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DIMENSION FREE L^P ESTIMATES FOR RIESZ TRANSFORMS ASSOCIATED WITH LAGUERRE FUNCTION EXPANSIONS OF HERMITE TYPE

Krzysztof Stempak and Błażej Wróbel

Abstract. We prove dimension free L^p estimates for Riesz transforms associated with multi-dimensional Laguerre function expansions of Hermite type. The range of the admissible Laguerre type multi-index α in these estimates depends on $p \in (1, \infty)$; for $1 < p \leq 2$ this range is almost optimal. The proof is based on suitably defined square functions with Poisson and modified Poisson semigroups involved.

1. INTRODUCTION

Dimension free L^p estimates for the classical Riesz transforms R_j, $j = 1, \dots, d$, on $\mathbb{R}^d$, were shown by E. M. Stein [18]. Later on it was found, see [6], that in fact the operator norms of R_j's on L^p spaces do not depend neither on d nor on j: $\|R_j\|_{p\to p} = \tan(\pi/2p)$ if $1 < p \leq 2$ and $\|R_j\|_{p\to p} = \cot(\pi/2p)$ if $2 \leq p < \infty$. Since then a similar phenomenon of dimension free L^p bounds was observed and analogous results were proved for Riesz transforms defined in different settings; see, for instance, [2, 7], where this was done in the context of Heisenberg groups and products of discrete abelian groups.

Similar efforts in proving dimension free bounds were undertaken in several settings of classical orthogonal expansions. Here Riesz transforms are suitably defined and correspond to an involved second order differential operator, a 'Laplacian', and associated first order operators, the 'derivatives'; see [12] for a unified approach to the theory of Riesz transforms and conjugacy in the setting of multi-dimensional orthogonal expansions.

We now briefly overview known results concerning dimension free L^p estimates for orthogonal expansions. The Hermite polynomial case, where the Ornstein-Uhlenbeck

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operator $-\Delta+2x\cdot\nabla$ on $\mathbb{R}^d$ plays the role of a 'Laplacian', was considered by Pisier [15] and Gutierrez [3], and the dimension free L^p bounds for considered Riesz transforms were proved. The Hermite function case (with the harmonic oscillator $-\Delta+|x|^2$ on $\mathbb{R}^d$) was recently treated by Harboure, de Rosa, Segovia and Torrea [5] (see also [8] for an independent proof). The Jacobi polynomial case was studied by Nowak and Sjögren [11]; they proved that the estimates depend neither on the dimension d nor on the Jacobi type multi-indices $\alpha, \beta \in [-1/2,\infty)^d$. The Laguerre polynomial case was initiated by Guttierrez, Incognito and Torrea [4], where the half-integer multi-indices were considered, and completed by Nowak [10] who considered the continuous range of type parameter α, i.e. $\alpha \in [-1/2,\infty)^d$.

In this paper we prove the dimension free L^p estimates for Riesz transforms R_j^α, $j = 1,\ldots,d$, naturally associated with multi-dimensional Laguerre expansions of Hermite type for the Laguerre type multi-index α. The main result of the paper is contained in Theorem 5.1. It says that for $1<p\le 2$ the dimension free L^p bounds hold for any $\alpha \in (\{-1/2\}\cup(1/2,\infty))^d$, while for $2<p<\infty$, due to the technique we use, the same happens for $\alpha \in (3/2,\infty)^d$. The fact that R_j^α are bounded on all $L^p(\mathbb{R}_+^d, dx)$, $1<p<\infty$, was proved by Nowak and Stempak [13]; in fact it was shown there that R_j^α, $j=1,\ldots,d$, are Calderón-Zygmund operators when $\alpha \in \mathcal{A}_d := (\{-1/2\}\cup[1/2,\infty))^d$. Clearly methods developed in [13] did not guarantee the d-independence of the bounds $\|R_j^\alpha\|_{L^p(\mathbb{R}_+^d)\to L^p(\mathbb{R}_+^d)}$. It should be noted that including the type parameter $-\mathbf{1/2} = (-1/2,\ldots,-1/2)$ into our result (such inclusion is expected due to a natural connection of the Laguerre case of $\alpha = -\mathbf{1/2}$ with the Hermite expansion setting, see Section 2) required additional efforts.

In the present paper we use a quite different technique, namely the method of g-functions. This technique, known as the Littlewood-Paley-Stein theory and presented in the seminal monograph [17], occured to be successful in treating the problem of dimension free L^p estimates in several settings. In short, the main ingredient of this method consists in constructing appropriate g-functions defined in terms of some semigroups, that properly relate a function and its Riesz transform, and proving dimension free L^p bounds for these g-functions. In our case the relevant g-functions are defined in terms of Poisson and modified Poisson semigroups, see Section 3, and the corresponding L^p bounds are stated in Theorem 3.1.

It is worth mentioning that the restrictions imposed on α, like $\alpha_j \notin (-1/2,1/2)$, $j=1,\ldots,d$, that appear in this paper were also present in [13] and [19] (and in other places), and the question of 'necessity' of these restrictions has been recently enlighten in [14]. It was proved there that that the heat semigroup that corresponds to the considered expansions of type $\alpha \in [-1/2,\infty)^d$ is a symmetric diffusion semigroup if and only if $\alpha \in \mathcal{A}_d$.

Throughout the paper $L^p = L^p(\mathbb{R}_+^d, dx)$ will mean the usual Lebesgue space of pth summable functions on $\mathbb{R}_+^d = (0,\infty)^d$ equipped with Lebesgue measure dx; $\|\cdot\|_p$

will denote the norm in L^p and $\langle \cdot, \cdot \rangle$ will stand for the usual inner product in L^2. For all facts concerning the setting of Laguerre expansions of Hermite type that are not properly explained below the reader may consult [13]. This research was inspired by [5] and, needless to say, our line of argument follows that proposed in [5]; this is further explicitely indicated in several places of the paper.

2. Preliminaries

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in (-1, \infty)^d$, and $\varphi_k^\alpha(x) = \varphi_{k_1}^{\alpha_1}(x_1) \cdots \varphi_{k_d}^{\alpha_d}(x_d)$ be the system of d-dimensional Laguerre functions,

$$\varphi_{k_i}^{\alpha_i}(x_i) = \left(\frac{2\Gamma(k_i+1)}{\Gamma(k_i+\alpha_i+1)} \right)^{1/2} L_{k_i}^{\alpha_i}(x_i^2) x_i^{\alpha_i+1/2} e^{-x_i^2/2}, \qquad x_i > 0, \quad i = 1, \ldots, d,$$

where $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, $\mathbb{N} = \{0, 1, \ldots\}$, and $L_{k_i}^{\alpha_i}$ denotes the Laguerre polynomial of degree k_i and order α_i. It is known that each φ_k^α is an eigenfunction of the differential operator

$$L_\alpha = -\Delta + V_\alpha(x), \quad \text{where} \quad V_\alpha(x) = |x|^2 + \sum_{i=1}^d \frac{1}{x_i^2} \left(\alpha_i^2 - \frac{1}{4} \right),$$

corresponding to the eigenvalue $\lambda_{|k|}^\alpha = 4|k| + 2|\alpha| + 2d$; here $|\alpha| = \alpha_1 + \ldots + \alpha_d$ (note that $|\alpha|$ may be negative) and $|k| = k_1 + \ldots + k_d$ is the length of k. Moreover, $\{\varphi_k^\alpha : k \in \mathbb{N}^d\}$ is an orthonormal basis in L^2. The operator

$$\mathcal{L}_\alpha f = \sum_{k \in \mathbb{N}^d} \lambda_{|k|}^\alpha \langle f, \varphi_k^\alpha \rangle \varphi_k^\alpha$$

on the domain

$$\operatorname{Dom}(\mathcal{L}_\alpha) = \left\{ f \in L^2 : \sum_{k \in \mathbb{N}^d} \left| \lambda_{|k|}^\alpha \langle f, \varphi_k^\alpha \rangle \right|^2 < \infty \right\}$$

is a natural self-adjoint extension of L_α, $C_c^\infty\left(\mathbb{R}_+^d\right) \subseteq \operatorname{Dom}(\mathcal{L}_\alpha)$, and the spectrum of $\mathcal{L}_\alpha$ is the discrete set $\{\lambda_n^\alpha : n \in \mathbb{N}\}$.

