TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 5, pp. 1365-1375, October 2014
DOI: 10.11650/tjm.18.2014.4137
This paper is available online at http://journal.taiwanmathsoc.org.tw

# FIXED POINTS FOR MULTIVALUED CONTRACTIONS IN $b$-METRIC SPACES WITH APPLICATIONS TO FRACTALS

Cristian Chifu and Gabriela Petruşel

**Abstract.** The purpose of this paper is to present some fixed point results for multivalued operators in $b$-metric spaces endowed with a graph, as well as, some existence results for multivalued fractals in b-metric spaces, using contractive conditions of Ćirić type with respect to the functional $H$.

## 1. Preliminaries

In this paper we will give some fixed point results for multivalued operators in b-metric spaces. Actually, the purpose of this paper is twofold.

First, we will present some fixed point results for Ćirić G-contraction in $b$-metric spaces endowed with a graph. The starting point for this is a result given by A. Nicolae, D. O'Regan and A. Petruşel in [7]. In this paper, the authors give some fixed point results for singlevalued and multivalued operators in metric spaces endowed with a graph. This approach, of using the context of metric spaces endowed with a graph, was recently introduced by J. Jachymski [5] and G. Gwóźdź-Lukawska, J. Jachymski [4].

The second purpose of this work is to give some existence results for the multivalued fractals in $b$-metric spaces. We will follow the approach given in [1].

Our results extend and complement some previous theorems given in [1, 2, 4, 5, 6], etc.

Since we will work in $b$-metric spaces, we start by presenting some notions about this type of spaces. For details see [3, 1, 2].

**Definition 1.1.** Let $(X, d)$ be a set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}$ is said to be a b-metric if and only if for all $x, y, z \in \mathbb{X}$, the following conditions are satisfied:

1. $d(x, y) = 0 \Longleftrightarrow x = y$;

Received December 4, 2013, accepted January 20, 2014.
Communicated by Boris Mordukhovich.
2010 *Mathematics Subject Classification*: 47H10, 54H25, 47H04.
*Key words and phrases*: Fixed point, $b$-Metric space, Connected graph, Multivalued fractal.

2. $d(x,y) = d(y,x)$;
3. $d(x,y) \leq s[d(x,z) + d(z,y)]$.

In this case, the pair $(X,d)$ is called $b-metric$ space with constant $s$.

**Remark 1.2.** The class of b-metric spaces is larger than the class of metric spaces since a b-metric space is a metric space when s=1.

**Example 1.3.** Let X=$\{0,1,2\}$ and $d : X \times X \to \mathbb{R}_+$ such that $d(0,1) = d(1,0) = d(0,2) = d(2,0) = 1, d(1,2) = d(2,1) = \alpha \geq 2, d(0,0) = d(1,1) = d(2,2) = 0$. Then

$$d(x,y) \leq \frac{\alpha}{2}[d(x,z) + d(z,y)], \text{ for } x,y,z \in X.$$

Then $(X,d)$ is a b-metric space. If $\alpha > 2$ the ordinary triangle inequality does not hold and $(X,d)$ is not a metric space.

**Example 1.4.** The set $l^p(\mathbb{R}) = \left\{(x_n) \subset \mathbb{R} | \lim_{n=1}^{\infty} |x_n|^p < \infty\right\}, 0 < p < 1$, together with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}_+, d(x,y) = \left(\lim_{n=1}^{\infty} |x-y|^p\right)^{1/p}$, is a b-metric space with constant $s = 2^{1/p}$.

**Definition 1.5.** Let $(X,d)$ be a $b-metric$ space with constant $s$. Then the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called:

1. Convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$;
2. Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to \infty$.

