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A Heat Conduction Problem on Some Semi-infinite Regions

Jong-Yi Chen and Yunshyong Chow*

Abstract. An infinite homogeneous d-dimensional medium initially is at zero temperature. A heat impulse is applied at the origin, raising the temperature there to a value greater than a constant value $u_0 > 0$. The temperature at the origin then decays, and when it reaches u_0, another equal-sized heat impulse is applied at time t_1. Subsequent equal-sized heat impulses are applied at the origin at times t_n, $n \geq 2$, when the temperature there has decayed to u_0. The waiting-time sequence $\{t_n - t_{n-1}\}$ can be defined recursively by a difference equation and its asymptotic behavior was first proposed as a conjecture by Myshkis in 1997.

In this paper we study the same heating-time problem set on semi-infinite regions $[-L, L] \times \mathbb{R}$ and $\{(x, y) : x^2 + y^2 \leq L\} \times \mathbb{R}$ with insulated boundary condition and all actions taking place at some point $\boldsymbol{p}$ which needs not be the origin.

1. Introduction

Myshkis [6] studied the following heat conduction problem: let $u(\boldsymbol{x}, t)$ be the temperature at position $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ and time t of a homogeneous medium filling up the whole $\mathbb{R}^d$. Suppose $u \equiv 0$ at $t = 0$ and a heat impulse of size b is applied at $\boldsymbol{x} = \mathbf{0}$. A heat impulse of the same size is applied again at $\boldsymbol{x} = \mathbf{0}$ at time t_1 when $u(\mathbf{0}, t_1) = u_0$, i.e., when the temperature at $\boldsymbol{x} = \mathbf{0}$ decreases to a given value $u_0 > 0$. This process is repeated indefinitely.

Denote by $t_0 = 0, t_1, t_2, \ldots$ the sequence of consecutive times that a heat impulse of size b is applied at $\boldsymbol{x} = \mathbf{0}$. By solving the heat equation

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = a \cdot \displaystyle\sum_{i=1}^{d} \dfrac{\partial^2 u}{\partial x_i^2}, \\ u(\boldsymbol{x}, t_{n-1}^{+}) = u(\boldsymbol{x}, t_{n-1}) + b \cdot \delta_{\mathbf{0}}(\boldsymbol{x}), \end{cases}$$

where a is the heat conduction coefficient of the medium and $\delta_{\mathbf{0}}(\boldsymbol{x})$ the Dirac function at $\boldsymbol{x} = \mathbf{0}$, it is easy to show by superposition principle that for $n \geq 0$ and $t_{n-1} < t \leq t_n$, $u(\boldsymbol{x}, t)$ is given by

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*Corresponding author.

$$u(\boldsymbol{x}, t) = b \sum_{j=0}^{n-1} f(\boldsymbol{x}, t - t_j). \tag{1.2}$$

Here $f(\boldsymbol{x}, t) = \left(\frac{1}{4\pi a t}\right)^{d/2} \exp\left(-\frac{\sum_{i=1}^{d} x_i^2}{4at}\right)$ is the fundamental solution to the heat equation (1.1) above. The heating condition

$$u(\mathbf{0}, t_n) = u_0 \quad \text{for } n \geq 1 \tag{1.3}$$

then implies

$$u_0 = u(\mathbf{0}, t_n) = b \sum_{j=0}^{n-1} f(\mathbf{0}, t_n - t_j) = b \sum_{j=0}^{n-1} \left(\frac{1}{4\pi a(t_n - t_j)}\right)^{d/2}. \tag{1.4}$$

For $j \geq 1$, define $\tau_j = 4\pi a(t_j - t_{j-1})(u_0/b)^{2/d}$ as the normalized waiting time between two consecutive heating times t_{j-1} and t_j. By a simple computation (1.4) can be rewritten as

$$\tau_1 = 1 \quad \text{and} \quad \sum_{j=1}^{n} \left(\sum_{s=j}^{n} \tau_s\right)^{-d/2} = 1 \quad \text{for } n \geq 2. \tag{1.5}$$

The sequence $\{\tau_n\}$ is thus recursively defined. Myshkis [6] conjectured that $\{\tau_n\}$ is increasing and $\tau_n/n \approx$ constant for $d = 1$. The following is known.

Theorem 1.1. [1,5] *Let $d \in \mathbb{N}$. The waiting-time sequence $\{\tau_n\}$ given in* (1.5) *is increasing and satisfies*

(i) $\lim_n \tau_n/n = \pi^2/2$ *for* $d = 1$,

(ii) $\lim_n \tau_n/\log n = 1$ *for* $d = 2$,

(iii) $\lim_n \tau_n = \{\zeta(d/2)\}^{2/d}$ *for* $d \geq 3$.