The jth partial derivative associated with L_α (Laguerre-type partial derivative) is given by

$$\delta_j = \frac{\partial}{\partial x_j} + v_j(x_j), \quad \text{where} \quad v_j(x_j) = x_j - \frac{1}{x_j}\big(\alpha_j + 1/2\big).$$

The formal adjoint of δ_j in $L^2(\mathbb{R}_+^d, dx)$ is

$$\delta_j^* = -\frac{\partial}{\partial x_j} + v_j(x_j).$$

Direct computation then shows that

$$L_\alpha = 2(|\alpha| + d) + \sum_{j=1}^{d} \delta_j^* \delta_j,$$

and this identity suggests $R_j^\alpha = \delta_j \mathcal{L}_\alpha^{-1/2}$ as a 'formal' definition of jth Riesz-Laguerre transform. Using $\frac{d}{dx} L_k^\alpha = -L_{k-1}^{\alpha+1}$, $\alpha > -1$, $k \in \mathbb{N}$, it can be easily seen that

$$\delta_j \varphi_k^\alpha = -2\sqrt{k_j}\varphi_{k-e_j}^{\alpha+e_j}, \qquad \delta_j^* \varphi_k^\alpha = -2\sqrt{k_j}\varphi_{k+e_j}^{\alpha-e_j}, \tag{2.1}$$

where e_j is the j-th coordinate vector in $\mathbb{R}_+^d$ and, by convention, $\varphi_{k-e_j}^{\alpha+e_j} = 0$ if $k_j = 0$. Therefore, the strict definition of R_j^α on L^2 is

$$R_j^\alpha f = -2\sum_{k=0}^{\infty} \left(\frac{k_j}{4|k| + 2|\alpha| + 2d}\right)^{1/2} \langle f, \varphi_k^\alpha \rangle \varphi_{k-e_j}^{\alpha+e_j}, \qquad f \in L^2. \tag{2.2}$$

Parseval's identity shows that R_j^α is a contraction on L^2.

The heat semigroup $\{T_t^\alpha\} = \{\exp(-t\mathcal{L}_\alpha)\}$ associated with $\mathcal{L}_\alpha$, according to the spectral theorem on L^2, is given by

$$T_t^\alpha f = \sum_{n=0}^{\infty} e^{-t\lambda_n^\alpha} \sum_{|k|=n} \langle f, \varphi_k^\alpha \rangle \, \varphi_k^\alpha, \qquad f \in L^2,$$

and it has the integral representation

$$T_t^\alpha f(x) = \int_{\mathbb{R}_+^d} \mathcal{G}_t^\alpha(x,y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d, \quad t > 0, \tag{2.3}$$

where

$$\begin{aligned}\mathcal{G}_t^\alpha(x,y) &= \sum_{n=0}^{\infty} e^{-t\lambda_n^\alpha} \sum_{|k|=n} \varphi_k^\alpha(x)\varphi_k^\alpha(y) \\ &= (\sinh 2t)^{-d} \exp\left(-\frac{1}{2}\coth 2t \left(|x|^2 + |y|^2\right)\right) \prod_{i=1}^{d} \sqrt{x_i y_i}\, I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right).\end{aligned}$$

Here I_ν, $\nu > -1$, is the modified Bessel function of the first kind and order ν. For $\alpha \in [-1/2, \infty)^d$ the right-hand side of (2.3) makes sense for any $f \in L^p$, $1 \le p \le \infty$ and in fact defines a family of operators $\{T_t^\alpha\}_{t>0}$ which are bounded on all L^p spaces, $1 \le p \le \infty$.

The Laguerre-Poisson semigroup $\{P_t^\alpha\} = \{\exp(-t(\mathcal{L}_\alpha)^{1/2})\}$ is defined spectrally on L^2 by

$$P_t^\alpha f = \sum_{n=0}^{\infty} e^{-t(\lambda_n^\alpha)^{1/2}} \sum_{|k|=n} \langle f, \varphi_k^\alpha \rangle \, \varphi_k^\alpha, \qquad f \in L^2, \tag{2.4}$$

and it has the integral representation

$$P_t^\alpha f(x) = \int_{\mathbb{R}_+^d} P_t^\alpha(x,y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d, \quad t > 0, \tag{2.5}$$

where

$$P_t^\alpha(x,y) = \sum_{n=0}^{\infty} e^{-t(\lambda_n^\alpha)^{1/2}} \sum_{|k|=n} \varphi_k^\alpha(x) \varphi_k^\alpha(y).$$

By the principle of subordination,

$$P_t^\alpha f(x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-t^2/4s}\, T_s f(x)\, ds,$$

and on the level of integral kernels,

$$P_t^\alpha(x,y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-t^2/4s}\, \mathcal{G}_s^\alpha(x,y)\, ds. \tag{2.6}$$

Again for $\alpha \in [-1/2,\infty)^d$ the right-hand side of (2.5) makes sense for any $f \in L^p$, $1 \le p \le \infty$ and also defines a family of operators $\{P_t^\alpha\}_{t>0}$ which are bounded on all L^p spaces, $1 \le p \le \infty$.

Apart of the Laguerre-Poisson semigroup $\{P_t^\alpha\}$ we shall use the *modified* Laguerre-Poisson semigroups

$$\{\widetilde{P}_t^{\alpha,j}\} = \{\exp(-t(\mathcal{L}_{\alpha+e_j}+2)^{1/2})\}, \qquad j = 1,\ldots,d,$$

which are given spectrally on L^2 by

$$\widetilde{P}_t^{\alpha,j} f = \sum_{n=0}^{\infty} e^{-t(\lambda_n^{\alpha+e_j}+2)^{1/2}} \sum_{|k|=n} \langle f, \varphi_k^{\alpha+e_j} \rangle \varphi_k^{\alpha+e_j}, \qquad f \in L^2. \tag{2.7}$$

See [13, Section 4] and [12, Section 5] for the definition of modified semigroups in a general framework. At this moment we should point out the indispensable role played by these semigroups in harmonic analysis of orthogonal expansions. Note that $\{\widetilde{P}_t^{\alpha,j}\}$ is subordinated (in the sense explained above) to $\{\widetilde{T}_t^{\alpha,j}\}$, the semigroup given on L^2 by $\{\widetilde{T}_t^{\alpha,j}\} = \{\exp(-t(\mathcal{L}_{\alpha+e_j}+2))\}$. Since the former semigroup has an integral representation with the kernels $\mathcal{G}_t^{\alpha+e_j,2}(x,y) := e^{-2t}\mathcal{G}_t^{\alpha+e_j}(x,y)$, it may be checked that also $\{\widetilde{P}_t^{\alpha,j}\}$ has an integral representation with kernels $\widetilde{P}_t^{\alpha,j}(x,y)$ subordinated (in the sense of (2.6)) to $\widetilde{T}_t^{\alpha,j}(x,y)$. It follows that for $\alpha \in [-1/2,\infty)^d$ the formula

$$\widetilde{P}_t^{\alpha,j} f(x) = \int_{\mathbb{R}_+^d} \widetilde{P}_t^{\alpha,j}(x,y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d, \quad t > 0,$$

initially valid for $f \in L^2$, extends to functions from all L^p, $1 \le p \le \infty$, and defines a bounded operator there.

