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The Minimal Cycles over Brieskorn Complete Intersection Surface Singularities

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Abstract. In this paper, we study a complete intersection surface singularity of Brieskorn type and provide a condition for the coincidence of the fundamental cycle and the minimal cycle on the minimal resolution space.

1. Introduction

Let (X, o) be a germ of a normal complex surface singularity and let $\pi\colon (\widetilde{X}, E) \to (X, o)$ be a resolution, where $E = \pi^{-1}(o)$ denotes the exceptional divisor. Let $E = \bigcup_{i=1}^{r} E_i$ be the irreducible decomposition of E. A formal sum $D = \sum_{i=1}^{r} d_i E_i$ ($d_i \in \mathbb{Z}$) is called a cycle on E. For any effective cycle D on E (i.e., $d_i \geq 0$ for any i), the arithmetic genus $p_a(D)$ of D is defined by $p_a(D) = 1 - \chi(D)$, where $\chi(D) = \dim_{\mathbb{C}} H^0(\widetilde{X}, \mathcal{O}_D) - \dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_D)$ and $\mathcal{O}_D = \mathcal{O}_{\widetilde{X}}/\mathcal{O}_{\widetilde{X}}(-D)$. From Riemann-Roch theorem, we have

$$\chi(D) = -\frac{1}{2}(D^2 + K_{\widetilde{X}} D), \tag{1.1}$$

where $K_{\widetilde{X}}$ is the canonical divisor on $\widetilde{X}$. If B, C are cycles, we have

$$p_a(B + C) = p_a(B) + p_a(C) - 1 + BC. \tag{1.2}$$

The fundamental cycle Z_E is by definition the smallest one among the cycles $F > 0$ such that $FE_i \leq 0$ for every irreducible component E_i of E. The arithmetic genus of Z_E is called the fundamental genus of (X, o) and denoted by $p_f(X, o)$. The minimal cycle A on E is the smallest one among the cycles $D > 0$ such that $p_a(D) = p_a(Z_E)$, $D \leq Z_E$. Clearly, we always have $A \leq Z_E$. It sometimes happens that $A = Z_E$. This equality holds on the minimal resolution for minimal Kulikov singularities (cf. [7]), hypersurface singularities of Brieskorn type with certain conditions (cf. [8]). However, even for a particular class of

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singularities, a more systematic study will be required in order to classify when such a coincidence of important cycles occurs.

In this paper, we consider a germ $(W, o) \subset (\mathbb{C}^m, o)$ of an isolated Brieskorn complete intersection singularity defined by

$$W = \{(x_i) \in \mathbb{C}^m \mid q_{j1}x_1^{a_1} + \cdots + q_{jm}x_m^{a_m} = 0,\ j = 3, \ldots, m\},$$

where $a_i \geq 2$ are integers. By Serre's criterion for normality, (W, o) is a normal surface singularity. Neumann [6] showed that the universal abelian cover of a weighted homogeneous normal surface singularity with rational homology sphere link is a complete intersection singularity of Brieskorn type. The aim of this paper is to give a condition for the coincidence of the fundamental cycle and the minimal cycle over these singularities.

This paper is organized as follows. In Section 2, we mention fundamental facts on cycles over a cyclic quotient singularity, and the minimal cycles over normal surface singularities. In Section 3, we consider the minimal cycles over Brieskorn complete intersection surface singularities and give a condition for the coincidence of the fundamental cycle and the minimal cycle on the minimal resolution space.

2. Preliminaries

Let us first introduce some notations which will be used throughout this paper. For $1 \leq i \leq m$, we define positive integers d_{im}, n_{im} and e_{im} as follows:

$$\begin{aligned} d_{im} &:= \operatorname{lcm}(a_1, \ldots, \widehat{a}_i, \ldots, a_m), \\ n_{im} &:= \frac{a_i}{\gcd(a_i, d_{im})}, \\ e_{im} &:= \frac{d_{im}}{\gcd(a_i, d_{im})}. \end{aligned}$$

