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EXISTENCE THEOREM OF IMPLICIT QUASIVARIATIONAL INEQUALITIES WITHOUT CONTINUITIES

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Abstract. This paper is to establish an existence result (Theorem 3.1) for the implicit quasivariational inequality without continuity assumptions in infinite-dimensional normed spaces.

1. Introduction

Let X and C be nonempty subsets of $\mathbf{R^n}$ and $\mathbf{R^m}$ respectively, $\Gamma : X \to 2^X$ and $\Phi : X \to 2^C$ two multifunctions, $\psi : X \times C \times X \to \mathbf{R}$ a single-valued map. The *implicit quasivariational inequality* is to find $(\hat{x}, \hat{z}) \in X \times C$ such that $\hat{x} \in \Gamma(\hat{x})$, $\hat{z} \in \Phi(\hat{x})$ and

$$\psi(\hat{x}, \hat{z}, y) \le 0, \quad \text{for all } y \in \Gamma(\hat{x}).$$

The above implicit quasivariational inequality covers the classical variational inequality problem and most of generalized problems of the classical variational inequalities as special cases. See, e.g., [10, 15-18, and the references there]. As a special case of the implicit quasivariational inequality, the quasivariational inequality problem was first introduced by Yao in [13]. It is remarkable that a great deal of finite-dimensional results to the quasivariational inequality problem have been found under continuity assumptions (see, e.g., [9, 13, 14]). Recently, the case involving discontinuity functions has come to many authors' attention and some interesting results have been obtained (see, e.g., [2, 5, 17]).

In [5], Cubiotti and Yao studied the implicit quasivariational inequality without assuming continuity of data mappings and gave some applications to generalized

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quasivariational inequalities with discontinuous fuzzy mappings. Their main existence result is the following [5, Theorem 3.2].

Theorem 1.1. *Let X be a nonempty compact convex subset of $\mathbf{R}^{\mathbf{n}}$, C a nonempty subset of $\mathbf{R}^{\mathbf{m}}$, $\Gamma : X \to 2^X$ and $\Phi : X \to 2^C$ two multifunctions, $\psi : X \times C \times X \to \mathbf{R}$ a single-valued map. Assume that:*

(i) Γ *is lower semicontinuous with nonempty convex values;*

(ii) *the set $E = \{x \in X : x \in \Gamma(x)\}$ is closed;*

(iii) $\operatorname{aff}(\Gamma(x)) = \operatorname{aff}(X)$*, for all $x \in E$;*

(iv) $\Phi(x)$ *is nonempty and compact for $x \in X$ and convex for $x \in E$;*

(v) *for each $y \in X$ the set $\{x \in E : \inf_{z\in\Phi(x)} \psi(x, z, y) \leq 0\}$ is closed;*

(vi) *for each $x \in E$ the set $\{y \in X : \inf_{z\in\Phi(x)} \psi(x, z, y) \leq 0\}$ is closed;*

(vii) *for each $x \in E$ one has $\inf_{z\in\Phi(x)} \psi(x, z, x) \leq 0$;*

($viii$) *for each $x \in E$ and each $z \in \Phi(x)$ the function $\psi(x, z, \cdot)$ is concave on $\Gamma(x)$;*

(ix) *for each $x \in E$ and each $y \in \Gamma(x)$ the function $\psi(x, \cdot, y)$ is lower semicontinuous (in the sense of single-valued maps) and convex on $\Phi(x)$.*

Then there exists $(\hat{x}, \hat{z}) \in X \times C$ such that $\hat{x} \in \Gamma(\hat{x})$, $\hat{z} \in \Phi(\hat{x})$ and

$$\psi(\hat{x}, \hat{z}, y) \leq 0 \quad \text{for all } y \in \Gamma(\hat{x}).$$

The purpose of this paper is to establish an existence result for the implicit quasivariational inequality without continuity assumptions in infinite-dimensional normed spaces. The approach is based on Theorem 1.1 and the proof of Theorem 1.2 in [7].

2. Preliminaries

We recall that if S and V are topological spaces and if $\Phi : S \to 2^V$ is a multifunction, then Φ is said to be *lower semicontinuous* at $x \in S$ if for each open set $\Omega \subset V$ with $\Phi(x) \cap \Omega \neq \emptyset$, the set

$$\Phi^-(\Omega) := \{y \in S : \Phi(y) \cap \Omega \neq \emptyset\}$$

is a neighborhood of x in S. We say that Φ is *lower semicontinuous in S* if it is lower semicontinuous at each point of S. We say that Φ has *open lower sections* if for each $y \in V$, the set $\Phi^-(\{y\})$ is open in S. If Φ has open lower sections and A is any subset of V, then the multifunction $\Phi_A : S \to 2^V$ defined by $\Phi_A(x) = \Phi(x) \cap A$ is lower semicontinuous in S.