The heat kernel $\mathcal{G}_t^\alpha(x,y)$ is for $\alpha \in [1/2,\infty)^d$ dominated pointwise on $\mathbb{R}_+^d \times \mathbb{R}_+^d$ by the heat kernel

$$G_t(x,y) = (2\pi)^{-d/2}(\sinh 2t)^{-d/2} \exp\left(-\frac{1}{4\tanh t}|x-y|^2 - \frac{\tanh t}{4}|x+y|^2\right), \ x,y \in \mathbb{R}^d$$

corresponding to the harmonic oscillator on $\mathbb{R}^d$, as the following lemma shows.

Lemma 2.1. *We have for* $\alpha \in [1/2,\infty)^d$

$$\mathcal{G}_t^\alpha(x,y) \le G_t(x,y), \qquad x,y \in \mathbb{R}_+^d, \quad t > 0.$$

Proof. Since for any fixed $z > 0$ the function $I_\nu(z)$ is decreasing for $\nu \ge 0$ (see the proof of [13, Lemma 2.1] and references given there), we have

$$\mathcal{G}_t^\alpha(x,y) \le \mathcal{G}_t^{\mathbf{1/2}}(x,y)$$

for all $\alpha \in [1/2,\infty)^d$, with the notation $\mathbf{1/2} = (1/2,\dots,1/2)$. But $I_{1/2}(z) = (2/\pi z)^{1/2}\sinh z$ and therefore

$$\begin{aligned}\sqrt{x_i y_i}\, I_{1/2}\Big(\frac{x_i y_i}{\sinh 2t}\Big) &= (2/\pi)^{1/2}(\sinh 2t)^{1/2}\sinh\Big(\frac{x_i y_i}{\sinh 2t}\Big)\\ &\le (1/2\pi)^{1/2}(\sinh 2t)^{1/2}\exp\Big(\frac{x_i y_i}{\sinh 2t}\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{G}_t^{\mathbf{1/2}}(x,y) &\le (2\pi)^{-d/2}(\sinh 2t)^{-d/2}\exp\Big(-\frac{1}{2}\coth 2t\,\big(|x|^2+|y|^2\big) + \sum_{i=1}^d \frac{x_i y_i}{\sinh 2t}\Big)\\ &= G_t(x,y).\end{aligned}$$

■

It is worth mentioning that the bound in Lemma 2.1 is valid, up to a multiplicative constant C_α, for any $\alpha \in [-1/2,\infty)^d$, see [20, Lemma 2.4] and also [13, Proposition 2.1]. It may happen, however, that for $\alpha \in [-1/2,\infty)^d \setminus [1/2,\infty)^d$, C_α depends on d as well.

Given $b \in \mathbb{R}$, consider the semigroup $\{T_t^{\alpha,b}\}$ defined on L^2 by $T_t^{\alpha,b} = \exp(-t(\mathcal{L}_\alpha + bI)) = e^{-tb}T_t^\alpha$ with $\mathcal{G}_t^{\alpha,b}(x,y) = e^{-tb}\mathcal{G}_t^\alpha(x,y)$ as the associated kernels. If $b \ge -2(|\alpha|+d)$, then the spectrum of $\mathcal{L}_\alpha + bI$ is non-negative and one may consider the corresponding 'Poisson' semigroup $\{P_t^{\alpha,b}\}$ defined on L^2 by $P_t^{\alpha,b} = \exp(-t(\mathcal{L}_\alpha + bI)^{1/2})$. Spectrally, $P_t^{\alpha,b}$ is given on L^2 by

$$P_t^{\alpha,b} f = \sum_{n=0}^{\infty} e^{-t(\lambda_n^\alpha+b)^{1/2}} \sum_{|k|=n} \langle f, \varphi_k^\alpha\rangle \, \varphi_k^\alpha, \qquad f \in L^2,$$

and again it may be checked that

$$\begin{aligned} P_t^{\alpha,b}(x,y) &= \sum_{n=0}^{\infty} e^{-t(\lambda_n^\alpha+b)^{1/2}} \sum_{|k|=n} \varphi_k^\alpha(x)\varphi_k^\alpha(y) \\ &= \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-t^2/4s}\, \mathcal{G}_s^{\alpha,b}(x,y)\, ds \end{aligned} \tag{2.8}$$

is the kernel corresponding to $P_t^{\alpha,b}$. Due to the subordination it follows that for $\alpha \in [-1/2,\infty)^d$ and $b \geq -2(|\alpha|+d)$, the formula

$$P_t^{\alpha,b} f(x) = \int_{\mathbb{R}_+^d} P_t^{\alpha,b}(x,y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d, \quad t>0,$$

initially valid for $f \in L^2$, extends to all $f \in L^p$, $1 \leq p \leq \infty$, and defines a bounded operator on each L^p. In what follows we shall use the notation

$$u_{\alpha,b}(x,t) = P_t^{\alpha,b} f(x).$$

As a matter of fact we will be interested only in $b \in \{-2,0,2\}$. Note that

$$P_t^{\alpha,2} = \widetilde{P}_t^{\alpha-e_j,j}, \qquad P_t^{\alpha,0} = P_t^\alpha,$$

and consequently,

$$u_{\alpha,2}(x,t) = \widetilde{P}_t^{\alpha-e_j,j} f(x), \qquad u_{\alpha,0}(x,t) = P_t^\alpha f(x).$$

Let $W_t(x) = (4\pi t)^{-d/2}\exp(-|x|^2/(4t))$, $x \in \mathbb{R}^d$, $t>0$, denote the usual Gauss-Weierstrass kernel in $\mathbb{R}^d$ and $\{W_t\}$ be the corresponding heat semigroup, $W_t h = W_t * h$, defined for functions $h \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$; by W_* we shall denote the associated maximal operator,

$$W_* h(x) = \sup_{t>0} W_t * |h|(x), \qquad x \in \mathbb{R}^d.$$

It is well known that $\|W_* h\|_{L^p(\mathbb{R}^d)} \leq A_p \|h\|_{L^p(\mathbb{R}^d)}$, with a universal constant A_p depending only on $1<p<\infty$ (and not on the dimension d). Given a function f on $\mathbb{R}_+^d$ let f_e denote its even extension on $\mathbb{R}^d$, i.e. $f_e(\varepsilon x) = f(x)$, $x \in \mathbb{R}_+^d$, $\varepsilon \in \mathcal{E}$, where $\mathcal{E} = \{(\varepsilon_1,\dots,\varepsilon_d) : \varepsilon_j = \pm 1\}$ and $\varepsilon x = (\varepsilon_1 x_1, \dots, \varepsilon_d x_d)$. We shall use the symbol W_*^+ to denote the maximal operator defined on functions from $L^p(\mathbb{R}_+^d)$, $1 \leq p \leq \infty$, by $W_*^+ f(x) = W_*(f_e)(x)$, $x \in \mathbb{R}_+^d$. Since $W_*(f_e)$ is $\mathcal{E}$-symmetric on $\mathbb{R}_+^d$ (in the sense that $W_*(f_e)(\varepsilon x) = W_*(f_e)(x)$, $x \in \mathbb{R}^d$, $\varepsilon \in \mathcal{E}$), it follows that

$$2^{d/p}\|W_*^+ f\|_p = \|W_*(f_e)\|_{L^p(\mathbb{R}^d)} \leq A_p \|f_e\|_{L^p(\mathbb{R}^d)} = A_p 2^{d/p}\|f\|_p,$$

hence

$$\|W_*^+ f\|_p \le A_p \|f\|_p. \tag{2.9}$$

The formula $\sinh 2t = 2\sinh t \cosh t$ leads to the estimate

$$G_t(x,y) \le (\cosh t)^{-d} W_{\tanh t}(x-y), \qquad x,y \in \mathbb{R}^d. \tag{2.10}$$