**Definition 1.6.** Let $(X,d)$ be a $b-metric$ space with constant $s$. If $Y$ is a nonempty subset of $X$, then the closure $\overline{Y}$ of $Y$ is the set of limits of all convergent sequences of points in $Y$, i.e.,

$$\overline{Y} := \{x \in X : \exists (x_n)_{n\in\mathbb{N}}, x_n \to x, \text{ as } n \to \infty\}.$$

**Definition 1.7.** Let $(X,d)$ be a $b-metric$ space with constant $s$. Then a subset $Y \subset X$ is called:

1. closed if and only if for each sequence $(x_n)_{n\in\mathbb{N}} \subset Y$ which converges to $x$, we have $x \in Y$;
2. compact if and only if for every sequence of elements of $Y$ there exists a subsequence that converges to an element of $Y$;
3. bounded if and only if $\delta(Y) := \{d(a,b) : a, b \in Y\} < \infty$.

**Definition 1.8.** The $b-metric$ space $(X,d)$ is complete if every Cauchy sequence in $X$ converges.

Let us consider the following families of subsets of a b-metric space $(X,d)$:

$$\mathcal{P}(X) = \{Y \,|Y \subset X\}, P(X) := \{Y \in \mathcal{P}(X) \,|Y \neq \emptyset\}; P_b(X)$$
$$:= \{Y \in P(X)|\ Y \text{ is bounded }\},$$
$$P_{cl}(X) := \{Y \in P(X)|\ Y \text{ is closed}\}; P_{cp}(X) := \{Y \in P(X)|\ Y \text{ is compact}\}$$

Let us define the gap functional $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,as:

$$D(A,B) = \inf\{d(a,b) \mid a \in A,\ b \in B\}.$$

In particular, if $x_0 \in X$, then $D(x_0, B) := D(\{x_0\}, B)$.

The excess generalized functional $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, as:

$$\rho(A,B) = \sup\{D(a,B) \mid a \in A\}.$$

The Pompeiu-Hausdorff generalized functional: $H : P(X)\times P(X) \to \mathbb{R}_+\cup\{+\infty\}$, as:

$$H(A,B) = \max\{\rho(A,B), \rho(B,A)\}.$$

The generalized diameter functional: $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$, as:

$$\delta(A,B) = \sup\{d(a,b) \mid a \in A,\ b \in B\}.$$

In particular $\delta(A) := \delta(A,A)$ is the diameter of the set $A$.

Let $T : X \to P(X)$ be a multivalued operator and $Graph(T) := \{(x,y) \in X\times X|y \in T(x)\}$ be the graphic of $T$. An element $x \in X$ is called a fixed point for $T$ if and only if $x \in T(x)$.

The set $Fix(T) := \{x \in X|\ x \in T(x)\}$ is called the fixed point set of $T$, while $SFix(T) = \{x \in X|\ \{x\} = T(x)\}$ is called the strict fixed point set of $T$. Notice that $SFix(T) \subseteq Fix(T)$.

The following properties of some of the functionals defined above will be used throughout the paper (see [2, 3] for details and proofs):

**Lemma 1.9.** *Let $(X,d)$ be a b-metric space with constant $s$ and $A, B \in P(X)$. Suppose that there exists $\eta > 0$ such that:*

$(i)$ *for each $a \in A$, there is $b \in B$ such that $d(a,b) \leq \eta$;*

$(ii)$ *for each $b \in B$, there is $a \in A$ such that $d(a,b) \leq \eta$.*

*Then, $H(A,B) \leq \eta$.*

**Lemma 1.10.** *Let (X,d) be a b-metric space with constant $s$. Then*

$$D(x, A) \le s\left[d(x, y) + D(y, A)\right], \text{ for all } x, y \in X, A \subset X.$$

**Lemma 1.11.** *Let (X,d) be a b-metric space with constant $s$ and $A, B \in P_b(X)$, $a \in A$. Then, for $\varepsilon > 0$, there exists $b \in B$ such that $d(a, b) \le H(A, B) + \varepsilon$.*