Here $\zeta(s) \equiv \sum_{k=1}^{\infty} k^{-s}$ *is the Riemann-Zeta function.*

Since $4\pi a(u_0/b)^{2/d} t_n = \sum_{s=1}^{n} \tau_s$, we get easily the following result.

Theorem 1.2. *The heating-time sequence $\{t_n : n \geq 0\}$ recursively defined by the heat equation* (1.1) *and the heating condition* (1.3) *satisfies:*

(i) $\lim_n \dfrac{t_n}{n^2} = \dfrac{\pi}{16a} \left(\dfrac{b}{u_0}\right)^2$ *for* $d = 1$,

(ii) $\lim_n \dfrac{t_n}{n \log n} = \dfrac{1}{4\pi a} \left(\dfrac{b}{u_0}\right)$ *for* $d = 2$,

(iii) $\lim_n \frac{t_n}{n} = \frac{1}{4\pi a}\left(\frac{b\zeta(\frac{d}{2})}{u_0}\right)^{\frac{2}{d}}$ *for* $d \geq 3$.

In particular, the conduction coefficient a can be determined without ever leaving the origin $\boldsymbol{x} = 0$ if one knows the impulse size b, the threshold temperature u_0 and the heating times $t_0 = 0, t_1, t_2, t_3, \ldots$.

In this paper we will study the same heating problem, but set on a semi-infinite region with insulated boundary condition and all actions taking place at point $\boldsymbol{p}$ which needs not be the origin. Two particular regions considered here are a slab in $\mathbb{R}^2$ and an infinite cylinder in $\mathbb{R}^3$ respectively. Let

$$\boldsymbol{D}_1 = [-L, L] \times \mathbb{R} \quad \text{and} \quad \boldsymbol{D}_2 = \left\{(x, y) : x^2 + y^2 \leq L^2\right\} \times \mathbb{R}. \tag{1.6}$$

By symmetry we may set $\boldsymbol{p} = (\mu, 0)$ for $\boldsymbol{D}_1$ and $\boldsymbol{p} = (\mu, 0, 0)$ for $\boldsymbol{D}_2$ respectively. As above, let $t_0 = 0, t_1, t_2, \ldots$ the sequence of consecutive times that a heat impulse of size b is applied at $\boldsymbol{p}$. For $t_{n-1} < t < t_n$, the temperature function u satisfies the following heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(\boldsymbol{x}, t) = a \cdot \Delta u(\boldsymbol{x}, t) \quad \text{for } (\boldsymbol{x}, t) \in \boldsymbol{D}_j \times \mathbb{R}^+, \\ \dfrac{\partial u}{\partial n}(\boldsymbol{x}, t)\Big|_{\partial \boldsymbol{D}_j} = 0 \quad \text{and} \quad u(\boldsymbol{x}, t_{n-1}^+) = u(\boldsymbol{x}, t_{n-1}) + b \cdot \delta_{\boldsymbol{p}}(\boldsymbol{x}). \end{cases} \tag{1.7}$$

Then t_n is determined by the heating condition

$$u(\boldsymbol{p}, t_n) = u_0 \quad \text{for } n \geq 1. \tag{1.8}$$

Equation (1.2) still holds except the fundamental solution f changes. In both cases, we require

$$f(\cdot, t) \text{ is bounded for any } t > 0 \quad \text{and} \quad \lim_{t\to\infty} \|f(\cdot, t)\|_\infty = 0. \tag{1.9}$$

We start with $\boldsymbol{D}_1$. The method of separation of variables and the superposition principle imply

$$f(x, y, t) = \int_{-\infty}^{\infty} d\beta \sum_{m=0}^{\infty} c_m(\beta) e^{\imath\beta y} e^{-a\left(\frac{m^2\pi^2}{4L^2} + \beta^2\right)t} \cdot \cos\frac{m\pi(x + L)}{2L}. \tag{1.10}$$

Using the impulse condition $f(\boldsymbol{x}, 0^+) = \delta_\mu(x)\delta_0(y)$, we can verify easily that

$$c_m(\beta) = \frac{2 - \delta_{m,0}}{4\pi L} \cos\frac{m\pi(\mu + L)}{2L}, \text{ which is independent of } \beta. \tag{1.11}$$

