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On a Porous Medium Equation with Weighted Inner Source Terms and a Nonlinear Nonlocal Boundary Condition

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Abstract. This paper deals with the asymptotic behavior for a porous medium equation with weighted inner source terms and a nonlinear nonlocal boundary condition. We find the influences of spacetime-varying weight functions and competitive relationship between the multiple nonlinearities on whether determining blow-up of solutions or not, and establish the blow-up rate estimate under some appropriate conditions. Especially, our results include the situation of slow, linear and fast diffusion.

1. Introduction

We consider a quasilinear parabolic equation with weighted inner source terms

$$(u^m)_t = \Delta u + c(x,t)u^p, \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

subject to Dirichlet-type weighted nonlinear nonlocal boundary and initial conditions

$$u(x,t) = \int_\Omega k(x,y,t)u^l(y,t)\,\mathrm{d}y, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with C^2 boundary $\partial\Omega$, constants $m,p,l > 0$. The weight function $c(x,t)$ is a nonnegative continuous function defined on $(x,t) \in \overline{\Omega} \times [0,T)$ and $k(x,y,t)$ is a nonnegative nontrivial continuous function defined on $y \in \overline{\Omega}$, $(x,t) \in \partial\Omega \times [0,T)$. Moreover, the initial data $u_0(x) \in C^{2+\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ is a nonnegative nontrivial function, which satisfies the compatibility condition $u_0(x) = \int_\Omega k(x,y,0)u_0^l(y)\,\mathrm{d}y$ for $x \in \partial\Omega$.

In fact, if we set $v(x,t) := u^m(x,t)$, then (1.1) can be rewritten as

$$v_t = \Delta v^{1/m} + c(x,t)v^{p/m}, \quad (x,t) \in \Omega \times (0,T).$$

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Therefore, the model (1.1) is of the *porous medium* type with weighted inner source terms, it appears in the model of Newton flux through a porous medium, the model of evolution of plasma temperature, and so on (cf. [21,33]). In the nonlinear diffusion theory, there exist obvious differences among the situations of slow ($m < 1$), fast ($m > 1$) and linear ($m = 1$) diffusion. For example, there is a finite speed propagation in the slow and linear diffusion situations, whereas an infinite speed propagation exists in the fast diffusion situation. Meanwhile, there are some important phenomena formulated as parabolic equations, which are coupled with weighted nonlocal boundary conditions in mathematical models, such as thermoelasticity theory. In this situation, the solution $u(x,t)$ describes entropy per volume of the material (see [5,6]).

To state our research motivation, let us recall some qualitative properties of solutions for the initial boundary value problems of parabolic equations with Dirichlet-type nonlocal boundary conditions. Concerning the research on the semilinear parabolic equation, one can refer to [7,13–15,18–20] and the references therein. For example, Friedman [13] first studied the linear parabolic equation

$$u_t - Au = 0, \quad (x,t) \in \Omega \times (0,T)$$

subject to the Dirichlet-type weighted linear nonlocal boundary condition, i.e., (1.2) with $l = 1$ and $k = k(x,y)$, where A is an elliptic operator,

$$A = \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x) \quad \text{and} \quad c(x) \le 0.$$

He showed that when $\int_\Omega |k(x,y)|\,\mathrm{d}y \le \rho < 1$, the solution tends to 0 monotonously and exponentially as $t \to \infty$. Deng [7] considered a more general form of semilinear parabolic equation

$$u_t = \Delta u + g(x,u), \quad (x,t) \in \Omega \times (0,T)$$

under the Dirichlet-type weighted linear nonlocal boundary condition, he established the local solvability, comparison principle and derived the long-time asymptotic behavior of solution. Afterwards, Gladkov and Kim [18,19] investigated a semilinear parabolic equation with weighted inner source terms

$$u_t = \Delta u + c(x,t)u^p, \quad (x,t) \in \Omega \times (0,T)$$

with $p > 0$ subject to Dirichlet-type weighted nonlinear nonlocal boundary condition (1.2). They proved the local well-posedness, global existence, blow-up phenomena of solutions by the method of super- and sub-solutions. Furthermore, with regard to the advances on local semilinear parabolic equation with weighted absorption source terms and local semilinear parabolic system with weighted source terms, we refer to [14,15,20].

For the studies on the porous medium equation with the local source term, Wang et al. [32] investigated a porous medium equation with power-like source term

$$u_t = \Delta u^m + u^p, \quad (x,t) \in \Omega \times (0,T),$$

where $m, p > 1$. Under the condition (1.2) with $k = k(x,y)$, they established the blow-up criteria and give the blow-up rate estimate. In addition, with respect to the research of local quasilinear coupled systems, nonlocal parabolic equation (system) under Dirichlet type nonlocal boundary conditions, as well as parabolic equation (system) under Neumann or Robin type nonlocal boundary conditions, we refer to [2,3,8,10,11,16,17,24–31,34,36].

As mentioned above, the research on the porous medium model with spacetime-varying weighted inner source term under weighted nonlinear nonlocal Dirichlet boundary condition (1.1)–(1.3) has not been carried out yet. A difficulty lies in finding influences of spacetime-varying weight functions and competitive relationship between the multiple nonlinearities on the asymptotic behavior of solutions. Motivated by this observation, based on the method of super- and sub-solutions, Kaplan's argument, the method of auxiliary problem and some ODE techniques, etc., we establish sufficient conditions to guarantee the solution globally exists or blow-up, and also give the blow-up rate estimate. Additionally, note that our results include the situation of slow, linear and fast diffusion.

The remainder of our paper is organized as follows. In Section 2, the definition of the solution to problem (1.1)–(1.3), comparison principle and the property of positiveness of the solution are introduced. In Section 3, under suitable assumptions on the exponents m, p, l and weighted functions, we proved the global existence and blow-up of solutions. The blow-up rate estimate is given in Section 4. Finally, we discuss in the last section the conclusions obtained in this paper.

2. Preliminaries

We begin with definitions of super- and sub-solution. Throughout the paper, we denote $Q_T := \Omega \times (0,T)$, $\overline{Q}_T := \overline{\Omega} \times [0,T)$, $S_T := \partial\Omega \times (0,T)$, $0 < T \leq \infty$.

Definition 2.1. A nonnegative function $\overline{u}(x,t) \in C(\overline{Q}_T) \cap C^{2,1}(Q_T)$ is called a super-solution of problem (1.1)–(1.3) in Q_T if

$$(\overline{u}^m)_t \geq \Delta \overline{u} + c(x,t)\overline{u}^p, \qquad (x,t) \in Q_T, \tag{2.1}$$

$$\overline{u}(x,t) \geq \int_\Omega k(x,y,t)\overline{u}^l(y,t)\,\mathrm{d}y, \qquad (x,t) \in S_T, \tag{2.2}$$

$$\overline{u}(x,0) \geq u_0(x), \qquad x \in \Omega. \tag{2.3}$$

Similarly, a nonnegative function $\underline{u}(x,t) \in C(\overline{Q}_T) \cap C^{2,1}(Q_T)$ is a sub-solution of problem (1.1)–(1.3) in Q_T if it satisfies (2.1)–(2.3) in the reverse order. We say that

$u(x,t)$ is a solution to problem (1.1)–(1.3) in Q_T if it is both a super-solution and a sub-solution to problem (1.1)–(1.3) in Q_T.

By the similar arguments in the literature [19], one can deduce the property of positiveness of the solution. The proof is more or less standard, so it is omitted here.

Lemma 2.2. *Let $u_0(x)$ is a nonnegative nontrivial function in Ω, and suppose that*

$$k(x,\cdot\,,t)\not\equiv 0,\quad \forall\,(x,t)\in S_T.$$

If $u(x,t)$ is a solution of problem (1.1)–(1.3) in Q_T, then $u(x,t)>0$ in $\overline{Q}_T$.

Next, the following modified comparison principle plays a crucial role in our proofs, which can be obtained by selecting a suitable test function and employing Gronwall's inequality.

Proposition 2.3 (Comparison principle)**.** *Suppose that $\underline{u}(x,t)$ and $\overline{u}(x,t)$ are the nonnegative sub- and super-solution of problem (1.1)–(1.3), respectively. If $\underline{u}(x,0)\le\overline{u}(x,0)$, $\underline{u}(x,0)\ge 0$ and $\overline{u}(x,0)\ge\delta>0$ in $\overline{\Omega}$, where δ is a positive constant. Then $\underline{u}(x,t)\le\overline{u}(x,t)$ in $\overline{Q}_T$.*

Proof. We first introduce a transformation. Let $v(x,t)=u^m(x,t)$. Then problem (1.1)–(1.3) becomes that

$$v_t=\Delta v^n+c(x,t)v^{np},\qquad (x,t)\in Q_T, \tag{2.4}$$

$$v^n(x,t)=\int_\Omega k(x,y,t)v^{nl}(y,t)\,\mathrm{d}y,\qquad (x,t)\in S_T, \tag{2.5}$$

$$v(x,0)=v_0(x)=u_0^m,\qquad x\in\Omega, \tag{2.6}$$

where $n=1/m>0$.

Let $\varphi(x,t)\in C^{2,1}(\overline{Q}_T)$ be a nonnegative function with $\varphi(x,t)=0$ and $\frac{\partial\varphi(x,t)}{\partial n}<0$ on S_T. Multiplying both sides of (2.4) by $\varphi(x,t)$ and integrating over Q_t for $0<t<T$, making use of (2.5) and Green's formula, one can see that

$$\begin{aligned}
&\int_\Omega v(x,t)\varphi(x,t)\,\mathrm{d}x\\
&=\int_\Omega v(x,0)\varphi(x,0)\,\mathrm{d}x+\iint_{Q_t} v\varphi_\tau\,\mathrm{d}x\mathrm{d}\tau+\iint_{Q_t}(\Delta v^n+c(x,\tau)v^{np})\varphi\,\mathrm{d}x\mathrm{d}\tau\\
&=\int_\Omega v(x,0)\varphi(x,0)\,\mathrm{d}x+\iint_{Q_t} v\varphi_\tau\,\mathrm{d}x\,\mathrm{d}v+\iint_{Q_t} v^n\Delta\varphi\,\mathrm{d}x\mathrm{d}\tau\\
&\quad+\iint_{Q_t} c(x,\tau)v^{np}\varphi\,\mathrm{d}x\mathrm{d}\tau-\iint_{S_t} v^n\frac{\partial\varphi}{\partial n}\,\mathrm{d}S\mathrm{d}\tau\\
&=\int_\Omega v(x,0)\varphi(x,0)\,\mathrm{d}x+\iint_{Q_t} v\varphi_\tau\,\mathrm{d}x\mathrm{d}\tau+\iint_{Q_t} v^n\Delta\varphi\,\mathrm{d}x\mathrm{d}\tau
\end{aligned}$$

$$+\iint_{Q_t} c(x,\tau)v^{np}\varphi\,\mathrm{d}x\mathrm{d}\tau - \iint_{S_t}\left(\int_\Omega k(x,y,\tau)v^{nl}(y,\tau)\,\mathrm{d}y\right)\frac{\partial\varphi}{\partial n}\,\mathrm{d}S\mathrm{d}\tau.$$