Let $(N, \|\cdot\|_N)$ be a real normed space. A multifunction $\Phi : S \to 2^N$ is said to be *Hausdorff lower semicontinuous* at $x \in S$ if given $\epsilon > 0$ there exists a neighborhood U of x in S such that

$$\Phi(x) \subset \Phi(y) + B(0, \epsilon), \qquad \text{for all } y \in U,$$

where $B(0, \epsilon)$ denotes an open ball in N centered at 0 with radius ϵ. We say that Φ is *Hausdorff lower semicontinuous in S* if it is Hausdorff lower semicontinuous at each point of S. In particular, Hausdorff lower semicontinuity implies lower semicontinuity and the converse is true if each set $\Phi(x)$ is nonempty and compact; see [11, Theorem 7.1.14].

For $x \in N$ and $r > 0$, let

$$B(x, r) = \{y \in N : \|y - x\| < r\},$$

$$\overline{B}(x, r) = \{y \in N : \|y - x\| \leq r\}.$$

Let $A \subset N$ be nonempty. The *closed convex hull* of A is denoted by $\overline{\mathrm{co}}(A)$ and the *affine hull* of A is denoted by $\mathrm{aff}(A)$, i.e.,

$$\mathrm{aff}(A) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \in \mathbf{N}, x_i \in A, \lambda_i \in \mathbf{R}, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

A subset $M \subset N$ is called an *affine manifold* if there exist $x \in N$ and a linear subspace H of N such that $M = x + H$. It is known that the set $\mathrm{aff}(A)$ is the smallest affine manifold containing A. If $A \subset E \subset N$, we will denote the interior of A in E by $\mathrm{int}_E(A)$. Recall that if A is a nonempty finite-dimensional convex set, then $\mathrm{int}_{\mathrm{aff}(A)}(A) \neq \emptyset$.

The following results will be used in the proof of Theorem 3.1.

Proposition 2.1. *Let X be a topological space, $(E, \|\cdot\|)$ a real normed space, and $\phi : X \to 2^E$ a multifunction. If ϕ is Hausdorff lower semicontinuous, then its closure $\overline{\phi}$, defined by $\overline{\phi}(x) = \overline{\phi(x)}$, is Hausdorff lower semicontinuous.*

Proof. Let $x_0 \in X$. Given $\epsilon > 0$ there exists a neighborhood U of x_0 in X such that

$$\phi(x_0) \subset \phi(x) + B(0, \epsilon/2), \qquad \text{for all } x \in U.$$

Let $y \in \overline{\phi(x) + B(0, \epsilon/2)}$. Then there exists $z \in \phi(x) + B(0, \epsilon/2)$ such that $\|y - z\| < \epsilon/2$. Hence $y - z \in B(0, \epsilon/2)$, and so $y \in \phi(x) + B(0, \epsilon/2) + B(0, \epsilon/2) = \phi(x) + B(0, \epsilon)$. We have

$$\overline{\phi(x) + B(0, \epsilon/2)} \subset \phi(x) + B(0, \epsilon)$$

from which it follows that

$$\overline{\phi(x_0)} \subset \overline{\phi(x) + B(0, \epsilon/2)} \subset \phi(x) + B(0, \epsilon) \subset \overline{\phi(x)} + B(0, \epsilon), \quad \text{for all } x \in U.$$

Hence $\overline{\phi}$ is Hausdorff lower semicontinuous. ∎

Proposition 2.2. *Let X be a topological space, $(E, \|\cdot\|)$ a real normed space, and M an affine manifold of E. Suppose that $\phi : X \to 2^M$ is a Hausdorff lower semicontinuous multifunction such that $\phi(x)$ is a convex set with nonempty interior, for all $x \in X$. Then for any $x_0 \in X$ and $y_0 \in \mathrm{int}_M \phi(x_0)$, there exists a neighborhood U of x_0 in X such that*

$$y_0 \in \mathrm{int}_M \phi(x), \quad \text{for all } x \in U.$$

Proof. Since ϕ is Hausdorff lower semicontinuous, it follows from Proposition 2.1 that $\overline{\phi}$ is also Hausdorff lower semicontinuous. Notice that for each $x \in X$, $\phi(x)$ is convex with nonempty interior; hence