Here, $\delta_{m,0}$ is the well-known Kronecker symbol. Putting it back to (1.10) and integrating out β, we get [7]

$$f(x, y, t) = \frac{e^{-\frac{y^2}{4at}}}{4L\sqrt{\pi a t}} \sum_{m=0}^{\infty} (2 - \delta_{m,0}) \cos\frac{m\pi(\mu + L)}{2L} \cos\frac{m\pi(x + L)}{2L} e^{-\frac{am^2\pi^2 t}{4L^2}}.$$

Here and in the derivation of (1.11) we have used the following formulas

$$\int_{-\infty}^{\infty} e^{\imath\beta y-a\beta^2 t}\,d\beta=\frac{\sqrt{\pi}e^{-\frac{y^2}{4at}}}{\sqrt{at}} \quad\text{and}\quad \int_{-\infty}^{\infty} e^{\imath(\beta-\beta')y}\,dy=2\pi\delta_0(\beta-\beta') \tag{1.12}$$

from the inverse Fourier transform. Let $v(t)=f(\mu,0,t)$. Then

$$v(t)=\frac{1}{4L\sqrt{\pi at}}\sum_{m=0}^{\infty}(2-\delta_{m,0})\cos^2\frac{m\pi(\mu+L)}{2L}\cdot e^{-\frac{am^2\pi^2t}{4L^2}}. \tag{1.13}$$

The heating condition (1.8) then implies that for $n\geq 1$,

$$u_0=u(\mu,0,t_n)=b\sum_{j=0}^{n-1}f(\mu,0,t_n-t_j)=b\sum_{j=0}^{n-1}v(t_n-t_j). \tag{1.14}$$

Remember $t_0=0$. This is the defining recursive relation for the heating times $\{t_n, n\geq 0\}$ of (1.7) set in $\boldsymbol{D}_1$.

As to $\boldsymbol{D}_2=\big\{(x,y):x^2+y^2\leq L\big\}\times\mathbb{R}\subseteq\mathbb{R}^3$, its fundamental solution f to (1.7) can be derived similarly. Using the cylindrical coordinate (r,θ,z) and symmetry, we expect f on $\boldsymbol{D}_2$ to be independent of θ. Hence,

$$\frac{\partial f}{\partial t}=a\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial z^2}\right)\quad\text{for } 0\leq r<L \text{ with } \left.\frac{\partial f}{\partial r}\right|_{r=L}=0. \tag{1.15}$$

By the method of separation of variables, we seek for particular solutions to (1.15) in the form $R(r)Z(z)T(t)$. Substituting into (1.15), we get

$$\begin{cases} R''(r)+\frac{1}{r}R'(r)+\alpha^2R(r)=0 \quad\text{and}\quad R'(L)=0,\\ Z''(z)+\beta^2Z(z)=0 \quad\text{and}\quad T'(t)=-a(\alpha^2+\beta^2)T(t), \end{cases} \tag{1.16}$$

where α,β are real constants. Hence, $Z(z)=e^{\imath\beta z}$, $T(t)=e^{-a(\alpha^2+\beta^2)t}$ and $R(r)=J_0(\alpha r)$. Here J_n means the Bessel function of order n. Note that the Weber function $Y_0(\alpha r)$ does not appear as $R(r)$ is required to be continuous at $r=0$. The insulated boundary condition in (1.16) implies that α satisfies $J_0'(\alpha L)=0$. Note that $J_0'(z)=-J_1(z)$, $J_1(0)=0$ and $J_1(-z)=-J_1(z)$. It is well-known [8] that all the zeros of $J_1(z)$ are real and simple. Let

$$S=\{\lambda_0=0<\lambda_1<\lambda_2<\cdots\}$$

be all the nonnegative zeros of $J_1(z)$. Then $R(r)=J_0(\lambda r/L)$ with $\lambda\in S$. By the superposition principle,

$$f(r,\theta,z,t)=\int_{-\infty}^{\infty}d\beta\sum_{k=0}^{\infty}c_k(\beta)e^{\imath\beta z}e^{-a\left(\frac{\lambda_k^2}{L^2}+\beta^2\right)t}J_0\left(\frac{\lambda_k r}{L}\right), \tag{1.17}$$

where the constant $c_k(\beta)$ is to be determined by the impulse condition

$$\delta_\mu(x)\delta_0(y)\delta_0(z) = f(r,\theta,z,0^+) = \int_{-\infty}^{\infty} d\beta \sum_{k=0}^{\infty} c_k(\beta) e^{\imath\beta z} J_0\left(\frac{\lambda_k r}{L}\right). \tag{1.18}$$