Then by the definitions of the sub- and super-solution, we obtain the following inequalities

$$\begin{aligned}
&\int_\Omega \underline{v}(x,t)\varphi(x,t)\,\mathrm{d}x\\
&\le \int_\Omega \underline{v}(x,0)\varphi(x,0)\,\mathrm{d}x + \iint_{Q_t}\underline{v}\varphi_\tau\,\mathrm{d}x\mathrm{d}\tau + \iint_{Q_t}\underline{v}^n\Delta\varphi\,\mathrm{d}x\mathrm{d}\tau\\
&\quad + \iint_{Q_t} c(x,\tau)\underline{v}^{np}\varphi\,\mathrm{d}x\mathrm{d}\tau - \iint_{S_t}\left(\int_\Omega k(x,y,\tau)\underline{v}^{nl}(y,\tau)\,\mathrm{d}y\right)\frac{\partial\varphi}{\partial n}\,\mathrm{d}S\mathrm{d}\tau,
\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}
&\int_\Omega \overline{v}(x,t)\varphi(x,t)\,\mathrm{d}x\\
&\ge \int_\Omega \overline{v}(x,0)\varphi(x,0)\,\mathrm{d}x + \iint_{Q_t}\overline{v}\varphi_\tau\,\mathrm{d}x\mathrm{d}\tau + \iint_{Q_t}\overline{v}^n\Delta\varphi\,\mathrm{d}x\mathrm{d}\tau\\
&\quad + \iint_{Q_t} c(x,\tau)\overline{v}^{np}\varphi\,\mathrm{d}x\mathrm{d}\tau - \iint_{S_t}\left(\int_\Omega k(x,y,\tau)\overline{v}^{nl}(y,\tau)\,\mathrm{d}y\right)\frac{\partial\varphi}{\partial n}\,\mathrm{d}S\mathrm{d}\tau.
\end{aligned}\tag{2.8}$$

Setting $\omega(x,t)=\underline{v}(x,t)-\overline{v}(x,t)$, combining (2.7) and (2.8), we get

$$\begin{aligned}
\int_\Omega \omega(x,t)\varphi(x,t)\,\mathrm{d}x \le{}& \int_\Omega \omega(x,0)\varphi(x,0)\,\mathrm{d}x\\
&+\iint_{Q_t}\big(\varphi_\tau+\phi_1(x,\tau)\Delta\varphi + c(x,\tau)\phi_2(x,\tau)\varphi\big)\omega\,\mathrm{d}x\mathrm{d}\tau\\
&-\iint_{S_t}\left(\int_\Omega k(x,y,\tau)\phi_3(y,\tau)\omega\,\mathrm{d}y\right)\frac{\partial\varphi}{\partial n}\,\mathrm{d}S\mathrm{d}\tau,
\end{aligned}$$

where

$$\begin{aligned}
\phi_1(x,\tau) &= \int_0^1 n\big(\theta\underline{v}+(1-\theta)\overline{v}\big)^{n-1}\,\mathrm{d}\theta,\\
\phi_2(x,\tau) &= \int_0^1 np\big(\theta\underline{v}+(1-\theta)\overline{v}\big)^{np-1}\,\mathrm{d}\theta,\\
\phi_3(x,\tau) &= \int_0^1 nl\big(\theta\underline{v}+(1-\theta)\overline{v}\big)^{nl-1}\,\mathrm{d}\theta.
\end{aligned}$$

Noticing that $\underline{v}(x,t)$, $\overline{v}(x,t)$ are bounded functions, it follows from $n\ge 1$, $nl\ge 1$ and $np\ge 1$ that ϕ_i $(i=1,2,3)$ are bounded nonnegative functions. If $n<1$, $nl<1$ and $np<1$, we have $\phi_1\le\delta^{n-1}$, $\phi_2\le\delta^{np-1}$, $\phi_3\le\delta^{nl-1}$ by the condition that $\underline{u}(x,0)\ge 0$ or $\overline{u}(x,0)\ge\delta>0$. Thus, we may choose appropriate function φ with $\frac{\partial\varphi}{\partial n}\le 0$ on $\partial\Omega$ as in [1, pp. 118–123] to obtain

$$\int_\Omega \omega(x,t)_+\,\mathrm{d}x \le C_1\int_\Omega \omega(x,0)_+\,\mathrm{d}x + C_2\iint_{Q_t}\omega(x,\tau)\,\mathrm{d}x\mathrm{d}\tau,$$

where $\omega_+ = \max\{\omega, 0\}$ and constants $C_1, C_2 > 0$. It follows from $\omega(x, 0)_+ \le 0$ that

$$\int_\Omega \omega(x,t)_+ \,\mathrm{d}x \le C_2 \iint_{Q_t} \omega(y,\tau)\,\mathrm{d}y\mathrm{d}\tau.$$

By Gronwall's inequality, we know that $\omega(x,t)_+ \le 0$, that is, $\underline{v}(x,t) \le \overline{v}(x,t)$ in Q_T.

On the other hand, for $(x,t) \in S_T$, we have

$$\underline{v}(x,t) - \overline{v}(x,t) \le \int_\Omega k(x,y,t)\big(\underline{v}^l(y,t) - \overline{v}^l(y,t)\big)\,\mathrm{d}y.$$

Since $k(x,y,t)$ is continuous function, one can see

$$\begin{aligned}\int_\Omega k(x,y,t)\big(\underline{v}^l(y,t) - \overline{v}^l(y,t)\big)_+\,\mathrm{d}y &= \int_\Omega k(x,y,t)\phi_3(y,t)\big(\underline{v}^l(y,t) - \overline{v}^l(y,t)\big)\,\mathrm{d}y\\ &\le C_3 \int_\Omega \big[\underline{v}(y,t) - \overline{v}(y,t)\big]_+\,\mathrm{d}y,\end{aligned}$$

where $\phi_3(x,\tau) = \int_0^1 nl\big(\theta\underline{v} + (1-\theta)\overline{v}\big)^{nl-1}\,\mathrm{d}\theta$ and constant $C_3 > 0$.

In view of $\int_\Omega \big[\underline{v}(y,t) - \overline{v}(y,t)\big]_+\,\mathrm{d}y = 0$, it can be easily seen that

$$\underline{v}(x,t) - \overline{v}(x,t) \le \int_\Omega k(x,y,t)\big(\underline{v}^l(y,t) - \overline{v}^l(y,t)\big)\,\mathrm{d}y \le 0,$$

and so $\underline{v}(x,t) \le \overline{v}(x,t)$ on S_T. In conclusion, we obtain $\underline{v}(x,t) \le \overline{v}(x,t)$ in $\overline{Q}_T$. The proof is completed. □

Finally, we state the local solvability theorem without proof.

Theorem 2.4 (Local existence theorem)**.** *Let $m > 0$, $p > 0$, $l > 0$ and $c(x,t)$ be a nonnegative continuous function in $\overline{Q}_T$ and $k(x,y,t)$ is nonnegative nontrivial continuous function in $\partial\Omega\times\overline{\Omega}\times[0,T)$. If the initial value $u_0(x) \in C^{2+\alpha}(\overline{\Omega})$ with $0 < \alpha < 1$ is a nonnegative nontrivial function satisfying the compatibility condition $u_0(x) = \int_\Omega k(x,y,0)u_0^l(y)\,\mathrm{d}y$ for $x \in \partial\Omega$, then there exists a constant $T > 0$ such that the problem* (1.1)–(1.3) *admits a nonnegative solution $u(x,t) \in C^{2,1}(\Omega\times(0,S))\cap C(\overline{\Omega}\times[0,S])$ for each $S < T$. Furthermore, either $T = \infty$ or*

$$\lim_{t\to T^-}\sup \|u(\,\cdot\,,t)\|_\infty = \infty.$$

Remark 2.5. Indeed, the proof of Theorem 2.4 can be obtained by using the Schauder's fixed point theorem (see [9, 34, 35]) or the regular theory to get the suitable estimate in a standard limiting process (see [4, 23]). For the sake of readers' understanding, a sketch outline of the proof process using Schauder's fixed point theorem is given below. Let

$$\mathbb{K} = \big\{v(x,t) : \|v\|_{L^\infty(Q_T)} \le K_0\big\},$$

where K_0 is an appropriate constant. For any given $v \in \mathbb{K}$, we introduce the following auxiliary problem

$$(u^m)_t = \Delta u + c(x,t)u^p, \qquad (x,t) \in \Omega \times (0,T), \tag{2.9}$$

$$u(x,t) = \int_\Omega k(x,y,t)v^l(y,t)\,\mathrm{d}y, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2.10}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{2.11}$$

According to the standard parabolic equation theory [12,22], problem (2.9)–(2.11) admits a solution $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$. Now, we define a mapping Υ as follows:

$$\Upsilon\colon v(x,t) \in \mathbb{K} \subset L^\infty(Q_T) \to \Upsilon[v] = u(x,t) \in C(\overline{Q}_T),$$

where $u(x,t)$ is the solution of (2.9)–(2.11). Next, proving the mapping Υ is continuous and precompact from $\mathbb{K}$ to $\mathbb{K}$. Therefore, Schauder's fixed-point theorem implies that the mapping Υ has a fixed point $u \in C(\overline{Q}_T)$, which is a solution to problem (1.1)–(1.3). Then, it follows from the regularity theory for parabolic equations that $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$.

Remark 2.6. By the comparison principle above, we can get the uniqueness of the solutions to the problem (1.1)–(1.3) in the case of $p \geq m$.

3. Global existence and blow-up criteria

In this section, by virtue of the method of super- and sub-solutions, Kaplan's argument, the method of auxiliary problem and some ODE techniques, etc., we present the suitable conditions of nonlinear exponents p, m, l and the weighted functions $c(x,t)$ and $k(x,y,t)$, which ensure solution exists globally or blows up in finite time. For the remainder of this paper, we denote

$$\mu := \sup_{\overline{\Omega}} u_0(x), \quad c_1(t) := \inf_{\overline{\Omega}} c(x,t), \quad c_2(t) := \sup_{\overline{\Omega}} c(x,t).$$

We first establish the results of global existence for solutions with arbitrary initial value.

Theorem 3.1. *Assume that $m > 0$, $p \leq m$ and $c(x,t)$ is a nonnegative continuous function. If one of the following conditions is satisfied:*

(i) *$l < 1$ and $k(x,y,t)$ is nonnegative continuous function or $l = 1$ and the weighted function $k(x,y,t)$ satisfies $\int_\Omega k(x,y,t)\,\mathrm{d}y \leq 1$, $\forall\,(x,t) \in S_T$;*

(ii) *$m \neq 1$, $l \leq 1$ and the weighted function $k(x,y,t)$ satisfies $\int_\Omega k(x,y,t)\,\mathrm{d}y \leq 1$, $\forall\,(x,t) \in S_T$,* *then problem* (1.1)–(1.3) *has global solutions for any nonnegative nontrivial initial data.*

Remark 3.2. Take the unit disc $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x|^2 = \sum_{i=1}^2 x_i^2 < 1\}$, $k(x, y, t) = \frac{3}{2\pi}\frac{y^2}{t+x^2}$ for $(x, t) \in S_T$, $y \in \Omega$, which satisfies the condition $\int_\Omega k(x, y, t)\,\mathrm{d}y \le 1$, $\forall\,(x, t) \in S_T$.

