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BLOW-UP FOR A SEMILINEAR PARABOLIC EQUATION WITH NONLINEAR MEMORY AND NONLOCAL NONLINEAR BOUNDARY

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Abstract. In this paper, we study a semilinear parabolic equation

$$u_t = \Delta u + \int_0^t u^p ds - ku^q, \quad x \in \Omega, \quad t > 0$$

with boundary condition $u(x,t) = \int_\Omega f(x,y)\, u^l(y,t) dy$ for $x \in \partial\Omega$, $t > 0$, where p, q, l, $k > 0$. The blow-up criteria and the blow-up rate are obtained under some appropriate assumptions.

1. Introduction

The main purpose of this paper is to study the blow-up properties of the nonnegative solutions for the following semilinear parabolic equation with nonlinear time-integral source and nonlocal nonlinear boundary condition

$$\begin{cases} u_t = \Delta u + \int_0^t u^p ds - ku^q, & x \in \Omega, \quad t > 0, \\ u(x,t) = \int_\Omega f(x,y)\, u^l(y,t) dy, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N for $N \geq 1$ with C^2 boundary $\partial\Omega$, p, q, l and k are positive parameters, the weight function $f(x,y)$ is nonnegative, nontrivial, continuous and defined for $x \in \partial\Omega$, $y \in \overline{\Omega}$, while the nonnegative nontrivial initial

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data $u_0(x) \in C^2(\overline{\Omega})$ satisfies the compatibility conditions $u_t(x,0) = \Delta u_0(x) - ku_0^q$ for $x \in \Omega$ and $u_0(x) = \int_\Omega f(x,y)u_0^l(y)\,dy$ for $x \in \partial\Omega$.

In [1], Bellout studied the following equation

$$u_t - \Delta u = \int_0^t (u+\lambda)^p\,ds + g(x), \quad x \in \Omega, \quad t > 0 \tag{1.2}$$

with homogeneous Dirichlet boundary condition, where $g(x) \geq 0$ is a smooth function and $\lambda > 0$. The author established the existence and the uniqueness of the local classical solution, and obtained some criteria for solutions to blow up in a finite time. Moreover, he obtained some results on the blow-up points under some suitable assumptions. In [26], Yamada investigated the stability properties of the global solutions of the following nonlocal Volterra diffusion equation

$$u_t - \Delta u = (a - bu)\,u - \int_0^t k(t-s)\,u(x,s)\,ds, \quad x \in \Omega, \quad t > 0. \tag{1.3}$$

In [15], Li and Xie considered the following single equation

$$\begin{cases} u_t = \Delta u + u^q \int_0^t u^p ds, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.4}$$

where $p,\ q \geq 0$. They gave a complete answer to the existence and the nonexistence of global solutions to problem (1.4) according to the different values of p and q. Furthermore, under the assumptions:

$$\text{There exists } t_0 \in (0,T^*) \text{ such that } u_t(x,t_0) \geq 0 \text{ for all } x \in \overline{\Omega}, \tag{1.5}$$

and

$$\Omega = \left\{x \in R^N : |x| < R\right\}, u_0(x) = u_0(|x|) \equiv u_0(r), u_0'(r) < 0 \text{ and } u_0''(0) < 0, \tag{1.6}$$

they derived the following blow-up rate for the special case $p > 1$ and $q = 0$,

$$C_1(T^* - t)^{-\frac{2}{p-1}} \leq \max_{x\in\overline{\Omega}} u(x,t) \leq C_2(T^* - t)^{-\frac{2}{p-1}}, \quad t \to T^*. \tag{1.7}$$

It is necessary to point out that assumption (1.5) seems to be reasonable, but unfortunately, the authors of [15] did not give a relationship between u_0 and (1.5). The characterization of the monotonicity condition (1.5) was given by Souplet in [22], he proved the existence of monotone in time solutions for problem (1.4) and obtained the blow-up rate (1.7) without assumption condition (1.6).

There have also been many other results for parabolic equations with nonlinear memory. We refer the readers to [27, 30, 29, 5, 12] and the references therein.

We note also that the nonlocal parabolic equations with space-integral source terms have been extensively studied by many authors (see [21] and the references therein). For example, Wang et al. [23] considered the following famous diffusion equation

$$u_t = d\Delta u + \int_\Omega u^p dx - ku^q, \quad x \in \Omega, \quad t > 0 \tag{1.8}$$

with homogeneous Dirichlet boundary condition and positive initial data. They concluded that the blow-up occurs for large initial data if $p > q \geq 1$, and that all solutions exist globally if $1 \leq p < q$. In the case of $p = q$, the issue depends on the comparison between $|\Omega|$ and k.

In [20], Soufi et al. investigated the heat equation with space-integral absorption of the form

$$\begin{cases} u_t = \Delta u + |u|^p - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |u|^p\, dx, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \displaystyle\int_\Omega u_0(x)\, dx = 0, & x \in \overline{\Omega}, \end{cases} \tag{1.9}$$

where $1 < p \leq 2$. Using the energy method and Gamma-convergence technique, they proved that all solutions blow up in a finite time if the energy of u_0 is nonpositive. Jazar and Kiwan [14] generalized the above result, and showed that the solution of problem (1.9) blows up in a finite time for all $p > 1$ if the initial energy is nonpositive.

As is well known, parabolic equations with nonlocal boundary conditions arise in various field theories such as the heat conduction within linear thermoelastcity. Day [2, 3] dealt with a heat equation which is subjected to the following boundary conditions

$$u(-R,t) = \int_{-R}^{R} f_1(x) u(x,t)\, dx, \quad u(R,t) = \int_{-R}^{R} f_2(x) u(x,t)\, dx.$$

Friedman [7] generalized Day's result to a general parabolic equation

$$u_t = \Delta u + g(x,u), \quad x \in \Omega, \quad t > 0, \tag{1.10}$$

which is subjected to the following nonlocal boundary condition

$$u(x,t) = \int_\Omega h(x,y)\, u(y,t)\, dy, \tag{1.11}$$

and studied the global existence of solutions and its monotonic decay property under some hypotheses on $h(x,y)$ and $g(x,u)$.

