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BOREL MAPS IN REAL REDUCTION THEORY

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Abstract. In [5], we gave a real reduction theory. It is the real analogue of J.von Neumann's (complex) reduction theory. In [4], E.G.Effros gave a natural explanation for (complex) reduction theory by Borel maps. In this note, we also use Borel maps to give an explanation for real measurable fields of Hilbert spaces, von Neumann algebras, and etc.

1. Introduction

The reduction theory of von Neumann algebras (VN algebra, in short) was set up in 1949 ([1]). Its main aim is to reduce the study of general VN algebras to the study of simpler VN algebras. Let N be a VN algebra in a Hilbert space K. Then there are a suitable Borel space $(E, \mathcal{B})$, a measurable field $K(\cdot)$ of Hilbert spaces on $(E, \mathcal{B})$, and a measurable field $N(\cdot)$ of VN algebras in $K(\cdot)$ such that

$$K \cong \int_E^{\oplus} K(t)d\mu(t), \ N \cong \int_E^{\oplus} N(t)d\mu(t),$$

where $\mu(\cdot)$ is a Borel measure on $(E, \mathcal{B})$. In particular, if K is separable, then

$$N \cong \int_{\mathbb{R}}^{\oplus} N(t)d\mu(t),$$

where $\mu(\cdot)$ is a finite Borel measure with compact support on $\mathbb{R}$, and $N(t)$ is a factor, a.e.μ. Thus, the study of VN algebras in a separable Hilbert space can be reduced to the study of factors. Of course, the concepts of measurable fields of Hilbert spaces, measurable fields of VN algebras, the direct integral $\int_E^{\oplus} N(t)d\mu(t)$,

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and etc., are very complicated, and seems artificial ([2, 3]). E.G.Effros ([4]) gave a natural explanation by Borel maps.

J. von Neumann's reduction theory is over complex field $\mathbb{C}$. In [5, 6], we gave a satisfactory real reduction theory: for a real VN algebra M in a real Hilbert space H, there are a suitable Borel bar space $(E, \mathcal{B}, -)$, a real measurable field $H(\cdot)$ of Hilbert spaces on $(E, \mathcal{B}, -)$, and a real measurable field $M(\cdot)$ of VN algebras in $H(\cdot)$ such that

$$H \cong \int_{(E,-)}^{\oplus} H(t)d\nu(t), \ M \cong \int_{(E,-)}^{\oplus} M(t)d\nu(t),$$

where $\nu(\cdot)$ is a Borel measure on $(E, \mathcal{B})$ and $\nu \circ - = \nu$. In particular, if H is separable, then

$$M \cong \int_{(\mathbb{C},-)}^{\oplus} M(t)d\nu(t),$$

where " $-$ " is the complex congugation in $\mathbb{C}$, $\nu(\cdot)$ is a finite Borel measure with compact support on $\mathbb{C}$ and $\nu \circ - = \nu$, and $M(t)$ is a factor, a.e. ν.

Similarily, the concepts of real measurable fields of Hilbert spaces, real measurable fields of VN algebras, the direct integral $\int_{(E,-)}^{\oplus} M(t)d\nu(t)$, and etc., are very complicated, and seems artifical ([5, 6]).

In this note, we also use Borel maps to give a natural explanation for these concepts.

Let H be a separable real Hilbert space, and let $H_c = H \dot{+} iH$ be its complexification (a separable complex Hilbert space [6]). Then we have bar " $-$ " operation on H_c, i.e.,

$$\overline{\xi + i\eta} = \xi - i\eta, \quad \forall \xi, \eta \in H.$$

Let $W(H)$ and $W(H_c)$ be the collections of all closed linear subspaces of H and H_c respectively. In $W(H)$ (or $W(H_c)$), there is a Borel structure generated by all subsets of $W(H)$(or $W(H_c)$) with the following form:

$$\{E \in W(H) : \ \|\xi + E\| < \lambda\}, \text{ where } \xi \in H \text{ and } \lambda > 0$$

$$(\text{or } \{E_c \in W(H_c) : \ \|\xi_c + E_c\| < \lambda\}, \text{ where } \xi_c \in H_c \text{ and } \lambda > 0, [2]).$$