*Proof.* Suppose that there exists $\varepsilon > 0$ such that for every $b \in B$

$$d(a, b) > H(A, B) + \varepsilon.$$

If we take, in the above inequality, the infimum with respect to $b \in B$, then

$$H(A, B) \ge D(a, B) \ge H(A, B) + \varepsilon.$$

Hence $\varepsilon \le 0$ which is a contradiction. ■

**Lemma 1.12.** *Let (X,d) be a b-metric space with constant $s$ and with $d : X \times X \to \mathbb{R}_+$ a continuous b-metric and let $A, B \in P_{cp}(X)$. Then for each $a \in, A$ there exists $b \in B$ such that*

$$d(a, b) \le H(A, B).$$

**Lemma 1.13.** *Let (X,d) be a b-metric space with constant $s$, $A \in P(X)$ and $x \in X$. Then $D(x, A) = 0$ if and only if $x \in \overline{A}$.*

## 2. Fixed Point Result in $b$-Metric Spaces Endowed with a Graph

Let $(X, d)$ be a $b$-metric space and $\Delta$ be the diagonal of $X \times X$. Let $G$ be a directed graph such that the set $V(G)$ of its vertices coincides with $X$ and $\Delta \subseteq E(G)$, $E(G)$ being the set of the edges of the graph. Assuming that $G$ has no parallel edges we will have that $G$ can be identified with the pair $(V(G), E(G))$.

If $x$ and $y$ are vertices of $G$, then a path in $G$ from $x$ to $y$ of length $k \in \mathbb{N}$ is a finite sequence $(x_n)_{n \in \{0,1,2,\cdots,k\}}$ of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1, 2, \cdots, k\}$.

Let us denote by $\tilde{G}$ the undirected graph obtained from $G$ by ignoring the direction of edges. Notice that a graph $G$ is connected if there is a path between any two vertices and it is weakly connected if $\tilde{G}$ is connected.

Let $G^{-1}$ be the graph obtained from $G$ by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Since it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention, we have that

$$E(\tilde{G}) = E(G) \cup E(G^{-1})$$

**Definition 2.1.** Let $(X, d)$ be a complete b-metric space with constant $s$ and $G$ be a directed graph. We say that the triple $(X, d, G)$ has the property $(A)$ if for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x,$ as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$.

In this section we will prove some fixed point results for a multivalued operator satisfying a contractive condition of Ćirić type with respect to the functional $H$. The data dependence of fixed point set is also studied.

In [7] the following set is defined:

$$X_T := \{x \in X : \text{ there exists } y \in T(x) \text{ such that } (x, y) \in E(G)\}.$$

**Definition 2.2.** Let $(X, d)$ be a complete b-metric space with constant $s$, $G$ be a directed graph and $T : X \to P_b(X)$ a multivalued mapping. The mapping $T$ is said to be a multivalued Ćirić G-contraction with constant $a$ if $0 < a < \frac{1}{s}$ and

(a) $H(T(x), T(y)) \leq a \max\left\{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2s}\left[D(x, T(y)) + D(y, T(x))\right]\right\}$, for all $(x, y) \in E(G)$;

(b) for $(x, y) \in E(G)$, if $u \in T(x)$ and $v \in T(y)$ are such that $d(u, v) \leq ad(x, y) + \alpha$, for some $\alpha > 0$, then $(u, v) \in E(G)$.

**Theorem 2.3.** *Let $(X, d)$ be a complete b-metric space with constant $s$ and $G$ be a directed graph such that the triple $(X, d, G)$ has the property $(A)$. If $T : X \to P_{b,cl}(X)$ is a Ćirić G-contraction, then:*

$(i)$ *For any $x \in X_T$, $T|_{[x]_{\tilde{G}}}$ has a fixed point;*

$(ii)$ *If $X_T \neq \emptyset$ and $G$ is weakly connected, then $T$ has a fixed point in $X$;*

$(iii)$ *If $Y = \cup\left\{[x]_{\tilde{G}}; x \in X_T\right\}$, then $T|_Y$ has a fixed point in $Y$;*

