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On the Unicity and the Ambiguity of Lusztig Parametrizations for Finite Classical Groups

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Abstract. The Lusztig correspondence is a bijective mapping between the Lusztig series indexed by the conjugacy class of a semisimple element s in the connected component $(G^*)^0$ of the dual group of G and the set of irreducible unipotent characters of the centralizer of s in G^*. In this article we discuss the unicity and ambiguity of such a bijective correspondence. In particular, we show that the Lusztig correspondence for a classical group can be made to be unique if we require it to be compatible with the parabolic induction and the finite theta correspondence.

1. Introduction

1.1.

Let $\mathbf{G}$ be a classical group defined over a finite field $\mathbf{F}_q$ of odd characteristic, and let F be the corresponding Frobenius endomorphism. Let $G = \mathbf{G}^F$ denote the finite subgroup of rational points, and let $\mathscr{E}(\mathbf{G})$ denote the set of irreducible characters (i.e., the characters of irreducible representations) of G.

Let $R^{\mathbf{G}}_{\mathbf{T}^*,s}$ denote the *Deligne–Lusztig virtual characters* (cf. [3,7]) indexed by conjugacy class of pair $(\mathbf{T}^*, s)$ where $\mathbf{T}^*$ is an F-stable maximal torus in the dual group $\mathbf{G}^*$ and s is a rational semisimple element contained in $\mathbf{T}^*$. Let $\mathscr{V}(\mathbf{G})$ be the space of (complex valued) class functions on G, and let $\mathscr{V}(\mathbf{G})^\sharp$ denote the subspace spanned by Deligne–Lusztig virtual characters. Note that $\mathscr{V}(\mathbf{G})$ is an inner product space with the inner product $\langle\,\cdot\,,\cdot\,\rangle_{\mathbf{G}}$ given by

$$\langle f_1, f_2\rangle_{\mathbf{G}} = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$$

for $f_1, f_2 \in \mathscr{V}(\mathbf{G})$, and $\overline{f_2(g)}$ denotes the complex conjugate of the value $f_2(g)$. For $f \in \mathscr{V}(\mathbf{G})$ the orthogonal projection of f onto $\mathscr{V}(\mathbf{G})^\sharp$ is denoted by $f^\sharp$ and called the *uniform projection.* A class function $f \in \mathscr{V}(\mathbf{G})$ is called *uniform* if $f^\sharp = f$.

A natural question is how much an irreducible character ρ of G can be determined by its uniform projection $\rho^\sharp$?

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If $\mathbf{G}$ is a general linear group GL_n or a unitary group U_n, then every irreducible character is uniform (i.e., $\rho = \rho^\sharp$) and the above question is trivial. However, if $\mathbf{G}$ is a symplectic group or an orthogonal group, the question is more subtle. Recall that $\rho \in \mathscr{E}(\mathbf{G})$ is called *unipotent* if $\langle \rho, R^{\mathbf{G}}_{\mathbf{T}^*,1}\rangle_{\mathbf{G}} \neq 0$ for some $\mathbf{T}^*$. If ρ is unipotent and $\mathbf{G}$ is connected, it is known that ρ is uniquely determined by its uniform projection, i.e., $\rho'^\sharp = \rho^\sharp$ if and only if $\rho' = \rho$ (cf. [5, Proposition 6.3] and [7, Theorem 4.4.23]). If ρ is unipotent and $\mathbf{G} = \mathrm{O}^\epsilon_{2n}$, it is also known that $\rho'^\sharp = \rho^\sharp$ if and only if $\rho' = \rho$ or $\rho' = \rho \cdot \mathrm{sgn}$ (cf. [18, Proposition 3.6]).