Proposition 1. *We have the following statements:*

(1) $W(H)$ *and* $W(H_c)$ *are standard Borel spaces;*

(2) $E \longrightarrow E^{\perp}$ *(or* $E_c \longrightarrow E_c^{\perp}$*),* $\forall E \in W(H)$ *(or* $E_c \in W(H_c)$*), is a Borel isomorphism on* $W(H)$ *(or* $W(H_c)$*);*

(3) $(W(H_c), -)$ *is a (standard) Borel bar space, i.e.,* "$-$" *is a Borel isomorphism on* $W(H_c)$ *and* $-^2 = id$*, where* $\overline{E}_c = \{\overline{\xi}_c : \ \xi_c \in E_c\}, \forall E_c \in W(H_c)$*;*

(4) *$E_c \in W(H_c)$ is said to be normal, if there is $E \in W(H)$ such that $E_c = E \dot{+} iE$. Then $E_c \in W(H_c)$ is normal, if and only if, $\overline{E}_c = E_c$.*

Moreover, $W(H_c)_N = \{E_c \in W(H_c) : E_c$ is normal$\}$ is a Borel subset of $W(H_c)$, and also is a standard Borel space by the inductive Borel structure;

(5) *$E \longrightarrow E_c = E \dot{+} iE$, $\forall E \in W(H)$, is a Borel isomorphism from $W(H)$ onto $W(H_c)_N$.*

Proof. (1) and (2) are just and similar to Proposition 11.1.8 in [2].

(3) By

$$\{E_c \in W(H_c) : \|\xi_c + \overline{E}_c\| < \lambda\} = \{E_c \in W(H_c) : \|\overline{\xi}_c + E_c\| < \lambda\}, \forall \xi_c \in H_c, \lambda > 0,$$

the conclusion is obvious.

(4) Clearly, $E_c (\in W(H_c))$ is normal, if and only, $\overline{E}_c = E_c$. Moreover, by Proposition 10.3.4 in [2] and (3), $W(H_c)_N$ is a Borel subset of $W(H_c)$. Further, by Proposition 10.3.15 in [2], $W(H_c)_N$ with the inductive Borel structure is a standard Borel space.

(5) For $\xi \in H, \lambda > 0$, under the map $E \longrightarrow E_c = E \dot{+} iE$, the Borel subset $\{E \in W(H) : \ \|\xi + E\| < \lambda\}$ of $W(H)$ becomes the subset

$$\{E_c = E \dot{+} iE \in W(H_c) : \ \|\xi + E\| < \lambda\}$$

of $W(H_c)$. Clearly, $\|\xi + E\| \geq \|\xi + E_c\|$. On the orther hand,

$$\begin{aligned} \|\xi + \xi' + i\eta'\|^2 &= \|\xi + \xi'\|^2 + \|\eta'\|^2 \\ &\geq \|\xi + \xi'\|^2 \geq \|\xi + E\|^2, \end{aligned}$$

$\forall \xi', \eta' \in E$. Thus, $\|\xi + E_c\| \geq \|\xi + E\|$, and $\|\xi + E\| = \|\xi + E_c\|$. It follows that $\{E_c = E \dot{+} iE \in W(H_c) : \ \|\xi + E\| < \lambda\} = \{E_c \in W(H_c)_N : \ \|\xi + E_c\| < \lambda\}$ is a Borel subset of $W(H_c)_N$. Conversely, let $\xi, \eta \in H$ and $\lambda > 0$. Under the map $E_c = E \dot{+} iE \longrightarrow E = H \cap E_c$, the Borel subset $\{E_c \in W(H_c)_N : \ \|\xi + i\eta + E_c\| < \lambda\}$ of $W(H_c)_N$ becomes the subset

$$\{E \in W(H) : \ \|\xi + i\eta + E_c\| < \lambda\}$$

of $W(H)$. Since $\|\xi + i\eta + E_c\|^2 = \|\xi + E\|^2 + \|\eta + E\|^2$, and

$$\begin{aligned} &\{E \in W(H) : \ \|\xi + E\|^2 + \|\eta + E\|^2 < \lambda^2\} \\ = &\bigcup_n (\{E \in W(H) : \ \|\xi + E\| < \lambda_n\} \bigcap \{E \in W(H) : \ \|\eta + E\| < (\lambda^2 - \lambda_n^2)^{\frac{1}{2}}\}), \end{aligned}$$

where $\{\lambda_n\}_n$ is a countably dense subset of $(0,\lambda)$(e.g., all rational numbers in $(0,\lambda)$), it follows that $\{E \in W(H) : \ \|\xi + i\eta + E_c\| < \lambda\}$ is also a Borel subset of $W(H)$. Therefore, $E \longrightarrow E_c = E \dot{+} iE, \forall E \in W(H)$, is a Borel isomorphism from $W(H)$ onto $W(H_c)_N$. ∎

