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MULTILINEAR ESTIMATES ON FREQUENCY-UNIFORM DECOMPOSITION SPACES AND APPLICATIONS

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Abstract. We study multilinear operators $T(f_1, f_2, ..., f_m)$ that commutes with simultaneous translations and prove that if T is bounded from $L^{p_1} \times L^{p_2} \times ... \times L^{p_m}$ to L^p, then for any $r \geqslant p$, $0 < p, q \leqslant \infty$ and

$$s > \begin{cases} n(1 - 1 \wedge \frac{1}{q}), & (\frac{1}{p}, \frac{1}{q}) \in D_1; \\ n(1 \vee \frac{1}{p} \vee \frac{1}{q} - \frac{1}{q}), & (\frac{1}{p}, \frac{1}{q}) \in \mathbb{R}^2_+ - D_1, \end{cases}$$

$(D_1 = \{(\frac{1}{p}, \frac{1}{q}) \in \mathbb{R}^2_+ : \frac{1}{q} \geqslant \frac{2}{p}, \frac{1}{p} \leqslant \frac{1}{2}\})$T is bounded from $M^s_{p_1,q} \times M^s_{p_2,q} \times ... \times M^s_{p_m,q}$ to $M^s_{r,q}$ (which improves the results obtained by [5], [6].), where $M^s_{p,q}$ is the modulation spaces. Besides, we also obtain the similar results for Triebel-type spaces $N^s_{p,q}$ introduced by [21] (T is bounded from $N^s_{p,q} \times N^s_{p,q} \times ... \times N^s_{p,q}$ to $N^s_{p,q}$). As applications, we obtain the boundedness on the modulation spaces for the bilinear Hilbert transform, bilinear fractional integral, the pointwise product of functions, and the bilinear oscillatory integral along parabolas. Also, in modulation spaces and $N^s_{p,q}$, we study the well-posedness of the Cauchy problem for the fractional heat and Schrödinger equations with some new nonlinear terms. Such nonlinear well-posedness problems are not studied in other function spaces.

1. Introduction and Notation

In recent years, the study of multilinear integrals has been received more attention, which is motivated not only as the generalization of the linear theory but also the natural appearance of multilinear singular theory. Historically, the interest in the multilinear operator initialed by Calderón about 50 years ago, in order to study the Cauchy integral and the Hilbert transform on some Lipschitz curves. In this paper, we focus on the boundedness of the multilinear operator on modulation spaces and Triebel-type spaces

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$N^s_{p,q}$. As applications, we obtain the boundedness on the modulation spaces for the bilinear Hilbert transform, bilinear fractional integral, the pointwise product of functions, and the bilinear oscillatory integral in along parabolas.

Let $T : S(\mathbb{R}^n) \times S(\mathbb{R}^n) \times ... \times S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ be a continuous multilinear operator, from the product of Schwarz spaces into the space of tempered distributions, which commutes with simultaneous translations. Then there exists a $\mu \in S'(\mathbb{R}^n \times \mathbb{R}^n \times ... \times \mathbb{R}^n)$, the symbol of T, such that $T(f_1, f_2, ..., f_m)(x)$ is the pair $\langle F_x, \mu\rangle$, with

$$F_x(\xi_1, ..., \xi_m) = \prod_{j=1}^{m} \widehat{f_j}(\xi_j) e^{2\pi i \langle \xi_j, x\rangle}.$$

More precisely, if μ is locally integrable function, we may write

$$T(f_1, f_2, ..., f_m)(x) = \int_{\mathbb{R}^{nm}} \prod_{j=1}^{m} \widehat{f_j}(\xi_j) e^{2\pi i \langle \xi_j, x\rangle} \cdot \mu(\xi_1, \xi_2, ..., \xi_m) d\xi_1 d\xi_2 ... d\xi_m. \tag{1}$$

This μ is called the symbol (or multiplier) of the operator T. We always assume this μ a locally integrable function in the following content. This assumption suffices in our applications and in the rest theorems in this paper.

Multilinear analysis is one of active and important subjects in the study of harmonic analysis and its related topics. [13], [19], and [20] give the boundedness of the bilinear fractional integral and bilinear Hilbert transform on $L^p(\mathbb{R}^n)$. [6] gives the boundedness of the multilinear operator $T(f_1, ..., f_m)$ on the modulation spaces as follows.

Theorem A. *Let T be defined by (1) with locally integrable $\mu \in S'(\mathbb{R}^n \times \mathbb{R}^n \times ... \times \mathbb{R}^n)$. If there exist $p, p_1, ..., p_m \in (0, +\infty]$ such that T can be extended to a bounded operator from $L^{p_1} \times L^{p_2} \times ... \times L^{p_m}$ to L^p*

$$\|T(f_1, f_2, ..., f_m)\|_{L^p(\mathbb{R}^n)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)},$$

then for any $r \geqslant max\{1, p\}$ and any $r_j \leqslant p_j$, we have

$$\|T(f_1, f_2, ..., f_m)\|_{M^s_{r,q}(\mathbb{R}^n)} \lesssim \prod_{j=1}^{m} \|f_j\|_{M^s_{r_j,q}(\mathbb{R}^n)},$$

where $0 < q \leqslant 1$, $s \geqslant 0$.

In this paper, the following theorem 1 modifies the technique in [6] with the Young inequality of number series to improve the indices of Theorem A.

Theorem 1. *Let T be defined by (1) with locally integrable* $\mu \in S'(\mathbb{R}^n \times \mathbb{R}^n \times ... \times \mathbb{R}^n)$*. If there exist* $p, p_1, ..., p_m \in (0, +\infty]$ *such that T can be extended to a bounded operator from* $L^{p_1} \times L^{p_2} \times ... \times L^{p_m}$ *to* L^p

$$\|T(f_1, f_2, ..., f_m)\|_{L^p(\mathbb{R}^n)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)},$$

then for any $r \geqslant p$, $0 < q \leqslant \infty$ *and any* $r_j \leqslant p_j$*, we have*

$$\|T(f_1, f_2, ..., f_m)\|_{M^s_{r,q}(\mathbb{R}^n)} \lesssim \prod_{j=1}^{m} \|f_j\|_{M^s_{r_j,q}(\mathbb{R}^n)},$$

where

$$s > \sigma(p,q) = \begin{cases} n(1 - 1 \wedge \frac{1}{q}), & (\frac{1}{p}, \frac{1}{q}) \in D_1; \\ n(1 \vee \frac{1}{p} \vee \frac{1}{q} - \frac{1}{q}), & (\frac{1}{p}, \frac{1}{q}) \in \mathbb{R}^2_+ - D_1 \end{cases}$$

and

$$D_1 = \{(\frac{1}{p}, \frac{1}{q}) \in \mathbb{R}^2_+ : \frac{1}{q} \geqslant \frac{2}{p}, \frac{1}{p} \leqslant \frac{1}{2}\}.$$

Recall the bilinear Hilbert transform

$$H(f,g)(x) = p.v. \int_{\mathbb{R}} \frac{f(x+t)g(x-t)}{t} dt$$

and bilinear fractional integral

$$B_\alpha(f,g) = \int_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\alpha}} dt.$$

As applications of Theorem 1, we easily obtain the boundedness on the modulation spaces for the bilinear Hilbert transform and the bilinear fractional integral.

Corollary 1. *Let* $0 < \alpha < n$, $1/p_1 + 1/p_2 > \alpha/n$, $1/p = 1/p_1 + 1/p_2 - \alpha/n$ *and* $p_1, p_2 > 1$*. We say that* (r, p_1, p_2) *is an* α*-triplet if*

$$1/r \leqslant 1/p_1 + 1/p_2 - \alpha/n.$$

If (r, p_1, p_2) *is an* α*-triplet, then for* $0 < q \leqslant \infty$ *and* $s > \sigma(p,q)$*, we have*

$$\|B_\alpha(f,g)\|_{M^s_{r,q}} \lesssim \|f\|_{M^s_{p_1,q}} \|g\|_{M^s_{p_2,q}}.$$

Proof. Since

$$B_\alpha(f,g)(x) \simeq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{f}(\xi_1)\widehat{g}(\xi_2)|\xi_1 - \xi_2|^{-\alpha} e^{2\pi i \langle x, (\xi_1 + \xi_2)\rangle} d\xi_1 d\xi_2.$$

Corollary 1 follows from $\|B_\alpha(f,g)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^r(\mathbb{R}^n)}$ (Grafakos [13] and Kenig and Stein [19]) and Theorem 1.