In this paper, the above question will be answered completely (cf. Corollaries 7.4, 9.4 and 8.4) for classical groups:

Theorem 1.1. *Let $\mathbf{G}$ be a symplectic group or an orthogonal group, and let $\rho, \rho' \in \mathscr{E}(\mathbf{G})$. Then $\rho'^\sharp = \rho^\sharp$ if and only of*

$$\rho' = \begin{cases} \rho & \text{if } \mathbf{G} = \mathrm{SO}_{2n+1}, \\ \rho, \rho^c & \text{if } \mathbf{G} = \mathrm{Sp}_{2n}, \\ \rho, \rho^c, \rho\cdot\mathrm{sgn}, \rho^c\cdot\mathrm{sgn} & \text{if } \mathbf{G} = \mathrm{O}^\epsilon_{2n}. \end{cases} \tag{1.1}$$

Here "sgn" denotes the sign character of an orthogonal group, and "ρ^c" denotes the character obtained from ρ by conjugating an element in the corresponding similitude group (cf. [23, §4.3, §4.10]). Note that if ρ is unipotent then $\rho^c = \rho$, and then (1.1) is reduced to the above known result.

1.2.

In fact, the above theorem is a consequence of a more precise result on the ambiguity of the Lusztig parametrization of irreducible characters of finite classical groups. From now on, we assume that $\mathbf{G}$ is a symplectic group or an orthogonal group. It is known by Lusztig that there is a partition

$$\mathscr{E}(\mathbf{G}) = \bigcup_{(s)} \mathscr{E}(\mathbf{G}, s),$$

where the union $\bigcup_{(s)}$ runs over G^*-conjugacy classes of semisimple elements in the connected component of the dual group G^* and $\mathscr{E}(\mathbf{G}, s)$ is called a (rational) *Lusztig series* which is defined by

$$\mathscr{E}(\mathbf{G}, s) = \{\rho \in \mathscr{E}(\mathbf{G}) \mid \langle \rho, R^{\mathbf{G}}_{\mathbf{T}^*,s}\rangle_{\mathbf{G}} \neq 0 \text{ for some } \mathbf{T}^* \text{ containing } s\}.$$

In particular, the subset $\mathscr{E}(\mathbf{G}, 1)$ consists of irreducible unipotent characters.

Now we first focus on the set of unipotent characters. Let $\mathscr{V}(\mathbf{G}, 1)$ denote the subspace spanned by elements in $\mathscr{E}(\mathbf{G}, 1)$, and let $\mathscr{V}(\mathbf{G}, 1)^\sharp$ denote the uniform projection of $\mathscr{V}(\mathbf{G},1)$. Following from Lusztig (cf. [13, 14]) we can define a set $\mathscr{S}_{\mathbf{G}}^\sharp$ (cf. (3.5)) and class functions $R_\Sigma^{\mathbf{G}}$ (cf. (3.4)) for $\Sigma\in\mathscr{S}_{\mathbf{G}}^\sharp$ such that the set $\{R_\Sigma^{\mathbf{G}}\mid\Sigma\in\mathscr{S}_{\mathbf{G}}^\sharp\}$ forms an orthonormal basis for $\mathscr{V}(\mathbf{G},1)^\sharp$. Lusztig constructs a set $\mathscr{S}_{\mathbf{G}}$ of similar classes of *symbols* (cf. (2.5)) and a bijective mapping $\mathscr{L}_1^{\mathbf{G}}\colon\mathscr{S}_{\mathbf{G}}\to\mathscr{E}(\mathbf{G},1)$ denoted by $\Lambda\mapsto\rho_\Lambda$ such that the value $\langle\rho_\Lambda,R_\Sigma^{\mathbf{G}}\rangle_{\mathbf{G}}$ is specified (cf. Proposition 3.5). Such a mapping $\mathscr{L}_1^{\mathbf{G}}$ is called a *Lusztig parametrization* of unipotent characters of $\mathbf{G}$. Because the uniform projection $\rho_\Lambda^\sharp$ can be obtained by the values $\langle\rho_\Lambda,R_\Sigma^{\mathbf{G}}\rangle_{\mathbf{G}}$, the problem of the uniqueness of $\mathscr{L}_1^{\mathbf{G}}$ is equivalent to the problem whether a unipotent character ρ_Λ can be uniquely determined by its uniform projection $\rho_\Lambda^\sharp$.