This estimate combined with that of Lemma 2.1, for $\alpha \in [1/2,\infty)^d$ produces

$$\mathcal{G}_t^\alpha(x,y) \le (\cosh t)^{-d} W_{\tanh t}(x-y), \qquad x,y \in \mathbb{R}_+^d. \tag{2.11}$$

If $b \ge 1-d$, then $(\cosh t)^{-d} \le C_b \exp(-(1-b)t)$. For $\alpha \in [1/2,\infty)^d$ this leads to

$$|u_{\alpha,b}(x,t)| \le C_b e^{-t} W_*^+ f(x), \qquad x \in \mathbb{R}_+^d, \quad t>0, \tag{2.12}$$

cf. [5, (2.8)]. If we consider more general $\alpha \in \mathcal{A}_d$, then (2.12) still holds. To see this observe first that $T_t^{-\mathbf{1/2}}(f) = T_t(f_e)$, where $\{T_t\}$ is the Hermite semigroup (see [13, (A.4), p.442]). Clearly, up to a permutation argument, it is enough to assume that $\alpha_1 = \ldots = \alpha_n = -1/2$, $\alpha_{n+1},\ldots,\alpha_d \ge 1/2$, for some $n \in \{1,\ldots,d\}$. Then $T_t^\alpha f = (T_t^{\alpha'} \otimes T_t')(f_e')$, where $\alpha' = (\alpha_{n+1},\ldots,\alpha_d)$, T_t' is the n-dimensional Hermite semigroup (acting on the first n variables), and f_e' is the $\mathcal{E}$-symmetrization of f in the first n variables. Now using the n-dimensional variant of (2.10), the $(d-n)$-dimensional variant of (2.11) and appropriate variant of $T_t^{-\mathbf{1/2}}(f) = T_t(f_e)$, we write

$$|T_t^\alpha f(x)| \le (\cosh t)^{-d} \int_{\mathbb{R}^n \times \mathbb{R}_+^{d-n}} W_{\tanh t}(x-y)|f_e'(y)|\,dy \le (\cosh t)^{-d} W_*^+ f(x).$$

From the latter inequality we proceed as in the case $\alpha \in [1/2,\infty)^d$.

Consequently, given $\alpha \in \mathcal{A}_d$ and $b \ge 1-d$, (2.12) applied to $f \equiv 1$ produces

$$\int_{\mathbb{R}_+^d} P_t^{\alpha,b}(x,y)\,dy \le C_b e^{-t}. \tag{2.13}$$

3. Square Functions

A thorough study of square functions in the setting of Laguerre function expansions of Hermite type, associated to the heat and Poisson semigroups has been performed in [19]. In the proof of our main result, Theorem 5.1, we shall use the following g-functions associated to the Poisson and modified Poisson semigroups:

$$g_j(f)(x) = \left(\int_0^\infty t\left|\delta_j P_t^\alpha f(x)\right|^2 dt\right)^{1/2}, \qquad j=1,\ldots,d,$$

and

$$\widetilde{g}_j(f)(x) = \left(\int_0^\infty t \big| \partial_t \widetilde{P}_t^{\alpha,j} f(x) \big|^2 \, dt \right)^{1/2}, \qquad j = 1, \dots, d.$$

It follows from [14, Proposition 4.2] that $\{P_t^\alpha\}$ and $\{\widetilde{P}_t^{\alpha,j}\}$, being subordinated to $\{T_t^\alpha\}$ and $\{\widetilde{T}_t^{\alpha,j}\}$, are symmetric diffusion semigroups whenever $\alpha \in \mathcal{A}_d$. Note however that the L^p-contractivity of $\{T_t^\alpha\}$ breaks down for $\alpha \in [-1/2, \infty)^d \setminus \mathcal{A}_d$. Since for $\alpha \in \mathcal{A}_d$ the semigroup $\{\widetilde{P}_t^{\alpha,j}\}$ is a symmetric diffusion semigroup, therefore, from a refinement of the general Littlewood-Paley-Stein theory included in [17], due to Coifman, Rochberg and Weiss [1], see also Meda [9, Theorem 2], we obtain for $\alpha \in \mathcal{A}_d$ and $j = 1, \dots, d$,

$$\widetilde{c}_p^{\,-1} \|f\|_p \le \|\widetilde{g}_j(f)\|_p \le \widetilde{c}_p \|f\|_p, \tag{3.1}$$

with a universal constant $\widetilde{c}_p$ depending only on $1 < p < \infty$. Note that the following fact is used here: if $\widetilde{P}_t^{\alpha,j} f = f$, then $f = 0$.

Given a function u on $\mathbb{R}_+^d \times (0, \infty)$, let

$$\delta u = (\delta_d^* u, \dots, \delta_1^* u, \partial_t u, \delta_1 u, \dots, \delta_d u)$$

mean the gradient vector and $|\delta u|$ mean its Euclidean norm in $\mathbb{R}^{2d+1}$. Each g_j, $j = 1, \dots, d$, is dominated pointwise by the full Laguerre gradient g-function,

$$g_\alpha(f)(x) = \left(\int_0^\infty t \big| \delta P_t^\alpha f(x) \big|^2 \, dt \right)^{1/2},$$

i.e. $g_j(f)(x) \le g_\alpha(f)(x)$, and thus analysis of g_j will be replaced by analysis of g_α.

Given $\alpha \in (\{-1/2\} \cup (1/2, \infty))^d$ set

$$M_\alpha = \max_j \frac{\alpha_j + 1/2}{\alpha_j - 1/2}$$

if $\alpha \ne -\mathbf{1/2}$ and $M_{-\mathbf{1/2}} = 1$. In what follows $\mathbf{1} = (1, \dots, 1)$. Our main tool is the following.

Theorem 3.1. *Given $1 < p < \infty$ there exists a constant c_p independent of d and α such that:*

(1) *for $1 < p \le 2$, $d \ge 1$ and $\alpha \in (\{-1/2\} \cup (1/2, \infty))^d$,*

$$\|g_\alpha(f)\|_p \le M_\alpha^{1/2} c_p \|f\|_p; \tag{3.2}$$

(2) *for $2 < p < \infty$, $d \ge 3$ and $\alpha \in (3/2, \infty)^d$,*

$$\|g_\alpha(f)\|_p \le M_{\alpha-\mathbf{1}}^{1/2} c_p \|f\|_p. \tag{3.3}$$

Consequently, for p, d and α as above, one has

$$\|g_j(f)\|_p \le c_{p,\alpha} \|f\|_p, \qquad j = 1, \dots, d, \tag{3.4}$$

with $c_{p,\alpha}$ equal either $M_\alpha^{1/2} c_p$ or $M_{\alpha-\mathbf{1}}^{1/2} c_p$, for $1 < p \le 2$ or $2 < p < \infty$, respectively.

To prove Theorem 3.1 we use methods from [5]. In fact we shall prove the bounds (3.2) and (3.3) only for f being a real-valued linear combination of the functions φ_k^α. Checking that this is enough (i.e. implies the same bounds for any $f \in L^p$ through a density-type argument) is fairly technical, and we decided to not include it here.

Below we consider u to be a real-valued function and assume that $f = \sum a_k \varphi_k^\alpha$ (finite sum, $a_k \in \mathbb{R}$). Then $u_{\alpha,b}(x,t) = P_t^{\alpha,b} f(x) = \sum a_k e^{-t(\lambda_{|k|}^\alpha + b)^{1/2}} \varphi_k^\alpha$. By $\Delta_{x,t}$ and $\nabla_{x,t}$ we denote the Laplacian and the gradient in $\mathbb{R}_+^d \times (0,\infty)$ respectively, and $|\nabla_{x,t} u|$ means the Euclidean norm of $\nabla_{x,t} u$ in $\mathbb{R}^{d+1}$. The following is an analogue of [5, (2.19)].