$(iv)$ *$FixT \neq \emptyset$ if and only if $X_T \neq \emptyset$.*

*Proof.* (i) Let $x_0 \in X_T$. There exists $x_1 \in T(x_0)$ such that $(x_0, x_1) \in E(G)$. Since $T$ is a Ćirić G-contraction there exists $a \in (0, \infty)$ with $a < \frac{1}{s}$ such that

$$\begin{aligned}
& H(T(x_0), T(x_1)) \\
\leq\ & a \max\{d(x_0, x_1), D(x_0, T(x_0)), D(x_1, T(x_1)), \\
& \frac{1}{2s}\left[D(x_0, T(x_1)) + D(x_1, T(x_0))\right]\} \\
\leq\ & a \max\{d(x_0, x_1), D(x_1, T(x_1)), \frac{1}{2s}\left[D(x_0, T(x_1)) + D(x_1, T(x_0))\right]\}
\end{aligned}$$

Leta us denote $\max\left\{d(x_0, x_1), D(x_1, T(x_1)), \frac{1}{2s}\left[D(x_0, T(x_1)) + D(x_1, T(x_0))\right]\right\} := M_1$

We have the following cases:

- If $M_1 = d(x_0, x_1)$, then $H(T(x_0), T(x_1)) \leq ad(x_0, x_1)$;
- If $M_1 = D(x_1, T(x_1))$, then

$$H(T(x_0), T(x_1)) \leq aD(x_1, T(x_1)) \leq aH(T(x_0), T(x_1)).$$

  Thus imply that $a \geq 1$ which is a contradiction.
- If $M_1 = \frac{1}{2s}[D(x_0, T(x_1)) + D(x_1, T(x_0))]$

$$\begin{aligned} H(T(x_0), T(x_1)) &\leq \frac{a}{2s}[D(x_0, T(x_1)) + D(x_1, T(x_0))] \\ &\leq \frac{a}{2s}[sd(x_0, x_1) + sD(x_1, T(x_1))] \\ &\leq \frac{a}{2}[d(x_0, x_1) + H(T(x_0), T(x_1))] \\ H(T(x_0), T(x_1)) &\leq \frac{a}{2-a}d(x_0, x_1) \leq ad(x_0, x_1). \end{aligned}$$

  Hence $H(T(x_0), T(x_1)) \leq ad(x_0, x_1)$. Using Lemma 1.3., for $\varepsilon = a$, we obtain that there exists $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq H(T(x_0), T(x_1)) + a \leq ad(x_{0,}x_1) + a.$$

We have: $(x_0, x_1) \in E(G)$, $x_1 \in T(x_0)$, $x_2 \in T(x_1)$ and $d(x_1, x_2) \leq ad(x_{0,}x_1) + a$. Using $(b)$ from Definition 2.2., we obtain that $(x_1, x_2) \in E(G)$.

Again we'll obtain that $H(T(x_1), T(x_2)) \leq ad(x_1, x_2) \leq a^2 d(x_{0,}x_1) + a^2$. Using Lemma 1.3., for $\varepsilon = a^2$, there exists $x_3 \in T(x_2)$ satisfying

$$d(x_2, x_3) \leq H(T(x_1), T(x_2)) + a^2 \leq a^2 d(x_{0,}x_1) + 2a^2.$$

Continuing this process we have $x_{n+1} \in T(x_n)$ such that $(x_n, x_{n+1}) \in E(G)$ and $d(x_n, x_{n+1}) \leq a^n d(x_{0,}x_1) + na^n$ for each $n \in \mathbb{N}$.

We have:

$$\begin{aligned} d(x_n, x_{n+p}) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^p d(x_{n+p-1}, x_{n+p}) \\ &\leq sa^n d(x_0, x_1) + sna^n + s^2 a^{n+1} d(x_0, x_1) + s^2(n+1)a^{n+1} ... + \\ &\quad + s^p a^{n+p-1} d(x_0, x_1) + s^p(n+p-1)a^{n+p-1} \\ &= sa^n \frac{1-(sa)^p}{1-sa} d(x_0, x_1) + nsa^n \frac{1-(sa)^p}{1-sa} + sa^n \lim_{k=1}^{p-1} k(as)^k. \end{aligned}$$

Hence, if $n \to \infty$, $d(x_n, x_{n+p}) \to 0$. Thus the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete *b-metric* space. Hence there exists $x \in X$ such that $x_n \to x$, as $n \to \infty$. By the property $(A)$ we have that $(x_n, x) \in E(G)$, for each $n \in \mathbb{N}$.