Multiplying both sides of (1.18) by $\exp(-\imath\beta' z)$ and then integrating over z,

$$\delta_\mu(x)\delta_0(y) = 2\pi \sum_{k=0}^{\infty} c_k(\beta') J_0\left(\frac{\lambda_k r}{L}\right), \tag{1.19}$$

where the second formula in (1.12) was used. Now we multiply both sides of (1.19) by $J_0(\lambda_\ell r/L)$ and then integrate over $\{(x,y) : x^2+y^2 \le L\}$. First changing to polar coordinate (r,θ) for the right-hand side integral and then using the following orthogonal relation for Dini's expansion [8, p. 580]

$$\int_0^L r J_0\left(\frac{\lambda_k r}{L}\right) J_0\left(\frac{\lambda_\ell r}{L}\right) dr = \frac{L^2}{2} J_0^2(\lambda_k)\cdot\delta_{k,\ell},$$

we obtain $J_0(\lambda_\ell\mu/L) = 2\pi^2 L^2 c_\ell(\beta') J_0^2(\lambda_\ell)$. Hence,

$$c_k(\beta') = \frac{J_0(\frac{\lambda_k\mu}{L})}{2\pi^2 L^2 J_0^2(\lambda_k)}, \quad \text{which is independent of } \beta'.$$

Putting it back to (1.17) and repeating the same procedure as in $\boldsymbol{D}_1$ case,

$$f(r,\theta,z,t) = \frac{1}{\pi L^2\sqrt{4\pi a t}} e^{-\frac{z^2}{4at}} \sum_{k=0}^{\infty} \frac{J_0(\frac{\lambda_k\mu}{L}) J_0(\frac{\lambda_k r}{L})}{J_0^2(\lambda_k)} e^{-\frac{a\lambda_k^2 t}{L^2}}.$$

This is the fundamental solution to (1.15). Let $w(t) = f(\boldsymbol{p},t)$. Then

$$w(t) = \frac{1}{\pi L^2\sqrt{4\pi a t}} \sum_{k=0}^{\infty} \frac{J_0^2(\frac{\lambda_k\mu}{L})}{J_0^2(\lambda_k)} e^{-\frac{a\lambda_k^2 t}{L^2}}. \tag{1.20}$$

Note that $J_0(\lambda_k) \neq 0$. The heating condition (1.8) then implies that

$$u_0 = u(\boldsymbol{p},t_n) = b\sum_{j=0}^{n-1} f(\boldsymbol{p}, t_n - t_j) = b\sum_{j=0}^{n-1} w(t_n - t_j) \quad \text{for } n \ge 1, \tag{1.21}$$

which is the defining recursive relation for the heating times $\{t_n, n \ge 0\}$ of (1.7) set on $\boldsymbol{D}_2$.

In comparison with (1.4), the heating times problem for semi-infinite regions $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ is more complicated than that for an infinite rod. It seems an awful task to verify the monotonicity of the waiting-time sequence $\{t_{n+1} - t_n; n \ge 0\}$ defined in (1.14) and (1.21). Fortunately, this can be done by using a result in [2], which can be applied to many similar problems in auto-regulated systems. Moreover, we will show that $\lim(t_{n+1} - t_n) = \infty$. Since $\lambda_0 = 0$ and $J_0(0) = 1$, the first term 1 of the infinite series in both (1.13) and (1.20) is the leading term as $t \to \infty$. By using $\lim_n(t_n - t_j) = \infty$, for $0 \le j < n$, to ignore all non-leading terms, both difference equations (1.14) and (1.21) look like (1.4) with $d = 1$. The following result, similar to Theorem 1.2(i), will be proved in Section 2.

Theorem 1.3. *For the heating times* $\{t_n : n \geq 0\}$ *recursively defined by the heat equation* (1.7) *and the heating condition* (1.8), *the waiting-time sequence* $\{t_{n+1} - t_n : n \geq 0\}$ *is increasing. Moreover,*

$$\lim_n \frac{t_n}{n^2} = \frac{\pi}{16a} \left(\frac{b}{u_0 \gamma_j} \right)^2, \tag{1.22}$$

where, depending on $j = 1$ *or* 2, $\gamma_j = 2L$ *or* πL^2 *and is the cross-section area of the semi-infinite region* $\boldsymbol{D}_j$ *given in* (1.6).