Lemma 3.2. *Let* $u = u(x,t) \in C^2(\mathbb{R}_+^d \times (0,\infty))$. *Then, for* $\alpha \in (\{-1/2\} \cup (1/2,\infty))^d$,

$$|\nabla_{x,t} u|^2 \le |\delta u|^2 \le 2M_\alpha (|\nabla_{x,t} u|^2 + V_\alpha(x) u^2). \tag{3.5}$$

Consequently, for $b \ge -2(|\alpha| + d)$,

$$|\delta u_{\alpha,b}|^2 \le M_\alpha \Big(\Delta_{x,t}(u_{\alpha,b}^2) - 2b u_{\alpha,b}^2 \Big). \tag{3.6}$$

Proof. Observe that

$$2|\partial_t u|^2 + \sum_{j=1}^d \left(|\delta_j^* u|^2 + |\delta_j u|^2 \right) = 2|\nabla_{x,t} u|^2 + 2u^2 \sum_{j=1}^d v_j(x_j)^2.$$

Since

$$v_j(x_j)^2 = x_j^2 + \frac{(\alpha_j + 1/2)^2}{x_j^2} - (2\alpha_j + 1) \le \frac{\alpha_j + 1/2}{\alpha_j - 1/2} \left(x_j^2 + \frac{\alpha_j^2 - 1/4}{x_j^2} \right),$$

we obtain (3.5). To prove (3.6) note that $\Delta_{x,t} u_{\alpha,b} = b u_{\alpha,b} + V_\alpha(x) u_{\alpha,b}$, hence we have

$$\begin{aligned} \Delta_{x,t}(u_{\alpha,b}^2) - 2b u_{\alpha,b}^2 &= 2|\nabla_{x,t} u_{\alpha,b}|^2 + 2u_{\alpha,b} \left(\Delta_{x,t} u_{\alpha,b} - b u_{\alpha,b} \right) \\ &= 2 \left(|\nabla_{x,t} u_{\alpha,b}|^2 + V_\alpha(x) u_{\alpha,b}^2 \right). \end{aligned}$$

Using this and (3.5) we deduce (3.6). ∎

From now on we assume $\varepsilon > 0$. The following is an analogue of [5, Lemma 1].

Lemma 3.3. *Let* $\alpha \in (\{-1/2\} \cup (1/2,\infty))^d$ *and* $b \ge -2(|\alpha| + d)$. *Then, for* $1 < p \le 2$, *denoting* $\rho_p = 2/(p(p-1))$ *we have*

$$|\delta u_{\alpha,b}|^2 \le M_\alpha \rho_p (u_{\alpha,b}^2 + \varepsilon)^{\frac{2-p}{2}} \Big(\Delta_{x,t}[(u_{\alpha,b}^2 + \varepsilon)^{p/2}] + p|b|(u_{\alpha,b}^2 + \varepsilon)^{p/2} \Big).$$

Proof. Straightforward calculations and the identity $|\nabla_{x,t}u^2|^2 = 4u^2|\nabla_{x,t}u|^2$ show that for $u \in C^2(\mathbb{R}^d_+ \times (0,\infty))$ one has

$$\begin{aligned}\Delta_{x,t}[(u^2+\varepsilon)^{\frac{p}{2}}] &= \frac{p(p-2)}{4}(u^2+\varepsilon)^{\frac{p-4}{2}}|\nabla_{x,t}(u^2)|^2 + \frac{p}{2}(u^2+\varepsilon)^{\frac{p-2}{2}}\Delta_{x,t}(u^2)\\ &= p(p-2)(u^2+\varepsilon)^{\frac{p-4}{2}}u^2\left(|\nabla_{x,t}u|^2 + V_\alpha(x)u^2\right)\\ &\quad + p(2-p)(u^2+\varepsilon)^{\frac{p-4}{2}}V_\alpha(x)u^4\\ &\quad + \frac{p}{2}(u^2+\varepsilon)^{\frac{p-2}{2}}\left(\Delta_{x,t}(u^2) - 2bu^2\right) + pb(u^2+\varepsilon)^{\frac{p-2}{2}}u^2.\end{aligned}$$

Since $1 < p \le 2$ and $V_\alpha \ge 0$, it follows that

$$\begin{aligned}&\Delta_{x,t}[(u^2_{\alpha,b}+\varepsilon)^{\frac{p}{2}}] + p|b|(u^2_{\alpha,b}+\varepsilon)^{\frac{p}{2}}\\ &\ge p(p-2)(u^2_{\alpha,b}+\varepsilon)^{\frac{p-4}{2}}u^2_{\alpha,b}\left(|\nabla_{x,t}u_{\alpha,b}|^2 + V_\alpha(x)u^2_{\alpha,b}\right)\\ &\quad + \frac{p}{2}(u^2_{\alpha,b}+\varepsilon)^{\frac{p-2}{2}}\left(\Delta_{x,t}(u^2_{\alpha,b}) - 2bu^2_{\alpha,b}\right).\end{aligned}$$

Now, using (3.5) and (3.6) we get the required estimate. ∎

4. Proof of Theorem 3.1

In the proof we follow the classical argument from [17] augmented by that from [5]. We prove (3.2) for $1 < p \le 2$ and then (3.3) for $p > 4$; the case $2 < p \le 4$ of (3.3) then follows by Marcinkiewicz' interpolation theorem. As already declared, throughout this section we assume that f is a real-valued linear combination of the functions φ_k^α, $f = \sum a_k\varphi_k^\alpha$ (finite sum, $a_k \in \mathbb{R}$).

Proof of (3.2). In fact we shall consider

$$g_{\alpha,b}(f)(x) = \left(\int_0^\infty t\left|\delta P_t^{\alpha,b}f(x)\right|^2 dt\right)^{1/2},$$

(so that $g_\alpha = g_{\alpha,0}$) and prove a slightly more general estimate,

$$\|g_{\alpha,b}(f)\|_p \le M_\alpha^{1/2}c_{p,b}\|f\|_p, \qquad 1 < p \le 2, \tag{4.1}$$

which is needed in the proof of (3.3) for $p > 4$ with $b = -2, 0, 2$. The bound (4.1) will be proved under the assumption $b \ge 1-d$; note that $b \ge 1-d$ implies $b \ge -2(|\alpha|+d)$ for $\alpha \in (\{-1/2\} \cup (1/2,\infty))^d$, which is required in Lemmas 3.2 and 3.3. Note also that for $b = -2$ the assumption $b \ge 1-d$ forces $d \ge 3$.

We shall use Lemma 3.3 and proceed by analogy with the proof of [5, Lemma 2]. Fix $R > 0$. Then, by Lemma 3.3, for fixed $x \in \mathbb{R}^d_+$ we have

$$\int_0^R t|\delta u_{\alpha,b}(x,t)|^2\,dt$$
$$\le M_\alpha\rho_p\int_0^R t(u_{\alpha,b}^2+\varepsilon)^{\frac{2-p}{2}}\Big(\Delta_{x,t}[(u_{\alpha,b}^2+\varepsilon)^{p/2}]+p|b|(u_{\alpha,b}^2+\varepsilon)^{p/2}\Big)\,dt$$
$$\le M_\alpha\rho_p\Big(\sup_{0<t\le R}u_{\alpha,b}^2+\varepsilon\Big)^{\frac{2-p}{2}}\left(\int_0^R t\Delta_{x,t}[(u_{\alpha,b}^2+\varepsilon)^{p/2}]\,dt+p|b|\int_0^R t(u_{\alpha,b}^2+\varepsilon)^{p/2}\,dt\right).$$