Hence by the above relation and the definition of Ćirić G-contraction, we obtain that

$$0 \leq \lim_{n\to\infty} H\left(Tx_n, Tx\right) \leq \lim_{n\to\infty} d\left(x_n, x\right) = 0.$$

Now, we prove that $x \in T\left(x\right)$.

We have:

$$\begin{aligned} D\left(x, T\left(x\right)\right) &\leq sd(x, x_{n+1}) + sD\left(x_{n+1}, T\left(x\right)\right) \\ &\leq sd(x, x_{n+1}) + sH\left(T\left(x_n\right), T\left(x\right)\right) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Thus $D\left(x, T\left(x\right)\right) = 0$, which implies that $x \in T\left(x\right)$.

On the other hand, since $(x_n, x) \in E(G)$, for $n \in \mathbb{N}$, we conclude that $(x_0, ..., x_{kn}, x)$ is a path in $G$, and thus $x \in [x_0]_{\widetilde{G}}$.

(ii) Since $X_T \neq \emptyset$, there exists $x_0 \in X_T$. As the graph $G$ is weakly connected we have that $[x_0]_{\widetilde{G}} = X$, then by **(i)**, $T$ has a fixed point in $X$.

(iii) it's a consequence of **(i)** and **(ii).**

(iv) If $FixT \neq \emptyset$, then there exists $x \in T\left(x\right)$. Since $\Delta \subset E\left(G\right)$ we have that $(x, x) \in E\left(G\right)$ and thus $x \in X_T$.

Now, if $X_T \neq \emptyset$, from **(i)** we have that $FixT \neq \emptyset$. ∎

**Theorem 2.4.** *Let $(X, d)$ be a complete b-metric space with constant $s$ and $G$ be a directed graph such that the triple $(X, d, G)$ has the property $(A)$. Let $T_1, T_2 : X \to P_{cp}(X)$ be two multivalued Ćirić G-contractions with the constants $a_1$and $a_2$. Suppose that:*

$(a)$ *there exists $\eta > 0$ such that $\delta(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$;*

$(b)$ $\forall\ x \in X$ , $X_{T_1} \neq \emptyset$ *and* $X_{T_2} \neq \emptyset$.

*In these conditions we have:*

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\eta s}{1 - s\max\left\{a_1, a_2\right\}} + \frac{s\max\left\{a_1, a_2\right\}}{\left(1 - s\max\left\{a_1, a_2\right\}\right)^2}.$$

*Proof.* We'll show that for every $x_1^* \in Fix(T_1)$, there exists $x_2^* \in Fix(T_2)$ such that

$$d(x_1^*, x_2^*) \leq \frac{s\eta}{1 - sa_2} + \frac{sa_2}{\left(1 - sa_2\right)^2}.$$

Let $x_1^* \in Fix(T_1)$ arbitrary. Then $(x_1^*, x_1^*) \in E(G)$ (since $\Delta \subseteq E(G)$) and so $x_1^* \in X_{T_1} = \left\{x \in X : \text{ there exists } y \in T_1\left(x\right) \text{ such that } (x, y) \in E\left(G\right)\right\}$. Thus, as in the proof of Theorem 2.1. we construct a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations of $T_2$, with $x_0 := x_1^*$ having the following properties:

(1) $(x_n, x_{n+1}) \in E(G) \cap Graph\,(T_2)$ , for each $n \in \mathbb{N}$;

(2) $d(x_n, x_{n+1}) \leq a_2^n d(x_0, x_1) + na_2^n$, for each $n \in \mathbb{N}$;

If we consider that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_2^*$, we have that $x_2^* \in Fix(T_2)$. Moreover, for each $n \geq 0$, we have:

$$d(x_n, x_{n+p}) \leq sa_2^n \frac{1-(sa_2)^p}{1-sa_2} d(x_0, x_1) + nsa_2^n \frac{1-(sa_2)^p}{1-sa_2} + sa_2^n \lim_{k=1}^{p-1} k(sa_2)^k,\ p \in \mathbb{N}^*.$$