As described in Subsection 1.1, $\mathscr{L}_1^{\mathbf{G}}$ is known to be unique if $\mathbf{G}=\mathrm{SO}_{2n+1}$ or Sp_{2n}. However, due to the disconnectedness of O_{2n}^ϵ, the mapping $\mathscr{L}_1^{\mathrm{O}_{2n}^\epsilon}$ is not uniquely determined. By using the result of "*cells*" by Lusztig, we determine how ambiguous a Lusztig parametrization $\mathscr{L}_1^{\mathrm{O}_{2n}^\epsilon}$ could be and we show in Proposition 5.6 that $\mathscr{L}_1^{\mathrm{O}_{2n}^\epsilon}$ can be chosen to be unique if we require it to be

- *compatible with parabolic induction* and
- *compatible with theta correspondence* on cuspidal characters or $\mathbf{1}_{\mathrm{O}_2^+}$, $\mathrm{sgn}_{\mathrm{O}_2^+}$.

In particular, $\mathscr{L}_1^{\mathrm{O}_{2n}^\epsilon}$ (and $\mathscr{L}_1^{\mathrm{Sp}_{2n'}}$) can be given so that $(\rho_\Lambda,\rho'_{\Lambda'})\in\Theta_{\mathbf{G},\mathbf{G}'}$ if and only if $(\Lambda,\Lambda')\in\mathscr{B}_{\mathbf{G},\mathbf{G}'}$ for $(\mathbf{G},\mathbf{G}')=(\mathrm{O}_{2n}^\epsilon,\mathrm{Sp}_{2n'})$ where $\mathscr{B}_{\mathbf{G},\mathbf{G}'}$ is a relation between $\mathscr{S}_{\mathbf{G}}$ and $\mathscr{S}_{\mathbf{G}'}$ defined in Subsection 5.2.

1.3.

For a general Lusztig series $\mathscr{E}(\mathbf{G},s)$, Lusztig shows (cf. [15]) that there exists a bijection

$$\mathfrak{L}_s\colon\mathscr{E}(\mathbf{G},s)\to\mathscr{E}(C_{\mathbf{G}^*}(s),1)$$

satisfying

$$\langle\rho,\epsilon_{\mathbf{G}}R_{\mathbf{T}^*,s}^{\mathbf{G}}\rangle_{\mathbf{G}}=\langle\mathfrak{L}_s(\rho),\epsilon_{C_{\mathbf{G}^*}(s)}R_{\mathbf{T}^*,1}^{C_{\mathbf{G}^*}(s)}\rangle_{C_{\mathbf{G}^*}(s)},\tag{1.2}$$

where $\epsilon_{\mathbf{G}}=(-1)^{\kappa(\mathbf{G})}$, $\kappa(\mathbf{G})$ denotes the rational rank of $\mathbf{G}$, and $C_{\mathbf{G}^*}(s)$ denotes the centralizer of s in $\mathbf{G}^*$. Such a bijection $\mathfrak{L}_s$ will be called a *Lusztig correspondence* in this article (it is called a *Jordan decomposition* in [6, 7]).

Now the question is to understand whether the Lusztig correspondence $\mathfrak{L}_s$ is uniquely determined by (1.2). If the answer is negative, then we want to know what kind of conditions need to be enforced to make $\mathfrak{L}_s$ unique. Some discussion on this problem can be founded in [7, Appendix A.5]. If $\mathbf{G}$ is a connected group with connected center, it is shown in [5, Theorem 7.1] that $\mathfrak{L}_s$ can be uniquely determined by (1.2).