Proof of Theorem 3.1. (i) Let T be any positive number. Since $c(x, t)$ and $k(x, y, t)$ are continuous functions, there exists positive constant M such that

$$c(x, t) \le M, \quad k(x, y, t) \le M \quad \text{in } \overline{Q}_T \text{ and } \partial\Omega \times \overline{Q}_T, \text{ respectively.} \tag{3.1}$$

We construct a super-solution of problem (1.1)–(1.3) in the form

$$\overline{u}(x, t) = \alpha^{1/m} e^{\beta t/m},$$

where

$$\alpha \ge \max\{1, \mu^m, (M|\Omega|)^{m/(1-l)}\}, \quad \beta > M. \tag{3.2}$$

Since $p \le m$, (3.1) and (3.2), it is easy to check that

$$(\overline{u}^m)_t - \Delta\overline{u} - c(x, t)\overline{u}^p = \alpha\beta e^{\beta t} - c(x, t)\alpha^{p/m} e^{p\beta t/m} \ge \alpha e^{\beta t}(\beta - M) \ge 0.$$

Moreover, by (3.1)–(3.2), $\overline{u}$ satisfies the following boundary and initial conditions:

$$\overline{u}(x, t) \ge M|\Omega|\alpha^{(l-1)/m}\overline{u}(x, t) \ge \int_\Omega k(x, y, t)\alpha^{l/m} e^{l\beta t/m}\,\mathrm{d}y = \int_\Omega k(x, y, t)\overline{u}^l\,\mathrm{d}y,$$
$$\overline{u}(x, 0) = \alpha^{1/m} \ge \mu \ge u_0(x).$$

Therefore, by virtue of Proposition 2.3, we know that $\overline{u}(x, t)$ is a super-solution of problem (1.1)–(1.3), which means that the solution of problem (1.1)–(1.3) exists globally.

In the case of $l = 1$, $\overline{u}(x, t)$ is a super-solution of (1.1)–(1.3) provided $\int_\Omega k(x, y, t)\,\mathrm{d}y \le 1$, $\forall\,(x, t) \in S_T$.

(ii) We divide the proof of Theorem 3.1(ii) into two cases.

Case 1*:* $0 < m < 1$. Let $h(x)$ be the positive solution of the linear problem

$$\Delta h(x) = \lambda,\ x \in \Omega; \quad h(x) = 1,\ x \in \partial\Omega, \tag{3.3}$$

with

$$0 < h_1 \le h(x) \le h_2 < 1 \tag{3.4}$$

for $x \in \Omega$ and some positive constants h_1 and h_2.

We construct a super-solution of problem (1.1)–(1.3), taken of the form

$$\overline{u}(x,t) = A_0 e^{k_0 t} \log\big[h(x) e^{k_0(m-1)t} + B_0\big],$$

where $h(x)$ is the solution of problem (3.3), constants $A_0 > 1$, $B_0 > e$, $k_0 > 0$ and satisfy

$$B_0 \log B_0 \geq 2(1-m)h_2, \tag{3.5}$$

$$k_0 \geq \frac{2MB_0\big[\log(h_2+B_0)\big]^p + 2A_0^{1-m}\lambda}{B_0 m (\log B_0)^m}, \tag{3.6}$$

$$A_0 \log(h_1 + B_0) \geq \mu. \tag{3.7}$$

By virtue of (3.5) and $p \leq m$, after simple calculation, we have

$$\begin{aligned} (\overline{u}^m)_t &= k_0 m A_0^m e^{k_0 m t}\big[\log\big(h(x)e^{k_0(m-1)t} + B_0\big)\big]^m \\ &\quad + A_0^m e^{k_0 m t} m \big[\log\big(h(x)e^{k_0(m-1)t} + B_0\big)\big]^{m-1} \frac{k(m-1)h(x)e^{k_0(m-1)t}}{h(x)e^{k_0(m-1)t} + B_0} \\ &\geq \frac{1}{2} k_0 m A_0^m e^{k_0 m t} (\log B_0)^m, \end{aligned} \tag{3.8}$$

$$\Delta \overline{u} = \frac{A_0 e^{k_0 m t}\Delta h(x)}{h(x)e^{k_0(m-1)t} + B_0} - \frac{A_0\big[\nabla\big(h(x)e^{k_0(m-1)t} + B_0\big)\big]^2}{\big(h(x)e^{k_0(m-1)t} + B_0\big)^2} \leq \frac{A_0 \lambda e^{k_0 m t}}{B_0}, \tag{3.9}$$

$$\overline{u}^p \leq A_0^m e^{k_0 m t}\big[\log(h_2 + B_0)\big]^p. \tag{3.10}$$

It follows from (3.6) and (3.8)–(3.10) that

$$(\overline{u}^m)_t - \Delta \overline{u} - c(x,t)\overline{u}^p \geq 0.$$

In addition, for $(x,t) \in S_T$, one can see

$$\begin{aligned} \overline{u}(x,t) &= A_0 e^{k_0 t} \log\big(e^{k_0(m-1)t} + B_0\big) \geq A_0 e^{k_0 t}\log\big(h_2 e^{k_0(m-1)t} + B_0\big) \\ &\geq \int_\Omega k(x,y,t) A_0^l e^{k_0 l t}\big[\log\big(h(y)e^{k_0(m-1)t} + B_0\big)\big]^l \,\mathrm{d}y = \int_\Omega k(x,y,t)\overline{u}^l \,\mathrm{d}y, \end{aligned}$$

and by (3.7), we have $\overline{u}(x,0) \geq A_0 \log(h_1 + B_0) \geq u_0(x)$. Consequently, $\overline{u}(x,t)$ is a super-solution of problem (1.1)–(1.3) by Proposition 2.3.

Case 2: $m > 1$. We introduce the eigenvalue problem of the operator $-\Delta$ under the homogeneous Dirichlet boundary condition

$$-\Delta\phi = \lambda_1 \phi,\ x \in \Omega, \quad \phi = 0,\ x \in \partial\Omega, \tag{3.11}$$

where λ_1 and $\phi(x)$ are the first eigenvalue and the corresponding eigenfunction, respectively. Then $\lambda_1 > 0$, $\phi(x) > 0$ in Ω, $\frac{\partial\phi}{\partial n} < 0$ on $\partial\Omega$ and normalized $\sup_{\overline{\Omega}} \phi(x) = 1$. Moreover, denote

$$c_3 := \max_{x\in\overline{\Omega}} |\nabla\phi(x)| > 0, \quad c_4 := \min_{x\in\partial\Omega}\left|\frac{\partial\phi}{\partial n}\right| > 0.$$

We construct a super-solution of problem (1.1)–(1.3) in the form

$$\overline{u}(x,t) = e^{k_1 t}\big[M_1 + (2M_1)^{\beta_1} L^{-1} c_4^{-1} e^{-L\phi(x)e^{k_1(m-1)t/2}}\big],$$

where $\phi(x)$ is the eigenfunction of problem (3.11), constants k_1, M_1 and L are defined by

$$k_1 \geq \frac{M(2M_1)^p + (\lambda - 1 + Lc_3^2)c_4^{-1}(2M_1)^{\beta_1}}{m}, \tag{3.12}$$

$$M_1 := \max\{2\|u_0\|_\infty, 2\}, \quad L := \max\left\{\frac{2^{\beta_1} M_1^{\beta_1 - 1}}{c_4}, \frac{(m-1)(2M_1)^{\beta_1}}{2e(M_1 - 1)c_4}\right\}. \tag{3.13}$$

By a straightforward calculation, one can see that

$$\begin{aligned}(\overline{u}^m)_t = {} & k_1 m e^{k_1 m t}\big[M_1 + (2M_1)^{\beta_1} L^{-1} c_4^{-1} e^{-L\phi(x)e^{k_1(m-1)t/2}}\big]^{m-1} \\ & \times \left[M_1 + (2M_1)^{\beta_1} L^{-1} c_4^{-1} e^{-L\phi(x)e^{k_1(m-1)t/2}} \left(1 + \frac{1}{2}(m-1)(-L\phi)e^{k_1(m-1)t/2}\right)\right].\end{aligned}$$

It follows from $-Ye^{-Y} \geq -e^{-1}$ for any $Y > 0$ that

$$-L\phi e^{k_1(m-1)t/2} e^{-L\phi(x)e^{k_1(m-1)t/2}} \geq -e^{-1},$$

and hence

$$(\overline{u}^m)_t \geq k_1 m e^{k_1 m t}. \tag{3.14}$$

Furthermore, according to $p \leq m$ and (3.13), we have

$$\begin{aligned}\Delta\overline{u} &= (2M_1)^{\beta_1} L^{-1} c_4^{-1} e^{k_1 t} \Delta e^{-L\phi(x)e^{k_1(m-1)t/2}} \\ &= (2M_1)^{\beta_1} c_4^{-1} e^{-L\phi(x)e^{k_1(m-1)t/2}} \big[\Delta(-\phi)e^{k_1(m+1)t/2} + L|\nabla\phi|^2 e^{k_1 m t/2}\big] \\ &\leq c_4^{-1}(2M_1)^{\beta_1} e^{k_1 m t}\big(\lambda_1 + Lc_3^2\big),\end{aligned} \tag{3.15}$$

$$\overline{u}^p = e^{k_1 p t}\big[M_1 + (2M_1)^{\beta_1} L^{-1} c_4^{-1} e^{-L\phi(x)e^{k_1(m-1)t/2}}\big]^p \leq (2M_1)^p e^{k_1 m t}. \tag{3.16}$$

Combining (3.12) and (3.14)–(3.16), we obtain

$$(\overline{u}^m)_t - \Delta\overline{u} - c(x,t)\overline{u}^p \geq \big[k_1 m - (\lambda_1 + Lc_3^2)c_4^{-1}(2M_1)^{\beta_1} - M(2M_1)^p\big]e^{k_1 m t} \geq 0.$$

On the other hand, for $(x,t) \in S_T$, one can see

$$\begin{aligned}\overline{u}(x,t) = e^{k_1 t}\big[M_1 + (2M_1)^{\beta_1} L^{-1} c_4^{-1}\big] &\geq \int_\Omega k(x,y,t) e^{k_1 t}\big[M_1 + (2M_1)^{\beta_1} L^{-1} c_4^{-1}\big]\,\mathrm{d}y \\ \geq \int_\Omega k(x,y,t) e^{k_1 l t}\big[M_1 + (2M_1)^{\beta_1} L^{-1} c_4^{-1} e^{-L\phi(y)e^{k_1(m-1)t/2}}\big]^l\,\mathrm{d}y &= \int_\Omega k(x,y,t)\overline{u}^l\,\mathrm{d}y,\end{aligned}$$

and by (3.13), we have $\overline{u}(x,0) = M_1 + (2M_1)^{\beta_1} L^{-1} c_4^{-1} \geq u_0(x)$. Now applying Proposition 2.3, we obtain the desired result. The proof is completed. □

Remark 3.3. Theorem 3.1(i) involves the result in [18, Theorem 2.5]. And Theorem 3.1(ii) is new conclusion, in which we adopts the technique of constructing the appropriate super-solution to deal with the quasilinear case.