In addition, parabolic equations with both space-integral source terms and nonlocal boundary conditions have been studied as well (see [4, 16, 19, 24] and the references therein). For instance, Lin and Liu [17] considered the problem of the form

$$u_t = \Delta u + \int_\Omega g(u)\,dx, \tag{1.12}$$

which is subjected to boundary condition (1.11). They established the local existence, the global existence and the nonexistence of solutions, and discussed the blow-up properties of solutions. Furthermore, they derived the uniform blow-up estimate for some special $g(u)$.

In particular, Wang et al. [25] studied the following problem

$$\begin{cases} u_t = \Delta u + \int_\Omega u^p dx - ku^q, & x \in \Omega, \quad t > 0, \\ u(x,t) = \int_\Omega \psi(x,y)\,u(y,t)dy, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.13}$$

and obtained the conditions for the existence and the nonexistence of global solutions. Moreover, they established the precise estimate of the blow-up rate under some suitable hypotheses.

However, as far as we know, there were only few articles which concerned with the blow-up behaviors of solutions for the parabolic equations coupled with nonlocal nonlinear boundary condition. Recently, Gladkov and Kim [10, 11] considered a semilinear heat equation as follows

$$\begin{cases} u_t = \Delta u + c(x,t)\,u^p, & x \in \Omega, \quad t > 0, \\ u(x,t) = \int_\Omega \varphi(x,y,t)\,u^l(y,t)\,dy, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.14}$$

where $p,\ l > 0$. In [11], they obtained the uniqueness and the nonuniqueness of the local solution. And in [10], according to the behavior of the coefficient functions $c(x,t)$ and $\varphi(x,y,t)$ as t tends to infinity, they gave some criteria for the existence of global solutions as well as for finite time blow-up solutions.

Mu et al. [18] considered the blow-up properties for the following problem

$$\begin{cases} u_t = \Delta u + \int_\Omega u^p dx - ku^q, & x \in \Omega, \quad t > 0, \\ u(x,t) = \int_\Omega \phi(x,y)\,u^l(y,t)dy, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.15}$$

where p, q, $l > 0$, and obtained some results as follows.

(i) Assume that $p \leq q$ and $l \leq 1$, then problem (1.15) has global solutions for any ϕ and u_0.

(ii) Assume that $\max\{p, l\} > q \geq 1$. If k is small enough, then for any positive function ϕ, u blows up in a finite time for sufficiently large u_0. While u exists globally for sufficiently small u_0 provided that $\int_\Omega \phi(x, y)\, dy \leq 1$.

Motivated by those of the above works, we consider the semilinear reaction diffusion Equation (1.1) with time-integral and nonlocal nonlinear boundary condition. In [15], the authors dealt with the blow-up behavior of Equation (1.4) by constructing some suitable self-similar subsolutions which blow up in a finite time. But this approach can not be extended to handle the blow-up property of Equation (1.1) due to the appearance of the nonlocal nonlinear boundary condition. Meanwhile, our method is very different from those previously used in [18] because the space-integral source term $\int_\Omega u^p dx$ is replaced by the time-integral term $\int_0^t u^p ds$. The proofs of our blow-up results are based on a variant of the eigenfunction method (Kaplan's method). We will show that the nonlinear memory term $\int_0^t u^p(x, s)\, ds$, the weight function $f(x, y)$ and the nonlinear term $u^l(y, t)$ in the boundary condition of problem (1.1) play substantial roles in determining whether the solution blows up or not. Moreover, we yield the blow-up rate for the special case $p > 1$ and $q = l = 1$ under some appropriate hypotheses.

Before starting the main results, we introduce some notations. Throughout this paper, we let λ_1 be the first eigenvalue and $\varphi(x)$ be the corresponding normalized eigenfunction of the problem

$$(1.16) \qquad \begin{cases} -\Delta\varphi(x) = \lambda\varphi, & x \in \Omega, \\ \varphi(x) = 0, & x \in \partial\Omega, \end{cases}$$

then

$$\lambda_1 > 0, \quad \varphi(x) > 0 \ \text{ and } \ \int_\Omega \varphi(x) dx = 1.$$

Further, for the sake of convenience, we denote

$$L = \max_{\overline{\Omega}} \varphi(x) \ \text{ and } m = \min_{\partial\Omega\times\overline{\Omega}} f(x, y).$$

The main results of this paper are stated as follows.

Theorem 1.1. *Assume that* $p \leq q$ *and* $l \leq 1$, *then problem (1.1) has global solutions for any nonnegative* $f(x, y)$ *and initial data* $u_0(x)$.

Theorem 1.2. *Assume that* $p > q \geq 1$ *and* $l > 0$, *then the solution of problem (1.1) blows up in a finite time for any nonnegative* $f(x, y)$ *and initial data* $u_0(x)$.

Remark 1.1. In [25], the authors showed that the behavior of the solution to problem (1.13) with $p > q \geq 1$ depends on the value of $\int_\Omega \psi(x, y)\, dy$ and initial data

u_0. Precisely, if $\int_\Omega \psi(x,y)\,dy \le 1$, then the solution exists globally for sufficiently small u_0, while the solution blows up in a finite time when $\int_\Omega \psi(x,y)\,dy > 1$ and u_0 is large enough. But thanks to the critical effect of the time-integral term $\int_0^t u^p(x,s)\,ds$, the solution of problem (1.1) fails to exist globally for any nonnegative weight function $f(x,y)$ and initial data $u_0(x)$ in the case of $p > q \ge 1$.