$$\mathrm{int}_M \overline{\phi(x)} = \mathrm{int}_M \phi(x), \quad \text{for all } x \in X,$$

by [12, p.38, Ch.II, Theorem 1.3]. For any $x_0 \in X$ and $y_0 \in \mathrm{int}_M \overline{\phi(x_0)}$, apply Proposition 2.4 [3] to $\overline{\phi}$ to choose a neighborhood U of x_0 in X such that

$$y_0 \in \mathrm{int}_M \left(\bigcap_{x \in U} \overline{\phi(x)} \right).$$

Therefore $y_0 \in \mathrm{int}_M \phi(x)$, for all $x \in U$. ∎

3. The Existence Result

The main result is stated and proved as follows.

Theorem 3.1. *Let M and N be real normed spaces. Let X be a nonempty closed convex subset of M, C a nonempty subset of N, $\Gamma : X \to 2^X$ and $\Phi : X \to 2^C$ two multifunctions, $\psi : X \times C \times X \to \mathbf{R}$ a single-valued map. Let K_1 and K_2 be two nonempty compact subsets of X such that $K_1 \subset K_2$, K_1 is finite-dimensional and $\overline{\mathrm{co}} K_2$ is compact. Suppose that the following conditions hold:*

(*i*) Γ *is Hausdorff lower semicontinuous with nonempty convex values.*

(*ii*) *The set $E = \{x \in X : x \in \Gamma(x)\}$ is closed.*

(*iii*) $\Gamma(x) \cap K_1 \neq \emptyset$, *for all $x \in X$.*

(*iv*) $\mathrm{int}_{\mathrm{aff}(X)} \Gamma(x) \neq \emptyset$, *for all $x \in X$.*

(*v*) $\Phi(x)$ *is nonempty and compact for $x \in X$ and convex for $x \in E$.*

(vi) *For any finite-dimensional subset A of X, there is a finite-dimensional linear subspace T of N with the projection map $p : N \to T$ such that $p(C) \subset C$ and $\psi(x, p(z), y) = \psi(x, z, y)$, for all $x, y \in A$ and $z \in \Phi(x)$.*

(vii) *For each $y \in X$ the set $\{x \in E : \inf_{z\in\Phi(x)} \psi(x, z, y) \leq 0\}$ is closed.*

($viii$) *For each $x \in E$ the set $\{y \in X : \inf_{z\in\Phi(x)} \psi(x, z, y) \leq 0\}$ is closed.*

(ix) *For each $x \in E$ one has $\inf_{z\in\Phi(x)} \psi(x, z, x) = 0$.*

(x) *For each $x \in E$ and each $z \in \Phi(x)$ the function $\psi(x, z, \cdot)$ is concave on $\Gamma(x)$.*

(xi) *For each $x \in E$ and each $y \in \Gamma(x)$ the function $\psi(x, \cdot, y)$ is lower semicontinuous (in the sense of single-valued maps) and convex on $\Phi(x)$.*

(xii) *For each $x \in X \setminus K_2$ and each $z \in \Phi(x)$, one has $\sup\limits_{y\in\Gamma(x)\cap K_1} \psi(x, z, y) > 0$.*

Then there exists $(\hat{x}, \hat{z}) \in X \times C$ such that $\hat{x} \in \Gamma(\hat{x})$, $\hat{z} \in \Phi(\hat{x})$ and

$$\psi(\hat{x}, \hat{z}, y) \leq 0 \quad \text{for all } y \in \Gamma(\hat{x}).$$

Proof. First observe that the set E is nonempty from part (c) of this proof. Let $H = \text{aff}(X)$ be the affine hull of X and let H_0 be the linear subspace of M corresponding to H. Assumption (iv) implies that $\text{int}_H\Gamma(x) \neq \emptyset$, for all $x \in X$. For each $a \in \overline{\text{co}}K_2$, choose any point $u_a \in \text{int}_H\Gamma(a)$. It follows from Proposition 2.2 that there exists an open ball V_a centered at a in M such that

$$u_a \in \text{int}_H\Gamma(x), \quad \text{for all } x \in V_a \cap X. \tag{3.1}$$

Since $\overline{\text{co}}K_2$ is compact, there exist $a_1, a_2, \ldots, a_n \in \overline{\text{co}}K_2$ such that

$$\overline{\text{co}}K_2 \subset \bigcup_{i=1}^{n}(V_{a_i} \cap H). \tag{3.2}$$

Let $W_1 = \cup_{i=1}^n(V_{a_i} \cap H)$ so that W_1 is bounded and hence $H \setminus W_1$ is nonempty and closed in H. From (3.2) we have

$$r = \inf\{d(x, H \setminus W_1) : x \in \overline{\text{co}}K_2\} > 0. \tag{3.3}$$

Let

$$W_2 = \overline{\text{co}}K_2 + [\overline{B}(0, r/2) \cap H_0]. \tag{3.4}$$

Then W_2 is convex and closed in H and $W_2 \subset W_1$.