Corollary 2. *Suppose $1 < p_1, p_2 \leqslant \infty$, $2/3 < p < \infty$, $r \geqslant p$ and $1/p = 1/p_1 + 1/p_2$. Then for $0 < q \leqslant \infty$ and $s > \sigma(p,q)$, we have*

$$\|H(f,g)\|_{M^s_{r,q}} \lesssim \|f\|_{M^s_{p_1,q}}\|g\|_{M^s_{p_2,q}}.$$

Proof. Since

$$H(f,g)(x) \simeq \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f}(\xi_1)\widehat{g}(\xi_2)sgn(\xi_1-\xi_2)e^{2\pi i\langle x,(\xi_1+\xi_2)\rangle}d\xi_1 d\xi_2.$$

Corollary 2 follows from $\|H(f,g)\|_{L^p(\mathbb{R})} \leqslant C\|f\|_{L^q(\mathbb{R})}\|g\|_{L^r(\mathbb{R})}$ (Lacey and Thiele[20]) and Theorem 1. For more details on the bilinear Hilbert transform and its many extensions and developments, one can refer to [22, 9, 12, 7, 10].

Corollary 3. *Suppose that $r \geqslant p$ with $1/p = \sum_{j=1}^m 1/p_j$. Then for $0 < q \leqslant \infty$ and $s > \sigma(p,q)$, we have*

$$\|\prod_{j=1}^m f_j\|_{M^s_{r,q}(\mathbb{R}^n)} \lesssim \prod_{j=1}^m \|f_j\|_{M^s_{p_j,q}(\mathbb{R}^n)}.$$

Proof. In the above operator $T(f_1, f_2, ..., f_m)$, if we let $\mu(\xi_1, \xi_2, ..., \xi_m) \equiv 1$, then

$$T(f_1, f_2, ..., f_m)(x) = \prod_{j=1}^m f_j(x).$$

By the Hölder inequality

$$\|\prod_{j=1}^m f_j\|_{L^p(\mathbb{R}^n)} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

So the theorem follows from Theorem 1.

We also study the bilinear oscillating integral along the parabola

$$T_\beta(f,g)(x) = \int_{-1}^1 f(x-t)g(x-t^2)e^{i|t|^{-\beta}}\frac{dt}{|t|},\ where\ \beta > 0.$$

Corollary 4. *Let $r \geqslant 2$, $0 < q \leqslant \infty$ and $0 < p_1 \leqslant \infty$, $0 < p_2 \leqslant 2$. If $\beta > 1$, for $s > n(1 \vee \frac{1}{q} - \frac{1}{q})$, we have*

$$\|T_\beta(f,g)\|_{M^s_{r,q}(\mathbb{R})} \lesssim \|f\|_{M^s_{p_1,q}(\mathbb{R})}\|g\|_{M^s_{p_2,q}(\mathbb{R})}.$$

Proof. In [1], Fan and Li showed

$$\|T_\beta(f,g)\|_{L^2(\mathbb{R})} \leqslant C\|f\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}$$

provided $\beta > 1$. Thus the theorem follows from the above inequality and Theorem 1.

Based on the above observation, we naturally want to know if there is a similar discussion as Theorem 1 on $N^s_{p,q}$. Unfortunately, because the multilinear operator T does not have an ℓ^r-valued extension, we cannot obtain results as better as Theorem 1. Despite this, we still can obtain the multilinear estimates on $N^s_{p,q}$ as follows.

Assume $T_\mu f = \mu^\vee * f$ and

$$T(f_1, f_2, ..., f_m)(x) = T_{\mu_1} f_1 ... T_{\mu_m} f_m. \tag{2}$$

Theorem 2. *Let T be defined by (2) with $\mu_i \in H^L(\mathbb{R}^n)$, $L > n(\frac{1}{min(1,p)} - \frac{1}{2})$, $i = 1, ..., m$. Then, for $0 < p < \infty$, $1 - \frac{1}{p+1} < q \leqslant \infty$, we have*

$$\|T(f_1, f_2, ..., f_m)(x)\|_{N^s_{p,q}} \lesssim \|f_1\|_{N^s_{p,q}} ... \|f_m\|_{N^s_{p,q}},$$

where $H^L(\mathbb{R}^n)$ denote the Sobolev spaces,

$$s > \begin{cases} n(1-\frac{1}{q}), & (\frac{1}{p},\frac{1}{q}) \in D_1; \\ 0, & (\frac{1}{p},\frac{1}{q}) \in D_2; \\ np(\frac{1}{p}-\frac{1}{q}), & (\frac{1}{p},\frac{1}{q}) \in D_3 \end{cases}$$

and

$$\begin{aligned} \mathbb{R}^2_+ &= \{(\frac{1}{p},\frac{1}{q}) \in \mathbb{R}^2 : \frac{1}{p} \geqslant 0, \frac{1}{q} \geqslant 0\};\ D_1 = (0,1] \times [0,1], \\ D_2 &= \{(\frac{1}{p},\frac{1}{q}) \in \mathbb{R}^2_+ : \frac{1}{q} - 1 < \frac{1}{p} < \frac{1}{q}, \frac{1}{q} > 1\}, \\ D_3 &= \{(\frac{1}{p},\frac{1}{q}) \in \mathbb{R}^2_+ : \frac{1}{p} \geqslant \frac{1}{q}, \frac{1}{p} > 1\}. \end{aligned}$$

Remark. Theorem 2 also holds if we replace the condition $\mu_i \in H^L(\mathbb{R}^n)$, $L > n(\frac{1}{min(1,p)} - \frac{1}{2})$, $i = 1, ..., m$ by $|\partial^\alpha \mu_i| < C_{\alpha,i}$, $|\alpha| > L$, $L > n(\frac{1}{min(1,p)} - \frac{1}{2})$, $i = 1, ..., m$(By the similar argument to the proof of the above Theorem 2).

This paper consists of six sections. Section 1 is the introduction. In Section 2, we introduce the definition of the modulation spaces and Triebel-type spaces $N^s_{p,q}$ and some necessary lemmas. The proof of Theorem 1 and Theorem 2 can be found in Section 3 and Section 4 respectively. Finally, in Section 5, we study the well-posedness of fractional heat equations and fractional Schrödinger equations.

2. FUNCTION SPACES

In recent decades, the well-posedness of nonlinear evolution equations is developed quickly and a large amount of work has devoted to the study of Besov and modulation spaces. For the well-posedness results in Besov and H^s spaces, one can refer to [17, 2, 3, 4, 18], etc.

Besov and Triebel spaces are two very important function spaces constructed in the 1960s. In recent years, these spaces are widely applied in the field of PDE. Besov and Triebel spaces are constructed by combining Littlewood-Paley decomposition with $\ell^q(L^p)$ and $L^p(\ell^q)$ respectively. Naturally, ones try to study the spaces by combining frequency-uniform decomposition with $\ell^q(L^p)$ and $L^p(\ell^q)$ respectively. Actually the spaces constructed by combining frequency-uniform decomposition with $\ell^q(L^p)$ are Modulation spaces. Many people have studied the well-posedness in these spaces (for example, see [27, 26, 28, 29], and [31]). In this paper, besides modulation spaces, we shall consider the spaces constructed by combining frequency-uniform decomposition and $L^p(\ell^q)$. Firstly, we will recall the definition of frequency-uniform decomposition and modulation spaces. In the 1930s, N. Wiener [25] first introduced the frequency-uniform decomposition. So, some time we call it Wiener decomposition of $\mathbb{R}^n$ that roughly denoted by

$$\Box_k \sim F^{-1}\chi_{Q_k}F, k \in \mathbb{Z}^n,$$

where χ_E is the characteristic function on E. Because Q_k (Q_k is the unit cube with the center at k) is just a translation of Q_0, we call this kind of operator frequency-uniform decomposition operator. But in this definition χ_{Q_k} is not smooth which was re-defined by smooth truncation function later in [30, 27, 28], and [11].