Therefore, denoting $A_R=\{x\in\mathbb{R}^d_+ : |x|\le R\}$, we obtain

$$\int_{A_R}\left(\int_0^R t|\delta u_{\alpha,b}|^2\,dt\right)^{p/2}dx\le M_\alpha^{p/2}\rho_p^{p/2}\int_{A_R}\Big(\sup_{0<t\le R}u_{\alpha,b}^2+\varepsilon\Big)^{\frac{p(2-p)}{4}}$$
$$\times\left(\int_0^R t\Delta_{x,t}[(u_{\alpha,b}^2+\varepsilon)^{\frac{p}{2}}]\,dt+p|b|\int_0^R t(u_{\alpha,b}^2+\varepsilon)^{\frac{p}{2}}\,dt\right)^{\frac{p}{2}}dx.$$

Using Hölder's inequality with the pair of conjugate exponents $2/(2-p)$ and $2/p$ gives

$$\begin{aligned}\int_{A_R}\left(\int_0^R t|\delta u_{\alpha,b}|^2\,dt\right)^{p/2}dx&\le M_\alpha^{p/2}\rho_p^{p/2}\left(\int_{A_R}\Big(\sup_{t>0}u_{\alpha,b}^2+\varepsilon\Big)^{p/2}dx\right)^{(2-p)/2}\\&\times\left(\int_{A_R}\left(\int_0^R t\Delta_{x,t}[(u_{\alpha,b}^2+\varepsilon)^{p/2}]dt+p|b|\int_0^R t(u_{\alpha,b}^2+\varepsilon)^{p/2}dt\right)dx\right)^{p/2}.\end{aligned}\tag{4.2}$$

Applying consecutively the dominated convergence theorem, (2.12) and (2.9) produces

$$\lim_{\varepsilon\to0^+}\left(\int_{A_R}\Big(\sup_{t>0}u_{\alpha,b}^2+\varepsilon\Big)^{p/2}dx\right)^{(2-p)/2}$$
$$=\left(\int_{A_R}\Big(\sup_{t>0}|u_{\alpha,b}|\Big)^{p}dx\right)^{(2-p)/2}$$
$$\le C_b^{\frac{p(2-p)}{2}}\left(\int_{A_R}|W_*^+f(x)|^p\,dx\right)^{(2-p)/2}$$
$$\le (A_pC_b)^{\frac{p(2-p)}{2}}\|f\|_p^{\frac{p(2-p)}{2}}.\tag{4.3}$$

We focus on getting a suitable bound for the second integral factor in (4.2). To simplify the notation, with no loss of generality we may assume that for some $n\in\{0,1,\dots,d\}$, $\alpha_1=\dots=\alpha_n=-1/2$, $\alpha_{n+1},\dots,\alpha_d>1/2$. To be precise, $n=0$ corresponds to $\alpha\in(1/2,\infty)^d$, while $n=d$ to $\alpha=-\mathbf{1/2}$. We know that for $x_i>0$, $\varphi_{k_i}^{-1/2}(x_i)$ coincides with $h_{2k_i}(x_i)$, i.e. the Hermite function of even degree $2k_i$. It follows that f and hence also $u_{\alpha,b}$ has a natural extension to $\mathbb{R}^n\times\mathbb{R}^{d-n}_+$, which is a C^∞ function in the first n variables. Moreover, since one-dimensional Hermite functions of even degree are even functions, both extensions are symmetric in the first n variables. Denoting the aforementioned extensions of f and $u_{\alpha,b}$ by the same symbols and setting $A_R^n = \mathbb{R}^n \times \mathbb{R}_+^{d-n} \cap \{x \in \mathbb{R}^d : |x| \le R\}$, we thus write

$$\int_{A_R}\int_0^R t\Delta_{x,t}[(u_{\alpha,b}^2+\varepsilon)^{p/2}]\,dt\,dx = 2^{-n}\int_{A_R^n}\int_0^R t\Delta_{x,t}[(u_{\alpha,b}^2+\varepsilon)^{p/2}]\,dt\,dx.$$

Consequently, by using Green's formula, we check that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+}\int_{A_R}\left(\int_0^R t\Delta_{x,t}[(u_{\alpha,b}^2+\varepsilon)^{p/2}]\,dt + p|b|\int_0^R t(u_{\alpha,b}^2+\varepsilon)^{p/2}\,dt\right)dx\\
&= \limsup_{\varepsilon\to 0^+}\left(2^{-n}\int_{A_R^n}\int_0^R t\Delta_{x,t}[(u_{\alpha,b}^2+\varepsilon)^{p/2}]\,dt\,dx + p|b|\int_{A_R}\int_0^R t(u_{\alpha,b}^2+\varepsilon)^{p/2}\,dt\,dx\right)\\
&\le 2^{-n}\int_{\partial Q_R^n}\left(tp|u_{\alpha,b}|^{p-1}\left|\partial_\nu u_{\alpha,b}\right| - |u_{\alpha,b}|^p\partial_\nu t\right)d\sigma(x,t) + p|b|\int_{A_R}\int_0^R t|u_{\alpha,b}|^p\,dt\,dx.
\end{aligned}\tag{4.4}$$

Indeed, let ∂Q_R^n be the boundary of $Q_R^n = A_R^n \times [0,R]$ in $\mathbb{R}^{d+1}$, σ be the surface measure on ∂Q_R^n, and ν be the unit normal vector field on ∂Q_R^n pointing out of Q_R^n. Then

$$\begin{aligned}
&\int_{A_R^n}\int_0^R t\Delta_{x,t}[(u_{\alpha,b}^2+\varepsilon)^{p/2}]\,dt\,dx\\
&= \int_{\partial Q_R^n}\left(t\partial_\nu[(u_{\alpha,b}^2+\varepsilon)^{p/2}] - (u_{\alpha,b}^2+\varepsilon)^{p/2}\partial_\nu t\right)d\sigma(x,t)\\
&= \int_{\partial Q_R^n}\left(tp(u_{\alpha,b}^2+\varepsilon)^{p/2-1}u_{\alpha,b}\partial_\nu u_{\alpha,b} - (u_{\alpha,b}^2+\varepsilon)^{p/2}\partial_\nu t\right)d\sigma(x,t)\\
&\le \int_{\partial Q_R^n}\left(tp(u_{\alpha,b}^2+\varepsilon)^{(p-1)/2}\left|\partial_\nu u_{\alpha,b}\right| - (u_{\alpha,b}^2+\varepsilon)^{p/2}\partial_\nu t\right)d\sigma(x,t),
\end{aligned}$$

and (4.4) follows.

Replacing the relevant expressions on the right-hand side of (4.2) by (4.3) and (4.4) we shall then let $R\to\infty$. This will require an analysis of the behavior of both summands in (4.4) when $R\to\infty$. To deal with the first summand decompose ∂Q_R^n as $\partial Q_R^n = S_R \cup \overline{A_R^n}\times\{R\}\cup\overline{A_R^n}\times\{0\}$, with

$$S_R = \{(x,t)\colon x\in A_R^n,\ |x|=R,\ 0<t\le R\}\cup\bigcup_{j=n+1}^{d}\{(x,t)\colon x\in\overline{A_R^n},\ x_j=0,\ 0<t\le R\},$$

where $\overline{A_R^n}$ denotes the closure of A_R^n in $\mathbb{R}^d$ (with appropriate adjustment when $d=1$). By assumption, $u_{\alpha,b}$ is a linear combination of functions of type $e^{-t(4|k|+2|\alpha|+2d+b)^{1/2}}\varphi_k^\alpha$.