We assume without loss of generality that $K_1 \cup \{u_{a_1}, \ldots, u_{a_n}\} \subset B(0, k)$, for all $k \in \mathbf{N}$. Let $\mathcal{F}$ be the family of all finite-dimensional linear subspaces of M containing the set $K_1 \cup \{u_{a_1}, \ldots, u_{a_n}\}$. Fix $k \in \mathbf{N}$ and $S \in \mathcal{F}$. Let

$Y_k = X \cap B(0,k)$ and consider the set $\overline{Y_k \cap S \cap W_2}$ which is nonempty since $K_1 \subset Y_k \cap S \cap W_2 \subset \overline{Y_k \cap S \cap W_2}$. Define the multifunction $\Gamma_S : \overline{Y_k \cap S \cap W_2} \to 2^{\overline{Y_k \cap S \cap W_2}}$ by

$$\Gamma_S(x) = \Gamma(x) \cap \overline{Y_k \cap S \cap W_2}.$$

Assumption (vi) states that there is a finite-dimensional linear subspace T_S of N with the projection map $p : N \to T_S$ such that $p(C) \subset C$ and $\psi(x, p(z), y) = \psi(x, z, y)$, for all $x, y \in Y_k \cap S$ and $z \in \Phi(x)$. Note that $\overline{Y_k \cap S \cap W_2} \subset Y_k \cap S$. Let the multifunction $\Phi_S : \overline{Y_k \cap S \cap W_2} \to 2^{C \cap T_S}$ be defined by

$$\Phi_S(x) = p(\Phi(x)), \quad \text{for } x \in \overline{Y_k \cap S \cap W_2}.$$

We now consider the finite-dimensional implicit quasivariational inequality problem corresponding to $(\overline{Y_k \cap S \cap W_2}, C \cap T_S, \Gamma_S, \Phi_S, \psi)$ and prove conditions (i) through (ix) in Theorem 1.1 are satisfied.

(a) The set $\overline{Y_k \cap S \cap W_2}$ is a nonempty compact convex subset of S.

(b) To prove the multifunction $\Gamma_S : \overline{Y_k \cap S \cap W_2} \to 2^{\overline{Y_k \cap S \cap W_2}}$ is lower semicontinuous, observe that

$$\text{(3.5)} \qquad \mathrm{int}_H \Gamma(x) \cap Y_k \cap S \cap W_2 \neq \emptyset, \quad \text{for all } x \in \overline{Y_k \cap S \cap W_2};$$

hence Γ_S has nonempty convex values. In fact, let $x \in \overline{Y_k \cap S \cap W_2}$ and choose $c \in Y_k \cap S \cap W_2$ such that $\|x - c\| \leq r/4$. Then $x - c \in \overline{B}(0, r/4) \cap H_0$. We obtain from (3.4) that

$$c \in \overline{\mathrm{co}} K_2 + [\overline{B}(0, r/2) \cap H_0],$$

so (3.3) implies that

$$x \in \overline{\mathrm{co}} K_2 + [\overline{B}(0, 3r/4) \cap H_0] \subset W_1.$$

Thus $x \in V_{a_i}$, for some $1 \leq i \leq n$. Especially (3.1) shows that $u_{a_i} \in \mathrm{int}_H \Gamma(x)$ and hence

$$u_{a_i} \in \mathrm{int}_H \Gamma(x) \cap Y_k \cap S \neq \emptyset.$$

By assumption (iii), $\Gamma(x) \cap K_1 \neq \emptyset$, for all $x \in X$. Fix $v \in \Gamma(x) \cap K_1$. By the convexity of $\Gamma(x)$ we have

$$\text{(3.6)} \qquad v + t(u_{a_i} - v) \in \mathrm{int}_H \Gamma(x) \cap Y_k \cap S, \quad \text{for all } t \in (0, 1].$$

On the other hand, it follows from (3.4) that

$$v + [\overline{B}(0, r/2) \cap H_0] \subset W_2,$$

and so there exists $\sigma \in (0,1]$ such that

$$v + t(u_{a_i} - v) \in W_2, \quad \text{for all } t \in (0, \sigma]. \tag{3.7}$$

Hence we obtain from (3.6) and (3.7) that

$$\operatorname{int}_H \Gamma(x) \cap Y_k \cap S \cap W_2 \neq \emptyset$$

as claimed.