When viewed from a faraway place, each $\boldsymbol{D}_j$ looks like an infinite rod. It is therefore expected that the order estimate of $\{t_n\}$ in (1.22) is consistent with that in Theorem 1.2(i), even if the temperature measurement is taken at a point different from the explosion point. See [4]. Certainly, constants b, u_0 and a should appear on the right-hand side of (1.22). The factor γ_j is of interest as it is the cross-section area of $\mathbf{D_j}$. We believe (1.22) holds for general regions like $\boldsymbol{G} \times \mathbb{R}$, where $\boldsymbol{G}$ is a smooth bounded domain in $\mathbb{R}^{d-1}$.

In contrast with the remark after Theorem 1.2, (1.22) is less satisfactory from the physical viewpoint as it fails to determine the conduction coefficient a which is now mixed up with the cross-section area γ_j of the semi-infinite region $\boldsymbol{D}_j$. Moreover, it is independent of the parameter μ in the action site $\boldsymbol{p}$. We wonder whether there is some way to determine the conduction coefficient a, the parameter μ and the geometric quantity γ_j from the heat-time sequence $\{t_n\}$.

Finally we remark that Theorem 1.3 is proved by modifying the method used in [5]. We first show that the waiting-time sequence $\{t_{n+1} - t_n; n \geq 0\}$ is increasing. By using some inequality shown in Lemma 2.2 below, we obtain that $\lim(t_{n+1} - t_n)/n$ exists by verifying $\liminf(t_{n+1} - t_n)/n = \limsup(t_{n+1} - t_n)/n$. It is then easy to find the limiting constant from the defining recursive formulas (1.14) and (1.21) respectively.

2. Proof of Theorem 1.3

For $j \geq 1$, define $\tau_j = t_j - t_{j-1}$ as the waiting time between two consecutive heatings. We first show $\{\tau_n : n \geq 1\}$ is increasing via the following result.

Lemma 2.1. [2] *Let sequence* $\{\tau_n\}$ *be recursively defined by*

$$\sum_{j=1}^{n} g\left(\sum_{s=j}^{n} \tau_s \right) = 1 \quad \textit{for } n \geq 1, \tag{2.1}$$

where g *is a continuous, strictly decreasing function on* $(0, \infty)$ *with* $g(0^+) \geq 1$ *and* $g(\infty) = 0$. *If* $\log g \in \mathcal{C}^1$ *and is convex, then the sequence* $\{\tau_n\}$ *is increasing. Moreover,* $\lim \tau_n = \beta < \infty$ *iff* $\sum_{n=1}^{\infty} g(n) < \infty$. *In that case, the constant* β *is uniquely determined by the equation* $\sum_{n=1}^{\infty} g(n\beta) = 1$.

Since $t_n - t_j = \sum_{s=j+1}^{n} \tau_s$, both the difference equations (1.14) for $\boldsymbol{D}_1$ and (1.21) for $\boldsymbol{D}_2$ can be rewritten in the form of (2.1) with

$$g(t) = \sum_{k=0}^{\infty} g_k(t), \quad \text{where } g_k(t) = c_k t^{-1/2} e^{-d_k t} \tag{2.2}$$

and all the constants c_k and d_k are positive except $d_0 = 0$. In particular,

$$c_0 = \begin{cases} \dfrac{b}{2Lu_0\sqrt{4\pi a}} & \text{for } \boldsymbol{D}_1, \\ \dfrac{b}{\pi L^2 u_0 \sqrt{4\pi a}} & \text{for } \boldsymbol{D}_2. \end{cases} \tag{2.3}$$

Note that $g(\infty) = 0$ by (1.9). For each $k \geq 0$, g_k is strictly decreasing on $(0, \infty)$ as $g_k' < 0$. Hence, g is strictly decreasing as well. It remains to show that $\log g$ is convex on $(0, \infty)$. First, each $\log g_k$ is convex as

$$\frac{g_k'' g_k - {g_k'}^2}{g_k^2} = (\log g_k)'' = \frac{1}{2} t^{-2} > 0. \tag{2.4}$$

Since $g_k''(t) = g_k(t)\left[d_k^2 + d_k \frac{1}{t} + \frac{3}{4t^2}\right] > 0$, it follows from (2.4) and Cauchy-Schwarz inequality that $(g_k + g_j)''(g_k + g_j) - (g_k + g_j)'^2$ is no less than

$$g_k'' g_j + g_j'' g_k - 2g_k' g_j' \geq 2\sqrt{g_k'' g_j g_j'' g_k} - 2g_k' g_j' \geq 2(|g_k' g_j'| - g_k' g_j') \geq 0.$$