Now we give an simple introduction. We first denote $|\xi|_\infty := max_{i=1,\ldots,n}|\xi_i|$, $Q_k : \{\xi \in \mathbb{R}^n : |\xi - k|_\infty \leqslant \frac{1}{2}\}$. By construction, we may assume that $\sigma_k(\xi)$ satisfies the following conditions

$$\left\{\begin{aligned} |\sigma_k(\xi)| &\geq c, \xi \in Q_k; \\ supp\sigma_k &\subset \{\xi : |\xi - k|_\infty \leq 1\}; \\ \Sigma_{k\in\mathbb{Z}^n}\sigma_k(\xi) &\equiv , \forall\xi \in \mathbb{R}^n; \\ |D^\alpha\sigma_k(\xi)| &\leq C_{|\alpha|}, \forall\xi \in \mathbb{R}^n, \alpha \in (N \cup \{0\})^n. \end{aligned}\right. \tag{3}$$

Denote

$$\Upsilon_n = \{\{\sigma_k\}_{k\in\mathbb{Z}^n} : \{\sigma_k\}_{k\in\mathbb{Z}^n}\ satisfies\ (1)\}.$$

Then Υ_n is nonempty. Let $\{\sigma_k\}_{k\in\mathbb{Z}^n} \in \Upsilon_n$ be a function sequence, denote

$$\Box_k := F^{-1}\sigma_k F,\ k \in \mathbb{Z}^n,$$

which is said to be the frequency-uniform decomposition operators. For any $k \in \mathbb{Z}^n$, we set $|k| = |k_1| + \ldots + |k_n|$, $\langle k \rangle = 1 + |k|$. For any $s \in \mathbb{R}, 0 < p, q \leq \infty$, we denote

$$M^s_{p,q}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{M^s_{p,q}} = (\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\Box_k f\|^q_{L^p})^{1/q} < \infty \right\}.$$

$M^s_{p,q} := M^s_{p,q}(\mathbb{R}^n)$ is said to be a modulation space, which was first introduced by Feichtinger [11] in the case $1 \leqslant p, q \leqslant \infty$.

If we combine these decomposition with $L^p(\ell^q)$, we can introduce a new spaces (denoted by $N^s_{p,q}$) as follows. If $0 < p < \infty,\ 0 < q \leqslant \infty$, for any $s \in \mathbb{R}$, we denote

$$N^s_{p,q}(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{N^s_{p,q}} = \|(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} |\Box_k f|^q)^{1/q}\|_p < \infty \right\}.$$

If $p = \infty,\ 0 < q \leqslant \infty$, for any $s \in \mathbb{R}$, we denote

$$\begin{aligned} N^s_{\infty,q}(\mathbb{R}^n) \quad := \quad & \{ f \in S'(\mathbb{R}^n) : \exists \{f_k(x)\}^\infty_{k=0} \subset L^\infty(\mathbb{R}^n) \ such\ that \\ & f = \Sigma^\infty_{k=0} F^{-1} \sigma_k F f_k \ in\ \mathcal{S}'(\mathbb{R}^n) \ and\ \|\langle k \rangle^s f_k\|_{L^\infty(\mathbb{R}^n, \ell^q)} < \infty \}, \end{aligned}$$

$$\|f\|_{N^s_{\infty,q}(\mathbb{R}^n)} = inf \|\langle k \rangle^s f_k\|_{L^\infty(\mathbb{R}^n, \ell^q)},$$

where the infimum is taken over all admissible representations of f in the sense of above definition ([21] discusses the semilinear estimates, dual estimates, Schwartz estimates on $N^s_{p,q}$ and embedding between $N^s_{p,q}$ and $F^s_{p,q}$. Also, it shows the well-posedness of NLS equation in $L^r(0, T; N^s_{p,2})$). Finally, we denote

$$L^r(\mathbb{R}; N^s_{p,q}) := \{ f(t,x) \in \mathcal{S}' : (\int_{\mathbb{R}} \|f\|^r_{N^s_{p,q}} dt)^{\frac{1}{r}} < \infty \}.$$

3. Proof of Theorem 1

Let $\xi^*_N = \sum_{j=1}^N \xi_j,\ N = 1, 2, ..., m$. By [6], we observe that

$$\begin{aligned} & FT(\Box_{j_1} f_1, ..., \Box_{j_m} f_m)(\xi) \\ = & \int_{\mathbb{R}^{(m-1)n}} (\prod_{j=1}^{m-1} \sigma_{k_j}(\xi_j) \widehat{f_j}(\xi_j)) \sigma_{k_m}(\xi - \xi^*_{m-1}) \widehat{f_m}(\xi - \xi^*_{m-1}) \\ & \cdot \mu(\xi_1, \xi_2, ..., \xi - \xi^*_{m-1}) d\xi_1 ... d\xi_m, \end{aligned}$$

and the support of $FT(\Box_{j_1} f_1, ..., \Box_{j_m} f_m)(\xi)$ is contained in the ball

$$B(\sum_{j=1}^m k_j, m\sqrt{n}) = \{ x \in \mathbb{R}^n : |\xi - \sum_{j=1}^m k_j| < m\sqrt{n} \}.$$

We will prove the Theorem 1 with several steps.

Step 1. $0 < p \leqslant \infty$, $q = \infty$. Recall that the choice of σ satisfies $\sum_{k\in\mathbb{Z}^n} \sigma_k = 1$. By seeing the proof of [6], we have

$$\begin{aligned} &\Box_k T(f_1, ..., f_m)(x) \\ = &\sum_{k_1,k_2,...,k_m\in\mathbb{Z}^n} \int_{\mathbb{R}^{mn}} (\prod_{j=1}^{m} \sigma_{k_j}(\xi_j)) \widehat{f}_j(\xi_j) \\ &\cdot \mu(\xi_1, \xi_2, ..., \xi_m) e^{2\pi i \langle x, \xi_m^* \rangle} \sigma_k(\xi_m^*) d\xi_1 ... d\xi_m. \end{aligned}$$

and that for $r \geqslant p$,

$$\begin{aligned} &\|T(f_1, f_2, ..., f_m)\|_{M^s_{r,q}(\mathbb{R}^n)} \\ \lesssim &(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \|\Box_k T(f_1, f_2, ..., f_m)\|^q_{L^p})^{\frac{1}{q}}. \end{aligned}$$

Let

$$A = (\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \|\Box_k T(f_1, f_2, ..., f_m)\|^q_{L^p})^{\frac{1}{q}}.$$

Here, by Appendix Theorem E (for $0 < p \leqslant 1$) and the Young's inequality (for $p \geqslant 1$), we have, in the case $q = \infty$,

$$\begin{aligned} A \,=\, &sup_{k\in\mathbb{Z}^n} \langle k\rangle^s \| \sum_{\substack{k_1,...,k_m\in\mathbb{Z}^n \\ |k_1+...+k_m-k|\leqslant(m+1)\sqrt{n}}} \Box_k T(\Box_{k_1} f_1, \Box_{k_2} f_2, ..., \Box_{k_m} f_m)\|_{L^p} \\ \lesssim\, &sup_{k\in\mathbb{Z}^n} \langle k\rangle^s \sum_{t=0}^{(m+1)\sqrt{n}} \| \sum_{\substack{k_1,...,k_m\in\mathbb{Z}^n \\ k_m=k-k^*_{m-1}\pm t}} T(\Box_{k_1} f_1, \Box_{k_2} f_2, ..., \Box_{k_m} f_m)\|_{L^p}, \end{aligned}$$

where $k^*_{m-1} = k_1 + k_2 + ... + k_{m-1}$. For

$$B = sup_{k\in\mathbb{Z}^n} \langle k\rangle^s \| \sum_{\substack{k_1,...,k_m\in\mathbb{Z}^n \\ k_m=k-k^*_{m-1}\pm t}} T(\Box_{k_1} f_1, \Box_{k_2} f_2, ..., \Box_{k_m} f_m)\|_{L^p},$$

we write

$$B \,=\, sup_{k\in\mathbb{Z}^n} a(k, k^*) b(k, k^*) \| \sum_{\substack{k_1,...,k_m\in\mathbb{Z}^n \\ k_m=k-k^*_{m-1}\pm t}} T(\Box_{k_1} f_1, \Box_{k_2} f_2, ..., \Box_{k_m} f_m)\|_{L^p},$$

with

$$\begin{aligned} a(k, k^*) \,&=\, \langle k - k^*_{m-1} \pm t\rangle^s \prod_{j=1}^{m-1} \langle k_j\rangle^s \\ b(k, k^*) \,&=\, \frac{\langle k\rangle^s}{\langle k - k^*_{m-1} \pm t\rangle^s \prod_{j=1}^{m-1} \langle k_j\rangle^s}. \end{aligned}$$

It is easy to check that $b(k,k^*) \lesssim 1$ uniformly on k and k_j, $j=1,2,...,m$. Thus the proof can be classified into the following two cases.