Letting $p \to \infty$ we get that

$$d(x_n, x_2^*) \leq \frac{sa_2^n}{1-sa_2} d(x_0, x_1) + \frac{nsa_2^n}{1-sa_2} + \frac{sa_2}{(1-sa_2)^2}, \forall n \in \mathbb{N}.$$

Choosing $n = 0$ in the above relation, we obtain

$$\begin{aligned} d(x_1^*, x_2^*) &\leq \frac{s}{1-sa_2} d(x_1^*, x_1) + \frac{sa_2}{(1-sa_2)^2} \leq \frac{s}{1-sa_2} \delta(T_1(x_1^*), T_2(x_1^*)) + \frac{sa_2}{(1-sa_2)^2} \\ &\leq \frac{s\eta}{1-sa_2} + \frac{sa_2}{(1-sa_2)^2}. \end{aligned}$$

Interchanging the roles of $T_1$ and $T_2$ we obtain that for every $u \in Fix\,(T_2)$ ,there exists $v \in Fix\,(T_1)$ such that

$$d(u, v) \leq \frac{s\eta}{1-sa_1} + \frac{sa_1}{(1-sa_1)^2}.$$

Thus, $H(Fix(T_1), Fix(T_2)) \leq \frac{\eta s}{1-s\max\{a_1,a_2\}} + \frac{s\max\{a_1,a_2\}}{(1-s\max\{a_1,a_2\})^2}$. ■

**Remark 2.5.** It is an open question to develop the above theorems to the case of multivalued operators satisfying Ćirić type conditions with respect to the functional $\delta$, see [3].

## 3. Fixed Point Results for Multivalued Fractals in $b$-Metric Spaces

Let $(X, d)$ be a $b-metric$ space with constant $s$ and $F_1, ..., F_m : X \to P\,(X)$ be multivalued operators. The system $F = (F_1, ..., F_m)$ is called an iterated multifunction system (IMS).

If $F = (F_1, ..., F_m)$ is such that $F_i : X \to P_{cp}\,(X)\,, i = \overline{1, m}$, are upper semicontinuos, then the operator $T_F$ defined as

$$T_F\,(Y) = \bigcup_{i=1}^{m} F_i\,(Y)\,,\ \text{for each } Y \in P_{cp}\,(X)\,,$$

is called the multi-fractal operator generated by the iterated multifunction system $F = (F_1, ..., F_m)$.

Since the operators $F_i : X \to P_{cp}(X), i = \overline{1, m}$, are upper semicontinuos, then $T_F : P_{cp}(X) \to P_{cp}(X)$.

A nonempty compact subset $A^* \subset X$ is said to be a multivalued fractals with respect to the iterated multifunction system $F = (F_1, ..., F_m)$ if and only if it is a fixed point for the associated multi-fractal operator, i.e. $T_F(A^*) = A^*$.

In this section we prove some existence and the uniqueness results for the self-similar set (fractal) of an iterated multifunction system in complete *b-metric* spaces.

**Definition 3.1.** Let $(X, d)$ be a b-metric space with constant $s$. $f : X \to X$ is said to be a Ćirić operator if there exists $a \in (0, \infty)$ with $a < \frac{1}{s}$ such that $d(f(x), f(y)) \leq a \max\left\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2s}\left[d(x, f(y)) + d(y, f(x))\right]\right\}$, for all $x, y \in X$.

**Definition 3.2.** Let $(X, d)$ be a b-metric space with constant $s$. $F : X \to P_{cp}(X)$ is said to be a multivalued Ćirić-type operator if there exists $a \in (0, \infty)$ with $a < \frac{1}{s}$ such that

$H(F(x), F(y)) \leq a \max\left\{d(x, y), \frac{1}{2s}\left[D(x, F(y)) + D(y, F(x))\right]\right\}$, for all $x, y \in X$.