Remark 1.2. For the case $p > q \ge 1$, the authors in [18] proved that problem (1.15) has blow-up solutions in a finite time as well as global solutions. More precisely, if k is small enough, then for any ϕ, the solution tends to infinity in a finite time when u_0 is large enough, while u exists globally for sufficiently small u_0 provided that $\int_\Omega \phi(x,y)\,dy \le 1$. From Theorem 1.2, we know that the property of the solution to problem (1.1) is very different from that of problem (1.15).

Theorem 1.3. *Assume that $l \ge q \ge p > 1$, for any $f(x,y) > 0$, if $\frac{m\lambda_1}{L} > k$, then the solution of problem (1.1) blows up in a finite time provided that the initial data $u_0(x)$ satisfies $\int_\Omega u_0(x)\varphi(x)\,dx >> 1$.*

Remark 1.3. Theorems 1.2 and 1.3 are still true when $k = 0$.

Remark 1.4. In the case of $l > 1$ and $q \ge p > 1$, we can not prove that the solutions of problem (1.1) exist globally or not for sufficiently small initial data $u_0(x)$ by the methods used in this paper.

Consider problem (1.1) with $q = l = 1$. In order to obtain the blow-up rate, we need to add the following assumption (assume that the solution of problem (1.1) blows up in finite time T^*).

(H) There exists constant $t_0 \in (0, T^*)$ such that $u_t(x, t_0) \ge 0$ for all $x \in \overline{\Omega}$.

Theorem 1.4. *Assume that $p > 1$, $q = l = 1$ and $\int_\Omega f(x,y)dy \le 1$ for all $x \in \partial\Omega$, assume also that (H) holds, then there exist constants $C > c > 0$ such that*

$$c\,(T^* - t)^{-\frac{2}{p-1}} \le \max_{x\in\overline{\Omega}} u(x,t) \le C\,(T^* - t)^{-\frac{2}{p-1}}, \quad t \to T^*.$$

The rest of this paper is organized as follows. In Section 2, we will establish the comparison principle and the local existence theorem for problem (1.1). Section 3 is mainly about the global existence of solutions and the proof of Theorem 1.1. The blow-up results of solutions and the proofs of Theorems 1.2 and 1.3 are given in section 4. Finally, we will estimate the blow-up rate in section 5.

2. Comparison Principle and Local Existence

The main goal of this section is to establish the local existence theorem and the comparison principle for problem (1.1). For simplicity, let us first denote $\Omega_T = \Omega \times (0,T)$ and $\overline{\Omega}_T = \overline{\Omega} \times [0,T]$ for $0 < T < +\infty$.

Theorem 2.1. (Local existence and uniqueness). *Assume that* p, q *and* $l > 0$, $u_0(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfies the compatibility condition* $u_0(x) = \int_\Omega f(x,y) u_0^l(y)\, dy$ *for* $x \in \partial\Omega$, *then there exists a small* $T > 0$ *such that problem (1.1) has a nonnegative solution* $u(x,t) \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$. *Furthermore, assume that the initial data* $u_0(x)$ *is strictly positive for the case* $\min\{p,q,l\} < 1$, *then the local solution of problem (1.1) is unique.*

Proof. Here we only give the sketch, one can see [1, 13] for more details. We divide our proof into three cases.

Case 1. We first consider the case that p, q, $l \geq 1$. Let $G(x,y;t)$ be the Green's function for

$$\mathcal{L}u = u_t - \Delta u, \quad x \in \Omega, \quad t > 0,$$

with boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0.$$

Then $u(x,t)$ is a solution of Equation (1.1) in $\overline{\Omega}_T$ if and only if for $(x,t) \in \overline{\Omega}_T$,

$$\begin{aligned} u(x,t) = &\int_\Omega G(x,y;t)\, u_0(y)\, dy \\ &+ \int_0^t \int_\Omega G(x,y;t-\eta) \left(\int_0^\eta u^p(y,\sigma)\, d\sigma - k u^q(y,\eta) \right) dy d\eta \\ &- \int_0^t \int_{\partial\Omega} \frac{\partial G(x,\xi;t-\eta)}{\partial n} \int_\Omega f(\xi,y)\, u^l(y,\eta)\, dy d\xi d\eta \\ \equiv\, &\mathcal{T}u(x,t). \end{aligned} \tag{2.1}$$

Notice $u_0(x) \in C(\overline{\Omega}) \subset L^\infty(\Omega)$, one can take a subset of $L^\infty(\Omega_T)$ as follows

$$\mathcal{B}_T = \left\{ u \in L^\infty(\Omega_T)\, ; \|u(t)\|_{L^\infty(\Omega)} \leq \widehat{M} + 1 \text{ for } t \in (0,T) \right\},$$

where $\widehat{M} = \sup_{x \in \overline{\Omega}} u_0(x)$, and prove that $\mathcal{T}$ is a strict contraction mapping from $\mathcal{B}_T$ into $\mathcal{B}_T$ for sufficiently small T. As a consequence of the contraction mapping principle, (2.1) is solvable in $L^\infty(\Omega_T)$ for small T. Meanwhile, we can show that if v is another solution, then

$$\sup_{\overline{\Omega}_T} |v - u| \leq \rho^{n-1} \sup_{\overline{\Omega}_T} |v - u|,$$

where $\rho \in (0,1)$, which yields that $v \equiv u$.

The continuity of the solution $u \in C(\overline{\Omega}_T)$ directly follows from the contraction mapping principle, and the asserted interior regularity follows from (2.1) and the properties of G (see [6] for more details).

