $U_1^{(n)}$ and $U_2^{(n)}$ in $\mathcal{M}$. We may assume that $U_j^{(n)}$ lies in a finite dimensional abelian subalgebra of $\mathcal{M}$ (otherwise, we replace $U_j^{(n)}$ by such an element close to it in trace norm). Since U_1 and U_2 are Haar unitary elements, we may assume that $U_1^{(n)}$ and $U_2^{(n)}$ have the same distribution and $U_1^{(n)} = \sum_{j=1}^{s_n} \lambda_j E_j^{(n)}$, $U_2^{(n)} = \sum_{j=1}^{s_n} \lambda_j F_j^{(n)}$ for $E_1^{(n)}, \ldots, E_{s_n}^{(n)}$ and $F_1^{(n)}, \ldots, F_{s_n}^{(n)}$ in $\mathcal{M}$ such that $\tau(E_j^{(n)}) = \tau(F_j^{(n)})$, $\sum_{j=1}^{s_n} E_j^{(n)} = \sum_{j=1}^{s_n} F_j^{(n)} = I$. From Lemma 4.1, there is a unitary element $U^{(n)}$ in $\mathcal{M}$ such that $(U^{(n)})^*E_j^{(n)}U^{(n)} = F_j^{(n)}$ for all $j = 1, \ldots, s_n$. Then $(U^{(n)})^*U_1^{(n)}U^{(n)} = U_2^{(n)}$. Let $U = \{U^{(n)}\}_n$ in $\mathcal{M}^\omega$. Then it is easy to see that $U^*U_1U = U_2$ and thus $U^*\mathcal{A}_1U = \mathcal{A}_2$. ■

Lemma 4.3. *Suppose ω is a free ultrafilter on $\mathbb{N}$ and $\mathcal{R}_\omega$ is the central sequence algebra of $\mathcal{R}$, the hyperfinite II_1 factor. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two non-atomic abelian von Neumann subalgebras of $\mathcal{R}_\omega$ with separable preduals. Then there is a unitary element U in $\mathcal{R}_\omega$ such that $U^*\mathcal{A}_1U = \mathcal{A}_2$.*

Proof. The proof of this lemma is similar to the above one. The only difference is that the resulting unitary element U lies in $\mathcal{R}_\omega$. Since $\mathcal{R}$ is hyperfinite, we may choose full matrix subalgebras $M_{2^k}(\mathbb{C})$ of $\mathcal{R}$ so that $M_{2^k}(\mathbb{C}) \subseteq M_{2^{k+1}}(\mathbb{C})$ and $\cup_{k=1}^\infty M_{2^k}(\mathbb{C})$ is ultraweakly dense in $\mathcal{R}$. From our assumption that $\mathcal{A}_1, \mathcal{A}_2$ are non-atomic abelian von Neumann algebras with separable preduals, they are isomorphic to $L^\infty[0,1]$. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are generated by Haar unitary elements U_1 and U_2 respectively. We write $U_1 = \{U_1^{(n)}\}$ and $U_2 = \{U_2^{(n)}\}$ for $U_1^{(n)}$ and $U_2^{(n)}$ in $\mathcal{R}$. Since U_1 and U_2 commute with $\mathcal{R}$, we may assume that $U_1^{(n)}$ and $U_2^{(n)}$ commute with $M_{2^n}(\mathbb{C})$ ($\subset \mathcal{R}$). We may also assume that $U_j^{(n)}$ lies in a finite dimensional abelian subalgebra of $M_{2^n}(\mathbb{C})' \cap \mathcal{R}$ for $j = 1, 2$ (otherwise, we replace $U_j^{(n)}$ by such an element). Since U_1 and U_2 are Haar unitary elements, we may assume that $U_1^{(n)}$ and $U_2^{(n)}$ have the same distribution and $U_1^{(n)} = \sum_{j=1}^{s_n} \lambda_j E_j^{(n)}$, $U_2^{(n)} = \sum_{j=1}^{s_n} \lambda_j F_j^{(n)}$ for $E_1^{(n)}, \dots, E_{s_n}^{(n)}$ and $F_1^{(n)}, \dots, F_{s_n}^{(n)}$ in $M_{2^n}(\mathbb{C})' \cap \mathcal{R}$ such that $\tau(E_j^{(n)}) = \tau(F_j^{(n)})$, $\sum_{j=1}^{s_n} E_j^{(n)} = \sum_{j=1}^{s_n} F_j^{(n)} = I$. Since $M_{2^n}(\mathbb{C})' \cap \mathcal{R}$ is again a factor of type II_1, there is a unitary element $U^{(n)}$ in it such that $(U^{(n)})^* E_j^{(n)} U^{(n)} = F_j^{(n)}$ for all $j = 1, \dots, s_n$. Then $(U^{(n)})^* U_1^{(n)} U^{(n)} = U_2^{(n)}$. Let $U = \{U^{(n)}\}_n$ in $\mathcal{R}^\omega$. Then it is easy to see that $U \in \mathcal{R}_\omega$ and $U^* U_1 U = U_2$ and thus $U^* \mathcal{A}_1 U = \mathcal{A}_2$. ∎