Corollary 3.4. *If $m > 1$, $p \le 1$, $l = 1$, $c(x,t)$ is a nonnegative continuous function and $\int_\Omega k(x,y,t)\,\mathrm{d}y \le 1$ is not valid for $(x,t) \in S_T$, then problem (1.1)–(1.3) exists global solution for any nonnegative nontrivial initial datum.*

Proof. Let $\lambda_1 > 0$ and $\phi(x)$ be the first eigenvalue and the corresponding eigenfunction of problem (3.11), respectively, and ϕ be chosen to satisfy for $0 < \varepsilon_1 < 1$ that

$$M \int_\Omega \frac{1}{\phi(y) + \varepsilon_1}\,\mathrm{d}y \le 1, \tag{3.17}$$

where the constant M is given in (3.1). We construct a super-solution of problem (1.1)–(1.3), taken of the form

$$\overline{u}(x,t) = \frac{Ce^{\gamma_1 t/m}}{\phi(x) + \varepsilon_1},$$

where $C \ge \sup_{\overline{\Omega}}(\phi + \varepsilon_1)$, $\gamma_1 > 0$ are constants to be chosen later. A direct calculation leads to

$$\begin{aligned}
&(\overline{u}^m)_t - \Delta\overline{u} - c(x,t)\overline{u}^p \\
&= \frac{C^m \gamma_1 e^{\gamma_1 t}}{(\phi(x) + \varepsilon_1)^m} - \left(\frac{\lambda_1 \phi}{\phi + \varepsilon_1} + \frac{2|\nabla\phi|^2}{(\phi + \varepsilon_1)^2}\right)\overline{u} - c(x,t)\overline{u}^p \\
&= \left[\left(\frac{C}{\phi + \varepsilon_1}\right)^{m-1} \gamma_1 e^{\left(\gamma_1 - \frac{\gamma_1}{m}\right)t} - \left(\frac{\lambda_1 \phi}{\phi + \varepsilon_1} + \frac{2|\nabla\phi|^2}{(\phi + \varepsilon_1)^2}\right)\right]\overline{u} - c(x,t)\overline{u}^p.
\end{aligned} \tag{3.18}$$

Combining $p \le 1$ and $C \ge \sup_{\overline{\Omega}}(\phi + \varepsilon_1)$, we have the inequality

$$\overline{u}^p(x,t) \le \overline{u}(x,t), \quad (x,t) \in Q_T. \tag{3.19}$$

According to (3.1) and (3.18)–(3.19), we obtain

$$\begin{aligned}
(\overline{u}^m)_t - \Delta\overline{u} - c(x,t)\overline{u}^p &\ge \left[\left(\frac{C}{\phi + \varepsilon_1}\right)^{m-1} \gamma_1 e^{\left(\gamma_1 - \frac{\gamma_1}{m}\right)t} - \left(\frac{\lambda_1 \phi}{\phi + \varepsilon_1} + \frac{2|\nabla\phi|^2}{(\phi + \varepsilon_1)^2}\right) - M\right]\overline{u} \\
&\ge \left[\gamma_1 - \left(\frac{\lambda_1 \phi}{\phi + \varepsilon_1} + \frac{2|\nabla\phi|^2}{(\phi + \varepsilon_1)^2}\right) - M\right]\overline{u} \ge 0,
\end{aligned}$$

provided that $\gamma_1 \ge \lambda_1 + \sup_{\overline{\Omega}} \frac{2|\nabla\phi|^2}{(\phi+\varepsilon_1)^2} + M$.

Furthermore, for $(x,t) \in S_T$, it follows from (3.1) and (3.17) that

$$\overline{u}(x,t) = \frac{Ce^{\gamma_1 t/m}}{\varepsilon_1} \ge \int_\Omega \frac{Ce^{\gamma_1 t/m} M}{\phi(y) + \varepsilon_1}\,\mathrm{d}y \ge \int_\Omega k(x,y,t)\overline{u}\,\mathrm{d}y.$$

It is clear from Proposition 2.3 that $\overline{u}(x,t)$ is a super-solution of problem (1.1)–(1.3) if $C \ge \max\left\{\sup_{\overline{\Omega}} u_0(x) \sup_{\overline{\Omega}}(\phi + \varepsilon_1), \sup_{\overline{\Omega}}(\phi + \varepsilon_1)\right\}$. This completes the proof. $\square$

Now we shall show the existence of global solutions to problem (1.1)–(1.3) for small initial data. We suppose in the following statement that

$$M_2 \geq \frac{b\lambda + b^p c_2(t) h_2^p - b^m c_2(t) h_1^m}{b^m h_1^m}, \quad t > 0, \tag{3.20}$$

where constants λ, h_1, h_2 are given in (3.3)–(3.4), $b > 0$ is determined later, and there exist a positive constant A_1 such that

$$\int_\Omega k(x,y,t)\,\mathrm{d}y e^{\frac{l-1}{m}\left(\int_0^t c_2(s)\,\mathrm{d}s + M_2 t\right)} \leq A_1, \quad \forall\,(x,t) \in S_T. \tag{3.21}$$

Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$ satisfying the property that $\Omega \Subset \widetilde{\Omega}$ and we introduce the eigenvalue problem of the operator $-\Delta$ given by

$$-\Delta\Phi = \widetilde{\lambda}_1 \Phi(x),\ x \in \widetilde{\Omega}, \quad \Phi(x) = 0,\ x \in \partial\widetilde{\Omega}, \tag{3.22}$$

where $\widetilde{\lambda}_1$ and $\Phi(x)$ are the first eigenvalue and the corresponding eigenfunction, respectively. Then $\widetilde{\lambda}_1 > 0$ and $\Phi(x) > 0$ in $\widetilde{\Omega}$. It is obvious that

$$\frac{\sup_{x\in\widetilde{\Omega}} \Phi(x)}{\inf_{x\in\overline{\Omega}} \Phi(x)} < a \tag{3.23}$$

for some $a > 1$. For convenience, we denote $\rho := \sup_{x\in\widetilde{\Omega}} \Phi(x)$. Meanwhile, suppose that

$$\int_0^\infty c_2(t) e^{-\gamma t}\,\mathrm{d}t < \infty \quad \text{for some } \gamma < \frac{p-m}{m}\rho^{1-m}\lambda_1, \tag{3.24}$$

and for some $A_2 > 0$ and $\sigma < \frac{l-1}{m}\rho^{1-m}\lambda_1$,

$$\int_\Omega k(x,y,t)\,\mathrm{d}y \leq A_2 e^{\sigma t}, \quad \forall\,(x,t) \in S_T, \tag{3.25}$$

where λ_1 is the first eigenvalue of problem (3.11).

Theorem 3.5. *Assume $m > 1$, $l > 1$. If one of the following conditions is satisfied:*

(i) *$p < m$, the weighted function $c(x,t)$ satisfies $\int_0^\infty c_2(t)\,\mathrm{d}t < \infty$ and hypotheses* (3.20)–(3.21) *hold;*

(ii) *$p > m$ and hypotheses* (3.24)–(3.25) *hold,*

then the solution of problem (1.1)–(1.3) *exists globally for small initial data.*

Remark 3.6. In fact, the weight functions $c(x,t)$ and $k(x,y,t)$ satisfying Theorem 3.5 can be selected. Take the unit ball $\Omega = \left\{x = (x_1,x_2,x_3) \in \mathbb{R}^3 \mid |x|^2 = \sum_{i=1}^3 x_i^2 < 1\right\}$, $c(x,t) = \frac{x^2}{1+t^2}$ for $(x,t) \in Q_T$, $M_2 = \frac{b\lambda + \left(b^p h_2^p - b^m h_1^m\right)}{b^m h_1^m}$, $k(x,y,t) = \frac{x^2}{1+y^2} e^{\frac{1-l}{m}(\arctan t + M_2 t)}$ for $(x,t) \in S_T$, $y \in \Omega$, then $c_2(t) = \frac{1}{1+t^2}$, which satisfies $\int_0^\infty c_2(t)\,\mathrm{d}t = \frac{\pi}{2} < \infty$, and the assumption (3.21) holds. Choose $c(x,t) = \frac{e^{(\gamma-1)t}}{1+x^2}$ for $(x,t) \in Q_T$, $k(x,y,t) = \frac{x^2}{1+y^2} e^{\sigma t}$ for $(x,t) \in S_T$, $y \in \Omega$, then $c_2(t) = e^{(\gamma-1)t}$, and the assumptions (3.24)–(3.25) are satisfied.

Proof of Theorem 3.5. (i) We construct a super-solution of problem (1.1)–(1.3) in the form

$$\overline{u}(x,t) = be^{\frac{1}{m}f(t)}h(x), \quad f(t) = \int_0^t c_2(s)\,\mathrm{d}s + M_2 t,$$

where $b<1$ is a positive constant and M_2 is defined in (3.20), and $h(x)$ is the solution of problem (3.3). By a straightforward calculation, we obtain

$$\begin{aligned}(\overline{u}^m)_t - \Delta\overline{u} - c(x,t)\overline{u}^p &= b^m h^m e^{f(t)} f'(t) - be^{\frac{1}{m}f(t)}\Delta h - b^p c(x,t) h^p e^{\frac{p}{m}f(t)} \\ &\geq e^{f(t)}\big[b^m h_1^m (c_2(t) + M_2) - b\lambda - b^p h_2^p c_2(t)\big] \geq 0.\end{aligned}$$

On the other hand, for $(x,t)\in S_T$, one can see

$$\overline{u}(x,t) = be^{\frac{1}{m}f(t)} \geq A_1 b^l e^{\frac{1}{m}f(t)} \geq \int_\Omega k(x,y,t) b^l e^{\frac{l}{m}f(t)} h^l(y)\,\mathrm{d}y = \int_\Omega k(x,y,t) u^l\,\mathrm{d}y,$$

provided $b \leq A_1^{1/(1-l)}$.

Hence, by Proposition 2.3, we know that there exists global solution of problem (1.1)–(1.3) provided $u_0(x) \leq bh(x)$.

(ii) Let $\Phi(x)$ is the eigenfunction corresponding to the first eigenvalue $\widetilde{\lambda}_1$ of problem (3.22), and satisfy $\Phi(x)>0$ in $\widetilde{\Omega}$, $\widetilde{\lambda}_1 < \lambda_1$. Now, choosing any ε which satisfies the inequality

$$0 < \varepsilon \leq (A_2 a^l)^{-1/(l-1)}, \tag{3.26}$$

where $a>1$, A_2 is given in (3.25), and taking

$$\sup_{x\in\widetilde{\Omega}} \Phi(x) = a\varepsilon, \tag{3.27}$$

then by (3.23) we have the inequality

$$\inf_{x\in\overline{\Omega}} \Phi(x) > \varepsilon. \tag{3.28}$$

We construct a super-solution of problem (1.1)–(1.3) in such a form that

$$\overline{u}(x,t) = \Phi(x) F^{1/m}(t),$$

where

$$\begin{aligned} F(t) &= e^{-\widetilde{\lambda}_1 (a\varepsilon)^{1-m} t}\left[B - \frac{p-m}{m}(a\varepsilon)^{p-m}\int_0^t c_2(s) e^{-\widetilde{\lambda}_1 \frac{p-m}{m}(a\varepsilon)^{1-m} s}\,\mathrm{d}s\right]^{-m/(p-m)}, \\ B &= 1 + \frac{p-m}{m}(a\varepsilon)^{p-m}\int_0^\infty c_2(s) e^{-\widetilde{\lambda}_1 \frac{p-m}{m}(a\varepsilon)^{1-m} s}\,\mathrm{d}s. \end{aligned}$$

Noticing $F(t) \leq e^{-\widetilde{\lambda}_1(a\varepsilon)^{1-m}t} \leq 1$ and $F(t)$ is a solution of the equation

$$F'(t) + \widetilde{\lambda}_1(a\varepsilon)^{1-m}F(t) - c_2(t)(a\varepsilon)^{p-m}F^{p/m}(t) = 0.$$

It can be shown by a simple calculation that

$$\begin{aligned}
&(\overline{u}^m)_t - \Delta\overline{u} - c(x,t)\overline{u}^p \\
&= \Phi^m F'(t) - \Delta\Phi F^{1/m} - c(x,t)\Phi^p F^{p/m} \geq \Phi^m F'(t) + \widetilde{\lambda}_1\Phi F^{1/m} - c_2(t)\Phi^p F^{p/m} \\
&\geq \Phi^m\big[F' + \widetilde{\lambda}_1\Phi^{1-m}F - c_2(t)\Phi^{p-m}F^{p/m}\big] \geq 0.
\end{aligned}$$

On the other hand, for $(x,t) \in S_T$, it follows from (3.25)–(3.28) that

$$\overline{u}(x,t) > \varepsilon F^{1/m}(t) \geq A_2(a\varepsilon)^l F^{1/m}(t) \geq \int_\Omega k(x,y,t)\Phi^l(y)F^{l/m}(t)\,\mathrm{d}y = \int_\Omega k(x,y,t)\overline{u}^l\,\mathrm{d}y.$$

As a result, by Proposition 2.3, there exists a global solution of problem (1.1)–(1.3) for any nonnegative initial data such that $u_0(x) \leq B^{-1/(p-m)}\Phi(x)$. The proof is completed. $\square$

Remark 3.7. Theorem 3.5 is concerned with the slow diffusion case and presents the new global existence conditions, such as hypothesis (3.21).