Case 2. Now, we consider the case that $p,\ q,\ l < 1$. Since the nonlinearities of problem (1.1) do not satisfy the Lipschitz condition in this case, we set

$$\phi_m(z) = \begin{cases} z^p, & \text{if } z > \frac{1}{m}, \\ \left[pm^{1-p}z + \frac{1-p}{m^p}\right]^+, & \text{otherwise,} \end{cases}$$

$$\psi_m(z) = \begin{cases} z^q, & \text{if } z > \frac{1}{m}, \\ \left[qm^{1-q}z + \frac{1-q}{m^q}\right]^+, & \text{otherwise,} \end{cases}$$

and

$$\zeta_m(z) = \begin{cases} z^l, & \text{if } z > \frac{1}{m}, \\ \left[lm^{1-l}z + \frac{1-l}{m^l}\right]^+, & \text{otherwise.} \end{cases}$$

It is clear that ϕ_m, ψ_m and ζ_m are nondecreasing, locally Lipschitz continuous with respect to z and monotone decreasing with respect to m, i.e.,

$$\phi_m \downarrow \left[|z|^{p-1} z\right]^+, \quad \psi_m \downarrow \left[|z|^{q-1} z\right]^+, \quad \text{and} \quad \zeta_m \downarrow \left[|z|^{l-1} z\right]^+ \quad \text{as} \quad m \to \infty.$$

Consider the following approximate problem

$$\text{(2.2)} \quad \begin{cases} (u_m)_t = \Delta u_m + \displaystyle\int_0^t \phi_m(u_m)(s)ds - k\psi_m(u_m), & x \in \Omega, \quad t > 0, \\ u_m(x,t) = \displaystyle\int_\Omega f(x,y)\,\zeta_m(u_m)(y,t)dy, & x \in \partial\Omega, \quad t > 0, \\ u_m(x,0) = u_0(x), & x \in \overline{\Omega}. \end{cases}$$

Similar to Case 1, we can claim that problem (2.2) admits a unique solution $u_m \in C\left(\overline{\Omega}_T\right) \cap C^{2,1}(\Omega_T)$. Notice that $\phi(0)$, $\psi(0)$, $\zeta(0)$ are nonnegative, we see that $u_m \geq 0$. Moreover, in view of the comparison principle (see Theorem 2.2), we find that $\{u_m\}$ is monotone decreasing. Hence, there exists a bounded nonnegative function $u = \lim_{m\to\infty} u_m$, which corresponds to the continuous solution of problem (1.1).

Moreover, $u_0(x) > 0$ tells us that $u(x,t)$ is strictly positive in $\overline{\Omega}_T$. In fact, from $u_t - \Delta u - \int_0^t u^p ds + ku^q \geq 0$ and the strong maximum principle, it follows that u can not take its minimum at an interior point of Ω, that is, the minimum point (x_0, t_0) must lie on the parabolic boundary. Since $f(x,y)$ is nontrivial for all $x \in \partial\Omega$, $y \in \overline{\Omega}$, we have $u(x,t) > 0$ for $x \in \partial\Omega$, $t \in (0,T]$. Therefore, $u(x,t) > 0$ in $\overline{\Omega}_T$. Up to now, we can immediately obtain the uniqueness of the local solution by combining the comparison principle with the strict positivity of the local solution.

Case 3. If $p < 1$, or $q < 1$, or $l < 1$. Based on Cases 1 and 2, we can obtain our result easily. The proof of Theorem 2.1 is completed. ∎

Next, we will give the following version of the comparison principle which plays a crucial role in our later proof.

Theorem 2.2. (Comparison principle). *Let* $\underline{u}$, $\overline{u} \in C\left(\overline{\Omega}_T\right) \cap C^{2,1}\left(\Omega_T\right)$ *be a nonnegative subsolution and supersolution of problem (1.1), respectively. In addition, assume that* $\underline{u}\,(x,t) > 0$ *in* $\overline{\Omega}_T$ *if* $\min\{p,q,l\} < 1$. *If* $\underline{u}\,(x,0) \le \overline{u}\,(x,0)$ *for* $x \in \overline{\Omega}$. *Then* $\underline{u} \le \overline{u}$ *in* $\overline{\Omega}_T$.

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1 in [11], we omit it here. ■

3. Global Existence of Solution

In this section, we discuss the global solvability of problem (1.1), and give the proof of Theorem 1.1. Our approach is a combination of the comparison principle and a supersolution technique.

Proof of Theorem 1.1. Let T be any positive number. In order to prove our conclusion, according to Theorem 2.2, we only need to construct a suitable explicit global supersolution of problem (1.1) in Ω_T. Remember that λ_1 is the first eigenvalue and φ is the corresponding normalized eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary condition. We choose δ and $\varepsilon \in (0,1)$ to satisfy

$$M \int_\Omega \frac{1}{\delta\varphi\,(y) + \varepsilon} dy \le 1, \tag{3.1}$$

where

$$M = \max_{\partial\Omega\times\overline{\Omega}} f\,(x,y)\,.$$

Now, let $v(x,t)$ be defined as

$$v\,(x,t) = \frac{ce^{\gamma t}}{\delta\varphi\,(x) + \varepsilon}$$

with

$$c = \sup_{\overline{\Omega}} \left(u_0 + 1\right)\left(\delta\varphi + \varepsilon\right), \quad \gamma = \max\left\{\frac{1}{kp}, \lambda_1 + \sup_{\overline{\Omega}} \frac{2\delta^2\left|\nabla\varphi\right|^2}{\left(\delta\varphi+\varepsilon\right)^2}\right\}. \tag{3.2}$$