The following lemma was proved by Popa in [14]:

Lemma 4.4. *Suppose $\mathcal{B}$ is a von Neumannn subalgebra of a type II_1 von Neumann algebra $\mathcal{M}$ (with trace τ) and U is a unitary operator in $\mathcal{M}$ such that, for any $\varepsilon > 0$, there is a finite dimensional von Neumann subalgebra $\mathcal{A}_\varepsilon$ of $\mathcal{B}$ such that $\tau(E) < \varepsilon$ for all minimal projections E in $\mathcal{A}_\varepsilon$, and $U\mathcal{A}_\varepsilon U^*$ and $\mathcal{B}$ are orthogonal with respect to τ, then U is orthogonal to the set of normalizers $\{V \in \mathcal{M} : V\mathcal{B}V^* = \mathcal{B},\ \ V \text{ unitary}\}$ of $\mathcal{B}$ in $\mathcal{M}$, in particular, U is orthogonal to to $\mathcal{B}$ and $\mathcal{B}' \cap \mathcal{M}$.*

Now we prove the main theorem in this section.

Theorem 4.5. *Suppose $\mathcal{M}$ is a factor of type II_1 and ω a free ultrafilter on $\mathbb{N}$. Let $\mathcal{M}^\omega$ be the ultrapower of $\mathcal{M}$. Then $\mathcal{M}^\omega$ is prime. Moreover, if $\mathcal{R}$ is the hyperfinite factor of type II_1, then $\mathcal{R}_\omega$ is also a prime factor of type II_1.*

Proof. Suppose on the contrary that $\mathcal{M}^\omega$ is not prime. Then $\mathcal{M}^\omega = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ for some factors $\mathcal{M}_1$ and $\mathcal{M}_2$ of type II_1. Choose non-atomic abelian subalgebras $\mathcal{A}_1$ of $\mathcal{M}_1$ and $\mathcal{A}_2$ of $\mathcal{M}_2$ such that $\mathcal{A}_1$ and $\mathcal{A}_2$ have separable preduals. From Lemma 4.2, there is a unitary U in $\mathcal{M}^\omega$ such that $U^* \mathcal{A}_1 U = \mathcal{A}_2$ which is orthogonal to $\mathcal{M}_1 \otimes \mathbb{C}I$. From the preceding lemma, U is orthogonal to the normalizers of $\mathcal{M}_1$ in $\mathcal{M}^\omega$. But the normalizers of $\mathcal{M}_1$ generate (linearly) $\mathcal{M}^\omega$ as a von Neumann algebra. This contradicts to the assumption that U lies in $\mathcal{M}^\omega$, which shows that $\mathcal{M}^\omega$ is prime. Similarly, using Lemma 4.3, we can show that $\mathcal{R}_\omega$ is also prime. ∎

From the above theorem, we see that the central sequence algebra $\mathcal{M}\overline{\otimes}\mathcal{N}$ of two type II_1 factors $\mathcal{M}$ and $\mathcal{N}$ may not be the tensor product of the corresponding central sequence algebras (since $\mathcal{R} \cong \mathcal{R}\overline{\otimes}\mathcal{R}$ and $\mathcal{R}_\omega$ is prime). This answers a question of D. McDuff [8] in the negative.