Case 1. $1 \leqslant p \leqslant \infty$, $s > n$;

Case 2. $0 < p < 1$, $s > n/p$. In the case 1, by the Minkowski and Hölder's inequalities, we have

$$\begin{aligned} B &\lesssim sup_{k\in\mathbb{Z}^n} \sum_{\substack{k_1,\dots,k_m\in\mathbb{Z}^n\\ k_m=k-k^*_{m-1}\pm t}} a(k,k^*)\|T(\Box_{k_1}f_1,\Box_{k_2}f_2,...,\Box_{k_m}f_m)\|_{L^p} \\ &\lesssim sup_{k\in\mathbb{Z}^n} \sum_{\substack{k_1,\dots,k_m\in\mathbb{Z}^n\\ k_m=k-k^*_{m-1}\pm t}} a(k,k^*)\prod_{j=1}^{m-1}\|\Box_{k_j}f_j\|_{L^{p_j}}\|\Box_{k_m}f_m\|_{L^{p_m}}. \end{aligned}$$

Let

$$\begin{aligned} \mho(k) &= \{(k_1,k_2)\in\mathbb{Z}^{2n} : k_1,k_2\in\mathbb{Z}^n; k_2 = k-k_1\pm t\}, \\ \Psi_0 &= \{(k_1,k_2)\in\mathbb{Z}^{2n} : \langle k_1\rangle \sim \langle k_2\rangle\}, \\ \Psi_1 &= \{(k_1,k_2)\in\mathbb{Z}^{2n} : \langle k_1\rangle \gg \langle k_2\rangle\}, \\ \Psi_2 &= \{(k_1,k_2)\in\mathbb{Z}^{2n} : \langle k_1\rangle \ll \langle k_2\rangle\}. \end{aligned}$$

For any subset $\Theta\subseteq\mathbb{Z}^n\times\mathbb{Z}^n$, we define

$$\begin{aligned} \Theta_1^{\perp} &= \{k_1\in\mathbb{Z}^n : \exists k_2\in\mathbb{Z}^n \ s.t.\ (k_1,k_2)\in\Theta\}, \\ \Theta_2^{\perp} &= \{k_2\in\mathbb{Z}^n : \exists k_1\in\mathbb{Z}^n \ s.t.\ (k_1,k_2)\in\Theta\}. \end{aligned}$$

Then we have (in the case $m=2$)

$$\begin{aligned} B &\lesssim sup_{k\in\mathbb{Z}^n} \sum_{\substack{k_1,k_2\in\mathbb{Z}^n\\ k_2=k-k_1^*\pm t}} a(k,k^*)\|\Box_{k_j}f_j\|_{L^{p_1}}\|\Box_{k_2}f_2\|_{L^{p_2}} \\ &\lesssim sup_{k\in\mathbb{Z}^n} \sum_{\mho(k)\cap\Psi_0} a(k,k^*)\|\Box_{k_1}f_1\|_{L^{p_1}}\|\Box_{k_2}f_2\|_{L^{p_2}} \\ &\quad + sup_{k\in\mathbb{Z}^n} \sum_{\mho(k)\cap\Psi_1} a(k,k^*)\|\Box_{k_1}f_1\|_{L^{p_1}}\|\Box_{k_2}f_2\|_{L^{p_2}} \\ &\quad + sup_{k\in\mathbb{Z}^n} \sum_{\mho(k)\cap\Psi_2} a(k,k^*)\|\Box_{k_1}f_1\|_{L^{p_1}}\|\Box_{k_2}f_2\|_{L^{p_2}} \\ &= sup_{k\in\mathbb{Z}^n} I + sup_{k\in\mathbb{Z}^n} II + sup_{k\in\mathbb{Z}^n} III. \end{aligned}$$

For any $k_2 \in (\mho(k) \cap (\Psi_0 \cup \Psi_1))_2^{\perp}$ with any fixed k, $s > n$, we have

$$
\begin{aligned}
(I+II) \lesssim & \sum_{(k_1,k_2)\in\mho(k)\cap(\Psi_0\cup\Psi_1)} a(k,k^*)\|\Box_{k_1}f_1\|_{L^{p_1}}\|\Box_{k_2}f_2\|_{L^{p_2}} \\
\lesssim & \, sup_{k_1\in(\mho(k)\cap(\Psi_0\cup\Psi_1))_1^{\perp}}\langle k_1\rangle^s\|\Box_{k_1}f_1\|_{L^{p_1}} \\
& \sum_{k_1\in(\mho(k)\cap(\Psi_0\cup\Psi_1))_1^{\perp}}\sum_{k_2=k-k_1\pm t}\|\Box_{k_2}f_2\|_{L^{p_2}} \\
\lesssim & \, sup_{k_1\in\mathbb{Z}^n}\langle k_1\rangle^s\|\Box_{k_1}f_1\|_{L^{p_1}}\sum_{k_2\in(\mho(k)\cap(\Psi_0\cup\Psi_1))_2^{\perp}}\sum_{k_1=k-k_2\pm t}\|\Box_{k_2}f_2\|_{L^{p_2}} \\
\lesssim & \, \|f_1\|_{M^s_{p_1,\infty}}\|f_2\|_{M_{p_2,1}} \lesssim \|f_1\|_{M^s_{p_1,\infty}}\|f_2\|_{M^s_{p_2,\infty}}.
\end{aligned}
$$

($M^{s_1}_{p,q_1} \subset M^{s_2}_{p,q_2}$, if $q_1 > q_2$, $s_1 - s_2 > n/q_2 - n/q_1$; [26])For any $k_2 \in (\mho(k) \cap \Psi_2)_2^{\perp}$ with any fixed k, $s > n$, we have

$$
\begin{aligned}
(III) \lesssim & \sum_{(k_1,k_2)\in\mho(k)\cap\Psi_2} a(k,k^*)\|\Box_{k_1}f_1\|_{L^{p_1}}\|\Box_{k_2}f_2\|_{L^{p_2}} \\
\lesssim & \, sup_{k_2\in(\mho(k)\cap\Psi_2)_2^{\perp}}\langle k_2\rangle^s\|\Box_{k_2}f_2\|_{L^{p_2}}\sum_{k_2\in(\mho(k)\cap\Psi_2)_2^{\perp}}\sum_{k_1=k-k_2}\|\Box_{k_1}f_1\|_{L^{p_1}} \\
\lesssim & \, \|f_2\|_{M^s_{p_2,\infty}}\|f_1\|_{M_{p_1,1}} \lesssim \|f_1\|_{M^s_{p_1,\infty}}\|f_2\|_{M^s_{p_2,\infty}} \\
& (M^{s_1}_{p,q_1} \subset M^{s_2}_{p,q_2},\ if\ q_1 > q_2, s_1 - s_2 > n/q_2 - n/q_1.)
\end{aligned}
$$

By the similar discussion, for general m, we have

$$\|T(f_1, f_2, ..., f_m)\|_{M^s_{p,\infty}} \lesssim \prod_{j=1}^{m} \|f_j\|_{M^s_{p_j,\infty}}.$$

Now we tune to estimate Case 2 for $0 < p < 1$ and $s > \frac{n}{p}$.

$$B \lesssim sup_{k\in\mathbb{Z}^n}\Big(\sum_{\substack{k_1,\dots,k_m\in\mathbb{Z}^n \\ k_m=k-k^*_{m-1}\pm t}} a(k,k^*)^p\|T(\Box_{k_1}f_1, \Box_{k_2}f_2, ..., \Box_{k_m}f_m)\|^p_{L^p}\Big)^{\frac{1}{p}}.$$

Then we have (in the case $m = 2$)

$$B \lesssim sup_{k\in\mathbb{Z}^n}\Big(\sum_{\mho_k\cap(\Psi_0\cup\Psi_1\cup\Psi_2)} a(k,k^*)^p\|T(\Box_{k_1}f_1, \Box_{k_2}f_2)\|^p_{L^p}\Big)^{\frac{1}{p}}$$