(The symbol $\widehat{\ }$ in the definition of d_{im} indicates an omitted term.) In addition, we define integers μ_{im} by the following conditions:

$$e_{im}\mu_{im} + 1 \equiv 0 \pmod{n_{im}}, \quad 0 \leq \mu_{im} < n_{im}.$$

For $1 \leq i \leq m$, we define integers $\widehat{g}$ and $\widehat{g}_i$ as follows:

$$\widehat{g} := \frac{a_1 \cdots a_m}{\operatorname{lcm}(a_1, \ldots, a_m)}, \quad \widehat{g}_i := \frac{a_1 \cdots \widehat{a}_i \cdots a_m}{\operatorname{lcm}(a_1, \ldots, \widehat{a}_i, \ldots, a_m)}.$$

2.1. Cyclic quotient singularities

For any $x \in \mathbb{R}$, we put $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$, and $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$. For integers $c_i \geq 2$, $i = 1, 2, \dots, r$, we put

$$[[c_1, \dots, c_r]] := c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_r}}}.$$

Let n and μ be positive integers that are relatively prime and $\mu < n$. Let ϵ_n denote the primitive n-th root of unity $\exp(2\pi\sqrt{-1}/n)$. Then the singularity of the quotient

$$\mathbb{C}^2 \Big/ \left\langle \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{\mu} \end{pmatrix} \right\rangle$$

is called the cyclic quotient singularity of type $C_{n,\mu}$. A non-singular point is regarded as of type $C_{1,0}$. It is known (cf. [1]) that if $E = \bigcup_{i=1}^{r} E_i$ is the exceptional divisor of the minimal resolution of $C_{n,\mu}$, then $E_i \simeq \mathbb{P}^1$ and the weighted dual graph of E is chain-shaped as in Figure 2.1, where $n/\mu = [[c_1, \dots, c_r]]$.

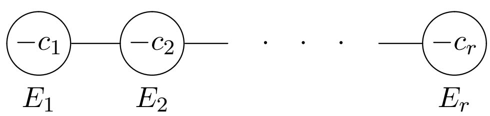


Figure 2.1: The weighted dual graph of $\bigcup_{i=1}^{r} E_i$

Lemma 2.1. [2, Lemma 1.2] *Let $e_i = [[c_i, \dots, c_r]]$. Take a positive integer λ_0 and define the sequence $\{\lambda_i\}_{i=0}^{r}$ by the recurrence formula $\lambda_i = \lceil \lambda_{i-1}/e_i \rceil$ for $1 \leq i \leq r$. Take relatively prime positive integers n_i and μ_i satisfying $n_i/\mu_i = e_i$ for $1 \leq i \leq r$. Put $\lambda_{r+1} := \lambda_r c_r - \lambda_{r-1}$.*

(1) *If $\lambda_{i-1} = \lambda_i c_i - \lambda_{i+1}$ holds for $1 \leq i \leq r$, then $\lambda_1 = (\mu_1\lambda_0 + \lambda_{r+1})/n_1$.*

(2) *If $\lambda_0 \equiv 0 \pmod{n_1}$, then $\lambda_i = \mu_i\lambda_{i-1}/n_i$ for $1 \leq i \leq r$. If $\mu_1\lambda_0 + 1 \equiv 0 \pmod{n_1}$, then $\lambda_i = (\mu_i\lambda_{i-1} + 1)/n_i$ for $1 \leq i \leq r$.*

(3) *If either $\lambda_0 \equiv 0 \pmod{n_1}$ or $\mu_1\lambda_0 + 1 \equiv 0 \pmod{n_1}$, then $\lambda_{i-1} = \lambda_i c_i - \lambda_{i+1}$ holds for $1 \leq i \leq r$. Furthermore, $\lambda_{r+1} = 0$ when $\lambda_0 \equiv 0 \pmod{n_1}$, and $\lambda_{r+1} = 1$ when $\mu_1\lambda_0 + 1 \equiv 0 \pmod{n_1}$.*

(4) *If $\lambda_0 \equiv 0 \pmod{n_1}$, then $\lambda_r = \lambda_0/n_1$. If $\mu_1\lambda_0 + 1 \equiv 0 \pmod{n_1}$, then $\lambda_r = \lceil \lambda_0/n_1 \rceil$.*

Example 2.2. Let $e_1 = [[2,2,2]] = \frac{4}{3}$ and take $\lambda_0 = 4$. Then $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$, $\lambda_4 = 0$ and $e_2 = \frac{3}{2}$, $e_3 = 2$, and $n_1 = 4$, $\mu_1 = 3$, $n_2 = 3$, $\mu_2 = 2$, $n_3 = 2$, $\mu_3 = 1$. Following Lemma 2.1, we have $\lambda_1 = (\mu_1\lambda_0 + \lambda_{r+1})/n_1 = (3 \times 4 + 0)/4 = 3$, $\lambda_2 = \mu_2\lambda_1/n_2 = (2 \times 3)/3 = 2$, $\lambda_3 = \mu_3\lambda_2/n_3 = \lambda_0/n_1 = 1$, $\lambda_4 = 0$.