Next let $x_0 \in \overline{Y_k \cap S \cap W_2}$ and let U be an open set in H such that

$$\Gamma_S(x_0) \cap U \neq \emptyset.$$

By (3.5) we can choose a point $v_0 \in \operatorname{int}_H \Gamma(x) \cap Y_k \cap S \cap W_2 \subset \Gamma_S(x_0)$. Fix $v_1 \in \Gamma_S(x_0) \cap U$. The convexity of $\Gamma(x_0)$ assures that

$$v_1 + t(v_0 - v_1) \in \operatorname{int}_H \Gamma(x_0) \cap \overline{Y_k \cap S \cap W_2}, \quad \text{for all } t \in (0,1]. \tag{3.8}$$

Since U is open in H, there exists $\rho > 0$ such that

$$v_1 + [\overline{B}(0,\rho) \cap H_0] \subset U. \tag{3.9}$$

By (3.8) and (3.9), there exists $\mu \in (0,1]$ such that

$$v_1 + \mu(v_0 - v_1) \in \operatorname{int}_H \Gamma(x_0) \cap \overline{Y_k \cap S \cap W_2} \cap U. \tag{3.10}$$

Proposition 2.2 implies that there is an open neighborhood D_{x_0} of x_0 in X such that

$$v_1 + \mu(v_0 - v_1) \in \operatorname{int}_H \Gamma(x), \quad \text{for all } x \in D_{x_0}. \tag{3.11}$$

We obtain from (3.10) and (3.11) that

$$v_1 + \mu(v_0 - v_1) \in \operatorname{int}_H \Gamma(x) \cap \overline{Y_k \cap S \cap W_2} \cap U, \quad \text{for all } x \in D_{x_0}.$$

In particular, $\Gamma_S(x) \cap U \neq \emptyset$, for all $x \in D_{x_0} \cap \overline{Y_k \cap S \cap W_2}$.

(c) Let $E_S = \{x \in \overline{Y_k \cap S \cap W_2} : x \in \Gamma_S(x)\}$ so that it is nonempty by [5, Proposition 3.1]. Also

$$\begin{aligned} E_S &= \{x \in \overline{Y_k \cap S \cap W_2} : x \in \Gamma_S(x)\} = \{x \in X : x \in \Gamma(x)\} \cap \overline{Y_k \cap S \cap W_2} \\ &= E \cap \overline{Y_k \cap S \cap W_2} \end{aligned}$$

is closed by assumption (ii).

(d) To prove

$$\operatorname{aff}(\Gamma_S(x)) = \operatorname{aff}(\overline{Y_k \cap S \cap W_2}), \quad \text{for all } x \in E_S,$$

fix $x \in E_S$. Since the set $\operatorname{int}_H \Gamma(x) \cap S$ is open in S, and since, by (3.5),

$$\begin{aligned} \emptyset \quad & \neq \operatorname{int}_H \Gamma(x) \cap Y_k \cap S \cap W_2 \\ & \subset (\operatorname{int}_H \Gamma(x) \cap S) \cap \overline{Y_k \cap S \cap W_2} = \operatorname{int}_H \Gamma(x) \cap \overline{Y_k \cap S \cap W_2} \\ & \subset \Gamma_S(x) \subset \overline{Y_k \cap S \cap W_2}, \end{aligned}$$

it follows from [4, Proposition 2.1] that

$$\operatorname{aff}(\overline{Y_k \cap S \cap W_2}) = \operatorname{aff}(\operatorname{int}_H \Gamma(x) \cap \overline{Y_k \cap S \cap W_2}) \subset \operatorname{aff}(\Gamma_S(x)) \subset \operatorname{aff}(\overline{Y_k \cap S \cap W_2}).$$

Therefore

$$\operatorname{aff}(\Gamma_S(x)) = \operatorname{aff}(\overline{Y_k \cap S \cap W_2}).$$

(e) It follows directly from assumption (v) and the definition of E_S that $\Phi_S(x)$ is nonempty and compact for $x \in \overline{Y_k \cap S \cap W_2}$ and convex for $x \in E_S$. Moreover, assumption (vi) implies that each $\Phi_S(x)$ is a finite-dimensional subset of C, for all $x \in \overline{Y_k \cap S \cap W_2}$.