For $s \in G^*$, we have a decomposition $s = s^{(0)} \times s^{(1)} \times s^{(2)}$ where $s^{(1)}$ (resp. $s^{(2)}$) is the part whose eigenvalues are all equal to -1 (resp. 1), and $s^{(0)}$ is the part whose eigenvalues do not contain 1 or -1. Then we can define groups $\mathbf{G}^{(0)}(s)$, $\mathbf{G}^{(-)}(s)$ and $\mathbf{G}^{(+)}(s)$ (cf. (6.6)) so that there is a bijective mapping

$$\mathscr{E}(C_{\mathbf{G}^*}(s), 1) \to \mathscr{E}(\mathbf{G}^{(0)}(s) \times \mathbf{G}^{(-)}(s) \times \mathbf{G}^{(+)}(s), 1).$$

Combining Lusztig parametrization $\mathscr{L}_1$ of unipotent characters for $\mathbf{G}^{(0)}$, $\mathbf{G}^{(-)}(s)$, $\mathbf{G}^{(+)}(s)$, and above bijection and the inverse $\mathfrak{L}_s^{-1}$ of a Lusztig correspondence, we obtain a bijective mapping

$$\mathscr{L}_s \colon \mathscr{S}_{\mathbf{G}^{(0)}(s)} \times \mathscr{S}_{\mathbf{G}^{(-)}(s)} \times \mathscr{S}_{\mathbf{G}^{(+)}(s)} \to \mathscr{E}(\mathbf{G}, s)$$

denoted by $(x, \Lambda_1, \Lambda_2) \mapsto \rho_{x,\Lambda_1,\Lambda_2}$ which is called a *modified Lusztig correspondence*. Then we prove the following results on the unicity and ambiguity of $\mathscr{L}_s$ (or equivalently, the unicity and ambiguity of $\mathfrak{L}_s$) for classical groups:

(1) Suppose that $\mathbf{G} = \mathrm{SO}_{2n+1}$. There is a unique modified Lusztig correspondence $\mathscr{L}_s$ (cf. Theorem 7.1). Note that SO_{2n+1} is a connected group with connected center, so this case is covered by [5, Theorem 7.1].

(2) Suppose that $\mathbf{G} = \mathrm{O}_{2n}^{\epsilon}$ where $\epsilon = +$ or $-$. There exists a unique modified Lusztig correspondence $\mathscr{L}_s$ which is compatible with the parabolic induction and some other conditions (on *basic characters*) (cf. Theorem 8.8).

(3) Suppose that $\mathbf{G} = \mathrm{Sp}_{2n}$. Here we provide two choices of the modified Lusztig correspondence $\mathscr{L}_s$:

   (a) There exists a unique modified Lusztig correspondence $\mathscr{L}_s$ which is compatible with the parabolic induction and compatible with the theta correspondence for the dual pair $(\mathbf{G}, \mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{SO}_{2n'+1})$, i.e., we show that $(\rho_{x,\Lambda_1,\Lambda_2}, \rho_{x',\Lambda_1',\Lambda_2'}) \in \Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$ if and only if
   - $s^{(0)} = -s'^{(0)}$ and $x = x'$,
   - $\Lambda_2 = \Lambda_1'$, and
   - $(\Lambda_1, \Lambda_2') \in \mathscr{B}_{\mathbf{G}^{(-)}(s),\mathbf{G}^{(+)}(s')}$,

   where $\rho_{x',\Lambda_1',\Lambda_2'}$ is given by the unique modified Lusztig correspondence in (1) (cf. Theorems 9.6 and 9.8).

   (b) There exists a unique modified Lusztig correspondence $\mathscr{L}_s$ which is compatible with the parabolic induction and compatible with the theta correspondence for the dual pair $(\mathbf{G}, \mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{O}_{2n'}^{\epsilon})$, i.e., $(\rho_{x,\Lambda_1,\Lambda_2}, \rho_{x',\Lambda_1',\Lambda_2'}) \in \Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$ if and only if

- $s^{(0)} = s'^{(0)}$ and $x = x'$,
- $\Lambda_1 = \Lambda'_1$, and
- $(\Lambda_2, \Lambda'_2) \in \mathscr{B}_{\mathbf{G}^{(+)}(s),\mathbf{G}^{(+)}(s')}$,

where $\rho_{x',\Lambda'_1,\Lambda'_2}$ is given by the unique modified Lusztig correspondence in (2) (cf. Theorems 9.9 and 9.10).