In order to show the following results, we suppose that

$$c_2(t)\rho^{p-1} - \widetilde{\lambda}_1 > 0, \quad \int_0^\infty \big[\rho^{p-1}c_2(t) - \widetilde{\lambda}_1\big]\,\mathrm{d}t < \infty, \tag{3.29}$$

where $\rho := \sup_{x\in\widetilde{\Omega}} \Phi(x)$, $\widetilde{\lambda}_1$ and $\Phi(x)$ are given in (3.22).

Theorem 3.8. *Assume $m > 0$, the weighted function $k(x,y,t)$ satisfies*

$$\int_\Omega k(x,y,t)\,\mathrm{d}y \leq 1, \quad \forall\,(x,t) \in S_T. \tag{3.30}$$

If one of the following conditions is satisfied:

(i) *$p > m$, $l \geq 1$, the weighted function $c(x,t)$ satisfies $\int_0^\infty c_2(t) < \infty$, and initial data is sufficiently small;*

(ii) *$p = m$, $l \leq 1$, the weighted function $c(x,t)$ satisfies $\int_0^\infty c_2(t) < \infty$, and initial data is arbitrary nonnegative function;*

(iii) *$m < 1$, $\min\{p,l\} > 1$, hypothesis (3.29) holds, and initial data is sufficiently small,*

then problem (1.1)–(1.3) admits global solutions.

Remark 3.9. Take the unit disc $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x|^2 = \sum_{i=1}^2 x_i^2 < 1\}$, $c(x,t) = x^2\big(\frac{\widetilde{\lambda}_1}{\rho^{p-1}} + \frac{1}{1+t^2}\big)$ for $(x,t) \in Q_T$, then $c_2(t) = \rho^{p-1} + \frac{1}{1+t^2}$, which satisfies the condition (3.29).

Proof of Theorem 3.8. (i) Consider the following Cauchy problem

$$\begin{cases} (g^m)'(t) = c_2(t) g^p(t), \quad t > 0, \\ g(0) = B_1^{1/(p-m)}, \end{cases} \tag{3.31}$$

where $B_1 = \big[1 + \frac{p-m}{m}\int_0^\infty c_2(t)\,\mathrm{d}t\big]$. Then we can write the explicit solution of problem (3.31) as

$$g(t) = \left[B_1 - \frac{p-m}{m}\int_0^t c_2(s)\,\mathrm{d}s\right]^{1/(m-p)},$$

and $g(t) \le 1$. It can be easily seen that $g(t)$ is a super-solution of problem (1.1)–(1.3) for initial data $u_0(x) \le B_1^{-1/(p-m)}$.

(ii) Since $p = m$ and $l \le 1$, it is not difficult to verify that

$$g_1(t) = g_1(0) e^{\frac{1}{m}\int_0^t c_2(s)\,\mathrm{d}s}$$

with $g_1(0) \ge \max\{1, \mu\}$ is a super-solution of problem (1.1)–(1.3).

(iii) As in the proof of Theorem 3.5(ii) we consider eigenfunction $\Phi(x)$, which satisfies (3.27) and (3.28), where

$$0 < \varepsilon \le a^{-l/(l-1)}. \tag{3.32}$$

We construct a super-solution of problem (1.1)–(1.3) in the form of separated variables

$$\overline{u}(x,t) = \Phi(x) G^{1/m}(t),$$

where $G(t)$ is the solution for the following initial value problem of ordinary differential equation

$$\begin{cases} G'(t) + \big[\widetilde{\lambda}_1 - (a\varepsilon)^{p-1} c_2(t)\big] G^{1/m}(t) = 0, \quad t > 0, \\ G(0) = B_2^{m/(m-1)}, \end{cases}$$

where

$$B_2 = 1 + \frac{1-m}{m}\int_0^\infty \big(c_2(t)(a\varepsilon)^{p-m} - \widetilde{\lambda}_1\big)\,\mathrm{d}t.$$

Then $G(t)$ can be written in an explicit form

$$G(t) = \left[B_2 + \frac{m-1}{m}\int_0^t \big(c_2(t)(a\varepsilon)^{p-1} - \widetilde{\lambda}_1\big)\,\mathrm{d}s\right]^{m/(m-1)}.$$

It can be easily seen that $G(t) < 1$. A direct calculation leads to

$$\begin{aligned}(\overline{u}^m)_t - \Delta\overline{u} - c(x,t)\overline{u}^p &= \Phi^m G'(t) - \Delta\Phi G^{1/m} - c(x,t)\Phi^p G^{p/m}\\ &\geq \Phi^m G'(t) + \widetilde{\lambda}_1 \Phi G^{1/m} - c_2(t)\Phi^p G^{p/m}\\ &\geq \Phi\big[G' + \big(\widetilde{\lambda}_1 - (a\varepsilon)^{p-1}c_2(t)\big)G^{1/m}\big] \geq 0.\end{aligned}$$

On the other hand, for $(x,t) \in S_T$, by (3.27)–(3.28), (3.30) and (3.32), we have

$$\overline{u}(x,t) > \varepsilon G^{1/m}(t) \geq (a\varepsilon)^l G^{1/m}(t) \geq \int_\Omega k(x,y,t)\Phi^l(y)G^{l/m}(t)\,\mathrm{d}y = \int_\Omega k(x,y,t)\overline{u}^l\,\mathrm{d}y.$$

Therefore, by employing Proposition 2.3, we know that there exists global solution of problem (1.1)–(1.3) for initial datum such that $u_0(x) \leq B_2^{m/(m-1)}\Phi(x)$. The proof is completed. □

Remark 3.10. For $p = m$, $l > 1$, if $\int_\Omega k(x,y,t)\,\mathrm{d}y \leq e^{\frac{1-l}{m}\int_0^t c_2(s)\,\mathrm{d}s}$, then problem (1.1)–(1.3) exists global solution.

We proceed to derive the blow-up phenomenon for large initial data.

Theorem 3.11. *Assume $p < m < 1$, $l > 1$. If the weighted functions $c(x,t)$ and $k(x,y,t)$ satisfy, respectively*

$$c(x,t) \geq M_3 \quad \text{for some } M_3 > 0, \tag{3.33}$$

and

$$\int_\Omega k(x,y,t)\,\mathrm{d}y \geq 2^{1/(1-p)} > 1, \quad \forall\,(x,t) \in S_T, \tag{3.34}$$

then the solution of problem (1.1)–(1.3) *blows up in finite time provided that initial data $u_0(x) \geq 2^{1/(1-p)}$.*

Remark 3.12. Take the unit ball $\Omega = \big\{x = (x_1,x_2,x_3) \in \mathbb{R}^3 \mid |x|^2 = \sum_{i=1}^3 x_i^2 < 1\big\}$, $c(x,t) = \frac{1+t}{2-x^2} \geq 1$ for $(x,t) \in Q_T$, $k(x,y,t) = 2^{1/(1-p)}\frac{1+t+y^2}{x^2}$ for $(x,t) \in S_T$, $y \in \Omega$, then $\int_\Omega k(x,y,t)\,\mathrm{d}y \geq \frac{4\pi}{5}2^{1/(1-p)} > 1$, $\forall\,(x,t) \in S_T$.

Proof of Theorem 3.11. We shall to construct a blow-up sub-solution in such form

$$\underline{u}(x,t) = \big[\widehat{a}h(x) + (1-ct)^{-k_2}\big]^\theta,$$

where $h(x)$ is the solution of problem (3.2) with $0 < h_1 \leq h(x) \leq h_2 < 1$, and constants $\widehat{a}$, k_2, θ, c are defined as

$$\widehat{a} := \frac{1}{h_2}, \quad k_2 := \frac{1-p}{p-m}, \quad \theta := \frac{k_2+1}{k_2(1-m)} = \frac{1}{1-p}, \quad c \geq \frac{2^{1-m\theta}\widehat{a}\lambda}{k_2 m}.$$

By a direct computation, one can see

$$(\underline{u}^m)_t = m\theta k_2 c\big[\widehat{a}h(x) + (1-ct)^{-k_2}\big]^{m\theta-1}(1-ct)^{-k_2-1} \le m\theta k_2 c 2^{m\theta-1}(1-ct)^{-k_2 m\theta-1}, \tag{3.35}$$

$$\begin{aligned}\Delta\underline{u} &= a\theta\big[\widehat{a}h(x) + (1-ct)^{-k_2}\big]^{\theta-1}\Delta h(x)\\ &\quad + \widehat{a}^2\theta(\theta-1)\big[\widehat{a}h(x) + (1-ct)^{-k_2}\big]^{\theta-2}|\nabla h|^2\\ &\ge \widehat{a}\lambda\theta(1-ct)^{-k_2\theta+k_2},\end{aligned} \tag{3.36}$$

$$\underline{u}^p \ge (1-ct)^{-k_2 p\theta}. \tag{3.37}$$

By virtue of (3.33) and (3.35)–(3.37), we obtain

$$(\underline{u}^m)_t - \Delta\underline{u} - c(x,t)\underline{u}^p \le \big[m\theta k_2 c 2^{m\theta-1} - \widehat{a}\lambda\theta - c(x,t)\big](1-ct)^{-k_2\theta+k_2} \le 0,$$

provided $M_3 = m\theta k_2 c 2^{m\theta-1} - \widehat{a}\lambda\theta$.

On the other hand, for $(x,t)\in S_T$, according to (3.34), we get

$$\begin{aligned}\underline{u}(x,t) &= \big[\widehat{a} + (1-ct)^{-k_2}\big]^{\theta} \le 2^{\theta}(1-ct)^{-k_3\theta} \le \int_\Omega k(x,y,t)\big[\widehat{a}h(y) + (1-ct)^{-k_2}\big]^{\theta}\,\mathrm{d}y\\ &\le \int_\Omega k(x,y,t)\big[\widehat{a}h(y) + (1-ct)^{-k_2}\big]^{l\theta}\,\mathrm{d}y = \int_\Omega k(x,y,t)\underline{u}^l\,\mathrm{d}y.\end{aligned}$$

Consequently, $\underline{u}$ is a sub-solution of problem (1.1)–(1.3) by Proposition 2.3, which implies $u(x,t)$ blows up before $1/c$. The proof is completed. □

Remark 3.13. Theorem 3.8 presents the conditions for global existence of solution when $\int_\Omega k(x,y,t)\,\mathrm{d}y \le 1$; while Theorem 3.11 gives the blow-up phenomenon for large initial data when $\int_\Omega k(x,y,t)\,\mathrm{d}y > 1$. Meanwhile, we point out that the conclusions and technique of constructing super- and sub-solution are new.

In addition, by virtue of Kaplan's technique, we present the sufficient conditions to ensure the solution of problem (1.1)–(1.3) blows up in finite time. For the sake of convenience, we denote

$$\phi_s := \sup_{\overline{\Omega}}\phi, \quad k_1(t) := \frac{\lambda_1}{\phi_s}\inf_{\partial\Omega\times\overline{\Omega}} k(x,y,t),$$

where ϕ is the eigenfunction of problem (3.11) corresponding to the first eigenvalue λ_1, and normalized $\int_\Omega \phi(x)\,\mathrm{d}x = 1$.