Hence, $\log(g_k + g_j)$ is convex on $(0, \infty)$ and then so does $\log\left(\sum_{k=0}^{m} g_k\right)$. Because $\sum_{k=0}^{m} g_k \uparrow g$, $\log g$ is convex as desired. Moreover, $\sum_{n=0}^{\infty} g(n) \geq \sum_{n=0}^{\infty} g_0(n) = \sum_{n=0}^{\infty} c_0 n^{-1/2} = \infty$. We have from Lemma 2.1 that

$$\{\tau_n\} \text{ is an increasing sequence and } \lim_{n\to\infty} \tau_n = \infty. \tag{2.5}$$

In order to show (1.22), we need to find an estimate for τ_n better than (2.5). In fact, we claim that under (2.1) and (2.2),

$$\lim_{n\to\infty} \frac{\tau_n}{n} = \frac{\pi^2 c_0^2}{2}, \tag{2.6}$$

where c_0 is given in (2.3). Then (1.22) follows easily by using $t_n = \sum_{s=1}^{n} \tau_s$.

We are going to mimic the proof in Theorem 1.1(i), where the following inequality plays a crucial role:

$$\sum_{j=1}^{n} \left(\sum_{s=j}^{n} s\right)^{-1/2} \leq \sum_{j=1}^{n+1} \left(\sum_{s=j}^{n+1} s\right)^{-1/2} \quad \text{for } n \geq 1. \tag{2.7}$$

The required counterpart for the present case is stated as follows. Its proof is left to the end of this section.

Lemma 2.2. *Let g be as given in* (2.1) *and* (2.2).

(i) *For* $n \geq 1$, $\sum_{j=1}^{n+1}\left(\sum_{s=j}^{n+1} s\right)^{-1/2} - \sum_{j=1}^{n}\left(\sum_{s=j}^{n} s\right)^{-1/2} \geq 1/(12n^2)$.

(ii) *For any* $c > 0$, *there exists an integer* n_0 *such that* $\left\{\sum_{j=1}^{n} g(c\sum_{s=j}^{n} s)\right\}$ *is an increasing sequence in* n *for* $n \geq n_0$.

Assume temporarily that Lemma 2.2 holds. Let $\widetilde{T}_j^k = \sum_{s=j}^{k} \tau_s$, $T_j^k = \sum_{s=j}^{k} s$ and for any $c > 0$,

$$S_c = \left\{k \in \mathbb{N} : k \geq n_0 \text{ and } \widetilde{T}_j^k \geq c\,T_j^k \text{ for } 1 \leq j \leq k\right\}, \tag{2.8}$$

where n_0 is given in Lemma 2.2(ii). We now verify (2.6) in the following three steps by modifying the proof of Theorem 1.1(i) in [5]:

(a) If $n \in S_c$ then $m \in S_c$ for all $m \geq n$.

(b) $\liminf \tau_n/n \geq \sup\{c > 0 : S_c \neq \emptyset\} \geq \limsup \tau_n/n$. Hence, $\lim_n \tau_n/n$ exists in $(0, \infty]$.

(c) $\lim \tau_n/n = \pi^2 c_0^2/2$.

Step (a). It suffices to show $n+1 \in S_c$ as we will have successively $n+2 \in S_c, n+3 \in S_c, \ldots$, and so on. Since $n \in S_c$ by assumption, (2.8) shows that

$$\widetilde{T}_j^n \geq c\,T_j^n \quad \text{holds for } 1 \leq j \leq n. \tag{2.9}$$

it suffices to show that $\widetilde{T}_{n+1}^{n+1} = \tau_{n+1} \geq c(n+1) = c\,T_{n+1}^{n+1}$. Adding it to (2.9), we will get $\widetilde{T}_j^{n+1} \geq c\,T_j^{n+1}$ for $1 \leq j \leq n$ as well and then $n+1 \in S_c$. Suppose the contrary. So

$$\tau_{n+1} < c(n+1) \quad \text{and then} \quad g(c(n+1)) < g(\tau_{n+1}) \tag{2.10}$$

as g is strictly decreasing on $(0, \infty)$. By (2.1), $g(\tau_{n+1}) = 1 - \sum_1^n g(\widetilde{T}_j^{n+1}) = \sum_1^n g(\widetilde{T}_j^n) - g(\widetilde{T}_j^{n+1})$. Using (2.10),

$$g(\tau_{n+1}) = \sum_1^n \int_{\widetilde{T}_j^n}^{\widetilde{T}_j^{n+1}} \left|g'(t)\right|\,dt \leq \sum_1^n \int_{\widetilde{T}_j^n}^{c(n+1)+\widetilde{T}_j^n} \left|g'(t)\right|\,dt. \tag{2.11}$$