One of the important results proved by Connes in [3] (and stated in [8]) says that the tensor product of two factors of type II_1 has no non-trivial central sequences if and only if any of the two factors has no non-trivial central sequences. When one of the central sequence algebra of the tensor factor is trivial, we can show that McDuff's question has a positive answer. Now we assume that $\mathcal{M}$ is a full factor of type II_1 and $\mathcal{N}$ is another type II_1 factor. We will show that the central sequence algebra of $\mathcal{M}\overline{\otimes}\mathcal{N}$ is canonically isomorphic to that of $\mathcal{N}$. This generalizes Connes's result. From the definition of property Γ (*for any positive constant and any given finitely many elements, there is a trace-zero unitary so that the unitary commutes with the given elements within the constant in trace norm*), we have the following lemma. Here we shall use $[X, U_j]$ to denote $XU_j - U_jX$.

Lemma 4.6. *Suppose $\mathcal{M}$ is a factor of type II_1 with a separable predual. Then $\mathcal{M}$ is a full factor if and only if there are unitary elements $U_1, \ldots, U_n$ in $\mathcal{M}$ and a positive constant K such that for any X in $\mathcal{M}$,*

$$\|X - \tau(X)I\|_2^2 \le K \sum_{j=1}^{n} \|[X, U_j]\|_2^2.$$

Theorem 4.7. *Let ω be a free ultrafilter on $\mathbb{N}$. Suppose $\mathcal{M}$ is a full factor of type II_1 and $\mathcal{N}$ is another type II_1 factor. Then $(\mathcal{M}\overline{\otimes}\mathcal{N})_\omega$ is canonically isomorphic to $\mathcal{N}_\omega$.*

Proof. Let Φ be the unique (trace-preserving) conditional expectation from $\mathcal{M}\overline{\otimes}\mathcal{N}$ onto $\mathcal{N}$ $(= \mathbb{C}I\overline{\otimes}\mathcal{N})$. Then Φ is ultraweakly continuous. Suppose that $\{Y_j\}$ in $\mathcal{M}\overline{\otimes}\mathcal{N}$ is a central sequence. We shall show that it is equivalent to a central sequence in $\mathbb{C}I\overline{\otimes}\mathcal{N}$. Without loss of generality, we may assume that each Y_j is a finite sum of simple tensor products in $\mathcal{M}\overline{\otimes}\mathcal{N}$ and write $Y_j = \sum_{l=1}^{m_j} A_l^{(j)} \otimes B_l^{(j)}$ for $A_l^{(j)}$ in $\mathcal{M}$ and $B_l^{(j)}$ in $\mathcal{N}$. Then $\Phi(Y_j) = \sum_{l=1}^{m_j} \tau(A_l^{(j)}) I \otimes B_l^{(j)}$ and $\|\Phi(Y_j)\| \le \|Y_j\|$. For any given j, we may choose $B_l^{(j)}$'s so that they are mutually orthogonal with respect to the trace and each has trace norm one. Suppose $U_1, \ldots, U_n$ and K are as given in Lemma 4.6 (for $\mathcal{M}$). From our assumption that $\{Y_j\}$ is central, we have that for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that when $j \ge N$,

$$\sum_{k=1}^{n} \|[Y_j, U_k \otimes I]\|_2^2 \le \varepsilon^2.$$

Then we have that

$$\begin{aligned}\|Y_j - \Phi(Y_j)\|_2^2 &= \|\sum_{l=1}^{m_j}(A_l^{(j)} - \tau(A_l^{(j)})I) \otimes B_l^{(j)}\|_2^2 \\ &= \sum_{l=1}^{m_j} \|A_l^{(j)} - \tau(A_l^{(j)})I\|_2^2 \|B_l^{(j)}\|_2^2 \\ &\le \sum_{l=1}^{m_j} K \sum_{k=1}^{n} \|[A_l^{(j)}, U_k]\|_2^2 = K \sum_{k=1}^{n}\sum_{l=1}^{m_j} \|[A_l^{(j)}, U_k]\|_2^2 \\ &= K \sum_{k=1}^{n} \|[Y_j, U_k \otimes I]\|_2^2 \le K\varepsilon^2.\end{aligned}$$