2.2. Minimal cycles over normal surface singularities

Let (X, o) be a germ of a normal complex surface singularity. Let $\pi\colon (\widetilde{X}, E) \to (X, o)$ be a resolution of (X, o), where $\pi^{-1}(o) = E = \bigcup_{i=1}^{r} E_i$ is the irreducible decomposition of E. Let D be a cycle with $0 \leq D < Z_E$, where Z_E is the fundamental cycle on E. Then we can construct a sequence of positive cycles

$$Z_0 = D,\ Z_1 = Z_0 + E_{i_1},\ \ldots,\ Z_j = Z_{j-1} + E_{i_j},\ \ldots,\ Z_l = Z_{l-1} + E_{i_l} = Z_E,$$

such that $Z_{j-1}E_{i_j} > 0$ for $j = \epsilon + 1, \ldots, l$, where E_{i_1} is arbitrary, and $\epsilon = 0$ if $D > 0$ and $\epsilon = 1$ if $D = 0$. This sequence is called a computation sequence from D to Z_E. When $D = 0$, it is a Laufer's computation sequence of Z_E. We can always construct a computation sequence from D to Z_E as in [3].

Lemma 2.3. [8, Lemma 1.1] *Let D be a cycle on E such that $0 \leq D \leq Z_E$. Then $p_a(D) \leq p_f(X, o)$.*

Proof. Let $Z_0 = D, Z_1 = Z_0 + E_{i_1}, \ldots, Z_{j+1} = Z_j + E_{i_{j+1}}, \ldots, Z_l = Z_E$ be a computation sequence from D to Z_E. Then for $j = 0, \ldots, l-1$, following (1.1) and (1.2), we have

$$p_a(Z_{j+1}) = p_a(Z_j) + p_a(E_{i_{j+1}}) - 1 + Z_jE_{i_{j+1}} \geq p_a(Z_j). \qquad \square$$

Definition 2.4. [8, Definition 1.2] Let A be a cycle on E satisfying $0 < A \leq Z_E$. Suppose $p_f(X, o) \geq 1$. Then A is said to be a minimal cycle on E if $p_a(A) = p_f(X, o)$ and $p_a(D) < p_f(X, o)$ for any cycle D with $D < A$.

In 1977, Laufer [4] showed that if (X, o) is an elliptic singularity (i.e., $p_f(X, o) = 1$), then A is the minimally elliptic cycle. In other words, if (X, o) is a minimally elliptic singularity, then $A = Z_E$ (cf. [4]). In fact, as Tomaru [8] said, for the definition of minimally elliptic cycle, we need not the assumption $A \leq Z_E$. However, in the case of $p_f(X, o) \geq 2$, we need the assumption $A \leq Z_E$. Further, as the minimally elliptic cycle, the existence and the uniqueness of the minimal cycle A can also be shown as in [4].

Lemma 2.5. [8, Lemma 1.4] *Let $Z_0 = A, Z_1 = Z_0 + E_{i_1}, \ldots, Z_E = Z_l = Z_{l-1} + E_{i_l}$ be a computation sequence from A to Z_E. Then E_{i_k} is a smooth rational cure and $Z_{k-1}E_{i_k} = 1$ for $k = 1, 2, \ldots, l$.*

Suppose that $E = \bigcup_{i=0}^{N} E_i$ whose dual graph is star-shaped with central curve E_0. Let $\bigcup_{i=1}^{s} E_i$ be a cyclic branch with $E_0 \cap E_1 \neq \varnothing$. Suppose that the weighted dual graph of $E_0 \cup (\bigcup_{i=1}^{s} E_i)$ is as in Figure 2.2, where $E_i^2 = -b_i$, $i = 1, 2, \dots, s$.