It seems that two choices in (a) and (b) of the modified Lusztig correspondence $\mathscr{L}_s$ for Sp_{2n} should be the same. However, we do not know how to obtain the conclusion yet.

1.4.

The contents of this article are as follows. In Section 2, we recall the notion and some basic result of "*symbols*" and "*special symbols*" by Lusztig from [11]. In Section 3 we first recall the notion of "*almost characters*" by Lusztig from [13, 14]. Then we record some results of cells from [17]. In Section 4 we show the uniqueness of $\mathscr{L}_1^{\mathbf{G}}$ for $\mathbf{G} = \mathrm{Sp}_{2n}$ and SO_{2n+1} by using the results in the previous section. Moreover, we also discuss the ambiguity of $\mathscr{L}_1^{\mathbf{G}}$ for $\mathbf{G} = \mathrm{O}^\epsilon_{2n}$. In Section 5 we discuss the relation between the theta correspondence $\Theta^\psi_{\mathbf{G},\mathbf{G}'}$ on unipotent characters for $(\mathbf{G},\mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{O}^\epsilon_{2n'})$ and Lusztig parametrizations $\mathscr{L}_1^{\mathbf{G}}\colon \mathscr{S}_{\mathbf{G}} \to \mathscr{E}(\mathbf{G},1)$ for $\mathbf{G} = \mathrm{Sp}_{2n}$, O^ϵ_{2n}. In Section 6 we discuss the relation between the theta correspondence $\Theta^\psi_{\mathbf{G},\mathbf{G}'}$ on certain Lusztig series for $(\mathbf{G},\mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{SO}_{2n'+1})$ or $(\mathrm{Sp}_{2n}, \mathrm{O}^\epsilon_{2n'})$ and the Lusztig correspondence $\mathfrak{L}_s\colon \mathscr{E}(\mathbf{G},s) \to \mathscr{E}(C_{\mathbf{G}^*}(s),1)$. In Section 7 we show that the Lusztig correspondence $\mathfrak{L}_s\colon \mathscr{E}(\mathbf{G},s) \to \mathscr{E}(C_{\mathbf{G}^*}(s),1)$ is unique for $\mathbf{G} = \mathrm{SO}_{2n+1}$. In Section 8 we show that $\mathfrak{L}_s$ can be chosen to be unique for $\mathbf{G} = \mathrm{O}^\epsilon_{2n}$ if we require $\mathfrak{L}_s$ to be compatible with the parabolic induction and some other conditions on "basic characters". In the final section we discuss the uniqueness of the Lusztig correspondence $\mathfrak{L}_s$ for $\mathbf{G} = \mathrm{Sp}_{2n}$. In particular, we show that a unique $\mathfrak{L}_s$ can be chosen to be compatible with the theta correspondence for the dual pair $(\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1})$ or for the dual pair $(\mathrm{Sp}_{2n}, \mathrm{O}^\epsilon_{2n})$.

2. Symbols and special symbols

2.1. Irreducible characters of Weyl groups

For a finite group G, let $\mathscr{E}(G)$ denote the set of irreducible (complex) characters of G. It is known that the set of irreducible characters $\mathscr{E}(S_n)$ of the symmetric group S_n is parametrized by the set $\mathscr{P}(n)$ of partitions of n. For $\lambda \in \mathscr{P}(n)$, we write $|\lambda| = n$, and the corresponding irreducible character of S_n is denoted by φ_λ.

Let W_n denote the Coxeter group of type B_n, i.e., W_n consists of all permutations on

$\{1, 2, \ldots, n, n^*, (n-1)^*, \ldots, 1^*\}$ which commutes with the involution

$$(1, 1^*)(2, 2^*) \cdots (n, n^*)$$

where (i, j) denote the transposition of i, j. For $i = 1, \ldots, n-1$, let

$$s_i = (i, i+1)(i^*, (i+1)^*) \quad \text{and} \quad \sigma_n = (n, n^*).$$

It is known that W_n is generated by $\{s_1, \ldots, s_{n-1}, \sigma_n\}$. Each element of W_n induces a permutation of $\{1, 2, \ldots, n\}$. So we have a surjective homomorphism $W_n \to S_n$ with kernel isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Therefore, φ_λ can be regarded as an irreducible character of W_n.