Definition 2. ([6, Ch.9]) Let $(E, \mathcal{B}, -)$ be a Borel bar space, i.e., $(E, \mathcal{B})$ is a Borel space, " $-$ " is a Borel isomorphism on $(E, \mathcal{B})$ and $-^2 = id$.

A real field $H(\cdot)$ of Hilbert spaces on $(E, \mathcal{B}, -)$ means that each $H(t)$ is a real Hilbert space, and $H(\overline{t}) = H(t), \forall t \in E$.

Let $(E, \mathcal{B}, -)$ and $H(\cdot)$ be as above. $\xi(\cdot)$ is called a real field of vectors, if $\xi(t) \in H(t)_c = H(t) \dot{+} iH(t)$, and $\xi(\overline{t}) = \overline{\xi(t)}, \forall t \in E$. $H(\cdot)$ is said to be measurable, if there is a sequence $\{\xi_n(\cdot)\}_n$ of real fields of vectors such that $\{\xi_n(t)\}_n$ is a total subset of $H(t)_c, \forall t \in E$, and

$$t \longrightarrow f_{n,m}(t) = \langle \xi_n(t), \xi_m(t) \rangle_t$$

is a (complex) measurable function on $(E, \mathcal{B})$, where $\langle , \rangle_t$ is the inner product in $H(t)_c, \forall t \in E, \forall n, m$. In this case, a real field $\xi(\cdot)$ of vectors is said to be measurable, if each function

$$t \longrightarrow \langle \xi(t), \xi_n(t) \rangle_t$$

is measurable on $(E, \mathcal{B}), \forall n$.

Let H be a fixed separable real Hilbert space. Then H becomes the constant field on $(E, \mathcal{B}, -)$, i.e., $H(t) = H, \forall t \in E$, and a real field $\xi(\cdot)$ of vectors is measurable if $t \longrightarrow \langle \xi(t), \eta \rangle$ is measurable on $(E, \mathcal{B}), \forall \eta \in H$ ([6]).

Proposition 3. *Let $(E, \mathcal{B}, -)$ be a Borel bar space, let $H(\cdot)$ be a real field of Hilbert spaces on E, and let H be a separable infinite-dimensional real Hilbert space.*

Then $H(\cdot)$ is measurable, if and only if, there is a field of operators $\mathcal{U}(\cdot) : H(\cdot)_c \longrightarrow H_c$ such that for each $t \in E, \mathcal{U}(t)$ is an isometry from $H(t)_c = H(t) \dot{+} iH(t)$ into $H_c = H \dot{+} iH, \mathcal{U}(\overline{t}) = \overline{\mathcal{U}(t)}$, and the map $t \longrightarrow \mathcal{U}(t)H(t)_c$ is a Borel map from $(E, \mathcal{B})$ into $W(H_c)_N$.

Moreover, in this case a real field $\xi(\cdot)$ of vectors is measurable, if and only if, $t \longrightarrow \langle \mathcal{U}(t)\xi(t), \eta \rangle$ is measurable, $\forall \eta \in H$.

Proof. Let $H(\cdot)$ be measurable, let $\{e_n(\cdot)\}_n$ be an orthogonal normalized basis of $H(\cdot)$([6]), and let $\{e_n\}_n$ be an orthogonal normalized basis of H. For any $t \in E$, let

$$\mathcal{U}(t)e_n(t) = \begin{cases} e_n, & \text{if } n \leq \dim H(t), \\ 0, & \text{if } n > \dim H(t). \end{cases}$$