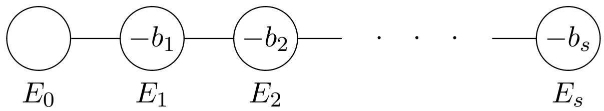


Figure 2.2: The weighed dual graph of $\bigcup_{i=1}^{s} E_i$

Let d, e be positive integers and $d/e = [[b_1, \dots, b_s]]$ satisfying $\gcd(d, e) = 1$. Let $c_0 = d$, $c_1 = e$ and let $c_2, c_3, \dots, c_s$ be the integers which are inductively defined by the relation $c_{i+1} = b_i c_i - c_{i-1}$ for $1 \leq i \leq s-1$, thus $c_s = 1$ by Lemma 2.1(4). Then we have the following lemma.

Lemma 2.6. [8, Lemma 3.2] *Suppose that the coefficient of E_0 in Z_E is dt, where t is a positive integer. Then the coefficient of E_i in Z_E is given by tc_i, $i = 1, 2, \dots, s$. In particular, $Z_E E_i = 0$ for $i = 1, 2, \dots, s$.*

Let d, e and $b_1, \dots, b_s$ be as above. Let l, μ be integers defined by $\mu d - el = 1$ with $0 < \mu < d$. Then $l/\mu = [[b_1, \dots, b_{s-1}, b_s - 1]]$. Put $\gamma_0 = l$, $\gamma_1 = \mu$ and define $\gamma_2, \dots, \gamma_s$ inductively by $\gamma_i = b_{i-1}\gamma_{i-1} - \gamma_{i-2}$ $(i = 2, \dots, s)$, then $\gamma_{s-1} = b_s - 1$ and $\gamma_s = 1$.

Lemma 2.7. [8, Lemma 3.3] *If the coefficient of E_0 in Z_E is l, then the coefficient of E_i in Z_E is given by γ_i, $1 \leq i \leq s$. In particular, $Z_E E_i = 0$ for $i = 1, \dots, s-1$ and $Z_E E_s = -1$. Furthermore, if $\lfloor \frac{d}{l} \rfloor = 1$, then $b_s \geq 3$.*

3. Minimal cycles over (W, o)

In this section, we consider the minimal cycles over Brieskorn complete intersection surface singularity (W, o) defined as in Section 1, and provide a condition for the coincidence of the fundamental cycle and the minimal cycle on the minimal resolution space. Let $\pi\colon (\widetilde{W}, E) \to (W, o)$ be the minimal good resolution of (W, o). Let $\alpha_i := n_{im}$, $\beta_i := \mu_{im}$ and $d_m = \operatorname{lcm}(a_1, \dots, a_m)$.

Theorem 3.1. [5, Theorem 4.4] *Let g and $-c_0$ denote the genus and the self-intersection number of E_0, respectively. Then the weighted dual graph of the exceptional set E is as in Figure 3.1, where the invariants are as follows:*

$$2g - 2 = (m-2)\widehat{g} - \sum_{i=1}^{m} \widehat{g}_i, \quad c_0 = \sum_{w=1}^{m} \frac{\widehat{g}_w \beta_w}{\alpha_w} + \frac{a_1 \cdots a_m}{d_m^2},$$

$$\beta_w/\alpha_w = \begin{cases} [[c_{w,1}, \dots, c_{w,s_w}]]^{-1} & \text{if } \alpha_w \geq 2, \\ 0 & \text{if } \alpha_w = 1. \end{cases}$$