Because g is convex and $g' < 0$ on $(0, \infty)$, $|g'|$ is decreasing. For any $0 < x \leq y$ and $v > 0$, we have

$$\int_x^{x+v} \left|g'(t)\right|\,dt - \int_y^{y+v} \left|g'(t)\right|\,dt = \left(\int_x^y - \int_{x+v}^{y+v}\right)\left|g'(t)\right|\,dt \geq 0. \tag{2.12}$$

Letting $x = cT_j^n$, $y = \widetilde{T}_j^n$ and $v = c(n+1)$ in (2.12) and then combining (2.10), (2.11) and (2.12) together,

$$g(c(n+1)) < \sum_1^n \int_{c\,T_j^n}^{c\,T_j^{n+1}} \left|g'(t)\right|\,dt = \sum_1^n g(c\,T_j^n) - g(c\,T_j^{n+1}).$$

Rearranging the terms, we have $\sum_1^n g(cT_j^n) > \sum_1^{n+1} g(cT_j^{n+1})$. This contradicts to Lemma 2.2(ii). The proof of Step (a) is thus completed. In particular, $\tau_m = \widetilde{T}_m^m \geq cT_m^m = cm$ for all $m \geq n$. Hence,

$$\liminf_n \tau_n/n \geq \sup\{c : S_c \neq \emptyset\} \stackrel{\text{def}}{=} \alpha. \tag{2.13}$$

Note that (2.5) implies that $S_c \neq \emptyset$ for some $c > 0$. Hence, $\alpha > 0$.

Step (b). Once Step (a) is done, Step (b) is routine. It suffices to show

$$\alpha \geq \beta \stackrel{\text{def}}{=} \limsup_n \tau_n/n. \tag{2.14}$$

Suppose the contrary that $\beta > \alpha$. In the following we only consider $\beta < \infty$. The case $\beta = \infty$ can be dealt with similarly. By continuity, first choose $\theta > 1$ and then $\epsilon > 0$ such that

$$\beta - \alpha\theta > 0 \quad \text{and} \quad ((\beta - \epsilon) - (\alpha + \epsilon)\theta)(\theta - 1) \geq \epsilon. \tag{2.15}$$

In particular, $\beta - \epsilon \geq (\alpha + \epsilon)\theta$. From (2.13) and Step (a), there exists $n_1 \geq n_0$ such that

$$\widetilde{T}_j^m \geq (\alpha - \epsilon)T_j^m \quad \text{for all } m \geq n_1 \text{ and } 1 \leq j \leq m. \tag{2.16}$$

By definition of β and (2.5), there is an $n > n_1$ such that

$$\tau_m \geq \tau_n \geq (\beta - \epsilon)n \quad \text{holds for } m \geq n. \tag{2.17}$$

We claim that

$$\widetilde{T}_j^{[\theta n]} \geq (\alpha + \epsilon)\sum_{s=j}^{[\theta n]} s \quad \text{for all } 1 \leq j \leq [\theta n] \tag{2.18}$$

which implies $S_{\alpha+\epsilon} \neq \phi$ as $[\theta n] \geq n \geq n_0$. It is a contradiction to (2.13) and thus (2.14) is verified. Since $\beta - \epsilon \geq (\alpha + \epsilon)\theta$, (2.18) for $n \leq j$ holds trivially by (2.17). Moreover,

$$\begin{aligned} \widetilde{T}_n^{[\theta n]} - (\alpha + \epsilon)\sum_{s=n}^{[\theta n]} s &\geq ((\beta - \epsilon)n - (\alpha + \epsilon)\theta n)([\theta n] - n + 1) \\ &\geq n((\beta - \epsilon) - (\alpha + \epsilon)\theta)(\theta - 1)n \geq \epsilon n^2 \end{aligned} \tag{2.19}$$

by (2.15). We have from (2.16) that for $1 \leq j < n$,

$$(\alpha + \epsilon)\sum_{s=j}^{n-1} s - \widetilde{T}_j^{n-1} \leq 2\epsilon\sum_{s=j}^{n-1} s \leq 2\epsilon\sum_{s=1}^{n-1} s \leq \epsilon n^2.$$

Adding up with (2.19), (2.18) for $1 \leq j < n$ follows immediately. This completes the proof of (2.18) and thus Step (b) as well. In particular, $\lim_n \tau_n/n$ exists.