(f) For each $y \in \overline{Y_k \cap S \cap W_2}$, assumption (vii) shows that the set

$$\begin{aligned} \{x \in E_S : \inf_{\bar{z} \in \Phi_S(x)} \psi(x, \bar{z}, y) \leq 0\} &= \{x \in \overline{Y_k \cap S \cap W_2} : \inf_{\bar{z} \in \Phi_S(x)} \psi(x, \bar{z}, y) \leq 0\} \cap E \\ &= \{x \in \overline{Y_k \cap S \cap W_2} : \inf_{z \in \Phi(x)} \psi(x, z, y) \leq 0\} \cap E \\ &= \{x \in E : \inf_{z \in \Phi(x)} \psi(x, z, y) \leq 0\} \cap \overline{Y_k \cap S \cap W_2} \end{aligned}$$

is closed.

(g) For each $x \in E_S$, assumption (viii) implies that the set

$$\begin{aligned} & \{y \in \overline{Y_k \cap S \cap W_2} : \inf_{\bar{z} \in \Phi_S(x)} \psi(x, \bar{z}, y) \leq 0\} \\ = \ & \{y \in \overline{Y_k \cap S \cap W_2} : \inf_{z \in \Phi(x)} \psi(x, z, y) \leq 0\} \\ = \ & \{y \in X : \inf_{z \in \Phi(x)} \psi(x, z, y) \leq 0\} \cap \overline{Y_k \cap S \cap W_2} \end{aligned}$$

is closed.

(h) For each $x \in E_S$, assumption (ix) implies that

$$\inf_{\bar{z} \in \Phi_S(x)} \psi(x, \bar{z}, x) = \inf_{z \in \Phi(x)} \psi(x, z, x) \leq 0.$$

(i) Let $x \in E_S$ and $\bar{z} = p(z) \in \Phi_S(x)$. For any $y_1, y_2 \in \Gamma_S(x) = \Gamma(x) \cap \overline{Y_k \cap S \cap W_2}$ and $t \in [0, 1]$, assumption (x) implies that

$$\begin{aligned} \psi(x, \bar{z}, ty_1 + (1-t)y_2) &= \psi(x, z, ty_1 + (1-t)y_2) \\ &\geq t\psi(x, z, y_1) + (1-t)\psi(x, z, y_2) \\ &= t\psi(x, \bar{z}, y_1) + (1-t)\psi(x, \bar{z}, y_2). \end{aligned}$$

Hence the function $\psi(x, \bar{z}, \cdot)$ is concave on $\Gamma_S(x)$.

(j) For each $x \in E_S$ and each $y \in \Gamma_S(x)$, assumption (xi) implies that the function $\psi(x, \cdot, y)$ is lower semicontinuous and convex on $\Phi(x)$. Thus it follows from the definition of Φ_S that the function $\psi(x, \cdot, y)$ is lower semicontinuous and convex on $\Phi_S(x)$.

Therefore, by Theorem 1.1 there exists $(x_S, \bar{z}_S) \in \left(\overline{Y_k \cap S \cap W_2}\right) \times (C \cap T_S)$ such that $x_S \in \Gamma_S(x_S)$, $\bar{z}_S \in \Phi_S(x_S)$ and

$$\psi(x_S, \bar{z}_S, y) \leq 0, \quad \text{for all } y \in \Gamma_S(x_S).$$

Let $z_S \in \Phi(x_S)$ such that $\bar{z}_S = p(z_S)$. Then we conclude that $x_S \in E$ and

$$\psi(x_S, z_S, y) \leq 0, \quad \text{for all } y \in \Gamma_S(x_S). \tag{3.12}$$

It is also immediate from assumption (ix) that $\psi(x_S, z_S, x_S) \geq 0$ and hence $\psi(x_S, z_S, x_S) = 0$ by (3.12). Moreover, $x_S \in K_2$ for all $S \in \mathcal{F}$ by assumption (xii). We shall prove that

$$\psi(x_S, z_S, y) \leq 0, \quad \text{for all } y \in \Gamma(x_S) \cap Y_k \cap S.$$