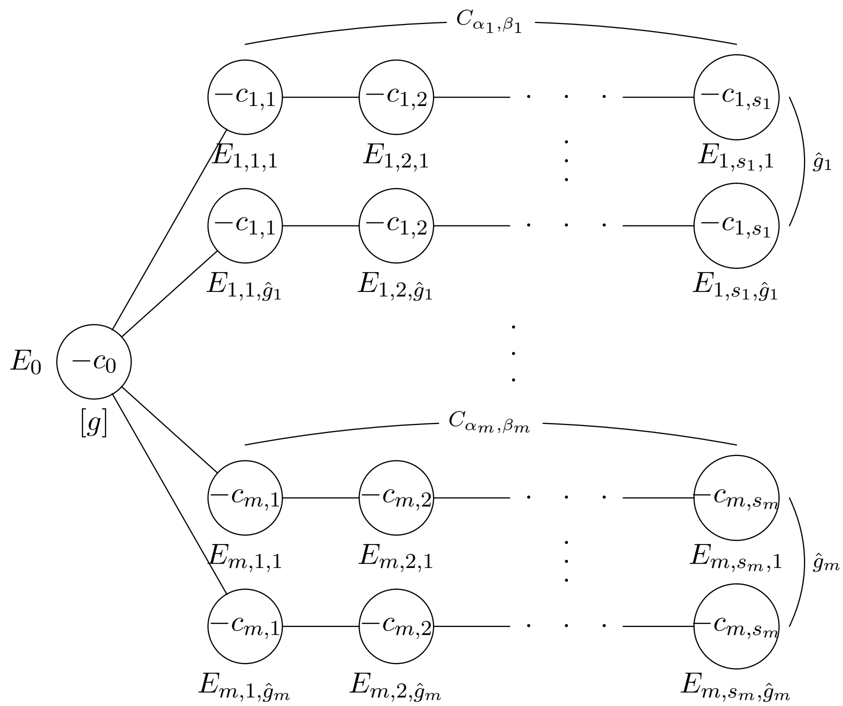


Figure 3.1: The weighted dual graph of the exceptional set E.

Theorem 3.2. [5, Theorem 5.1] *Let* $\epsilon_{w,\nu} = [[c_{w,\nu}, \ldots, c_{w,s_w}]]$ *if* $s_w > 0$*, and let*

$$Z_E = \theta_0 E_0 + \sum_{w=1}^{m} \sum_{\nu=1}^{s_w} \sum_{\xi=1}^{\widehat{g}_w} \theta_{w,\nu,\xi} E_{w,\nu,\xi}.$$

Then θ_0 *and the sequence* $\{\theta_{w,\nu,\xi}\}$ *are determined by the following:*

$$\theta_{w,0,\xi} := \theta_0 := \min\left(e_{mm}, \alpha_1 \cdots \alpha_m\right),$$
$$\theta_{w,\nu,\xi} = \lceil \theta_{w,\nu-1,\xi}/\epsilon_{w,\nu} \rceil \quad (1 \le \nu \le s_w).$$

Theorem 3.3. *Let* $\pi' \colon (\widehat{W}, E) \to (W, o)$ *be the minimal resolution of* (W, o)*. Assume* $\mathrm{lcm}(a_1, \ldots, a_{m-1}) \le a_m < 2 \cdot \mathrm{lcm}(a_1, \ldots, a_{m-1})$*, then* $Z_E = A$ *on* E*.*

Proof. Following the proof of Lemma 2.3, by Definition 2.4, we need only to prove that $p_a(Z_E - E_i) < p_f(W, o)$ for any irreducible component E_i of E. By (1.1) and (1.2), we have

$$p_a(Z_E) = p_a(Z_E - E_i + E_i) = p_a(Z_E - E_i) + p_a(E_i) - 1 + (Z_E - E_i)E_i,$$

which implies that

$$p_a(Z_E - E_i) = p_a(Z_E) - p_a(E_i) + 1 - Z_E E_i + E_i^2. \tag{3.1}$$

Assume that π' is the minimal good resolution, then $E_0^2 \leq -2$ (or $E_0^2 = -1$ and $g(E_0) \geq 1$) and the weighted dual graph of the minimal good resolution of (W, o) is given as in Figure 3.1. Let B be any irreducible component of $E - E_0 - \bigcup_{w=1}^{m}(\bigcup_{\xi=1}^{\widehat{g}_w} E_{w,s_w,\xi})$, by Lemma 2.1, Theorem 3.2 and (3.1), we have $Z_E B = 0$ and

$$p_a(Z_E - B) < p_f(W, o).$$

Since $\mathrm{lcm}(a_1, \ldots, a_{m-1}) \leq a_m$, $e_{mm} \leq \alpha_m \leq \alpha_1 \cdots \alpha_m$. In particular, in this case, $Z_E = M_E = (x_m)_E$ obtained by Meng-Okuma (cf. [5]), where M_E is the maximal ideal cycle on E. From Theorem 3.2, the coefficient of E_0 in Z_E is e_{mm}. It follows from Theorem 3.2, Lemma 2.6, Lemma 2.7 and Lemma 2.1(3) that for $w \in \{1, \ldots, m\}$ and $\xi \in \{1, \ldots, \widehat{g}_w\}$, we have

$$Z_E E_{w,s_w,\xi} = \begin{cases} 0 & \text{if } w \neq m, \\ -1 & \text{if } w = m. \end{cases}$$