Thus as elements in $(\mathcal{M}\overline{\otimes}\mathcal{N})^\omega$, $\{Y_j\} = \{\Phi(Y_j)\}$, which corresponds to an element in $\mathcal{N}^\omega$. This shows that $(\mathcal{M}\overline{\otimes}\mathcal{N})_\omega$ is contained in $\mathcal{N}_\omega$ and therefore they are equal. ∎

5. Non-isomorphic Ultrapowers

In this section, we shall show that certain properties of a factor hold true for its ultrapower. For example, we shall show that the ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1 factor has property Γ while $\mathcal{L}_{F_2}^\omega$ does not. Thus $\mathcal{R}^\omega$ is not isomorphic to $\mathcal{L}_{F_2}^\omega$ although $\mathcal{L}_{F_2}$ can be embedded into $\mathcal{R}^\omega$ as a subfactor.

First we prove a lemma.

Lemma 5.1. *Suppose $\mathcal{M}$ is a subfactor of $\mathcal{R}^\omega$ with a separable predual. Then $\mathcal{M}' \cap \mathcal{R}^\omega$ contains a 2×2 full matrix algebra.*

Proof. Suppose $M_1, M_2, \ldots$ are in the unit ball of $\mathcal{M}$ so that they are ultraweakly dense in the ball. Write $M_j = \{X_n^{(j)}\}$ with $X_n^{(j)}$ in $\mathcal{R}$. For any given n and $\{X_k^{(l)} : 1 \le k, l \le n\}$, there is a 2×2 matrix unit system $E_n^{(11)}, E_n^{(12)}, E_n^{(21)}, E_n^{(22)}$ in $\mathcal{R}$ such that $\|X_k^{(l)} E_n^{(st)} - E_n^{(st)} X_k^{(l)}\|_2 \le \frac{1}{n}$, for $1 \le k, l \le n$ and $1 \le s, t \le 2$. Let $E_{st} = \{E_n^{(st)}\}$ in $\mathcal{R}^\omega$. Then E_{st}'s commute with $M_1, M_2, \ldots$ and they form a 2×2 matrix unit system in $\mathcal{R}^\omega$. This completes the proof. ∎

The proof of the following corollary is similar to the above proof of the lemma.

Corollary 5.2. *Suppose $\mathcal{M}$ is a type II_1 factor with a separable predual. Then $\mathcal{M}^\omega$ has property Γ if and only if $\mathcal{M}$ has.*

The following corollary is an immediate consequence of Lemma 5.1 and a result in [8].

Corollary 5.3. *Suppose $\mathcal{M}$ is a type II_1 factor with a separable predual and $\mathcal{M}^\omega \cong \mathcal{R}^\omega$. Then $\mathcal{M} \cong \mathcal{M}\overline{\otimes}\mathcal{R}$.*

For two non-isomorphic factors, to determine whether their ultrapowers are isomorphic or not seems to be a hard problem. We end the section with the following theorem.

Theorem 5.4. *Suppose $\mathcal{M}$ is a type II_1 factor with a separable predual and $\mathcal{M}^\omega \cong \mathcal{R}^\omega$. Then the following are equivalent:*

(1) *$\mathcal{M} \cong \mathcal{R}$;*

(2) *For any two embeddings Φ and Ψ of $\mathcal{M}$ into $\mathcal{R}^\omega$, there is a unitary operator U in $\mathcal{R}^\omega$ such that $U^*\Phi(X)U = \Psi(X)$ for any X in $\mathcal{M}$;*

(3) *For any embedding Ψ of $\mathcal{M}\overline{\otimes}\mathcal{M}$ into $\mathcal{R}^\omega$, there is a unitary U in $\mathcal{R}^\omega$ such that $U^*\Psi(X \otimes Y)U = \Psi(Y \otimes X)$ for all X, Y in $\mathcal{M}$;*

(4) *For any embedding Ψ of $\mathcal{M}\overline{\otimes}\mathcal{M}$ into $\mathcal{R}^\omega$, there is a unitary U in $\mathcal{R}^\omega$ such that $U^*\Psi(X \otimes I)U = \Psi(I \otimes X)$ for all X in $\mathcal{M}$.*

Proof. The implications of (1) to (2), (2) to (3) and (3) to (4) are clear. We only need to show that (4) implies (1).