A direct computation shows

$$\begin{aligned}
Pv &\equiv v_t - \Delta v - \int_0^t v^p ds + kv^q \\
&= \gamma v - v\left(\frac{\lambda_1\delta\varphi}{\delta\varphi+\varepsilon} + \frac{2\delta^2\left|\nabla\varphi\right|^2}{\left(\delta\varphi+\varepsilon\right)^2}\right) - \int_0^t \frac{c^p e^{p\gamma s}}{\left(\delta\varphi+\varepsilon\right)^p} ds + \frac{kc^q e^{q\gamma t}}{\left(\delta\varphi+\varepsilon\right)^q} \\
&= \gamma v - v\left(\frac{\lambda_1\delta\varphi}{\delta\varphi+\varepsilon} + \frac{2\delta^2\left|\nabla\varphi\right|^2}{\left(\delta\varphi+\varepsilon\right)^2}\right) + \frac{kc^q e^{q\gamma t}}{\left(\delta\varphi+\varepsilon\right)^q} + \frac{c^p}{\gamma p\left(\delta\varphi+\varepsilon\right)^p} - \frac{c^p e^{p\gamma t}}{\gamma p\left(\delta\varphi+\varepsilon\right)^p} \\
&\ge \gamma v - v\left(\frac{\lambda_1\delta\varphi}{\delta\varphi+\varepsilon} + \frac{2\delta^2\left|\nabla\varphi\right|^2}{\left(\delta\varphi+\varepsilon\right)^2}\right) + \frac{kc^q e^{q\gamma t}}{\left(\delta\varphi+\varepsilon\right)^q} - \frac{c^p e^{p\gamma t}}{\gamma p\left(\delta\varphi+\varepsilon\right)^p}.
\end{aligned}$$

From (3.2), it follows that

$$(3.3)\quad Pv \geq \begin{cases} v\left[\gamma-\left(\frac{\lambda_1\delta\varphi}{\delta\varphi+\varepsilon}+\frac{2\delta^2\left|\nabla\varphi\right|^2}{(\delta\varphi+\varepsilon)^2}\right)\right]+v^p\left(k-\frac{1}{\gamma p}\right)\geq 0, & \text{if } p=q,\\ v\left[\gamma-\left(\frac{\lambda_1\delta\varphi}{\delta\varphi+\varepsilon}+\frac{2\delta^2\left|\nabla\varphi\right|^2}{(\delta\varphi+\varepsilon)^2}\right)\right]+v^p\left(\frac{kc^{(q-p)}e^{(q-p)\gamma t}}{(\delta\varphi+\varepsilon)^{q-p}}-\frac{1}{\gamma p}\right)\geq 0, & \text{if } p<q,\end{cases}$$

and

$$(3.4)\qquad v(x,0)=\frac{c}{\delta\varphi(x)+\varepsilon}\geq\frac{\sup\limits_{\overline{\Omega}}\left(u_0(x)+1\right)\left(\delta\varphi(x)+\varepsilon\right)}{\delta\varphi(x)+\varepsilon}>u_0(x).$$

On the other hand, for any $(x,t)\in\partial\Omega\times(0,T)$, (3.1), $v(x,t)>1$ and $l\leq 1$ guarantee

$$(3.5)\qquad \begin{aligned} v(x,t)&=\frac{ce^{\gamma t}}{\varepsilon}>ce^{\gamma t}\geq\int_\Omega f(x,y)\frac{ce^{\gamma t}}{\delta\varphi(y)+\varepsilon}dy=\int_\Omega f(x,y)\,v(y,t)\,dy\\ &\geq\int_\Omega f(x,y)\,v^l(y,t)\,dy.\end{aligned}$$

Combining now from (3.3) to (3.5), we know that $v(x,t)$ is a supersolution of problem (1.1) in Ω_T and the solution $u(x,t)<v(x,t)$ by the comparison principle. Consequently, problem (1.1) has global solutions. The proof of Theorem 1.1 is completed. ■

4. Blow-up of Solution

In this section, we turn our attention to the blow-up properties of problem (1.1). Due to the complication of the nonlocal nonlinear boundary condition, the approaches used in [15, 30] can not be extended to handle the blow-up behaviors of solutions of Equation (1.1). The proofs of Theorems 1.2 and 1.3 rely on the modified eigenfunction method combined with the properties of some special differential inequalities.

Proof of Theorem 1.2. We will use a modification of an argument in the proof of Theorem 5.1 in [21] to prove our blow-up result of the case $p>q\geq 1$. Let $u(x,t)$ be the unique solution to problem (1.1). We first define two auxiliary functions as follows

$$J_1(t)=\int_\Omega\varphi(x)u(x,t)\,dx\quad\text{and}\quad J_2(t)=\int_0^t\int_\Omega u^p(x,s)\varphi(x)dxds,\quad 0\leq t<T.$$

Taking the derivative of $J_1(t)$ with respect to t, and using Green's formula we could obtain

$$
\begin{aligned}
&J_1{}'(t) \\
&= \int_\Omega \varphi\left(\Delta u + \int_0^t u^p ds - ku^q\right)dx \\
&= J_2(t) + \int_{\partial\Omega} \frac{\partial u}{\partial n}\varphi dS - \int_\Omega \nabla\varphi\cdot\nabla u dx - k\int_\Omega \varphi u^q dx \qquad (4.1)\\
&= J_2(t) + \int_\Omega u\Delta\varphi dx - \int_{\partial\Omega}\frac{\partial\varphi}{\partial n}udS - k\int_\Omega \varphi u^q dx \\
&= -\lambda_1 J_1(t) + J_2(t) - \int_{\partial\Omega}\frac{\partial\varphi}{\partial n}\left(\int_\Omega f(x,y)\,u^l(y,t)\,dy\right)dS - k\int_\Omega \varphi u^q dx,
\end{aligned}
$$

where n denotes the unit outer normal vector on $\partial\Omega$. Moreover, we have $\frac{\partial\varphi}{\partial n} < 0$ for all $x \in \partial\Omega$, and

$$\int_{\partial\Omega}\frac{\partial\varphi}{\partial n}dS = -\lambda_1\int_\Omega \varphi dx = -\lambda_1.$$

It follows immediately from (4.1) that

$$J_1{}'(t) \geq -\lambda_1 J_1(t) + J_2(t) + \frac{m\lambda_1}{L}\int_\Omega \varphi u^l dx - k\int_\Omega \varphi u^q dx. \tag{4.2}$$

By (4.2) and Jensen's inequality, we have

$$\begin{cases} J_2{}'(t) \geq J_1^p(t), \\ J_1{}'(t) \geq -\lambda_1 J_1(t) + J_2(t) - k\,(J_2{}'(t))^{\frac{q}{p}}. \end{cases} \tag{4.3}$$