Since $\mathrm{lcm}(a_1, \ldots, a_{m-1}) \leq a_m < 2 \cdot \mathrm{lcm}(a_1, \ldots, a_{m-1})$, $e_{mm} \leq \alpha_m < 2e_{mm}$, which implies $\left\lfloor \frac{\alpha_m}{e_{mm}} \right\rfloor = 1$. Following Lemma 2.7, we have $(E_{m,s_m,\xi})^2 < -2$, $\xi \in \{1, \ldots, \widehat{g}_m\}$. Then by (3.1), we have

$$p_a(Z_E - E_{w,s_w,\xi}) < p_f(W, o), \quad w = 1, \ldots, m;\ \xi = 1, \ldots, \widehat{g}_w.$$

From Theorem 3.1, we have

$$\begin{aligned} -Z_E \cdot E_0 &= c_0 e_{mm} - \sum_{w=1}^{m-1} \frac{\widehat{g}_w e_{mm} \beta_w}{\alpha_w} - \frac{\widehat{g}_m (e_{mm} \beta_m + 1)}{\alpha_m} \\ &= e_{mm} \left(c_0 - \sum_{w=1}^{m} \frac{\widehat{g}_w \beta_w}{\alpha_w} \right) - \frac{\widehat{g}_m}{\alpha_m} \\ &= \frac{e_{mm} a_1 \cdots a_m}{d_m^2} - \frac{\widehat{g}_m}{\alpha_m} \\ &= \frac{e_{mm} \widehat{g}}{d_m} - \frac{\widehat{g}_m}{\alpha_m} = 0. \end{aligned}$$

Therefore, by (3.1) and the adjunction formula, we also have

$$p_a(Z_E - E_0) = p_a(Z_E) - g(E_0) + 1 + E_0^2 < p_f(W, o).$$

Similar as the proof of Theorem 4.4 in [8], we assume that the minimal resolution does not coincide the minimal good resolution. Let $\pi := \phi \circ \pi' \colon (\overline{W}, \overline{E}) \xrightarrow{\phi} (\widehat{W}, E) \xrightarrow{\pi'} (W, o)$ be the minimal good resolution, where ϕ is a birational morphism obtained by iterating monoidal transforms centered at a point. We may assume that E has at least two irreducible components, otherwise $Z_E = A$ obviously. It suffices to show that $p_a(Z_E - E_i) < p_f(W, o)$ for any $E_i \subset E$. Suppose that $p_a(Z_E - E_i) = p_f(W, o) = p_a(Z_E)$ for some

$E_i \subset E$. Since $Z_E = Z_E - E_i + E_i$ is a part of a computation sequence for Z_E, it follows from Lemma 2.5 that E_i is a smooth rational curve and

$$Z_E E_i = (Z_E - E_i + E_i)E_i = (Z_E - E_i)E_i + E_i^2 = 1 + E_i^2.$$

Since E_i is smooth, $g(E_i) = 0$. Hence by (1.1) and the adjunction formula $K_{\widehat{W}} E_i = -E_i^2 + 2g(E_i) - 2$ for any $E_i \subset E$, where $K_{\widehat{W}}$ is the canonical divisor on $\widehat{W}$, we have

$$\begin{aligned} p_a(Z_E - E_i) - p_a(Z_E) &= 1 + \frac{1}{2}\left((Z_E - E_i)^2 + K_{\widehat{W}}(Z_E - E_i)\right) \\ &\quad + 1 + \frac{1}{2}(Z_E^2 + K_{\widehat{W}} Z_E) \\ &= -1 - Z_E E_i = 0, \end{aligned}$$

which implies $Z_E E_i = -1$. Thus $E_i^2 = -2$. Let $\overline{E}_i$ be the proper transform of E_i by ϕ. Then $Z_E E_i = Z_{\overline{E}} \overline{E}_i = -1$ by (0.2.2) in [9], which implies that $\overline{E}_i = E_{m,s_m,\xi}$, $\xi \in \{1, \dots, \widehat{g}_m\}$ and the coefficient of $\overline{E}_i$ in $Z_{\overline{E}}$ is 1 by Lemma 2.7. From Proposition 2.9 in [9], the coefficient of E_i in Z_E is 1. It follows that there exists only one irreducible component $E_j \subset E$ that intersects E_i transversely, which implies that ϕ doesn't contain any monoidal transform centered at a point of E_i. Then $E_{m,s_m,\xi}^2 = \overline{E}_i^2 = E_i^2 = -2$, which contradicts Lemma 2.7. Hence we complete the proof. □