Meanwhile, in the case of $0 < m \le 1$, we assume that

$$c_1(t) > 2\lambda_1, \quad \int_0^\infty c_1(t)\,\mathrm{d}t = \infty \quad \text{or} \quad k_1(t) > 2\lambda_1, \quad \int_0^\infty k_1(t)\,\mathrm{d}t = \infty. \tag{3.38}$$

In the case of $m > 1$, we need another assumption that

$$\int_0^\infty c_1(t)e^{\lambda_1(1-np)t}\,\mathrm{d}t = \infty \quad \text{or} \quad \int_0^\infty k_1(t)e^{\lambda_1(1-nl)t}\,\mathrm{d}t = \infty. \tag{3.39}$$

Moreover, in order to discuss the following results, let $v(x,t) = u^m(x,t)$ is a solution of problem (2.4)–(2.6), and we define the auxiliary functions

$$J(t) := \int_\Omega v(x,t)\phi(x)\,\mathrm{d}x = \int_\Omega u^m(x,t)\phi(x)\,\mathrm{d}x, \tag{3.40}$$
$$A(t) := e^{\lambda_1 t}\int_\Omega v(x,t)\phi(x)\,\mathrm{d}x = e^{\lambda_1 t}\int_\Omega u^m(x,t)\phi(x)\,\mathrm{d}x,$$

where $\phi(x)$ is the eigenfunction of (3.11).

Theorem 3.14. *Assume $m > 0$.*

(i) *If $0 < m \le 1$, $\max\{p,l\} > 1$, hypothesis* (3.38) *holds, and initial data u_0 satisfies $\int_\Omega u_0\phi\,\mathrm{d}x > 1$, then solution of problem* (1.1)–(1.3) *blow-up in the measure of $J(t)$ at finite time.*

(ii) *If $\max\{p,l\} > m > 1$, hypothesis* (3.39) *holds, and initial data u_0 satisfies $\int_\Omega u_0^m\phi\,\mathrm{d}x > 1$, then solution of problem* (1.1)–(1.3) *blow-up in the measure of $A(t)$ at finite time.*

Remark 3.15. Take the unit ball $\Omega = \big\{x = (x_1,x_2,x_3) \in \mathbb{R}^3 \mid |x|^2 = \sum_{i=1}^3 x_i^2 < 1\big\}$, $c(x,t) = 2\lambda_1\frac{1+t}{2-x^2}$ for $(x,t) \in Q_T$, $k(x,y,t) = 2\phi_s + \frac{x^2(1+y^2)}{t^3}$ for $(x,t) \in S_T$, $y \in \Omega$, then $c_1(t) = 2\lambda_1(1+t)$, $k_1(t) = 2\lambda_1 + \frac{\lambda_1}{\phi_s t^3}$, and the assumption (3.38) holds. Choose $c(x,t) = \frac{e^{-\lambda_1(1-np)t}}{2-x^2}$ for $(x,t) \in Q_T$, $k(x,y,t) = \frac{e^{-\lambda_1(1-nl)t}y^3}{\sin(\pi x^2/2)}$ for $(x,t) \in S_T$, $y \in \Omega$, then $c_1(t) = e^{-\lambda_1(1-np)t}$, $k_1(t) = e^{-\lambda_1(1-nl)t}$, and the assumption (3.39) is fulfilled.

Proof of Theorem 3.14. (i) Noticing that $\max\{p,l\} > 1$, we suppose for the definiteness that $\max\{p,l\} = p > 1$, since the proof of other cases is similar. Differentiating the function $J(t)$ and using (2.4)–(2.5), Green's formula and the equality $\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\,\mathrm{d}S = -\lambda_1$, we obtain

$$\begin{aligned} J'(t) &= \int_\Omega v_t\phi(x)\,\mathrm{d}x \\ &= \int_\Omega \big(\Delta v^n + c(x,t)v^{np}\big)\phi(x)\,\mathrm{d}x \\ &= \int_\Omega v^n\Delta\phi\,\mathrm{d}x - \int_{\partial\Omega}\frac{\partial\phi}{\partial n}v^n\,\mathrm{d}S + \int_\Omega c(x,t)v^{np}\phi\,\mathrm{d}x \\ &= -\int_\Omega \lambda_1 v^n\phi\,\mathrm{d}x - \int_{\partial\Omega}\frac{\partial\phi}{\partial n}\left(\int_\Omega k(x,y,t)v^{nl}\,\mathrm{d}y\right)\mathrm{d}S + \int_\Omega c(x,t)v^{np}\phi\,\mathrm{d}x \\ &\ge -\lambda_1\int_\Omega v^n\phi\,\mathrm{d}x + k_1(t)\int_\Omega v^{nl}\phi\,\mathrm{d}x + c_1(t)\int_\Omega v^{np}\phi\,\mathrm{d}x. \end{aligned} \tag{3.41}$$

According to $np > n > 1$ and $\int_\Omega u_0\phi\,\mathrm{d}x > 1$ and applying Jensen's inequality to the second and third terms on the right side of (3.41), we arrive at

$$J'(t) \ge \left(\frac{1}{2}c_1(t) - \lambda_1\right)J^n + \frac{1}{2}c_1(t)J^{np} \ge \frac{1}{2}c_1(t)J^{np}. \tag{3.42}$$

Integrating the inequality (3.42) from 0 to t, one can derive the inequality

$$J(t) \geq \left[J^{1-np}(0) - \frac{np-1}{2} \int_0^t c_1(s)\,\mathrm{d}s \right]^{-1/(np-1)}. \tag{3.43}$$

Therefore, it follows from $np > 1$ and (3.38) that $u(x,t)$ blows up in measure of $J(t)$ at finite time $T \leq T_1$, where T_1 satisfies the following identity

$$J^{1-np}(0) = \frac{np-1}{2} \int_0^{T_1} c_1(s)\,\mathrm{d}s.$$

(ii) Noticing that $\max\{p,l\} > m$, we suppose for the definiteness that $\max\{p,l\} = p > 1$, since the proof of other cases is similar. Differentiating the function $A(t)$ and using Green's formula and the equality $\int_{\partial\Omega} \frac{\partial\phi}{\partial n}\,\mathrm{d}S = -\lambda_1$, we obtain

$$\begin{aligned} A'(t) &= \lambda_1 e^{\lambda_1 t} \int_\Omega v\phi(x)\,\mathrm{d}x + e^{\lambda_1 t} \int_\Omega v_t \phi(x)\,\mathrm{d}x \\ &= \lambda_1 A(t) + e^{\lambda_1 t} \int_\Omega \big(\Delta v^n + c(x,t) v^{np}\big)\phi(x)\,\mathrm{d}x \\ &= \lambda_1 A(t) + e^{\lambda_1 t} \left[-\lambda_1 \int_\Omega v^n \phi\,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial\phi}{\partial n} \left(\int_\Omega k(x,y,t) v^{nl}\,\mathrm{d}y \right) \mathrm{d}S \right] \\ &\quad + e^{\lambda_1 t} \int_\Omega c(x,t) v^{np}\phi\,\mathrm{d}x \\ &\geq \lambda_1 A(t) + e^{\lambda_1 t} \int_\Omega \big(-\lambda_1 v^n + k_1(t) v^{nl} + c_1(t) v^{np} \big)\phi\,\mathrm{d}x. \end{aligned} \tag{3.44}$$

Applying Jensen's inequality to the second term on the right side of (3.41), we get

$$\begin{aligned} A'(t) &\geq \lambda_1 A(t) - \lambda_1 e^{\lambda_1 t} \left(\int_\Omega v\phi\,\mathrm{d}x \right)^n + e^{\lambda_1 t} c_1(t) \left(\int_\Omega v\phi\,\mathrm{d}x \right)^{np} \\ &\geq \lambda_1 A(t) - \lambda_1 e^{\lambda_1 t} \left(\int_\Omega v\phi\,\mathrm{d}x \right) + e^{\lambda_1 t} c_1(t) \left(\int_\Omega v\phi\,\mathrm{d}x \right)^{np} \\ &= e^{\lambda_1 (1-np) t} c_1(t) A^{np}(t). \end{aligned} \tag{3.45}$$

Integrating the inequality (3.45) from 0 to t, one can derive the inequality

$$A(t) \geq \left[A^{1-np}(0) - (np-1) \int_0^t c_1(s) e^{\lambda_1(1-np)s}\,\mathrm{d}s \right]^{-1/(np-1)}.$$

Therefore, it follows from $np > 1$ and (3.39) that $u(x,t)$ blows up in measure of $A(t)$ at finite time $T \leq T_2$, where T_2 satisfies the following identity

$$A^{1-np}(0) = (np-1) \int_0^{T_2} c_1(s) e^{\lambda_1(1-np)s}\,\mathrm{d}s.$$

The proof is completed. $\square$

We proceed to analyze the blow-up phenomena for any nonnegative nontrivial initial data. For the simplicity of our notation, we denote $P(v,t) := -\lambda_1 v^n + k_1(t)v^{nl} + c_1(t)v^{np}$, where $v(x,t) = u^m(x,t)$, $n = 1/m$. Let $\eta(t)$ be any nonnegative function such that

$$\int_0^\infty \eta(t)\,\mathrm{d}t = \infty. \tag{3.46}$$

Theorem 3.16. *Assume that* $\max\{p,l\} > m > 0$, $P(v,t) \geq \eta(t)v^{n\max\{p,l\}}$ *for any* $v \geq 0$ *and* $t \geq 0$*. Then any solution of problem* (1.1)–(1.3) *blows up in finite time for arbitrary nonnegative nontrivial initial data.*

Proof. We employ the same arguments in Theorem 3.14 and discuss the blow-up of solution in two different measures. We suppose for the definiteness that $\max\{p,l\} = p > m$, then we know $np > 1$, other cases can be discussed in the same way.

We first prove solution blows up in the measure of $J(t)$. By (3.41) and Jensen's inequality, we obtain

$$J'(t) \geq \int_\Omega P(v,t)\phi(x)\,\mathrm{d}x \geq \int_\Omega \eta(t)v^{np}\phi\,\mathrm{d}x \geq \eta(t)J^{np}.$$

By using the arguments similar to (3.43), we obtain the desired result.

Next, we prove solution blows up in the measure of $A(t)$. Combining (3.44) and Jensen's inequality, we have

$$\begin{aligned} A'(t) &\geq \lambda_1 A(t) + e^{\lambda_1 t}\int_\Omega P(v,t)\phi(x)\,\mathrm{d}x \\ &\geq \lambda_1 A(t) + \eta(t)e^{\lambda_1 t}\int_\Omega v^{np}\phi(x)\,\mathrm{d}x \\ &\geq \lambda_1 A(t) + e^{\lambda_1(1-np)t}\eta(t)A^{np}(t). \end{aligned} \tag{3.47}$$

By virtue of Bernoulli's technique to ordinary differential inequality (3.47), we deduce

$$A(t) \geq e^{-\lambda_1 t}\left[A^{1-np}(0) - (np-1)\int_0^t \eta(s)e^{\lambda_1(np-1)s}\,\mathrm{d}s\right]^{-1/(np-1)}.$$

In view of $np > 1$ and (3.46), it can be easily seen that $u(x,t)$ blows up in the measure of $A(t)$. The proof is completed. □

Remark 3.17. Suppose that $c_1(t) \geq \widehat{c}_1$ and $k_1(t) \geq \widehat{k}_1$, where constant $\widehat{c}_1$, $\widehat{k}_1 > 0$. We can get the similar conclusions as in Theorem 3.16 for $\max\{p,l\} > m$,

$$-\lambda_1 v^n + \widehat{k}_1 v^{nl} + \widehat{c}_1 v^{np} > 0 \quad \text{for } v > 0. \tag{3.48}$$

In fact, it is not hard to verify from (3.48) that $P(v,t) \geq \widehat{\varepsilon}v^{np}$ if $np > 1$ and $P(v,t) \geq \widehat{\varepsilon}v^{nl}$ if $nl > 1$ for some $\widehat{\varepsilon} > 0$ and for all $v \geq 0$ and $t \geq 0$.