From Corollary 5.3, $\mathcal{M} \cong \mathcal{M}\overline{\otimes}\mathcal{R}$. Choose a generator X for $\mathcal{M}$ and denote the isomorphism from $\mathcal{M}^\omega$ to $\mathcal{R}^\omega$ by α. Then X can be viewed as an element in $\mathcal{M}^\omega$ and let $Z = \alpha(X)$. Now we consider $\mathcal{M}\overline{\otimes}\mathcal{M}$ and define an embedding $\varphi : \mathcal{M}\overline{\otimes}\mathcal{M} \to (\mathcal{M}\overline{\otimes}\mathcal{R})^\omega$ by $\varphi(X \otimes I) = X \otimes I$ and $\varphi(I \otimes X) = I \otimes Z$. Note that $(\mathcal{M}\overline{\otimes}\mathcal{R})^\omega \cong \mathcal{M}^\omega \cong \mathcal{R}^\omega$. Thus we may view φ as an embedding from $\mathcal{M}\overline{\otimes}\mathcal{M}$ into $\mathcal{R}^\omega$. From our assumption, there is a unitary U in $\mathcal{R}^\omega$ such that $X \otimes I = U^*(I \otimes Z)U$. Since X is a generator for $\mathcal{M}$, from [3] we know that $\mathcal{M} \cong \mathcal{R}$. ∎

6. Embedding Into the Ultrapower of the Hyperfinite II_1 Factor

Connes asks whether every factor of type II_1 with a separable predual can be embedded into the ultrapower of the hyperfinite II_1 factor. Suppose a factor $\mathcal{M}$ can be embedded into $\mathcal{R}^\omega$ as a subfactor. It is also interesting to know when two embeddings are equivalent. The following theorem answers this question for a hyperfinite von Neumann algebra. It is a consequence of the fact that two embeddings of a finite dimensional algebra into $\mathcal{R}$ are approximately equivalent.

Theorem 6.1. *Suppose $\mathcal{R}_1$ is a finite hyperfinite von Neumann algebra with a separable predual and Φ, Ψ are two embeddings of $\mathcal{R}_1$ into $\mathcal{R}^\omega$. Then there is a unitary element U in $\mathcal{R}^\omega$ such that $U^*\Phi(X)U = \Psi(X)$, for all X in $\mathcal{R}_1$.*

Popa [14] showed that $\mathcal{R}^\omega$ does not have a Cartan subalgebra. Similar method implies that $\mathcal{R}_\omega$ has the same property. We state the result in the following corollary and omit its proof here.

Corollary 6.2. *The central sequence algebra $\mathcal{R}_\omega$ of $\mathcal{R}$ has no Cartan subalgebras.*

The central sequence algebra $\mathcal{R}_\omega$ of $\mathcal{R}$ is clearly not hyperfinite. It shares many similar properties to those of $\mathcal{R}^\omega$. We do not know if they are isomorphic to each other. But one can easily prove the following result.

Theorem 6.3. *The ultrapower $\mathcal{R}^\omega$ can be embedded into $\mathcal{R}_\omega$.*

Proof. Since $\mathcal{R} \cong \otimes_1^\infty \mathcal{R}$, we shall show that $\mathcal{R}^\omega$ can be embedded into $(\otimes_1^\infty \mathcal{R})_\omega$. For any $X = \{X_n\}$ in $\mathcal{R}^\omega$ with X_n in $\mathcal{R}$, define $\varphi(X)$ to be an element in $(\otimes_1^\infty \mathcal{R})^\omega$ corresponding to the sequence $X_1 \otimes I \otimes I \otimes \cdots, I \otimes X_2 \otimes I \otimes \cdots, \ldots$ in $(\otimes_1^\infty \mathcal{R})^\omega$. It is easy to see that $\varphi(X)$ is a central sequence and thus φ induces an embedding from $\mathcal{R}^\omega$ into $(\otimes_1^\infty \mathcal{R})_\omega$. ■

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Junsheng Fang
Department of Mathematics,
University of New Hampshire,
Durham, NH 03824,
U.S.A.

Liming Ge and Weihua Li
Academy of Mathematics and System Science,
Chinese Academy of Science,
Beijing 100080,
P. R. China.