By virtue of Lemma 5.3 in [21] with $r = \frac{q}{p}$ and $\alpha = 1$, we can easily conclude that there exist $T \in (0,\infty)$ and positive constants c_1 and c_2 such that

$$\lim_{t\to T}\chi(t) = \lim_{t\to T}\left(c_1 J_2^\xi(t) + c_2 J_1(t)\right) = \infty, \tag{4.4}$$

where $\xi \in (0,1)$, which implies that $u(x,t)$ blows up in a finite time for any nonnegative nontrivial initial data $u_0(x)$. The proof of Theorem 1.2 is completed. ■

Proof of Theorem 1.3. The first part of the arguments is the same as in Theorem 1.2. Since $l \geq q \geq 1$, we know that

$$u^l + 1 > u^q. \tag{4.5}$$

Employing this inequality into (4.2), we obtain

$$J_1{}'(t) \geq -\lambda_1 J_1(t) + \left(\frac{m\lambda_1}{L} - k\right)\int_\Omega \varphi u^l dx - k. \tag{4.6}$$

Further, since $l > 1$ and $\frac{m\lambda_1}{L} - k > 0$, Jensen's inequality can be used to (4.6) to get

$$J_1{}'(t) \geq -\lambda_1 J_1(t) + \left(\frac{m\lambda_1}{L} - k\right) J_1^l(t) - k. \tag{4.7}$$

Since the function $f(J_1) = J_1^l$ is convex, then there exists $\eta > 1$ such that

$$\left(\frac{m\lambda_1}{L} - k\right) J_1^l \geq 2(\lambda J_1 + k)$$

holds for $J_1 \geq \eta$.

It follows easily that if $J_1(0) > \eta$, then $J_1(t)$ is increasing on its interval of the existence and

$$J_1{}'(t) \geq \frac{1}{2} J_1^l. \tag{4.8}$$

From the above inequality it follows that

$$\lim_{t \to T_0^-} J_1(t) = +\infty, \tag{4.9}$$

where

$$T_0 = \frac{2}{(l-1) J_1^{l-1}(0)}.$$

Then by the assumption on the initial data in Theorem 1.3, the solution $u(x,t)$ becomes infinite in a finite time. The proof of Theorem 1.3 is completed. ■

5. Blow-up Rate Estimate

In this section, we will derive the blow-up rate of the blow-up solution for the following special case of problem (1.1),

$$\begin{cases} u_t = \Delta u + \int_0^t u^p(x,s)ds - ku, & x \in \Omega, \quad t > 0, \\ u(x,t) = \int_\Omega f(x,y)\, u(y,t)dy, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{5.1}$$

where $p > 1$. By Theorem 1.2, for any nonnegative nontrivial initial data u_0, u blows up in a finite time. In the next, we will first follow the general idea in [8] to estimate an upper bounder of the blow-up rate near the blow-up time.

Let $T^* < \infty$ be the maximal time of the existence of a blowing up solution. We have the following Lemma.

Lemma 5.1. *Suppose that* $\int_\Omega f(x,y)dy \le 1$ *for all* $x \in \partial\Omega$ *and assumption* (H) *holds, then for any* $t_1 \in (t_0, T^*)$*, the blow-up solution of problem (5.1) satisfies*

$$u(x,t) \le C(T^* - t)^{-\frac{2}{p-1}}, \quad t_1 < t < T^*, \tag{5.2}$$

where $C > 0$ *is a constant.*

Proof. Let $v = u_t$, then it is easy to verify that v satisfies the following problem

$$\begin{cases} v_t = \Delta v + u^p - kv, & x \in \Omega, \quad t \in (0, T^*), \\ v(x,t) = \displaystyle\int_\Omega f(x,y)\, v(y,t)dy, & x \in \partial\Omega, \quad t \in (0,T^*), \\ v(x,0) = \Delta u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{5.3}$$

Further, thanks to $u_0(x) \in C^2(\overline{\Omega})$, one can prove that v is actually in $C^{2,1}(\Omega_{T^*}) \cap C(\overline{\Omega}_{T^*})$. Let

$$J(x,t) = u_t - \delta \int_0^t u^p ds \quad \text{for} \quad (x,t) \in \Omega \times (t_1, T^*),$$

where δ is a sufficiently small positive constant, then we have $J \in C^{2,1}(\Omega \times (t_1, T^*)) \cap C(\overline{\Omega} \times (t_1, T^*))$. A straightforward computation yields

$$\begin{aligned} & J_t - \Delta J \\ &= u_{tt} - \delta u^p - \Delta u_t + \delta p \int_0^t u^{p-1} \Delta u ds + \delta p(p-1) \int_0^t u^{p-2} |\nabla u|^2 ds \\ &\ge (1-\delta)u^p - ku_t + \delta p \int_0^t u^{p-1} \left(u_s - \int_0^s u^p d\tau + ku\right) ds \\ &= \delta p \int_0^t u^{p-1} \left(u_s - \int_0^s u^p d\tau\right) ds - k\left(u_t - \delta \int_0^t u^p ds\right) \\ &\quad + (1-\delta)u^p + k\delta(p-1) \int_0^t u^p ds \\ &\ge p \int_0^t u^{p-1} \left(u_s - \delta \int_0^s u^p d\tau\right) ds - k\left(u_t - \delta \int_0^t u^p ds\right) \\ &\quad + (1-\delta)u^p - p(1-\delta) \int_0^t u^{p-1} u_s ds \\ &= p \int_0^t u^{p-1} J ds - kJ + (1-\delta)u_0^p \\ &\ge p \int_0^t u^{p-1} J ds - kJ. \end{aligned} \tag{5.4}$$

For fixed $(x,t) \in \partial\Omega \times (t_1, T^*)$, we have

$$
\begin{aligned}
J(x,t) &= u_t - \delta \int_0^t u^p ds \\
&= \int_\Omega f(x,y) u_t(y,t) dy - \delta \int_0^t \left(\int_\Omega f(x,y) u(y,s) dy \right)^p ds.
\end{aligned}
$$