Remark 3.18. Theorems 3.14 and 3.16 include the results in [18, Theorems 3.1 and 3.3]. In addition, we introduce new auxiliary functions $J(t)$ and $A(t)$ to overcome the difficulties caused by slow and fast diffusion situations, respectively.

Finally, we obtain more information for global existence and blow-up in a finite time of solutions for (1.1)–(1.3) when either $p = 1$ or $l = 1$. The following statement deals with the case that $p = 1$, $l > m$ or $l > 1$ and needs two new assumptions that

$$c(x,t) \geq M_4 \quad \text{and} \quad \int_0^\infty k_1(t)e^{(nl-1)(M_4-\lambda_1)t}\,\mathrm{d}t = \infty, \tag{3.49}$$

$$c(x,t) \leq M_5 < \gamma_0 \quad \text{and} \quad \int_\Omega k(x,y,t)\,\mathrm{d}y \leq A_3 e^{-(l-1)K(\gamma_0-\lambda_1)t}, \tag{3.50}$$

where constants $M_4, \gamma_0, A_3, K > 0$, $M_5 > \lambda_1$ and λ_1 is defined as (3.11).

Theorem 3.19. *Let* $p = 1$.

(i) *Assume that* $m > 0$, $l > m$, *hypothesis* (3.49) *holds, and initial data* u_0 *satisfies* $\int_\Omega u_0^m \phi\,\mathrm{d}x > 1$, *then the solution of problem* (1.1)–(1.3) *blows up in finite time.*

(ii) *Assume that* $m > 1$, $l \leq 1$ *and hypothesis* (3.50) *holds, then the solution of problem* (1.1)–(1.3) *exists globally for small initial data.*

Remark 3.20. Take the unit ball $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x|^2 = \sum_{i=1}^3 x_i^2 < 1\}$, $c(x,t) = 2\pi^2 \frac{1+t}{2-x^2}$ for $(x,t) \in Q_T$, $k(x,y,t) = \frac{1+y^2}{\pi^2(1+x^2)} e^{-(nl-1)\pi^2 t}$ for $(x,t) \in S_T$, $y \in \Omega$, then $c(x,t) \geq 2\pi^2$, $k_1(t) = \frac{e^{-(nl-1)\pi^2 t}}{\phi_s}$, and the assumption (3.49) holds. Choose $c(x,t) = \pi^2 \frac{2+x^2}{1+t}$ for $(x,t) \in Q_T$, $k(x,y,t) = \frac{e^{-K(l-1)t} y^3}{\sin(\pi x^2/2)}$ for $(x,t) \in S_T$, $y \in \Omega$, then $c(x,t) \leq 3\pi^2 < \gamma_0 = 1 + 3\pi^2$, the assumption (3.50) is satisfied.

Proof of Theorem 3.19. (i) Note that from (3.41) and (3.49), one can see

$$J'(t) \geq \int_\Omega \left[(M_4 - \lambda_1)v^n + k_1(t)v^{nl}\right]\phi\,\mathrm{d}x. \tag{3.51}$$

Let $(m-1)(M_4 - \lambda_1) < 0$. Making use of Jensen's inequality to (3.51), we arrive at

$$J'(t) \geq (M_4 - \lambda_1)J^n + k_1(t)J^{nl},$$

where $J(t)$ is defined in (3.40). Now changing the function $J(t) = w(t)e^{(M_4-\lambda_1)t}$, a simple calculation yields

$$\begin{aligned} J'(t) &= w'(t)e^{(M_4-\lambda_1)t} + w(t)(M_4-\lambda_1)e^{(M_4-\lambda_1)t} \\ &= w'(t)e^{(M_4-\lambda_1)t} + (M_4-\lambda_1)J(t) \\ &\geq (M_4-\lambda_1)J^n + k_1(t)J^{nl}. \end{aligned} \tag{3.52}$$

Then from (3.52) we derive the following inequality

$$w'(t) \geq k_1(t) J^{nl} e^{-(M_4-\lambda_1)t} = k_1(t) e^{(nl-1)(M_4-\lambda_1)t} w^{nl}.$$

Applying the same arguments of (3.45), the conclusion holds.

(ii) Now we prove the result of the global existence of solutions. Let

$$\lambda_1 - (\gamma - M_5) < \widetilde{\lambda}_1 < \lambda_1. \tag{3.53}$$

As in the proof of Theorem 3.5(ii) we consider eigenfunction $\Phi(x)$ which satisfies (3.27) and (3.28), where

$$0 < \varepsilon \leq (A_3 a^l)^{-1/(l-1)}. \tag{3.54}$$

We construct a super-solution of problem (1.1)–(1.3) in the form

$$\overline{u}(x,t) = \Phi(x) e^{K(M_5-\widetilde{\lambda}_1)t},$$

where K is a positive constant satisfies $mK\varepsilon^{m-1} > 1$. After a simple calculation, one obtain

$$\begin{aligned}
&(\overline{u}^m)_t - \Delta\overline{u} - c(x,,t)\overline{u} \\
&= Km(M_5 - \widetilde{\lambda}_1)\Phi^m e^{mK(M_5-\widetilde{\lambda}_1)t} - \Delta\Phi(x) e^{K(M_5-\widetilde{\lambda}_1)t} - c(x,t)\overline{u} \\
&\geq mK(M_5 - \widetilde{\lambda}_1)\varepsilon^{m-1}\overline{u} + \big(\widetilde{\lambda}_1 - c(x,t)\big)\overline{u} \geq (mK\varepsilon^{m-1} - 1)(M_5 - \widetilde{\lambda}_1)\overline{u} \geq 0.
\end{aligned}$$

On the other hand, for $(x,t) \in S_T$, by (3.27)–(3.28), (3.50) and (3.53)–(3.54), one can see that

$$\begin{aligned}
\overline{u}(x,t) &\geq e^{K(M_5-\widetilde{\lambda}_1)t}\varepsilon \geq A_3 e^{K(M_5-\widetilde{\lambda}_1)t}(a\varepsilon)^l \geq A_3 e^{-(l-1)K(\gamma-\lambda_1)t + lK(M_5-\widetilde{\lambda}_1)t}(a\varepsilon)^l \\
&\geq \int_\Omega k(x,y,t)\Phi^l(y) e^{lK(M_5-\widetilde{\lambda}_1)t}\,\mathrm{d}y = \int_\Omega k(x,y,t)\overline{u}^l(y,t)\,\mathrm{d}y.
\end{aligned}$$

Therefore, $\overline{u}$ is the super-solution of problem (1.1)–(1.3) for initial data $u_0(x) \leq \Phi(x)$. The proof is completed. □

The following Theorem 3.21 is the case when $l = 1$ and $p > m$. We consider the blow-up phenomena of solution to problem (1.1)–(1.3) for any initial datum.

Theorem 3.21. *Assume that $m > 0$, $p > m$, $l = 1$ and the weighted function $k(x,y,t)$ satisfies*

$$\int_\Omega k(x,y,t)\,\mathrm{d}y \geq 1, \quad \forall\,(x,t) \in S_T.$$

If one of the following conditions is satisfied:

(i) *the weighted function* $c(x,t)$ *satisfies* $\int_0^\infty c_1(t)\,\mathrm{d}t = \infty$*;*

(ii) *the weighted function* $c(x,t) \geq 1$*,*

then the solution of problem (1.1)–(1.3) *blow-up in finite time for any nonnegative nontrivial initial data.*

Proof. (i) Let $t_0 > 0$ and $u(x,t)$ be a solution of problem (1.1)–(1.3). By Lemma 2.2, there exists $\epsilon > 0$ such that

$$u(x,t_0) > \epsilon \quad \text{for any } x \in \overline{\Omega}.$$

It is easy to verify that

$$H(t) = \left[\epsilon^{-(p-m)} - \frac{p-m}{m}\int_{t_0}^{t} c_1(\tau)\mathrm{d}\tau\right]^{-1/(p-m)}$$

is the sub-solution of problem (1.1)–(1.3) in $Q_T \cap \{t > t_0\}$ for any $t_0 < T < T^*$, where T^* satisfies the equality

$$\int_{t_0}^{T^*} c_1(\tau)\mathrm{d}\tau = \frac{m}{(p-m)\epsilon^{p-m}}.$$

Since $H(t)$ blows up in finite time, we draw the conclusion by Proposition 2.3.

(ii) Let

$$\underline{u} = \frac{1}{(b_0 - c_0 t)^k}, \qquad t \in [0, b_0/c_0),$$

where $k = 1/(p-m)$, $b_0 = \underline{\sigma}^{-1/k}$, $c_0 = \frac{p-m}{m}\min\left\{1, \underline{\sigma}^p\right\}$, $\underline{\sigma} := \min_{x\in\overline{\Omega}} u_0(x)$. It is easy to check that

$$(\underline{u}^m)_t - \Delta\underline{u} - c(x,t)\underline{u}^p = \frac{kmc_0}{(b_0 - c_0 t)^{km+1}} - c(x,t)\frac{1}{(b_0 - c_0 t)^{kp}} \leq 0,$$

$$\underline{u} = \frac{1}{(b_0 - c_0 t)^k} \leq \int_\Omega k(x,y,t)\underline{u}\,\mathrm{d}y,$$

$$\underline{u}(x,0) = \frac{1}{b_0^k} = \min_{x\in\overline{\Omega}} u_0(x) \leq u_0(x).$$

Thus, $\underline{u}(x,t)$ is a sub-solution of problem (1.1)–(1.3) by Proposition 2.3, which implies the solution of problem (1.1)–(1.3) will blow up before b_0/c_0. □

Remark 3.22. It is worth noting that the divergence of the integral $\int_0^\infty c_1(t)\,\mathrm{d}t$ plays an important role in the conclusion of Theorem 3.21.

Remark 3.23. The conclusion in [18, Theorem 4.6] is included by Theorem 3.21(i). Moreover, we derive the new blow-up phenomenon in the case of $c(x,t) \geq 1$.

For the case of linear boundary flux, we show that when $\int_\Omega k(x,y,t)\,\mathrm{d}y \leq 1$, the solution still can blow-up under appropriate conditions.