Let $y \in \Gamma(x_S) \cap Y_k \cap S$. Notice that

$$x_S \in K_2 \cap Y_k \subset \overline{\text{co}} K_2 \cap Y_k \subset Y_k \subset H$$

and

$$y \in \Gamma(x_S) \cap Y_k \subset Y_k \subset H.$$

Since Y_k is convex and $H - H \subset H_0$, there is a sufficiently small number $t \in (0, 1)$ such that

$$x_S + t(y - x_S) \in Y_k \cap \left[\overline{\text{co}} K_2 + \left(\overline{B}(0, r/2) \cap H_0\right)\right] = Y_k \cap W_2.$$

Moreover, since $x_S \in \Gamma(x_S) \cap Y_k \cap S$ and $\Gamma(x_S)$ is convex, we have

$$x_S + t(y - x_S) \in \Gamma(x_S) \cap Y_k \cap S \cap W_2 \subset \Gamma(x_S) \cap \overline{Y_k \cap S \cap W_2} = \Gamma_S(x_S).$$

We obtain from (3.12) and assumptions (ix), (x) that

$$\begin{aligned} 0 \quad &\geq \psi(x_S, z_S, x_S + t(y - x_S)) \\ &\geq t\psi(x_S, z_S, y) + (1-t)\psi(x_S, z_S, x_S) = t\psi(x_S, z_S, y), \end{aligned}$$

so $\psi(x_S, z_S, y) \leq 0$ as desired. Consequently, given any fixed $k \in \mathbf{N}$, for each $S \in \mathcal{F}$, there exist $x_S \in \overline{Y_k \cap S \cap W_2}$ and $z_S \in \Phi(x_S)$ such that $x_S \in \Gamma_S(x_S)$ and

(3.13) $$\psi(x_S, z_S, y) \leq 0, \quad \text{for all } y \in \Gamma(x_S) \cap Y_k \cap S.$$

Now we fix $k \in \mathbf{N}$ and consider the net $\{x_S : S \in \mathcal{F}\}$ with $\mathcal{F}$ ordered by the set inclusion $\subset$. It follows from the compactness of K_2 that the net $\{x_S : S \in \mathcal{F}\}$ has a cluster point $\hat{x}_k \in K_2$. Since the set E is closed, we have $\hat{x}_k \in E$ and thus $\hat{x}_k \in \Gamma(\hat{x})$. Assumption (iv) states that $\mathrm{int}_H \Gamma(\hat{x}_k) \neq \emptyset$. We next claim that

(3.14) $$\inf_{z \in \Phi(\hat{x}_k)} \psi(\hat{x}_k, z, y) \leq 0, \quad \text{for all } y \in \mathrm{int}_H \Gamma(\hat{x}_k) \cap Y_k.$$

On the contrary, assume that there exists $y_0 \in \mathrm{int}_H \Gamma(\hat{x}_k) \cap Y_k$ such that

(3.15) $$\inf_{z \in \Phi(\hat{x}_k)} \psi(\hat{x}_k, z, y_0) > 0.$$

By Proposition 2.2 there exists $\epsilon > 0$ such that

(3.16) $$y_0 \in \mathrm{int}_H \Gamma(x), \quad \text{for all } x \in B(\hat{x}_k, \epsilon) \cap X.$$

It is seen from (3.15) and assumption (vii) that there exists a positive number $\alpha < \epsilon$ such that

(3.17) $$B(\hat{x}_k, \alpha) \cap X \subset \{x \in E : \inf_{z \in \Phi(x)} \psi(x, z, y_0) > 0\}.$$

By construction there exists $S_0 \in \mathcal{F}$ such that $y_0 \in S_0$ and $x_{S_0} \in B(\hat{x}_k, \alpha)$. Then we have $y_0 \in \mathrm{int}_H \Gamma(x_{S_0}) \cap Y_k \cap S_0$ by (3.16). Therefore (3.13) implies that

(3.18) $$\psi(x_{S_0}, z_{S_0}, y_0) \leq 0.$$

However, (3.17) shows that

$$\inf_{z \in \Phi(x_{S_0})} \psi(x_{S_0}, z, y_0) > 0.$$

In particular,

$$\psi(x_{S_0}, z_{S_0}, y_0) > 0$$

which contradicts (3.18). Hence (3.14) holds.