For $(k_1, k_2) \in \mho_k \cap \Psi_0$, we have

$$
\begin{aligned}
&\Big(\sum_{\mho_k\cap\Psi_0} a(k,k^*)^p\|T(\Box_{k_1}f_1,\Box_{k_2}f_2)\|^p_{L^p}\Big)^{\frac{1}{p}}\\
&\lesssim\Big(\sum_{\mho_k\cap\Psi_0}\langle k_1\rangle^{sp}(\|\Box_{k_1}f_1\|_{L^{p_1}}\|\Box_{k_2}f_2\|_{L^{p_2}})^p\Big)^{\frac{1}{p}}\\
&\lesssim\Big(sup_{k_1\in\mathbb{Z}^n}\langle k_1\rangle^{sp}\|\Box_{k_1}f_1\|^p_{L^{p_1}}\sum_{k_2\in\mathbb{Z}^n}\|\Box_{k_2}f_2\|^p_{L^{p_2}}\Big)^{\frac{1}{p}}\\
&\lesssim\|f_1\|_{M^s_{p_1,\infty}}\|f_2\|_{M_{p_2,p}}\lesssim\|f_1\|_{M^s_{p_1,\infty}}\|f_2\|_{M^s_{p_2,\infty}}\ (s>n/p).
\end{aligned}
$$

For any $(k_1,k_2)\in\mho_k\cap(\Psi_1\cup\Psi_2)$ with every fixed k, imitating the process as in the discussion above, we have, for $s>n/p$

$$
\sum_{i=1}^{2}\Big(\sum_{\mho_k\cap\Psi_i} a(k,k^*)^p\|T(\Box_{k_1}f_1,\Box_{k_2}f_2\Big)\|^p_{L^p})^{\frac{1}{p}}\lesssim\|f_1\|_{M^s_{p_1,\infty}}\|f_2\|_{M^s_{p_2,\infty}}.
$$

Step 2. $1\leqslant p\leqslant\infty$, $q\leqslant 1$, $s\geqslant 0$. Similar to the estimate of Step 1, we have

$$
A\lesssim\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}\sum_{t=0}^{(m+1)\sqrt{n}}\|\sum_{\substack{k_1,\dots,k_m\in Z^n\\ k_m=k-k^*_{m-1}\pm t}}T(\Box_{k_1}f_1,\Box_{k_2}f_2,\dots,\Box_{k_m}f_m)\|^q_{L^p}\Big)^{\frac{1}{q}}
$$

Now, for $m=2$, we have

$$
\begin{aligned}
A&\lesssim\Big(\sum_{k\in\mathbb{Z}^n}\sum_{\substack{k_1,k_2\in\mathbb{Z}^n\\ k_2=k-k_1\pm t}}\langle k\rangle^{sq}\|T(\Box_{k_1}f_1,\Box_{k_2}f_2)\|^q_{L^p}\Big)^{\frac{1}{q}}\\
&\lesssim\Big(\sum_{k\in\mathbb{Z}^n}\sum_{\substack{k_1,k_2\in\mathbb{Z}^n\\ k_2=k-k_1\pm t}}(\langle k_1\rangle\langle k_2\rangle)^{sq}(\|\Box_{k_1}f_1\|_{L^{p_1}}\|\Box_{k_2}f_2\|_{L^{p_2}})^q\Big)^{\frac{1}{q}}\\
&\lesssim\|f_1\|_{M^s_{p_1,q}}\|f_2\|_{M^s_{p_2,q}}.
\end{aligned}
$$

In the above estimate, we use the Young inequality $\|a_i*b_i\|_{\ell^1}\lesssim\|a_i\|_{\ell^1}\|b_i\|_{\ell^1}$. A similar estimate for $m\geqslant 3$.

Step 3. $0<p<1,\ p\geqslant q,\ s\geqslant 0$. By imitating the process as in Step 2 and Appendix Theorem E, we have

$$
A\lesssim\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}\sum_{t=0}^{(m+1)\sqrt{n}}\|\sum_{\substack{k_1,\dots,k_m\in\mathbb{Z}^n\\ k_m=k-k^*_{m-1}\pm t}}T(\Box_{k_1}f_1,\Box_{k_2}f_2,\dots,\Box_{k_m}f_m)\|^q_{L^p}\Big)^{\frac{1}{q}}.
$$

Moreover, (we prove $m=2$ for simplicity) by Young's inequality again, we have

$$
\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}\|\sum_{\substack{k_1,k_2\in\mathbb{Z}^n\\ k_2=k-k_1\pm t}}T(\Box_{k_1}f_1,\Box_{k_2}f_2)\|_{L^p}^q\Big)^{\frac{1}{q}}\\
&\lesssim\Big(\sum_{k\in\mathbb{Z}^n}\sum_{\substack{k_1,k_2\in\mathbb{Z}^n\\ k_2=k-k_1\pm t}}\langle k\rangle^{sq}\|T(\Box_{k_1}f_1,\Box_{k_2}f_2)\|_{L^p}^q\Big)^{\frac{1}{q}}\\
&\lesssim\Big(\sum_{k\in\mathbb{Z}^n}\sum_{\substack{k_1,k_2\in\mathbb{Z}^n\\ k_2=k-k_1\pm t}}(\langle k_1\rangle\langle k_2\rangle)^{sq}(\|\Box_{k_1}f_1\|_{L^{p_1}}\|\Box_{k_2}f_2\|_{L^{p_2}})^q\Big)^{\frac{1}{q}}\\
&\lesssim\|f_1\|_{M^s_{p_1,q}}\|f_2\|_{M^s_{p_2,q}}.
\end{aligned}
$$

This finishes the proof of this step.

Step 4. For $0<p<1$, by Step 1 and Step 3, we have

$$\|T(f_1,f_2,...,f_m)\|_{M^{s_1}_{p,\infty}}\leqslant C_0\prod_{j=1}^m\|f_i\|_{M^{s_1}_{p_i,\infty}},\ s_1>\frac{n}{p};$$

and

$$\|T(f_1,f_2,...,f_m)\|_{M^{s_2}_{p,p}}\leqslant C_1\prod_{j=1}^m\|f_i\|_{M^{s_2}_{p_i,p}},\ s_2\geqslant 0.$$

Then by the complex interpolation theorem (Appendix Theorem A), we can obtain that for $0<p<1$, $p\leqslant q\leqslant\infty$ and $s>n(\frac{1}{p}-\frac{1}{q})$,

$$\|T(f_1,f_2,...,f_m)\|_{M^s_{p,q}}\leqslant C_2\prod_{j=1}^m\|f_i\|_{M^s_{p_i,q}}.$$

For $1\leqslant p\leqslant\infty$, by Step 1 and Step 2, we have

$$\|T(f_1,f_2,...,f_m)\|_{M^{s_1}_{p,\infty}}\leqslant C_0\prod_{j=1}^m\|f_i\|_{M^{s_1}_{p_i,\infty}},\ s_1>n;$$

and

$$\|T(f_1,f_2,...,f_m)\|_{M^{s_2}_{p,1}}\leqslant C_1\prod_{j=1}^m\|f_i\|_{M^{s_2}_{p_i,1}},\ s_2\geqslant 0.$$

Then by the complex interpolation theorem (Appendix Theorem A), we can obtain that for $1\leqslant p\leqslant\infty$, $1\leqslant q\leqslant\infty$ and $s>n(1-\frac{1}{q})$,

$$\|T(f_1,f_2,...,f_m)\|_{M^s_{p,q}}\leqslant C_2\prod_{j=1}^m\|f_i\|_{M^s_{p_i,q}}.$$

Through the above discussion and the embedding relationship of Modulation space ([30]), we can obtain that for $s > \sigma(p,q)$,

$$\|T(f_1, f_2, ..., f_m)\|_{M^s_{r,q}(\mathbb{R}^n)} \lesssim \prod_{j=1}^{m} \|f_j\|_{M^s_{r_j,q}(\mathbb{R}^n)}, (for\ r \geqslant p; r_j \leqslant p_j).$$