**Theorem 3.3.** *Let $(X, d)$ be a $b-metric$ space with constant $s$ and let $f : X \to X$ be a Ćirić type operator with constant $a \in \left(0, \frac{1}{s}\right)$. Then*

$(i)$ $Fixf = \{x^*\}$;

$(ii)$ $\forall x \in X$ *the sequence* $(f^n(x))_{n\in\mathbb{N}} \xrightarrow{d} x^*$, *as* $n \to \infty$.

*Proof.* The existence of the fixed point and (ii) follows from Theorem 2.3. So we have to prove the uniqueness.

Suppose that $x^*, y^* \in Fixf, x^* \neq y^*$.

Then $d(x^*, y^*) = d(f(x^*), f(y^*)) \leq a \max\left\{d(x^*, y^*), \frac{1}{2s}\left[d(x^*, f(y^*)) + d(y^*, f(x^*))\right]\right\}$.

If $\max\left\{d(x^*, y^*), \frac{1}{2s}\left[d(x^*, f(y^*)) + d(y^*, f(x^*))\right]\right\} = d(x^*, y^*)$, then we obtain that $a \geq 1$ which is a contradiction.

If $\max\left\{d(x^*, y^*), \frac{1}{2s}\left[d(x^*, f(y^*)) + d(y^*, f(x^*))\right]\right\} = \frac{1}{2s}\left[d(x^*, f(y^*)) + d(y^*, f(x^*))\right]$, then $d(x^*, y^*) \leq \frac{a}{s} d(x^*, y^*)$. Thus, $\frac{a}{s} \geq 1$ again a contradiction. ■

**Theorem 3.4.** *Let $(X, d)$ be a complete b-metric space with constant $s$, such that $d : X \times X \to \mathbb{R}_+$ is a continuous b-metric. Let $F_i : X \to P_{cp}(X), i = \overline{1, m}$, be upper semicontinuos multivalued Ćirić-type operators. Then, the multivalued operator $T_F$ generated by the iterated multifunction system $F = (F_1, ..., F_m)$, by the relation $T_F(Y) = \bigcup_{i=1}^{m} F_i(Y)$, for each $Y \in P_{cp}(X)$, verify the following conditions:*

$(i)$ $T_F : (P_{cp}(X), H) \to (P_{cp}(X), H)$;

$(ii)$ $T_F$ *is a Ćirić type operator, in the sense that*

$$H(T_F(Y_1), T_F(Y_2)) \leq a \max\{H(Y_1, Y_2), H(Y_1, T_F(Y_1), H(Y_2, T_F(Y_2)), \frac{1}{2s}[H(Y_1, T_F(Y_2) + H(Y_2, T_F(Y_1)]\}$$

$(iii)$ *There exists a unique multivalued fractal* $A^*_{T_F} \in P_{cp}(X)$ *such that* $(T^n_F(A))_{n\in\mathbb{N}} \xrightarrow{H} A^*_{T_F}$, *as* $n \to \infty$, *for every* $A \in P_{cp}(X)$.

*Proof.* **(**i) By the upper semicontinuity of the operators $F_i, i = \overline{1,m}$, we have that $T_F : (P_{cp}(X), H) \to (P_{cp}(X), H)$.

(ii) We will prove first that for any $Y_1, Y_2 \in P_{cp}(X)$ we have

$$H(F_i(Y_1), F_i(Y_2)) \leq a_i \max\{H(Y_1, Y_2), H(Y_1, F_i(Y_1), H(Y_2, F_i(Y_2)), \frac{1}{2s}[H(Y_1, F_i(Y_2) + H(Y_2, F_i(Y_1)]\}, i = \overline{1,m}.$$