Since $\alpha_j > 1/2$, $j = n+1, \dots, d$, we have $\varphi_{k_j}^{\alpha_j}(0) = 0$. Moreover, from the very definition of φ_k^α it is easy to verify that for $\alpha \in \mathcal{A}_d$, $|\varphi_k^\alpha(x)| \le C_k^\alpha e^{-|x|^2/4}$ and $|\nabla \varphi_k^\alpha(x)| \le D_k^\alpha e^{-|x|^2/4}$. Hence, we check that

$$\lim_{R\to\infty} \int_{S_R} \big(tp|u_{\alpha,b}|^{p-1}\big|\partial_\nu u_{\alpha,b}\big| - |u_{\alpha,b}|^p \partial_\nu t\big)\ d\sigma(x,t) = 0,$$
$$\lim_{R\to\infty} \int_{\overline{A_R}\times\{R\}} \big(tp|u_{\alpha,b}|^{p-1}\big|\partial_\nu u_{\alpha,b}\big| - |u_{\alpha,b}|^p \partial_\nu t\big)\ d\sigma(x,t) = 0.$$

Since $u_{\alpha,b}(x,0) = f(x)$, $x \in \mathbb{R}^n \times \mathbb{R}_+^{d-n}$, we finally obtain

$$\lim_{R\to\infty} 2^{-n} \int_{\partial Q_R^n} \big(tp|u_{\alpha,b}|^{p-1}\big|\partial_\nu u_{\alpha,b}\big| - |u_{\alpha,b}|^p \partial_\nu t\big)\ d\sigma(x,t)$$
$$= 2^{-n} \int_{\mathbb{R}^n\times\mathbb{R}_+^{d-n}} |f(x)|^p\, dx = \|f\|_p^p.$$

To treat the second summand in (4.4) note that (2.12) ($b \ge 1-d$ is guaranteed) and (2.9) produce

$$\begin{aligned} p|b| \int_{A_R} \int_0^R t|u_{\alpha,b}(x,t)|^p\, dt\, dx &\le p|b| C_b^p \int_0^R te^{-pt}\, dt \cdot \int_{\mathbb{R}_+^d} |W_*^+ f(x)|^p\, dx \\ &\le p|b| C_b^p I_p A_p^p \|f\|_p^p, \end{aligned}$$

with $I_p = \int_0^\infty te^{-pt}\, dt$.

Summarizing, (4.4) is bounded by a constant depending only on p and b, times $\|f\|_p^p$. This bound together with (4.3) shows the required estimate (4.1) and thus (3.2).

Proof of (3.3). *The case* $p \ge 4$.. Recall that the constant C_b appears in (2.12) and (2.13). The technical lemma we shall use is the following (cf. [5, Lemma 3]).

Lemma 4.1. *Let* $\alpha \in [\frac{3}{2}, \infty)^d$ *and* $D = \max\{C_{-2}, C_0, C_2\}$. *Then, for* $x \in \mathbb{R}_+^d$ *and* $t > 0$,

$$\begin{aligned} &\big|\delta u_\alpha(x,t)\big|^2 \\ &\le D \int_{\mathbb{R}_+^d} \Big[P_{t/2}^{\alpha-\mathbf{1},-2}(x,y) + P_{t/2}^{\alpha-\mathbf{1},2}(x,y) + P_{t/2}^{\alpha-\mathbf{1}}(x,y)\Big] \big|\delta u_\alpha(y,t/2)\big|^2\, dy. \end{aligned} \tag{4.5}$$

Proof. The monotonicity argument for Bessel functions, already invoked in the proof of Lemma 2.1, and (2.8) show that $P_{t/2}^{\mu,b}(x,y) \le P_{t/2}^{\alpha-\mathbf{1},b}(x,y)$, for $x, y \in \mathbb{R}_+^d$, $\mu = \alpha - e_j, \alpha + e_j, \alpha$, and $b = -2, 2, 0$, respectively. By using this fact (4.5) is an immediate consequence of the bounds

$$|\delta_j^* u_\alpha(x,t)|^2 \le C_{-2} \int_{\mathbb{R}_+^d} P_{t/2}^{\alpha-e_j,-2}(x,y) |\delta_j^* u_\alpha(y,t/2)|^2 \, dy,$$
$$|\delta_j u_\alpha(x,t)|^2 \le C_2 \int_{\mathbb{R}_+^d} P_{t/2}^{\alpha+e_j,2}(x,y) |\delta_j u_\alpha(y,t/2)|^2 \, dy,$$
$$|\partial_t u_\alpha(x,t)|^2 \le C_0 \int_{\mathbb{R}_+^d} P_{t/2}^{\alpha}(x,y) \big|\partial_t u_\alpha(y,t/2)\big|^2 \, dy,$$

$j = 1, \ldots, d$, (actually they hold under the weaker assumption: $\alpha \in (\{1/2\} \cup [3/2, \infty))^d$ and $|\alpha| + d \ge 2$). To prove the first bound (the second and third follow analogously), note that from (2.1) it follows that

$$\delta_j^* u_\alpha(x,t) = P_{t/2}^{\alpha-e_j,-2}(\delta_j^* u_\alpha(\cdot, t/2))(x).$$

Using this and Schwarz' inequality we obtain

$$|\delta_j^* u_\alpha(x,t)|^2 \le \Big(\int_{\mathbb{R}_+^d} P_{t/2}^{\alpha-e_j,-2}(x,y)\, dy \Big) \Big(\int_{\mathbb{R}_+^d} P_{t/2}^{\alpha-e_j,-2}(x,y) |\delta_j^* u_\alpha(y,t/2)|^2 \, dy \Big),$$

which, by (2.13), implies the required bound (note that the factor $e^{-t/2}$ was neglected). ■

Let $2/p + 1/q = 1$ and $\phi \in L^q$ be a nonnegative function. Since $P_t^{\alpha-\mathbf{1},b}(x,y)$ ($b = -2, 2, 0$) is symmetric in x and y, by (4.5) and the inequality from Lemma 3.3 taken with $b = 0$ and $p = 2$, we have

$$\begin{aligned} &\int_{\mathbb{R}_+^d} g_\alpha(f)(x)^2 \, \phi(x) \, dx \\ &\le 4D \int_{\mathbb{R}_+^d} \int_0^\infty t |\delta u_\alpha(x,t)|^2 \Big(P_t^{\alpha-\mathbf{1},-2}\phi(x) + P_t^{\alpha-\mathbf{1},2}\phi(x) + P_t^{\alpha-\mathbf{1}}\phi(x) \Big) \, dt \, dx \\ &\le 4D M_\alpha \rho_2 \int_{\mathbb{R}_+^d} \int_0^\infty t \Delta_{x,t}\big(u_\alpha^2\big) \Big(P_t^{\alpha-\mathbf{1},-2}\phi(x) + P_t^{\alpha-\mathbf{1},2}\phi(x) + P_t^{\alpha-\mathbf{1}}\phi(x) \Big) \, dt \, dx. \end{aligned}$$

Let $\mathcal{I}(\phi)$ denote the latter double integral taken over $\mathbb{R}_+^d \times (0, \infty)$. Since

$$\Delta_{x,t}\big(P_t^{\alpha-\mathbf{1},b}\phi\big) = (V_{\alpha-\mathbf{1}} + bI) P_t^{\alpha-\mathbf{1},b}\phi, \qquad b = -2, 2, 0,$$

and $V_{\alpha-\mathbf{1}}(x) \ge 0$ (this is since $\alpha \in (3/2, \infty)^d$), we see that

$$\begin{aligned} \Delta_{x,t}(u_\alpha^2 P_t^{\alpha-\mathbf{1},b}\phi) &= \big(\Delta_{x,t} u_\alpha^2\big) P_t^{\alpha-\mathbf{1},b}\phi + 4u_\alpha (\nabla_{x,t} u_\alpha \cdot \nabla_{x,t} P_t^{\alpha-\mathbf{1},b}\phi) + u_\alpha^2 \Delta_{x,t} P_t^{\alpha-\mathbf{1},b}\phi \\ &\ge \big(\Delta_{x,t} u_\alpha^2\big) P_t^{\alpha-\mathbf{1},b}\phi + 4u_\alpha (\nabla_{x,t} u_\alpha \cdot \nabla_{x,t} P_t^{\alpha-\mathbf{1},b}\phi) + b u_\alpha^2 P_t^{\alpha-\mathbf{1},b}\phi, \end{aligned}$$

where the dot denotes the inner product in $\mathbb{R}^{d+1}$. Restating the above gives

$$(\Delta_{x,t}u_\alpha^2)P_t^{\alpha-\mathbf{1},b}\phi \le \Delta_{x,t}(u_\alpha^2 P_t^{\alpha-\mathbf{1},b}\phi) - 4u_\alpha(\nabla_{x,t}u_\alpha \cdot \nabla_{x,t}P_t^{\alpha-\mathbf{1},b}\phi) - bu_\alpha^2 P_t^{\alpha-\mathbf{1},b}\phi.$$