Then $\mathcal{U}(t)$ is an isometry from $H(t)_c$ into H_c, $\forall t \in E$. Since $H(\cdot)_c$ is also measurable on $(E, \mathcal{B})$([6]), it follows from the proof of Proposition 12.1.6 in [2] and $\{e_n\}_n \subset H$ that $t \longrightarrow \mathcal{U}(t)H(t)_c$ is a Borel map from $(E, \mathcal{B})$ into $W(H_c)_N$. Moveover,

$$\begin{aligned} \mathcal{U}(\bar{t}) \sum_n \alpha_n e_n(\bar{t}) &= \sum_n \alpha_n e_n = \overline{\sum_n \overline{\alpha}_n e_n} \\ &= \overline{\mathcal{U}(t) \sum_n \overline{\alpha}_n e_n(t)} = \overline{\mathcal{U}(t)} \sum_n \alpha_n e_n(\bar{t}) \end{aligned}$$

since $e_n(\bar{t}) = \overline{e_n(t)}, \forall t \in E$, where $\alpha_n \in \mathbb{C}$, $\forall n$, and $\sum_n |\alpha_n|^2 < +\infty$. Therefore, $\mathcal{U}(\bar{t}) = \overline{\mathcal{U}(t)}, \forall t \in E$.

In this case, by [6] a real field $\xi(\cdot)$ of vectors is measurable, if and only if, each function

$$t \longrightarrow \langle \xi(t), e_n(t) \rangle_t$$

is measurable on $(E, \mathcal{B}), \forall n$. Since $\mathcal{U}(t)$ is a unitary operator from $H(t)_c$ onto the complex span of $\{e_n : \ n \le \dim H(t)\}$ (a normal subspace of H_c), and $e_n(t) = 0$ if $t > \dim H(t)$, it follows that

$$\begin{aligned} \langle \xi(t), e_n(t) \rangle_t &= \langle \mathcal{U}(t)\xi(t), \mathcal{U}(t) e_n(t) \rangle \\ &= \begin{cases} \langle \mathcal{U}(t)\xi(t), e_n \rangle, & \text{if } n \le \dim H(t), \\ 0, & \text{if } n > \dim H(t), \end{cases} \end{aligned}$$

$\forall t \in E$. Therefore, $\xi(\cdot)$ is measurable, if and only if, $t \longrightarrow \langle \mathcal{U}(t)\xi(t), \eta \rangle$ is measurable on $(E, \mathcal{B}), \forall \eta \in H$.

Conversely, let such $\mathcal{U}(\cdot)$ exist. By Proposition 12.1.6 in [2], the field $H(\cdot)_c$ is measurable, and each field $\xi(\cdot)$ of vectors in $H(\cdot)_c$ is measurable, if and only if,

$$t \longrightarrow \langle \mathcal{U}(t)\xi(t), \eta \rangle$$

is measurable on $(E, \mathcal{B}), \forall \eta \in H$. If $\xi(\cdot)$ is a measurable field of vectors in $H(\cdot)_c$, then by $\mathcal{U}(\bar{t}) = \overline{\mathcal{U}(t)}, \forall t \in E$,

$$\begin{aligned} t \longrightarrow \langle \mathcal{U}(t)\overline{\xi(\bar{t})}, \eta \rangle &= \langle \overline{\mathcal{U}(\bar{t})}\ \overline{\xi(\bar{t})}, \eta \rangle \\ &= \langle \overline{\mathcal{U}(\bar{t})\xi(\bar{t})}, \eta \rangle = \langle \eta, \mathcal{U}(\bar{t})\xi(\bar{t}) \rangle \end{aligned}$$

is still measurable on $(E, \mathcal{B}), \forall \eta \in H$, i.e., $\overline{\xi(\bar{\cdot})}$ is also a measurable field of vectors in $H(\cdot)_c$, and $\frac{1}{2}(\xi(\cdot) + \overline{\xi(\bar{\cdot})})$ is a real measurable field of vectors. Therefore, the real field $H(\cdot)$ is measurable. ∎

Definition 4. Let H be a separable real Hilbert space, and $H_c = H \dot{+} iH$. Let $\mathrm{VN}(H)$ and $\mathrm{VN}(H_c)$ be the collections of all real and complex von Neumann (VN,simply) algebras in H and H_c respectively.