4. Proof of Theorem 2

Proof. By the above observation and the Appendix Theorem F, we have

$$\begin{aligned}
&\|T(f_1, f_2, ..., f_m)\|_{N^s_{p,q}(\mathbb{R}^n)} \\
&\lesssim \|(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} (\sum_{\substack{k_1,...,k_m\in\mathbb{Z}^n \\ |k_1+...+k_m-k|\leqslant(m+1)\sqrt{n}}} |T(\Box_{k_1}f_1, \Box_{k_2}f_2, ..., \Box_{k_m}f_m)|)^q)^{\frac{1}{q}}\|_{L^p} \\
&\lesssim \sum_{t=0}^{(m+1)\sqrt{n}} \|(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} (\sum_{\substack{k_1,...,k_m\in\mathbb{Z}^n \\ k_m=k-k^*_{m-1}\pm t}} |\mu_1^\vee * (\Box_{k_1}f_1)|...|\mu_m^\vee * (\Box_{k_m}f_m)|)^q)^{\frac{1}{q}}\|_{L^p}.
\end{aligned}$$

Again, we will only estimate the case $m = 2$, for simplicity. Let $A' = \|(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq}$ $(\sum_{\substack{k_1,...,k_m\in\mathbb{Z}^n \\ k_m=k-k^*_{m-1}\pm t}} |\mu_1^\vee * (\Box_{k_1}f_1)|...|\mu_m^\vee * (\Box_{k_m}f_m)|)^q)^{\frac{1}{q}}\|_{L^p}$. We have

$$\begin{aligned}
A' &\lesssim \|(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} (\sum_{\mho_k\cap\Psi_0} |\mu_1^\vee * (\Box_{k_1}f_1)||\mu_2^\vee * (\Box_{k_2}f_2)|)^q)^{\frac{1}{q}}\|_{L^p} \\
&\quad + \|(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} (\sum_{\mho_k\cap\Psi_1} |\mu_1^\vee * (\Box_{k_1}f_1)||\mu_2^\vee * (\Box_{k_2}f_2)|)^q)^{\frac{1}{q}}\|_{L^p} \\
&\quad + \|(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} (\sum_{\mho_k\cap\Psi_2} |\mu_1^\vee * (\Box_{k_1}f_1)||\mu_2^\vee * (\Box_{k_2}f_2)|)^q)^{\frac{1}{q}}\|_{L^p} \\
&= I + II + III
\end{aligned}$$

Next, we will sketch the proof in the following three cases.

Step 1. $1 \leqslant p < \infty, q = 1, s \geqslant 0$. By Hölder's inequality, Young's inequality ($\|a_i * b_i\|_{\ell^1} \leqslant \|a_i\|_{\ell^1}\|b_i\|_{\ell^1}$) and Appendix Theorem F, we have

$$\begin{aligned}
A' &\lesssim \|\sum_{k\in\mathbb{Z}^n} \sum_{\substack{k_1,k_2\in\mathbb{Z}^n \\ k_2=k-k_1\pm t}} \langle k\rangle^s |\mu_1^\vee * (\Box_{k_1}f_1)||\mu_2^\vee * (\Box_{k_2}f_2)|\|_{L^p} \\
&\lesssim \|\sum_{k_1\in\mathbb{Z}^n} \langle k_1\rangle^s |\mu_1^\vee * (\Box_{k_1}f_1)|\|_{L^{p_1}} \|\sum_{k_2\in Z^n} \langle k_2\rangle^s |\mu_2^\vee * (\Box_{k_2}f_2)|\|_{L^{p_2}} \\
&\lesssim \|\sum_{k_1\in\mathbb{Z}^n} \langle k_1\rangle^s |\Box_{k_1}f_1|\|_{L^{p_1}} \|\sum_{k_2\in\mathbb{Z}^n} \langle k_2\rangle^s |\Box_{k_2}f_2|\|_{L^{p_2}} \\
&= \|f_1\|_{N^s_{p_1,1}} \|f_2\|_{N^s_{p_2,1}} \lesssim \|f_1\|_{N^s_{p,1}} \|f_2\|_{N^s_{p,1}}
\end{aligned}$$

(see Appendix Theorem B, $1/p = 1/p_1 + 1/p_2$.)

Step 2. $0 < p < \infty, q = \infty$. Case 1, $1 \leqslant p < \infty, s > n$. By Appendix Theorem F, with the same notation as in the Proof of Theorem 1, we have

$$
\begin{aligned}
I &\lesssim \|sup_{k\in\mathbb{Z}^n}\langle k\rangle^s \sum_{\mho_k\cap\Psi_0} |\mu_1^\vee * (\Box_{k_1} f_1)||\mu_2^\vee * (\Box_{k_2} f_2)|\|_{L^p} \\
&\lesssim \|sup_{k_1\in\mathbb{Z}^n}\langle k_1\rangle^s |\mu_1^\vee * (\Box_{k_1} f_1)|\|_{L^{p_1}} \|\sum_{k_2\in\mathbb{Z}^n} |\mu_2^\vee * (\Box_{k_2} f_2)|\|_{L^{p_2}} \\
&\lesssim \|sup_{k_1\in\mathbb{Z}^n}\langle k_1\rangle^s |\Box_{k_1} f_1|\|_{L^{p_1}} \|\sum_{k_2\in\mathbb{Z}^n} |\Box_{k_2} f_2|\|_{L^{p_2}} \\
&= \|f_1\|_{N^s_{p_1,\infty}} \|f_2\|_{N_{p_2,1}} \lesssim \|f_1\|_{N^s_{p,\infty}} \|f_2\|_{N^s_{p,\infty}}.
\end{aligned}
$$

Here we use Appendix Theorem H and Young's inequality in the above estimate.

Case 2, $0 < p < 1, s > n$. By Appendix Theorem B,F,G, we have

$$
\begin{aligned}
I &\lesssim |sup_{k\in\mathbb{Z}^n}\langle k\rangle^s \sum_{\mho_k\cap\Psi_0} |\mu_1^\vee * (\Box_{k_1} f_1)||\mu_2^\vee * (\Box_{k_2} f_2)|\|_{L^p} \\
&\lesssim \|sup_{k\in\mathbb{Z}^n} \sum_{\mho_k\cap\Psi_0} \langle k_1\rangle^s |\mu_1^\vee * (\Box_{k_1} f_1)||\mu_2^\vee * (\Box_{k_2} f_2)|\|_{L^p} \\
&\lesssim \|sup_{k_1\in\mathbb{Z}^n}\langle k_1\rangle^s |\mu_1^\vee * (\Box_{k_1} f_1)|\|_{L^p} \|\sum_{k_2\in\mathbb{Z}^n} |\mu_2^\vee * (\Box_{k_2} f_2)|\|_{L^\infty} \\
&\lesssim \|sup_{k_1\in\mathbb{Z}^n}\langle k_1\rangle^s |\Box_{k_1} f_1|\|_{L^p} \|\sum_{k_2\in\mathbb{Z}^n} (|\sigma_{k_2}^\vee| * |\mu_2^\vee * (\Box_{k_2} f_2)|)\|_{L^\infty} \\
&\lesssim \|sup_{k_1\in\mathbb{Z}^n}\langle k_1\rangle^s |\Box_{k_1} f_1|\|_{L^p} \|\sum_{k_2\in\mathbb{Z}^n} (|\mu_2^\vee * (\Box_{k_2} f_2)|)\|_{L^1} \\
&\lesssim \|f_1\|_{N^s_{p,\infty}} \|f_2\|_{N_{1,1}} \lesssim \|f_1\|_{N^s_{p,\infty}} \|f_2\|_{N^s_{p,\infty}},
\end{aligned}
$$

where, we use the relations $\|f\|_{N_{1,1}} = \|f\|_{M_{1,1}} \lesssim \|f\|_{M_{p,1}} \lesssim \|f\|_{M^s_{p,\infty}} \lesssim \|f\|_{N^s_{p,\infty}}$, for $s > n$. By the similar discussion, we can dominate the term II and III, i.e.,

$$
\begin{aligned}
&\sum_{i=1}^{2} \|sup_{k\in\mathbb{Z}^n}\langle k\rangle^s \sum_{\mho_k\cap\Psi_i} |\mu_1^\vee * (\Box_{k_1} f_1)||\mu_2^\vee * (\Box_{k_2} f_2)|\|_{L^p} \\
&\lesssim \|f_1\|_{N^s_{p,\infty}} \|f_2\|_{N^s_{p,\infty}} \ (for\ s > n).
\end{aligned}
$$