In fact, as Tomaru [8] said, in elliptic case, i.e., $(a_1, a_2) = (2, 3)$ or $(2, 4)$ or $(3, 3)$, the result of Theorem 3.3 is already known by the classification of minimally elliptic singularities (cf. [4]).

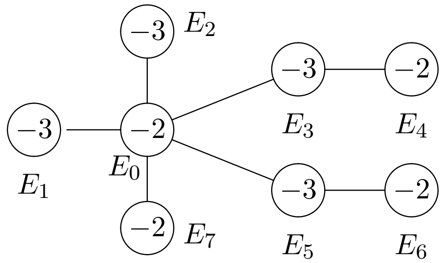
Let $\pi\colon (\widetilde{W}, E) \to (W, o)$ be a resolution of (W, o). We define the $\mathbb{Q}$-coefficient cycle K on E by the relation:

$$-KE_i = K_{\widetilde{W}} E_i$$

for any irreducible component $E_i \subseteq E$, where $K_{\widetilde{W}}$ is a canonical divisor of $\widetilde{W}$. We call K the canonical cycle on E (cf. [10, Definition 2.18]). Since (W, o) is a Gorenstein singularity, there exists a cycle K such that $-K$ is a canonical divisor of $\widetilde{W}$.

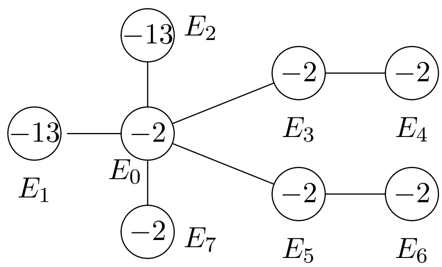
Theorem 3.4. [8, Theorem 1.6] *Let $\pi\colon (\widetilde{W}, E) \to (W, o)$ be the minimal good resolution and A the minimal cycle on E. Suppose $p_f(W, o) \geq 2$. Then $-K \geq Z_E + A$.*

Example 3.5. Let $(W, o) = \left\{x_1^2 + x_2^3 + x_3^4 = 0, 2x_1^2 + 3x_2^3 + x_4^5 = 0\right\} \subset \mathbb{C}^4$. Note that $\operatorname{lcm}(2, 3, 4) \not< 5 < 2 \cdot \operatorname{lcm}(2, 3, 4)$. The minimal resolution graph (is also the minimal good resolution graph) of (W, o) is given as follows:



Then the fundamental cycle $Z_E = 12E_0 + 4E_1 + 4E_2 + 5E_3 + 3E_4 + 5E_5 + 3E_6 + 6E_7$ and $p_f(W, o) = 7$. The minimal cycle $A = 12E_0 + 4E_1 + 4E_2 + 5E_3 + 2E_4 + 5E_5 + 2E_6 + 6E_7$ and $-K = 44E_0 + 15E_1 + 15E_2 + 18E_3 + 9E_4 + 18E_5 + 9E_6 + 22E_7$. It is clear that $Z_E \neq A$ and $-K > Z_E + A$.

Example 3.6. Let $(W, o) = \left\{x_1^2 + x_2^3 + x_3^4 = 0, 3x_1^2 + 5x_2^3 + x_4^{13} = 0\right\} \subset \mathbb{C}^4$. Note that $\mathrm{lcm}(2, 3, 4) < 13 < 2 \cdot \mathrm{lcm}(2, 3, 4)$. The minimal resolution graph (is also the minimal good resolution graph) of (W, o) is given as follows:



Then we have $Z_E = A = 12E_0 + E_1 + E_2 + 8E_3 + 4E_4 + 8E_5 + 4E_6 + 6E_7$, $p_f(W, o) = 11$ and $-K = 132E_0 + 11E_1 + 11E_2 + 95E_3 + 48E_4 + 95E_5 + 48E_6 + 66E_7$. It is easy to see that $-K > Z_E + A$.

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