Let $B(H), B(H_c)$ be the collections of all (real, complex)linear bounded operators in H, H_c respectively, and let $T(H), T(H_c)$ be the collections of all trace class operators in H, H_c respectively. Then

$$T(H_c) = T(H) \dot{+} iT(H),\ B(H) = T(H)^*,\ B(H_c) = B(H) \dot{+} iB(H),\ B(H_c) = T(H_c)^*$$

(see [2,6]). For each $M \in \mathrm{VN}(H)$ and $N \in \mathrm{VN}(H_c)$, let

$$M_\perp = \{a \in T(H) :\ \mathrm{tr}(ab) = 0,\ \forall b \in M\}$$

and

$$N_\perp = \{a_c \in T(H_c) :\ \mathrm{tr}(a_c b_c) = 0,\ \forall b_c \in N\}.$$

Then $M \cong T(H)/M_\perp$, and $N \cong T(H_c)/N_\perp$ (see [2,6]).

In $\mathrm{VN}(H)$ (or $\mathrm{VN}(H_c)$), there is a Borel structure generated by all subsets of $\mathrm{VN}(H)$ (or $\mathrm{VN}(H_c)$) with following form:

$$\{M \in \mathrm{VN}(H) :\ \|t + M_\perp\|_1 < \lambda\},\ \text{where } t \in T(H) \text{ and } \lambda > 0$$

$$(\text{or } \{N \in \mathrm{VN}(H_c) :\ \|t_c + N_\perp\|_1 < \lambda\},\ \text{where } t_c \in T(H_c) \text{ and } \lambda > 0, [2]).$$

Proposition 5. *Let H be a separable real Hilbert space, and $H_c = H \dot{+} iH$. Then*

(1) $\mathrm{VN}(H)$ *and* $\mathrm{VN}(H_c)$ *are standard Borel spaces;*

(2) $M \longrightarrow M'$, $\forall M \in \mathrm{VN}(H)$, *is a Borel isomorphism of* $\mathrm{VN}(H)$*;*

(3) $(\mathrm{VN}(H_c), -)$ *is a (standard) Borel bar space, where* $\overline{N} = \{\overline{a} :\ a \in N\}, \forall N \in \mathrm{VN}(H_c)$*;*

(4) $M_c \in \mathrm{VN}(H_c)$ *is said to be normal, if there is a* $M \in \mathrm{VN}(H)$ *such that* $M_c = M \dot{+} iM$. *Then* $M_c \in \mathrm{VN}(H_c)$ *is normal, if and only if,* $\overline{M}_c = M_c$.

Moreover, $\mathrm{VN}(H_c)_N = \{M_c \in \mathrm{VN}(H_c) \colon M_c \text{ is normal}\}$ *is a Borel subset of* $\mathrm{VN}(H_c)$*, and is also a standard Borel space by the inductive Borel structure;*

(5) $M \longrightarrow M_c = M \dot{+} iM, \forall M \in \mathrm{VN}(H)$, *is a Borel isomorphism from* $\mathrm{VN}(H)$ *onto* $\mathrm{VN}(H_c)_N$.

Proof.

(1) It is just and similar to Theorem 11.3.2 in [2].

(2) It is Proposition 11.3.1 and 11.3.5 in [2].

(3) By $\{M_c \in \mathrm{VN}(H_c) : \ \|t_c + (\overline{M}_c)_\perp\|_1 < \lambda\} = \{M_c \in \underline{\mathrm{VN}(H_c) : \ \|\overline{t}_c} + (M_c)_\perp\|_1 < \lambda\}, \forall t_c \in T(H_c), \lambda > 0$ (clearly $(\overline{M}_c)_\perp = \overline{(M_c)_\perp}$), the conclusion is obvious (see Proposition 10.3.2.1) in [2]).

(4) Clearly, $M_c \in \mathrm{VN}(H_c)$ is normal, if and only if, $\overline{M}_c = M_c$.

Now by Proposition 10.3.4.1) in [2], $\mathrm{VN}(H_c)_N$ is a Borel subset of $\mathrm{VN}(H_c)$. Moreover, by Proposition 10.3.15 in [2], $\mathrm{VN}(H_c)_N$ with the inductive Borel structure is a standard Borel space.