Step 3. $0 < p \leqslant 1, p = q, s \geqslant 0$. By Appendix Theorem D,C,B and Young's inequality, we have

$$
\begin{aligned}
A' &\lesssim \|(\sum_{k\in\mathbb{Z}^n}\sum_{\substack{k_1,k_2\in\mathbb{Z}^n\\ k_2=k-k_1\pm t}}\langle k\rangle^{sp}(|\mu_1^\vee*(\Box_{k_1}f_1)||\mu_2^\vee*(\Box_{k_2}f_2)|)^p)^{\frac{1}{p}}\|_{L^p}\\
&\lesssim (\sum_{k\in\mathbb{Z}^n}\sum_{\substack{k_1,k_2\in\mathbb{Z}^n\\ k_2=k-k_1\pm t}}(\langle k_1\rangle\langle k_2\rangle)^{sp}\|\Box_{k_1}f_1\|_{L^p}^p\|\Box_{k_2}f_2\|_{L^\infty}^p)^{\frac{1}{p}}\\
&\lesssim (\sum_{k_1\in\mathbb{Z}^n}\langle k_1\rangle^{sp}\|\Box_{k_1}f_1\|_{L^p}^p)^{\frac{1}{p}}(\sum_{k_2\in\mathbb{Z}^n}\langle k_2\rangle^{sp}\|\Box_{k_2}f_2\|_{L^\infty}^p)^{\frac{1}{p}}\\
&\lesssim \|f_1\|_{M^s_{p,p}}\|f_2\|_{M^s_{p,p}}=\|f_1\|_{N^s_{p,p}}\|f_2\|_{N^s_{p,p}}
\end{aligned}
$$

Step 4. Let $(\frac{1}{p},\frac{1}{q})\in D_1$. It is easy to see that $(\frac{1}{p},\frac{1}{q})$ is a point at the line segment connecting $(\frac{1}{p},0)$ and $(\frac{1}{p},1)$. At the point $(\frac{1}{p},0)$, in step 2, we have shown that $\|T(f_1,f_2)\|_{N^s_{p,\infty}}\lesssim\|f_1\|_{N^s_{p,\infty}}\|f_2\|_{N^s_{p,\infty}}$ if $s>n$. For $(\frac{1}{p},1)$, in step 1, we have shown that $\|T(f_1,f_2)\|_{N^s_{p,1}}\lesssim\|f_1\|_{N^s_{p,1}}\|f_2\|_{N^s_{p,1}}$ if $s\geqslant 0$. Using the complex interpolation (Appendix Theorem A), we can obtain that for $(\frac{1}{p},\frac{1}{q})\in D_1$, $\|T(f_1,f_2)\|_{N^s_{p,q}}\lesssim\|f_1\|_{N^s_{p,q}}\|f_2\|_{N^s_{p,q}}$ if $s>n(1-\frac{1}{q})$.

If $(\frac{1}{p},\frac{1}{q})\in D_2$, then it belongs to the segment by connecting $(\frac{1}{p_0},1)$ and $(\frac{1}{\bar p},\frac{1}{\bar p})$, where $\frac{1}{p_0}<\frac{1}{p}-\frac{1}{q}+1$ and $\bar p=1-\frac{(1-q)pp_0}{q(p_0-p)}$. In Step 1, we see that for $s\geqslant 0$, $\|T(f_1,f_2)\|_{N^s_{p_0,1}}\lesssim\|f_1\|_{N^s_{p_0,1}}\|f_2\|_{N^s_{p_0,1}}$. In Step 3, we see that $\|T(f_1,f_2)\|_{N^s_{\bar p,\bar p}}\lesssim\|f_1\|_{N^s_{\bar p,\bar p}}\|f_2\|_{N^s_{\bar p,\bar p}}$. The complex interpolation between them gives that for $(\frac{1}{p},\frac{1}{q})\in D_2$ and $s\geqslant 0$, $\|T(f_1,f_2)\|_{N^s_{p,q}}\lesssim\|f_1\|_{N^s_{p,q}}\|f_2\|_{N^s_{p,q}}$.

If $(\frac{1}{p},\frac{1}{q})\in D_3$, then one can make a line segment connecting $(\frac{1}{p},\frac{1}{p})$ and $(\frac{1}{p},0)$. For $(\frac{1}{p},\frac{1}{p})$, we see that once $s\geqslant 0$, $\|T(f_1,f_2)\|_{N^s_{p,p}}\lesssim\|f_1\|_{N^s_{p,p}}\|f_2\|_{N^s_{p,p}}$. For $(\frac{1}{p},0)$, we see that once $s\geqslant n$, $\|T(f_1,f_2)\|_{N^s_{p,\infty}}\lesssim\|f_1\|_{N^s_{p,\infty}}\|f_2\|_{N^s_{p,\infty}}$. Then we use complex interpolation to obtain that $\|T(f_1,f_2)\|_{N^s_{p,q}}\lesssim\|f_1\|_{N^s_{p,q}}\|f_2\|_{N^s_{p,q}}$ if $s>np(\frac{1}{p}-\frac{1}{q})$.

By the above discussion, we can obtain that for

$$
s>\begin{cases} n(1-\frac{1}{q}), & (\frac{1}{p},\frac{1}{q})\in D_1;\\ 0, & (\frac{1}{p},\frac{1}{q})\in D_2;\\ np(\frac{1}{p}-\frac{1}{q}), & (\frac{1}{p},\frac{1}{q})\in D_3,\end{cases}
$$

$$
\|T(f_1,f_2,...,f_m)\|_{N^s_{p,q}(\mathbb{R}^n)}\lesssim\prod_{j=1}^m\|f_j\|_{N^s_{p,q}(\mathbb{R}^n)}.
$$

5. Applications on Cauchy Problems

In this section we study the well-posedness of the following fractional Heat equation

$$
u_t+|\Delta|^{-\alpha}u=f(u);\ u(0,x)=u_0(x),\ \alpha\in(0,\infty).\ (*)
$$

This problem has an equivalent form of the integral equation

$$u(t) = \Re(t)u_0 - i\int_0^t \Re(t-\tau)f(u(\tau))d\tau,$$

where $\Re(t) = F^{-1}e^{-t|\xi|^{2\alpha}}F$. The Cauchy Problem (*) has been extensively studied in recent years (see [16, 23, 32, 8], etc.). For the semigroup estimate, [8] established the following result.

Theorem B. (Theorem 3.1 in [8]). *Suppose* $1 \leqslant p \leqslant \infty$ *and* $s \in \mathbb{R}$. *Then*

$$\|e^{-t(-\Delta)^\alpha}u_0\|_{M^s_{p,q}} \leqslant C\|u_0\|_{M^s_{p,q}},$$

where the constant C is independent of t (This result is also true, if we replace $M^s_{p,q}$ *by* $N^s_{p,q}$*, i.e., suppose* $1 \leqslant p \leqslant \infty$ *and* $s \in \mathbb{R}$*, we have* $\|e^{-t(-\Delta)^\alpha}u_0\|_{N^s_{p,q}} \leqslant C\|u_0\|_{N^s_{p,q}}$*.).*

As an application of Theorem 1, we now assume that $f(u)$ is a more general nonlinear function

$$\begin{aligned} f(u)(x,t) &= T(u(t,\cdot), u(t,\cdot), ..., u(t,\cdot))(x) \\ &= \int_{\mathbb{R}^{nm}} \prod_{j=1}^m \widehat{u}(t,\xi_j)e^{2\pi i\langle \xi_j, x\rangle}\mu(\xi_1,...,\xi_m)d\xi_1...d\xi_m, \end{aligned}$$

where $\widehat{u}(t,\xi_j)$ is the Fourier transform of $u(t,x)$ on the x-variable. We have the following theorem.