For this purpose, let $Y_1, Y_2 \in P_{cp}(X)$. For each $i = \overline{1,m}$ we have:

$$\begin{aligned}
&\rho(F_i(Y_1), F_i(Y_2)) = \sup_{x\in Y_1} \rho(F_i(x), F_i(Y_2)) \\
&= \sup_{x\in Y_1}(\inf_{y\in Y_2}(\rho(F_i(x), F_i(y)))) \leq \sup_{x\in Y_1}(\inf_{y\in Y_2}(H(F_i(x), F_i(y)))) \\
&\leq \sup_{x\in Y_1}(\inf_{y\in Y_2}(a_i \max\{d(x,y), \frac{1}{2s}[D(x, F_i(y)) + D(y, F_i(x))]\})) \\
&\leq a_i \max\{H(Y_1, Y_2), \frac{1}{2s}[\rho(Y_1, F_i(Y_2)) + \rho(Y_2, F_i(Y_1))]\} \\
&\leq a_i \max\{H(Y_1, Y_2), \frac{1}{2s}[H(Y_1, F_i(Y_2)) + H(Y_2, F_i(Y_1))]\} \\
&\leq a_i \max\{H(Y_1, Y_2), H(Y_1, F_i(Y_1), H(Y_2, F_i(Y_2)), \\
&\quad \frac{1}{2s}[H(Y_1, F_i(Y_2)) + H(Y_2, F_i(Y_1))]\}
\end{aligned}$$

Hence, for each $i = \overline{1,m}$ we have

$$H(F_i(Y_1), F_i(Y_2)) \leq a_i \max\{H(Y_1, Y_2), H(Y_1, F_i(Y_1), H(Y_2, F_i(Y_2)), \frac{1}{2s}[H(Y_1, F_i(Y_2) + H(Y_2, F_i(Y_1)]\}.$$

Using the following property

$$H\left(\bigcup_{i=1}^m F_i(Y_1), \bigcup_{i=1}^m F_i(Y_2)\right) \leq \max\{H(F_i(Y_1), F_i(Y_2)), ..., H(F_i(Y_1), F_i(Y_2))\},$$

we obtain that

$$\begin{aligned} H(T_F(Y_1), T_F(Y_2)) &\leq \max_{i\in\{1,2,\cdots,m\}} \{H(F_i(Y_1), F_i(Y_2))\} \\ &\leq a \max\{H(Y_1, Y_2), H(Y_1, T_F(Y_1)), H(Y_2, T_F(Y_2)), \\ &\quad \frac{1}{2s}\left[H(Y_1, T_F(Y_2)) + H(Y_2, T_F(Y_1))\right]\}, \end{aligned}$$

where $a := \max_{i\in\{1,2,\cdots,m\}} a_i$.

(iii) from (ii) we have $T_F$ is a (singlevalued) Ćirić type operator on the complete *b-metric* space $(P_{cp}(X), H)$, and thus, by Theorem 3.1., we obtain that $Fix(T_F) = \left\{A^*_{T_F}\right\}$ and $T^n_F(A) \to A^*_{T_F}$, as $n \to \infty$, for each $A \in P_{cp}(X)$. ■

## References


1. M. Boriceanu, M. Bota and A. Petruşel, Multivalued fractals in b-metric spaces, *Cent. Eur. J. Math.*, **8(2)** (2010), 367-377.
2. M. Boriceanu, A. Petruşel and A. I. Rus, Fixed point theorems for some multivalued generalized contraction in $b$-metric spaces, *Internat. J. Math. Statistics*, **6** (2010), 65-76.
3. S. Czerwik, Nonlinear set-valued contraction mappings in $b$-metric spaces, *Atti Sem. Mat. Univ. Modena*, **46** (1998), 263-276.
4. G. Gwóźdź-Lukawska and J. Jachymski, IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem, *J. Math. Anal. Appl.*, **356** (2009), 453-463.
5. J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Amer. Math. Soc.*, **136** (2008), 1359-1373.
6. J. Harjani and K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, *Nonlinear Anal.*, **71** (2009), 3403-3410.
7. A. Nicolae, D. O'Regan and A. Petruşel, Fixed point theorems for singlevalued and multivalued generalized contractions in metric spaces endowed with a graph, *J. Georgian Math. Soc.*, **18** (2011), 307-327



Cristian Chifu and Gabriela Petruşel
Babeş-Bolyai University Cluj-Napoca
Faculty of Business
Department of Business
Horea street, No. 7, Cluj-Napoca
Romania
E-mail: Cristian.Chifu@tbs.ubbcluj.ro
Gabi.Petrusel@tbs.ubbcluj.ro