(5) Let $M \in \mathrm{VN}(H), M_c = M \dot{+} iM (\in \mathrm{VN}(H_c))$, and let $M_\perp = \{a \in T(H) : \mathrm{tr}(ab) = 0, \forall b \in M\}$, $(M_c)_\perp = \{a_c \in T(H_c) : \ \mathrm{tr}(a_c b_c) = 0, \forall b_c \in M_c\}$. Clearly, $(M_c)_\perp = M_\perp \dot{+} iM_\perp$. If $a \in T(H)$, then clearly

$$\|a + M_\perp\|_1 \geq \|a + (M_c)_\perp\|_1.$$

On the other hand,

$$\begin{aligned} \|a+b\|_1 &\leq \frac{1}{2}(\|a+b+ic\|_1 + \|a+b-ic\|_1) \\ &= \|a+b+ic\|_1, \quad \forall b, c \in M_\perp. \end{aligned}$$

Thus

$$\|a + M_\perp\|_1 \leq \|a + (M_c)_\perp\|_1 \text{ and } \|a + M_\perp\|_1 = \|a + (M_c)_\perp\|_1,$$

$\forall a \in T(H), M \in \mathrm{VN}(H)$.

By Proposition 10.3.2.1) and Theorem 10.3.2 in [2], the map $M_c = M \dot{+} iM \longrightarrow M$ is Borel from $\mathrm{VN}(H_c)_N$ onto $\mathrm{VN}(H)$. Further, by Theorem 10.3.12 in [2], $M \longrightarrow M_c = M \dot{+} iM, \forall M \in \mathrm{VN}(H)$, is a Borel isomorphism from $\mathrm{VN}(H)$ onto $\mathrm{VN}(H_c)_N$. ■

Definition 6. ([6]) Let $H(\cdot)$ and $K(\cdot)$ be two real measurable fields of Hilbert spaces on a Borel bar space $(E, \mathcal{B}, -)$. A field $a(\cdot)$ of operators from $H(\cdot)$ to $K(\cdot)$ is said to be real measurable, if $a(t) \in B(H(t)_c, K(t)_c), \forall t \in E,$ and $a(\cdot)\xi(\cdot)$ is real measurable field of vectors in $K(\cdot)$ for each real measurable field $\xi(\cdot)$ of vectors in $H(\cdot)$.

Remark. Let $H(\cdot)$ be a real measurable field of Hilbert spaces on a Borel bar space $(E, \mathcal{B}, -)$, and let H be a separable infinite-dimensional real Hilbert space. If we see H as the constant field on E, then it is obvious that the field $\mathcal{U}(\cdot)$ of operators from $H(\cdot)$ to H in Proposition 3 is real measurable.

Definition 7. ([6]) Let $H(\cdot)$ be a real measurable field of Hilbert spaces on a Borel bar space $(E, \mathcal{B}, -)$. A field $M(\cdot)$ of (complex) VN algebras in $H(\cdot)$

is said to be real measurable, if $M(t)$ is a (complex) VN algebra in $H(t)_c = H(t) \dot{+} iH(t), \forall t \in E$, and there is a sequence $\{a_n(\cdot)\}_n$ of real measurable fields of operators in $H(\cdot)$ such that $M(t)$ is weakly generated by the sequence $\{a_n(t)\}_n$ of operators, $\forall t \in E$.

Theorem 8. *Let $(E, \mathcal{B}, -)$ be a Borel bar space.*

(1) *If H is a separable real Hilbert space, $H_c = H \dot{+} iH$, and $M(\cdot)$ is a field of (complex) VN algebras in the constant field H on $(E, \mathcal{B}, -)$, then $M(\cdot)$ is real measurable, if and only if, $t \longrightarrow M(t)$ is a Borel map from $(E, \mathcal{B})$ into VN(H_c), and $M(\bar{t}) = \overline{M(t)}, \forall t \in E$;*

(2) *Let $H(\cdot)$ be a real measurable field of Hilbert spaces on $(E, \mathcal{B}, -)$, H be a separable infinite-dimensional real Hilbert space, $\mathcal{U}(\cdot) : \ H(\cdot)_c = H(\cdot) \dot{+} iH(\cdot) \longrightarrow H_c = H \dot{+} iH$ be a field of operators as in Proposition 3, and let $M(\cdot)$ be a field of (complex) VN algebras in $H(\cdot)$. Then $M(\cdot)$ is real measurable, if and only if, $t \longrightarrow \widetilde{M}(t)$ is a Borel map from $(E, \mathcal{B})$ into VN(H_c), where $\widetilde{M}(t) = \mathcal{U}(t)M(t)\mathcal{U}(t)^* \bigoplus \mathbb{C}1(t)$(a VN algebra in H_c), and $\mathcal{U}(t)^*$ is the adjoint (or the inverse) of $\mathcal{U}(t)$ as a unitary operator from $H(t)_c$ onto $\mathcal{U}(t)H(t)_c$, and $1(t)$ is the identity operator on $(\mathcal{U}(t)H(t)_c)^\perp$ (the orthogonal part of $\mathcal{U}(t)H(t)_c$ in H_c), $\forall t \in E$, and $M(\bar{t}) = \overline{M(t)}, \forall t \in E$.*