Next consider the sequence $\{\hat{x}_k\}$ of points in K_2. By the compactness of K_2 there exists a subsequence of $\{\hat{x}_k\}$, still denoted by $\{\hat{x}_k\}$, which converges to a point $\hat{x}$ of K_2. We will prove that

$$\inf_{z\in\Phi(\hat{x})} \psi(\hat{x}, z, y) \le 0, \quad \text{for all } y \in \mathrm{int}_H\Gamma(\hat{x}). \tag{3.19}$$

Suppose on the contrary that there exists $y_1 \in \mathrm{int}_H\Gamma(\hat{x})$ such that

$$\inf_{z\in\Phi(\hat{x})} \psi(\hat{x}, z, y_1) > 0. \tag{3.20}$$

Again, by Proposition 2.2 there exists $\delta > 0$ such that

$$y_1 \in \mathrm{int}_H\Gamma(x), \quad \text{for all } x \in B(\hat{x}, \delta) \cap X. \tag{3.21}$$

By (3.20) and assumption (vii) there exists a positive number $\beta < \delta$ such that

$$B(\hat{x}, \beta) \cap X \subset \{x \in E : \inf_{z\in\Phi(x)} \psi(x, z, y_1) > 0\}. \tag{3.22}$$

Choose an integer k such that $\hat{x}_k \in B(\hat{x}, \beta)$ and $y_1 \in Y_k$. It follows from (3.21) that

$$y_1 \in \mathrm{int}_H\Gamma(\hat{x}_k) \cap Y_k;$$

hence by (3.14) we have

$$\inf_{z\in\Phi(\hat{x}_k)} \psi(\hat{x}_k, z, y_1) \le 0.$$

However, (3.22) implies that

$$\inf_{z\in\Phi(\hat{x}_k)} \psi(\hat{x}_k, z, y_1) > 0,$$

a contradiction. Consequently, (3.19) holds. Therefore

$$\sup_{y\in\mathrm{int}_H\Gamma(\hat{x})} \inf_{z\in\Phi(\hat{x})} \psi(\hat{x}, z, y) \le 0.$$

As the supremum of a family of lower semicontinuous functions on $\Phi(\hat{x})$, the function $z \to \sup_{y\in\mathrm{int}_H\Gamma(\hat{x})} \psi(\hat{x}, z, y)$ is lower semicontinuous on the compact set $\Phi(\hat{x})$, so there exists $\hat{z} \in \Phi(\hat{x})$ such that

$$\sup_{y\in\mathrm{int}_H\Gamma(\hat{x})} \psi(\hat{x}, \hat{z}, y) = \inf_{z\in\Phi(\hat{x})} \sup_{y\in\mathrm{int}_H\Gamma(\hat{x})} \psi(\hat{x}, z, y). \tag{3.23}$$

Applying [8, Theorem 2]fan and assumptions (v), (x) and (xi), it follows that

$$\inf_{z\in\Phi(\hat{x})} \sup_{y\in \mathrm{int}_H\Gamma(\hat{x})} \psi(\hat{x}, z, y) = \sup_{y\in \mathrm{int}_H\Gamma(\hat{x})} \inf_{z\in\Phi(\hat{x})} \psi(\hat{x}, z, y). \tag{3.24}$$

Hence (3.23) and (3.24) imply that

$$\sup_{y\in \mathrm{int}_H\Gamma(\hat{x})} \psi(\hat{x}, \hat{z}, y) \leq 0.$$

Let $y \in \Gamma(\hat{x})$. Choose a point $w \in \mathrm{int}_H\Gamma(\hat{x})$. We infer from the convexity of $\Gamma(\hat{x})$ that

$$tw + (1-t)y \in \mathrm{int}_H\Gamma(\hat{x}), \qquad \text{for all } t \in (0, 1].$$

Since the function $\psi(\hat{x}, \hat{z}, \cdot)$ is concave on $\Gamma(\hat{x})$, we have

$$t\psi(\hat{x}, \hat{z}, w) + (1-t)\psi(\hat{x}, \hat{z}, y) \leq \psi(\hat{x}, \hat{z}, tw + (1-t)y) \leq 0, \qquad \text{for all } t \in (0, 1];$$

hence $\psi(\hat{x}, \hat{z}, y) \leq 0$ by letting t approach 0. Therefore

$$\sup_{y\in\Gamma(\hat{x})} \psi(\hat{x}, \hat{z}, y) \leq 0.$$

This completes the proof. ■

Remark. (a) The reader may notice that the set $\overline{\mathrm{co}}(K_2)$ is compact when M is a Banach space; see [1, Theorem,p. 174].

(b) If N is a finite-dimensional space, then condition (vi) of Theorem 3.1 is satisfied by letting $T = N$ and p the identity map.

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