Step (c). Let $\lim_n \tau_n/n = \alpha \in (0, \infty]$. Then $\tau_n \approx \alpha n$ for n no less than some number $M \geq n_0$. Hence,

$$\sum_{s=j}^{n} \tau_s \approx \alpha \sum_{s=j}^{n} s = \alpha(n+j)(n-j+1)/2 \approx \alpha(n^2 - j^2)/2 \quad \text{for } j \geq M. \tag{2.20}$$

By (1.14) and (1.21), we have $d_k \geq d_1 > 0 = d_0$ for all $k \geq 1$. Obviously, $\lim_{t\to\infty} t^4 e^{-d_k t/2} = 0$. By (2.2) and (1.9) we have that for t large,

$$t^4 \sum_{k=1}^{\infty} g_k(t) \leq \sum_{k=1}^{\infty} g_k(t/2) \leq g(t/2) \xrightarrow{t\to\infty} 0. \tag{2.21}$$

It follows from (2.1), (2.2), (2.5) and (2.21) that

$$1 \approx \sum_{j=1}^{n} g_0 \left(\sum_{s=j}^{n} \tau_s \right) = c_0 \sum_{j=1}^{n} \left(\sum_{s=j}^{n} \tau_s \right)^{-1/2}, \tag{2.22}$$

which is almost the same as (1.5). We proceed as in [5]. Since

$$\max_{1\leq j<M} (\widetilde{T}_j^n)^{-1/2} \leq \tau_n^{-1/2} \xrightarrow{n} 0$$

by (2.5), we get from (2.20) and (2.22) that

$$1 \approx c_0 \sqrt{\frac{2}{\alpha}} \sum_{j=L}^{n} \frac{1}{n\sqrt{1-(j/n)^2}} \xrightarrow{n} c_0 \sqrt{\frac{2}{\alpha}} \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \frac{c_0 \pi}{\sqrt{2\alpha}}.$$

It follows that not only $\alpha < \infty$ but also $\alpha = (c_0\pi)^2/2$ as claimed in (2.6).

Proof of Lemma 2.2. **Part (i).** Let $D_n = \sum_{j=1}^n \left(\sum_{s=j}^n s \right)^{-1/2}$. Define

$$A_0 = 0 \quad \text{and} \quad A_j = \sum_{s=1}^{j} s \quad \text{for } j \geq 1.$$

By the Binomial Theorem,

$$\left(\sum_{s=j}^{n} s \right)^{-1/2} = (A_n - A_{j-1})^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} A_{j-1}^k / A_n^{k+1/2}.$$

Hence,

$$D_n = \sum_{j=1}^{n} (A_n - A_{j-1})^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \left[\sum_{j=1}^{n} A_{j-1}^k / A_n^{k+1/2} \right].$$

For each $k \geq 0$, the sum inside the bracket above is increasing in n by Lemma 1.2(vii) in [3]. Since $(-1)^k \binom{-\frac{1}{2}}{k} > 0$ for $k \geq 0$, we get (2.7). By keeping only the term $k = 0$,

$$\begin{aligned} D_{n+1} - D_n &\geq \frac{n+1}{A_{n+1}^{1/2}} - \frac{n}{A_n^{1/2}} \\ &= \frac{n+1}{\sqrt{(n+1)(n+2)/2}} - \frac{n}{\sqrt{n(n+1)/2}} \\ &\geq \sqrt{\frac{n+1}{n+2}} - \sqrt{\frac{n}{n+1}} = \frac{\frac{1}{(n+2)(n+1)}}{\sqrt{\frac{n+1}{n+2}} + \sqrt{\frac{n}{n+1}}} \geq \frac{1}{12n^2}. \end{aligned} \tag{2.23}$$

Part (ii). Let $H_n = \sum_{j=1}^n g(c\sum_{s=j}^n s)$. Since $d_0 = 0$ and all g_k in (2.2) are positive, a simple rearrangement after singling out the function g_0 shows $H_{n+1} \geq H_n$ holds if

$$c_0(D_{n+1} - D_n) \geq \sum_{j=1}^n \sum_{k=1}^\infty g_k \left(c \sum_{s=j}^n s \right). \tag{2.24}$$

By (2.21), the right-hand side above is bounded by $\sum_{j=1}^n (c\sum_{s=j}^n s)^{-4} \leq n(cn)^{-4} = c^{-4}n^{-3}$ when n is large. In view of (2.23), (2.24) holds for n large. The conclusion follows. □

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Jong-Yi Chen
Department of Applied Mathematics, National Dong-Hwa University, Hwalien 973, Taiwan
E-mail address: jongyi@mail.ndhu.edu.tw

Yunshyong Chow
Institute of Mathematics, Academia Sinica, Taipei 106, Taiwan
E-mail address: chow@math.sinica.edu.tw