Proof. (1) By Proposition 12.3.2 in [2] and Proposition 9.3.2 in [6], the conclusion is obvious.

(2) For each $t \in E$, since $\mathcal{U}(t)H(t)_c \in W(H_c)_N$ we can write

$$\mathcal{U}(t)H(t)_c = K(t) \dot{+} iK(t),$$

where $K(t)$ is a closed subspace of H (indeed, $K(t) = \mathcal{U}(t)H(t)_c \cap H$). By $\mathcal{U}(\bar{t}) = \overline{\mathcal{U}(t)}$ and $H(\bar{t}) = H(t)$, we have

$$\begin{aligned} K(\bar{t}) = \mathcal{U}(\bar{t})H(\bar{t})_c \cap H &= \overline{\mathcal{U}(t)}\ \overline{H(t)}_c \cap H \\ &= \overline{\mathcal{U}(t)H(t)}_c \cap H = \mathcal{U}(t)H(t)_c \cap H = K(t), \end{aligned}$$

$\forall t \in E$. Therefore, $t \longrightarrow K(t)$ and $t \longrightarrow K(t)^\perp$ (the orthogonal part of $K(t)$ in H) are real fields of Hilbert spaces on $(E, \mathcal{B}, -)$, and $(\mathcal{U}(t)H(t)_c)^\perp = K(t)^\perp \dot{+} iK(t)^\perp, \forall t \in E$. Since any real field $\xi(\cdot)$ of vectors in $H(\cdot)$ is measurable if and only if $t \longrightarrow \langle \mathcal{U}(t)\xi(t), \eta \rangle$ is measurable on $(E, \mathcal{B}), \forall \eta \in H$ (Proposition 3), it follows that the real fields $K(\cdot)$ and $K(\cdot)^\perp$ of Hilbert spaces are measurable (i.e., any real field $\xi'(\cdot)$ of vectors in $K(\cdot)$ or $K(\cdot)^\perp$ is measurable if and only if $t \longrightarrow \langle \xi'(t), \eta \rangle$ is measurable on $(E, \mathcal{B})$, $\forall \eta \in H$).

Clearly, $\mathbb{C}1(\cdot)$ is a real measurable field of (complex) VN algebras in $K(\cdot)^\perp$.

By Proposition 3 and Remark following Definition 6, $M(\cdot)$ is a real measurable field of (complex) VN algebras in $H(\cdot)$ if and only if $\mathcal{U}(\cdot)M(\cdot)\mathcal{U}(\cdot)^*$ is a real measurable field of (complex) VN algebras in $K(\cdot)$. Therefore, $M(\cdot)$ is measurable if and only if $\widetilde{M}(\cdot)$ is measurable. Moreover, clearly $M(\bar{t}) = \overline{M(t)}$ if and only if $\widetilde{M}(\bar{t}) = \overline{\widetilde{M}(t)}, \forall t \in E$.

Now by the conclusion (1), our desired result (2) is obvious. ■

References

1. J. von Neumann, On rings of operators: Reduction theory, *Ann. Math.*, **50** (1949), 401-485.
2. B. R. Li, *Introduction to operator algebras*, World Scientific, Singapore, 1992.
3. M. Takesaki, *Theory of operator algebras, I*, Springer-Verlag, New York, 1979.
4. E. G. Effros, The Borel space of von Neumann algebras on a separable Hilbert space, *Pacific J. Math.*, **15** (1965), 1153-1164.
5. B. R. Li, Real reduction theory, *Sci. in China, Ser. A*, **41** (1998), 574-581.
6. B. R. Li, *Real operator algebras*, World Scientific, Singapore, 2003.

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