By using $u_t(y,t) = J(y,t) + \delta \int_0^t u^p ds$, we have

$$
\begin{aligned}
&\int_\Omega f(x,y) u_t(y,t) dy - \delta \int_0^t \left(\int_\Omega f(x,y) u(y,s) dy \right)^p ds \\
= &\int_\Omega f(x,y) \left(J(y,t) + \delta \int_0^t u^p ds \right) dy - \delta \int_0^t \left(\int_\Omega f(x,y) u(y,s) dy \right)^p ds \\
= &\int_\Omega f(x,y) J(y,t) dy + \delta \int_0^t \left[\int_\Omega f(x,y) u^p(x,s) dy - \left(\int_\Omega f(x,y) u(y,s) dy \right)^p \right] ds.
\end{aligned}
$$

Noticing that $0 < F(x) = \int_\Omega f(x,y) dy \le 1$ for $x \in \partial\Omega$, we can apply Jensen's inequality to the last integral in the above inequality,

$$
\begin{aligned}
&\int_\Omega f(x,y) u^p(x,s) dy - \left(\int_\Omega f(x,y) u(y,t) dy \right)^p \\
&\ge F(x) \left(\int_\Omega f(x,y) u(y,t) \frac{dy}{F(x)} \right)^p - \left(\int_\Omega f(x,y) u(y,t) dy \right)^p \\
&\ge 0.
\end{aligned}
$$

Here, we used $p > 1$ and $0 < F(x) \le 1$ in the last inequality. Hence,

$$
J(x,t) \ge \int_\Omega f(x,y) J(y,t) dy \tag{5.5}
$$

holds for all $(x,t) \in \partial\Omega \times (t_1, T^*)$.

On the other hand, the assumption condition (H) implies that

$$
J(x,t_1) = u_t(x,t_1) - \delta \int_0^{t_1} u^p(x,s)\, ds \ge 0 \text{ in } \overline{\Omega}. \tag{5.6}
$$

Since f and u are nonnegative bounded continuous for $(x,t) \in \Omega \times (t_1, T^*)$, it follows from (5.4), (5.5) and (5.6) that $J(x,t) \ge 0$ for $(x,t) \in \Omega \times (t_1, T^*)$, which means

$$
u_t \ge \delta \int_0^t u^p(x,s) ds. \tag{5.7}
$$

Multiplying both sides of the inequality (5.7) by u^p and integrating over (t_1, t), we have

$$u^p(x,t) \geq [\delta(1+p)]^{\frac{p}{1+p}} \left(\int_{t_1}^{t} u^p(x,s)\, ds \right)^{\frac{2p}{1+p}}, \quad t_1 < t < T^*. \tag{5.8}$$

Dividing both sides of (5.8) by $\left(\int_{t_1}^{t} u^p(x,s)\, ds\right)^{\frac{2p}{1+p}}$ and integrating above inequality from t to T^*, we deduce that

$$\int_{t_1}^{t} u^p(x,s)\, ds \leq [\delta(1+p)]^{-\frac{p}{p-1}} (T^* - t)^{-\frac{p+1}{p-1}}, \quad t_1 < t < T^*. \tag{5.9}$$

Taking a special $t' = \frac{T^*+t}{2}$ and applying $u_t \geq 0$ for $t \in [t_0, T^*)$, we see that from (5.9)

$$\begin{aligned} \frac{T^*-t}{2} u^p(x,t) &\leq \int_t^{t'} u^p(x,s) ds \leq \int_{t_1}^{t'} u^p(x,s) ds \\ &\leq [\delta(1+p)]^{-\frac{p}{p-1}} \left(T^* - t'\right)^{-\frac{p+1}{p-1}} \\ &\leq [\delta(1+p)]^{-\frac{p}{p-1}} \left(\frac{T^*-t}{2}\right)^{-\frac{p+1}{p-1}}, \end{aligned}$$

which yields

$$u(x,t) \leq C(T^* - t)^{-\frac{2}{p-1}}, \quad t_1 < t < T^*. \tag{5.10}$$

where $C = \left[\frac{4}{\delta(1+p)}\right]^{\frac{1}{p-1}}$. The proof of Lemma 5.1 is completed. ■

Remark 5.1. If one can show that the solution $u(x,t)$ to problem (5.1) blows up everywhere under some suitable hypotheses, that is, total blow-up phenomenon occurs, then Lemma 5.1 can be proved for all $l \geq 1$ by using the fact $\lim\limits_{t \to T^*} \left(\inf\limits_{x \in \Omega} u(x,t) \right) = \infty$.

Proof of Theorem 1.4. Let $M(t) = \max\limits_{x \in \overline{\Omega}} u(x,t)$, then by the similar manners as in the proof of Theorem 4.5 in [8], we know that $M(t)$ is Lipschitz continuous, and thus it is differential almost everywhere. Moreover, the following estimate holds for $0 < t < T^*$,

$$M'(t) \leq \int_0^t M^p ds - kM \leq \int_0^t M^p ds.$$

By the analogous way as for (5.9), we get

$$\int_0^t M^p(x,s)\, ds \geq c_1 (T^* - t)^{-\frac{p+1}{p-1}}, \quad 0 < t < T^*. \tag{5.11}$$

For $t_1 \leq \eta < t < T^*$, by exploiting (5.10), (5.11) and M being nondecreasing on $[t_1, T^*)$, we obtain

$$c_1 (T^* - t)^{-\frac{p+1}{p-1}} \leq \int_0^\eta M^p ds + \int_\eta^t M^p ds \leq C_1 (T^* - \eta)^{-\frac{p+1}{p-1}} + (t - \eta) M^p (t).$$