Theorem 3.24. *Let $0 < m < 1$, $p = 1$, $l = 1$. If $0 < \int_\Omega k(x,y,t)\,\mathrm{d}y \le 1$ for $(x,t) \in S_T$, and the weighted function $c(x,t)$ satisfies*

$$c_1(t) \ge 2\lambda_1, \quad \int_0^\infty c_1(t)\,\mathrm{d}t = \infty, \tag{3.55}$$

where λ_1 is given in (3.11), *then there exists blow-up solution of problem* (1.1)–(1.3) *for any nonnegative nontrivial initial data.*

Proof. Let $t_0 > 0$ and $u(x,t)$ be a solution of problem (1.1)–(1.3). By Lemma 2.2, there exists $\epsilon > 0$ such that

$$u(x,t_0) > 2\left(\frac{\epsilon}{2}\right)^{1/m} \quad \text{for } x \in \overline{\Omega}. \tag{3.56}$$

Let $\alpha(t)$ be a smooth function, which satisfies the following relations:

$$\alpha(0) = \frac{1}{\phi_s}, \quad \alpha(t) > 0, \quad \alpha'(t) \le 0,$$

where ϕ_s is defined in (3.39). Set

$$k(x,y,t) = \frac{1}{\alpha(t) + |\Omega|},$$

then obviously,

$$\int_\Omega k(x,y,t)\,\mathrm{d}y < 1, \quad \forall\,(x,t) \in S_T.$$

Now, we construct a sub-solution in such form

$$\underline{u}(x,t) = M^{1/m}(t)[\alpha(t)\phi(x) + 1],$$

where $f(t)$ is the solution to the following initial value problem of ordinary differential equation

$$\begin{cases} M'(t) = \frac{1}{2}c_1(t)M^{1/m}(t), & t > t_0, \\ M(t_0) = \frac{\epsilon}{2} > 0. \end{cases}$$

Then $M(t)$ can be written in an explicit form

$$M(t) = \left[\left(\frac{2}{\epsilon}\right)^{(1-m)/m} - \frac{1-m}{2m}\int_{t_0}^t c_1(s)\,\mathrm{d}s\right]^{-m/(1-m)},$$

and from (3.55) we derive $M(t)$ blows up in finite time T_3 with

$$\left(\frac{2}{\epsilon}\right)^{(1-m)/m} - \frac{1-m}{2m}\int_{t_0}^{T_3} c_1(s)\,\mathrm{d}s = 0.$$

By a straightforward calculation, one can see that

$$\begin{aligned}
&(\underline{u})_t^m - \Delta \underline{u} - c(x,t)\underline{u}^p \\
&\le M'(t)[\alpha(t)\phi(x)+1]^m + \lambda_1\phi_s\alpha(t)M^{1/m}(t) - c_1(t)M^{1/m}(t)[\alpha(t)\phi(x)+1] \\
&\le 2^m\big[M'(t) + (\lambda_1 - c_1(t))M^{1/m}(t)\big] \\
&\le 2^m\left[M'(t) + \frac{1}{2}c_1(t)M^{1/m}(t)\right] = 0.
\end{aligned}$$

On the other hand, for $x \in S_T$, we have

$$\begin{aligned}
\underline{u}(x,t) = M^{1/m}(t) &= \int_\Omega \frac{M^{1/m}(t)\alpha(t)\phi(y)}{\alpha(t)+|\Omega|} + \frac{M^{1/m}(t)}{\alpha(t)+|\Omega|}\,\mathrm{d}y \\
&= \int_\Omega \frac{1}{\alpha(t)+|\Omega|}M^{1/m}(t)[\alpha(t)\phi(y)+1]\,\mathrm{d}y = \int_\Omega k(x,y,t)\underline{u}\,\mathrm{d}y.
\end{aligned}$$

Moreover, using (3.56), we find that $\underline{u}(x,t_0) < u(x,t_0)$. Therefore, by Proposition 2.3, the conclusion follows. This completes the proof. □

4. Blow-up rate estimate

In order to show blow-up rate estimate of the blow-up solution, we need the following assumptions on the initial data $u_0(x)$:

(H_1) $\Delta u_0(x) + c(x,0)u_0^p > 0$, $x \in \Omega$;

(H_2) there exists $\delta' > 0$ such that

$$\Delta u_0(x) + c(x,0)u_0^p(x) - \delta' u_0^p(x) \ge 0, \quad x \in \Omega.$$

Theorem 4.1. *Assume that $p > m$, $l = 1$. If the weighted functions $c(x,t)$ and $k(x,y,t)$ satisfy, respectively*

$$\begin{gathered}
c_t(x,t) \ge 0, \quad c(x,t) \le M, \quad \forall\,(x,t) \in S_T, \\
k_t(x,y,t) \ge 0, \quad \int_\Omega k(x,y,t)\,\mathrm{d}y \le 1, \quad \forall\,(x,t) \in S_T,\ y \in \partial\Omega,
\end{gathered}$$

and initial data satisfies conditions (H_1)–(H_2), *and $u(x,t)$ is the blow-up solution of problem* (1.1)–(1.3) *in finite time T, then*

$$\widehat{c}(T-t)^{-1/(p-m)} \le u(x,t) \le \widehat{C}(T-t)^{-1/(p-m)},$$

where $\widehat{c} = \big(\frac{M(p-m)}{m}\big)^{-1/(p-m)}$, $\widehat{C} = (\delta(p-m))^{-1/(p-m)}$, δ is a positive constant.

Suppose that the solution $u(x,t)$ of problem (1.1)–(1.3) blows up in finite time T, and let $U(t) = \max_{x\in\overline{\Omega}} u(x,t)$, then we have the following lemma.

Lemma 4.2. *Assume $u_0(x)$ satisfies conditions* (H_1)–(H_2)*, then there exists a positive constant $\widehat{c} = \left(\frac{M(p-m)}{m}\right)^{1/(m-p)}$ such that $U(t) \geq \widehat{c}(T-t)^{-1/(p-m)}$.*

Proof. It is obvious that $U(t)$ is Lipschitz continuous and differentiable almost everywhere. By (1.1) and $\Delta U(t) \leq 0$, it is easy to know that

$$(U^m)_t \leq MU^p,$$

Integrating the above inequality over (t, T), we obtain the left conclusion. □

We are ready to give a proof of the main Theorem 4.1.

Proof of Theorem 4.1. Let $J = u_t - \delta u^{p+1-m}$ for some $\delta > 0$. Since $(u^m)_t = mu^{m-1}u_t$, we know that

$$u_t = \frac{1}{m}u^{1-m}(u^m)_t,$$

and

$$\begin{aligned} u_{tt} &= \frac{1-m}{m}u^{-m}u_t(u^m)_t + \frac{1}{m}u^{1-m}(\Delta u + c(x,t)u^p)_t \\ &= \frac{1-m}{m}u^{-m}u_t(u^m)_t + \frac{1}{m}u^{1-m}(\Delta u_t + c_t(x,t)u^p + pc(x,t)u^{p-1})u_t. \end{aligned}$$

A straightforward computation yields

$$\begin{aligned} &J_t - \frac{1}{m}u^{1-m}\Delta J \\ &= (1-m)u^{-1}u_t^2 + \frac{c_t(x,t)}{m}u^{p+1-m} + \frac{pc(x,t)}{m}u^{p-m}u_t - \delta(p+1-m)u^{p-m}u_t \\ &\quad + \delta(p+1-m)(p-m)u^{p-m-1}|\nabla u|^2 + \frac{\delta}{m}(p+1-2m)u^{p-m}\Delta u \\ &\geq (1-m)u^{-1}u_t^2 + \frac{pc(x,t)}{m}u^{p-m}u_t - \delta(p+1-m)u^{p-m}u_t + \frac{\delta}{m}(p+1-2m)u^{p-m}\Delta u \\ &\geq (1-m)u^{-1}u_t^2 + \frac{pc(x,t)}{m}u^{p-m}u_t - \frac{1}{m}\delta(p+1-m)u^{2p+1-2m}c(x,t). \end{aligned}$$

For sufficient small $\delta > 0$, we have

$$J_t - \frac{1}{m}u^{1-m}\Delta J \geq \left[(1-m)u^{-1}u_t + \left(\frac{pc(x,t)}{m} + \frac{c(x,t)}{m}\delta(p+1-m)\right)u^{p-m}\right]J. \tag{4.1}$$

On the other hand, for $(x,t) \in S_T$, we obtain

$$\begin{aligned} &J = u_t - \delta u^{p+1-m} \\ &\quad = \int_\Omega k_t(x,y,t)u \,\mathrm{d}y + \int_\Omega k(x,y,t)u_t \,\mathrm{d}y - \delta\left(\int_\Omega k(x,y,t)u \,\mathrm{d}y\right)^{p+1-m} \\ &\quad \geq \int_\Omega k(x,y,t)uJ \,\mathrm{d}y + \delta\left[\int_\Omega k(x,y,t)u^{p+1-m} \,\mathrm{d}y - \left(\int_\Omega k(x,y,t)u \,\mathrm{d}y\right)^{p+1-m}\right]. \end{aligned}$$

Noticing that $0 < F(x,t) = \int_\Omega k(x,y,t)\,\mathrm{d}y \le 1$ for $(x,t) \in S_T$, $p+1-m > 1$, and applying Jensen's inequality to the last part in the above inequality, we can get

$$\begin{aligned}
&\int_\Omega k(x,y,t)u^{p+1-m}\,\mathrm{d}y - \left(\int_\Omega k(x,y,t)u\,\mathrm{d}y\right)^{p+1-m} \\
&\ge F(x,t)\left(\int_\Omega k(x,y,t)u\,\frac{\mathrm{d}y}{F(x,t)}\right)^{p+1-m} - \left(\int_\Omega k(x,y,t)u\,\mathrm{d}y\right)^{p+1-m} \ge 0.
\end{aligned}$$

Hence, we have

$$J \ge \int_\Omega k(x,y,t)uJ\,\mathrm{d}y \tag{4.2}$$

for all $(x,t) \in S_T$. Moreover,

Since $u_0(x)$ satisfies the conditions (H_1)–(H_2), and

$$\begin{aligned}
J(x,t) &= u_t(x,t) - \delta u^{p+1-m}(x,t) \\
&= \frac{1}{m}u^{1-m}(x,t)\big[\Delta u(x,t) + c(x,t)u^p(x,t)\big] - \delta u^{p+1-m}(x,t) \\
&= \frac{1}{m}u^{1-m}(x,t)\big[\Delta u(x,t) + (c(x,t) - \delta')u^p(x,t)\big],
\end{aligned}$$

where $\delta' = \delta/m$, we can obtain $J(x,0) \ge 0$. It follows from (4.1)–(4.2) that $J(x,t) \ge 0$ in $\overline{Q}_T$, that is,

$$u_t \ge \delta u^{p+1-m}.$$

Integrating it over (t,T), we get

$$u(x,t) \le \widehat{C}(T-t)^{-1/(p-m)}, \tag{4.3}$$

where $\widehat{C} = (\delta(p-m))^{1/(m-p)}$. Combining (4.3) with Lemma 4.2, we obtain the blow-up rate estimate. The proof is completed. □

5. Conclusion

The model (1.1)–(1.3) considered in this paper is the Dirichlet initial boundary value problem of quasilinear parabolic equation with weighted source term under the nonlinear nonlocal boundary condition. To our knowledge, [18] has studied the linear diffusion case ($m = 1$), but the research of slow and fast diffusion case with weighted source term $c(x,t)u^p$ has not been carried yet. Indeed, the multiple nonlinearities and two weighted functions that appear in our model (1.1)–(1.3) pose greater difficulties and challenges. On the other hand, the methods used in [18] (linear diffusion case) can not be directly applied in our quasilinear model (1.1)–(1.3), which makes it necessary to improvement and innovation. In addition, it is worth mentioning that the obtained main results are more detailed and complex, which involve the partial results of existing literature [18], and many results in our theorems are new, the specific instructions can refer to the aforementioned Remarks 3.3, 3.7, 3.13, 3.18 and 3.23.

In this paper, our blow-up criteria can be roughly summarized as follows: (i) the larger reaction source or boundary flow or weighted functions or initial data and smaller diffusion term benefit the occurrence of blow-up of solutions; (ii) the global existence and blow-up results depend on the behavior of the coefficients $c(x,t)$ and $k(x,y,t)$ as t tends to infinity; (iii) the size relationship between $\int_\Omega k(x,y,t)\,\mathrm{d}y$ and 1 is also an important criterion that divides the global existence and blow-up of solutions.

Finally, we point out that the methods and techniques used in this article by improving can be applied to a large class of diffusion model with weighted nonlocal source terms, such as space integral source term $c(x,t)u^p\int_\Omega u^q\,\mathrm{d}x$, memory source term $c(x,t)u^p\int_0^t u^q\,\mathrm{d}s$ or moving localized source term $c(x,t)u^p(x_0(t),t)$ etc.

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