Consequently, $\mathcal{I}(\phi) \le \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$, where

$$\mathcal{I}_1 \equiv \int_{\mathbb{R}^d_+}\int_0^\infty t\Big(\Delta_{x,t}(u_\alpha^2 P_t^{\alpha-\mathbf{1},-2}\phi) + 4|u_\alpha||\nabla_{x,t}u_\alpha||\nabla_{x,t}P_t^{\alpha-\mathbf{1},-2}\phi| \\ +2u_\alpha^2 P_t^{\alpha-\mathbf{1},-2}\phi\Big)\,dt\,dx,$$

and similarly for $\mathcal{I}_2$ and $\mathcal{I}_3$ with replacement of -2 by 2 and 0, respectively.

We now estimate $\mathcal{I}_1$. Since $u_\alpha(x,0) = f(x)$ and $P_0^{\alpha-\mathbf{1},-2}\phi(x) = \phi(x)$, by Green's formula and Hölder's inequality with $2/p + 1/q = 1$ it follows that

$$\mathcal{I}_{1,1} \equiv \int_{\mathbb{R}^d_+}\int_0^\infty t\Delta_{x,t}(u_\alpha^2 P_t^{\alpha-\mathbf{1},-2}\phi)\,dt\,dx = \int_{\mathbb{R}^d_+} f(x)^2\phi(x)\,dx \le \|f\|_p^2\|\phi\|_q.$$

Moreover, by (2.12), the left-hand side inequality in (3.5) applied to $u = u_\alpha$ and $u = u_{\alpha-\mathbf{1},-2}$, (2.13) and Schwarz' inequality,

$$\begin{aligned}\mathcal{I}_{1,2} &\equiv \int_{\mathbb{R}^d_+}\int_0^\infty t\Big(4|u_\alpha||\nabla_{x,t}u_\alpha||\nabla_{x,t}P_t^{\alpha-\mathbf{1},-2}\phi| + 2u_\alpha^2 P_t^{\alpha-\mathbf{1},-2}\phi\Big)\,dt\,dx \\ &\le 4C_0\int_{\mathbb{R}^d_+} W_*^+f(x)g_\alpha(f)(x)g_{\alpha-\mathbf{1},-2}(\phi)(x)\,dx + 2C_0^2C_{-2}I_3 \\ &\quad \int_{\mathbb{R}^d_+} W_*^+f(x)^2W_*^+\phi(x)\,dx.\end{aligned}$$

Consequently, applying Hölder's inequality for three functions with $1/p+1/p+1/q = 1$ (note that $q \le 2$), (2.12) and (2.9) together with (4.1) applied to $g_{\alpha-\mathbf{1},-2}$ gives

$$\mathcal{I}_{1,2} \le M_{\alpha-\mathbf{1}}^{1/2}c_p'(\|f\|_p\|g_\alpha(f)\|_p\|\phi\|_q + \|f\|_p^2\|\phi\|_q).$$

(Note that the condition $\alpha \in (3/2,\infty)^d$ assures $\alpha - \mathbf{1} \in (1/2,\infty)^d$.)

Summarizing, from estimates of $\mathcal{I}_{1,2}$ and $\mathcal{I}_{1,2}$ we conclude that

$$\mathcal{I}_1 \le M_{\alpha-\mathbf{1}}^{1/2}c_p''(\|f\|_p\|g_\alpha(f)\|_p\|\phi\|_q + \|f\|_p^2\|\phi\|_q). \tag{4.6}$$

The same reasoning leads to analogous bounds for $\mathcal{I}_2$ and $\mathcal{I}_3$. Thus, we arrive at

$$\|g_\alpha(f)\|_p^2 \le 4DM_\alpha\rho_2 \sup_{\|\phi\|_q=1}\mathcal{I}(\phi) \le M_\alpha M_{\alpha-\mathbf{1}}^{1/2}c_p'''\big(\|g_\alpha(f)\|_p\|f\|_p + \|f\|_p^2\big).$$

It follows that $\|g_\alpha(f)\|_p \le M_{\alpha-\mathbf{1}}^{1/2}c_p\|f\|_p$, as desired. The proof of (3.3) for $p > 4$ is completed and thus the proof of Theorem 3.1 is finished.

5. Riesz Transforms

Recall that the Riesz-Laguerre transform R_j^α is defined on L^2 by

$$R_j^\alpha f = -2\sum_{k=0}^{\infty}\left(\frac{k_j}{4|k|+2|\alpha|+2d}\right)^{1/2}\langle f,\varphi_k^\alpha\rangle\varphi_{k-e_j}^{\alpha+e_j}.$$

It was shown in [13, Theorem 3.3] that (among other things), for $\alpha \in \mathcal{A}_d$, R_j^α extend uniquely to bounded linear operators on L^p, $1<p<\infty$; we use the same symbols to denote these extensions. Our main theorem reads as follows.

Theorem 5.1. *Let $1<p<\infty$ and $\varepsilon>0$. Assume that $d\geq 1$ and $\alpha \in (\{-1/2\}\cup(1/2+\varepsilon,\infty))^d$ when $1<p\leq 2$, or $d\geq 3$ and $\alpha\in(3/2+\varepsilon,\infty)^d$ when $2<p<\infty$. Then there exists a constant $C_{p,\varepsilon}$ not depending on the dimension d, such that for all $j=1,\dots,d$,*

$$\|R_j^\alpha f\|_p \leq C_{p,\varepsilon}\|f\|_p, \qquad f\in L^p.$$

Proof. Due to the aforementioned result from [13] it is convenient (and enough) to consider functions of the form $f=\sum a_k\varphi_k^\alpha$ (finite sum). Using the definitions of R_j^α, P_t^α, $\widetilde{P}_t^{\alpha,j}$ and (2.1) shows that $\partial_t\widetilde{P}_t^{\alpha,j}(R_j^\alpha f)(x) = -\delta_j P_t^\alpha f(x)$, and hence

$$\widetilde{g}_j(R_j^\alpha f)(x) = g_j(f)(x).$$

Applying (3.1) and then (3.4) leads to

$$\|R_j^\alpha f\|_p \leq \widetilde{c}_p\|\widetilde{g}_j(R_j^\alpha f)\|_p = \widetilde{c}_p\|g_j(f)\|_p \leq \widetilde{c}_p c_{p,\alpha}\|f\|_p.$$

Clearly, with the given assumption on α one has $\widetilde{c}_p c_{p,\alpha}\leq C_{p,\varepsilon}$, where $C_{p,\varepsilon}$ is appropriately chosen, see the structure of the constant $c_{p,\alpha}$ appearing in (3.4). ■

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Krzysztof Stempak
Instytut Matematyki i Informatyki
Politechnika Wrocławska
Wyb. Wyspiańskiego 27,
50-370 Wrocław, Poland
and

Katedra Matematyki i Zastosowań Informatyki
Politechnika Opolska
ul. Luboszycka 3, 45-036 Opole
Poland
E-mail: krzysztof.stempak@pwr.wroc.pl

Błażej Wróbel
Instytut Matematyczny
Uniwersytet Wrocławski,
Pl. Grunwaldzki 2-4, 50-384 Wrocław
Poland
E-mail: blazej.wrobel@math.uni.wroc.pl