For t close enough to T^*, taking $\eta = \rho t + (1 - \rho)T^*$ with $\rho = \left(\frac{2C_1}{c_1}\right)^{\frac{p-1}{p+1}} > 1$, we deduce that

$$M(t) \geq c (T^* - t)^{-\frac{2}{p-1}}, \tag{5.12}$$

which proves the lower estimate. Combining it with Lemma 5.1, we obtain the blow-up rate estimate. The proof of the Theorem 1.4 is completed. ■

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REFERENCES

1. H. Bellout, Blow-up of solutions of parabolic equation with nonlinear memory, *J. Differential Equations*, **70** (1987), 42-68.
2. W. A. Day, A decreasing property of solutions of parabolic equations with applications to thermoelasticity, *Quart. Appl. Math.*, **40** (1983), 468-475.
3. W. A. Day, *Heat Conduction within Linear Thermoelasticity*, Springer-Verlag, New York, USA, 1985.
4. K. Deng, Comparison principle for some nonlocal problems, *Quart. Appl. Math.*, **50** (1992), 517-522.
5. L. L. Du and C. L. Mu, Global existence and blow-up analysis to a degenerate reaction diffusion system with nonlinear memory, *Nonlinear Anal. Real World Appl.*, **9** (2008), 303-315.
6. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hill, Englewood Cliffs, NJ, 1983.
7. A. Friedman, Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions, *Quart. Appl. Math.*, **44** (1986), 401-407.
8. A. Friedman and B. McLeod, Blow-up of positive solutions of semilinear heat equations, *Indiana Univ. Math. J.*, **34** (1985), 425-447.

9. A. Gladkov and M. Guedda, Semilinear heat equation with absorption and a nonlocal boundary condition, *Appl. Anal.*, **91** (2012), 2267-2276.

10. A. Gladkov and K. I. Kim, Blow-up of solutions for semilinear heat equation with nonlinear nonlocal boundary condition, *J. Math. Anal. Appl.*, **338** (2008), 264-273.

11. A. Gladkov and K. I. Kim, Uniqueness and nonuniqueness for reaction-diffusion equation with nonlocal boundary, *Adv. Math. Sci. Appl.*, **19** (2009), 39-49.

12. Y.-J. L. Guo and M. Shimojo, Blow-up for parabolic equations and systems with non-negative potential, *Taiwanese J. Math.*, **15** (2011), 995-1005.

13. D. Hirata, Blow-up for a class of semilinear integro-differential equations of parabolic type, *Math. Methods Appl. Sci.*, **22** (1999), 1087-1100.

14. M. Jazar and R. Kiwan, Blow-up of a nonlocal semilinear parabolic equation with Neumann boundary conditions, *Ann. Inst. H. Poincaré-Anal. Non Lineaire*, **25** (2008), 215-218.

15. Y. X. Li and C. H. Xie, Blow-up for semilinear parabolic equations with nonlinear memory, *Z. Angew. Math. Phys.*, **55** (2004), 15-27.

16. J. Liang, H. Y. Wang and T. J. Xiao, On a comparison principle for delay coupled systems with nonlocal and nonlinear boundary conditions, *Nonlinear Anal.*, **71** (2009), e359-e365.

17. Z. G. Lin and Y. R. Liu, Uniform blow-up profiles for diffusion equations with nonlocal source and nonlocal boundary, *Acta Math. Sci. Ser. B*, **24** (2004), 443-450.

18. C. L. Mu, D. M. Liu and S. M. Zhou, Properties of positive solutions for a nonlocal reaction diffusion equation with nonlocal nonlinear boundary condition, *J. Korean Math. Soc.*, **47** (2010), 1317-1328.

19. C. V. Pao, Asymptotic behavior of solutions of reaction diffusion equations with nonlocal boundary conditions, *J. Comput. Appl. Math.*, **88** (1998), 225-238.

20. A. E. Soufi, M. Jazar and R. Monneau, A Gamma-convergence argument for the blow-up of a nonlocal semilinear parabolic equation with Neumann boundary conditions, *Ann. Inst. H. Poincaré-Anal. Non Lineaire*, **24** (2007), 17-39.

21. P. Souplet, Blow-up in nonlocal reaction diffusion equations, *SIAM J. Math. Anal.*, **29** (1998), 1301-1334.

22. P. Souplet, Monotonicity of solutions and blow-up for semilinear parabolic equations with nonlinear memory, *Z. Angew. Math. Phys.*, **55** (2004), 28-31.

23. M. X. Wang and Y. M. Wang, Properties of positive solutions for nonlocal reaction diffusion problems, *Math. Methods Appl. Sci.*, **19** (1996), 1141-1156.

24. Y. L. Wang, C. L. Mu and Z. Y. Xiang, Blow up of solutions to a porous medium equation with nonlocal boundary condition, *Appl. Math. Comput.*, **192** (2007), 579-585.

25. Y. L. Wang, C. L. Mu and Z. Y. Xiang, Properties of positive solution for nonlocal reaction diffusion with nonlocal boundary, *Bound. Value Prob.*, **2007** (2007), Article ID 64579, 12 pages.

26. Y. Yamada, On a certain class of semilinear Volterra diffusion equations, *J. Math. Anal. Appl.*, **88** (1982), 443-457.

27. Y. Yamada, Asymptotic stablity for some system of semilinear Voterra diffusion equations, *J. Differential Equations*, **52** (1984), 295-326.

28. H. M. Yin, On a class of parabolic equations with nonlocal boundary conditions, *J. Math. Anal. Appl.*, **294** (2004), 712-728.

29. J. Zhou, C. L. Mu and M. S. Fan, Global existence and blow-up to a degenerate reaction diffusion system with nonlinear memory, *Nonlinear Anal. Real World Appl.*, **9** (2008), 1578-1534.

30. J. Zhou, C. L. Mu and F. Lu, Blow-up and global existence to a degenerate reaction diffusion equation with nonlinear memory, *J. Math. Anal. Appl.*, **333** (2007), 1138-1152.

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