Theorem 3. *Suppose that the multiplier* $\mu(\xi_1,...,\xi_m)$ *ensures*

$$\|T(f_1,f_2,...,f_m)\|_{L^p(\mathbb{R}^n)} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

for

$$p_j \geqslant p \ and \ j = 1,2,3,...,m.$$

Assume also $p,q \geqslant 1$, $r \geqslant m$. *Then there exists* T^* *such that for any* $u_0 \in M^s_{p,q}(\mathbb{R}^n)$ *and* $s > n(1-\frac{1}{q})$*, the initial value problem (*) has a unique solution*

$$u \in L^r(0,T^*; M^s_{p,q}).$$

Moreover, if $T^* < \infty$*, then*

$$\|u\|_{L^r(0,T^*;M^s_{p,q})} = \infty.$$

Proof. By the estimates that for $p \geqslant 1$ and $s > n(1 \vee \frac{1}{q} - \frac{1}{q})$,

$$\|f(u)\|_{M^s_{p,q}} \lesssim \|u\|^m_{M^s_{p,q}}; \ \|\Re(t)u_0\|_{M^s_{p,q}} \lesssim C\|u_0\|_{M^s_{p,q}},$$

we have

$$\|\int_0^t \Re(t-\tau)f(u(\tau))d\tau\|_{L^r(0,T;M^s_{p,q})} \lesssim T^{1-\frac{m-1}{r}}\|u\|^m_{L^r(0,T;M^s_{p,q})}.$$

Then by the similar argument as [6] and the fixed-point theorem, we can obtain the following results (Let $T_\mu f = \mu^\vee * f$).

Theorem 4. Let $\mu_i \in H^L(\mathbb{R}^n)$, $L > n(\frac{1}{min(1,p)} - \frac{1}{2})$, $i = 1, ..., m$ and

$$T(f_1, f_2, ..., f_m)(x) = T_{\mu_1}f_1...T_{\mu_m}f_m.$$

Assume also $1 \leqslant p < \infty$, $q \geqslant 1$, $r \geqslant m$. Then there exists T^* such that for any $u_0 \in N^s_{p,q}(\mathbb{R}^n)$ and $s > n(1-\frac{1}{q})$, the initial value problem (*) with $f(u) = T(u(t,.), u(t,.), ..., u(t,.))(x)$ has a unique solution

$$u \in L^r(0, T^*; N^s_{p,q}).$$

Moreover, if $T^* < \infty$, then

$$\|u\|_{L^r(0,T^*;N^s_{p,q})} = \infty.$$

Proof. Let $\mathcal{D} = \{u \in L^r([0,T); N^s_{p,q}) : \|u\|_{L^r([0,T);N^s_{p,q})} < \delta\}$ be equipped the metric with the distance $d(u,v) = \|u - v\|_{L^r(0,T;N^s_{p,q})}$. Consider the map

$$\Im : u(t) \to \Re(t)u_0 - i\int_0^t \Re(t-\tau)f(u(\tau))d\tau.$$

By the similar discussion as Theorem 3, we can obtain that the map $\Im : (\mathcal{D}, d) \to (\mathcal{D}, d)$ is a strict contraction map. By Banach's fixed-point theorem, there exists T^* and a unique solution $u \in \mathcal{D}$ satisfies the conditions in the theorem.

APPENDIX

Theorem A. *Let T be a continuous multi-linear operator from* $A^1_0 \times A^2_0 \times ... \times A^m_0$ *to* B_0 *and From* $A^1_1 \times A^2_1 \times ... \times A^m_1$ *to* B_1*, satisfying*

$$\|T(f^{(1)}, f^{(2)}, ..., f^{(m)})\|_{B_0} \leqslant C_0 \prod_{j=1}^m \|f^j\|_{A_0^{(j)}};$$
$$\|T(f^{(1)}, f^{(2)}, ..., f^{(m)})\|_{B_1} \leqslant C_1 \prod_{j=1}^m \|f^j\|_{A_1^{(j)}};\ for\ f^j \in A_0^{(j)} \cap A_1^{(j)}.$$

Then T is continuous from $(A_0^{(1)}, A_1^{(1)})_\theta \times (A_0^{(2)}, A_1^{(2)})_\theta \times ... \times (A_0^{(m)}, A_1^{(m)})_\theta$ *to* $(B_0, B_1)_\theta$ *with norm at most* $C_0^{1-\theta}C_1^\theta$*, provided* $0 \leqslant \theta \leqslant 1$.

Proof. One can refer to [15] (For the complex interpolation of modulation spaces (Triebel-type space $N^s_{p,q}$), one can refer to [15] ([21])).

Theorem B. *Assume* $1 \leqslant p_2 \leqslant p_1 < \infty,\ 1 \leqslant q \leqslant \infty$, *then we have*

$$\|u\|_{N^s_{p_1,q}} \leqslant \|u\|_{N^s_{p_2,q}}.$$

Theorem C. (Generalized Bernstein inequality). *Let* $\Omega \subset R^n$ *be a compact set,* $0 < r \leqslant \infty$. *Let us denote* $\sigma_r = n(1/(r \wedge 1) - 1/2)$ *and assume that* $s > \sigma_r$. *Then there exists a constant* $C > 0$ *such that*

$$\|F^{-1}\varphi F f\|_{L^r} \leqslant C\|\varphi\|_{H^s}\|f\|_{L^r}$$

holds for all $f \in L^r_\Omega := \{f \in L^p : supp\widehat{f} \subset \Omega\}$ *and* $\varphi \in H^s$. *Moreover, if* $r \geqslant 1$, *then the above estimate holds for all* $f \in L^r$.

Proof. One can refer to [15].

Theorem D. *Assume* $0 < p \leqslant q \leqslant \infty$, *Let* $\Omega \subset \mathbb{R}^n$ *be compact set,* $diam\Omega < 2R$. *Then there exists* $C(p, q, R) > 0$, *such that*

$$\|f\|_{L^q} \leqslant C\|f\|_{L^p},\ \forall f \in L^p_\Omega.$$

Where $L^p_\Omega = \{f \in L^p : supp\hat{f} \subset \Omega\}$.

Proof. One can refer to [29], [30].

Theorem E. (Convolution in L^p with $p < 1$). *Let* $0 < p \leqslant 1$. $L^p_{B(x_0,R)} = \{f \in L^p(\mathbb{R}^n) : supp\widehat{f} \subset B(x_0, R)\}$, $B(x_0, R) = \{x : |x - x_0| \leqslant R\}$. *Suppose that* $f, g \in L^p_{B(x_0,R)}$, *then there exists a constants* $C > 0$ *which is independent of* x_0 *and* $R > 0$ *such that*

$$\|f * g\|_{L^p} \leqslant CR^{n(1/p-1)}\|f\|_{L^p}\|g\|_{L^p}$$

Proof. One can refer to [24].

Theorem F. *Assume* $0 < p < \infty,\ 0 < q \leqslant \infty$, *and* $\Omega = \{\Omega_k\}_{k=0}^\infty$ *is a sequence with compact support in* $\mathbb{R}^n$, *let* $d_k > 0$ *is the diameter of* Ω_k. *If* $s > n(\frac{1}{min(1,p,q)} - \frac{1}{2})$, *then there exists a constant C such that*

$$\|F^{-1}M_k F f_k\|_{L^p(\ell^q)} \leqslant C sup_i\|M_i(d_i\cdot)\|_{H^s}\|f_k\|_{L^p(\ell^q)}$$

where $\{f_k\}_{k=0}^\infty \in L^p_\Omega(\ell^q), \{M_k(x)\}_{k=0}^\infty \subset H^s$.

Proof. One can refer to [24].($L^p_\Omega(\ell^q) = \{f | f = \{f_k\}_{k=0}^\infty \subset S', suppFf_k \subset \Omega_k\ if\ k = 0, 1, 2...$ and $\|f_k\|_{L^p(\ell^q)} < \infty\}$)

Theorem G. *Let $0 < p, q \leqslant \infty$ and $(X, \mu), (Y, \upsilon)$ be two measure space. Let T be a positive linear operator mapping $L^p(X)$ into $L^q(Y)$ [respectively, into $L^{q,\infty}(Y)$] with norm A. Let B be a Banach space. Then T has a B-valued extension $\vec{T}$ that maps $L^p(X, B)$ into $L^q(Y, B)$ [respectively, into $L^{q,\infty}(Y, B)$] with the same norm.*

Proof. One can refer to [14] (An operator T acting on measurable functions is called positive if it satisfies $f \geqslant 0 \Rightarrow T(f) \geqslant 0$).

Theorem H. *Assume $s_1, s_2 \in \mathbb{R}$, $0 < p < \infty$, $0 < q_1, q_2 \leqslant \infty$, then, for $q_2 < q_1$, $s_1 - s_2 > n/q_2 - n/q_1$, we have*

$$N^{s_1}_{p,q_1} \subset N^{s_2}_{p,q_2}.$$

Proof. One can refer to [21] for the proof of the above theorems.

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