An ordered pair $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]$ of two partitions is called a *bi-partitions*. Let $\mathscr{P}_2(n)$ denote the set of bipartitions of n, i.e.,

$$\mathscr{P}_2(n) = \left\{ \begin{bmatrix}\mu\\ \nu\end{bmatrix} \mid |\mu| + |\nu| = n \right\}.$$

For a bi-partition $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]$, we define its *transpose* by $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]^{\mathrm{t}} = \left[\begin{smallmatrix}\nu\\ \mu\end{smallmatrix}\right]$. A bi-partition $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]$ is called *degenerate* if $\mu = \nu$, and it is called *non-degenerate* otherwise. For $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right] \in \mathscr{P}_2(n)$ such that $\mu \in \mathscr{P}(k)$ and $\nu \in \mathscr{P}(l)$ with $k + l = n$, we define

$$\varphi_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]} = \mathrm{Ind}_{W_k \times W_l}^{W_n}(\varphi_\mu \otimes (\varepsilon_l \varphi_\nu)) \tag{2.1}$$

where $\varepsilon_l \colon W_l \to \{\pm 1\}$ is given by $s_i \mapsto 1$ for each i and $\sigma_l \mapsto -1$. It is known that $\varphi_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}$ is an irreducible character of W_n, and the mapping $\mathscr{P}_2(n) \to \mathscr{E}(W_n)$ by $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right] \mapsto \varphi_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}$ gives a parametrization of $\mathscr{E}(W_n)$ such that

- $\varphi_{\left[\begin{smallmatrix}n\\ -\end{smallmatrix}\right]} = \mathbf{1}_{W_n}$,
- $\varphi_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]} \cdot \varepsilon_n = \varphi_{\left[\begin{smallmatrix}\nu\\ \mu\end{smallmatrix}\right]}$, in particular $\varphi_{\left[\begin{smallmatrix}-\\ n\end{smallmatrix}\right]} = \varepsilon_n$

(cf. [8, Theorem 5.5.6]).

The kernel W_n^+ of ε_n is a subgroup of index two generated by $\{s_1, \ldots, s_{n-1}, \sigma_n s_{n-1} \sigma_n\}$. Let $W_n^- = W_n \backslash W_n^+$. It is well-known that if $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]$ is non-degenerate, then $\varphi_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}|_{W_n^+} = \varphi_{\left[\begin{smallmatrix}\nu\\ \mu\end{smallmatrix}\right]}|_{W_n^+}$ which is an irreducible character of W_n^+; if $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]$ is degenerate, then $\varphi_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}|_{W_n^+}$ is a sum of two irreducible characters of W_n^+.

2.2. Lusztig's symbols

In this subsection, we recall some basic notations of "symbols" from [11]. A *symbol* Λ is an ordered pair

$$\Lambda = \binom{A}{B} = \binom{a_1, a_2, \ldots, a_{m_1}}{b_1, b_2, \ldots, b_{m_2}} \tag{2.2}$$

of two finite sets A, B (possibly empty) of nonnegative integers. The sets A, B are also denoted by Λ^*, Λ_* and called the *first row*, the *second row* of Λ respectively. A symbol Λ is called *reduced* if $0 \notin A \cap B$. If $\Lambda = \binom{A}{B}$, then we define its *transpose* by $\Lambda^{\mathrm{t}} = \binom{B}{A}$. We denote $\Lambda_1 \subset \Lambda_2$ and call Λ_1 a *subsymbol* of Λ_2 if both $\Lambda_1^* \subset \Lambda_2^*$ and $(\Lambda_1)_* \subset (\Lambda_2)_*$. If $\Lambda_1 \subset \Lambda_2$, their difference is defined by $\Lambda_2 \setminus \Lambda_1 = \binom{\Lambda_2^* \setminus \Lambda_1^*}{(\Lambda_2)_* \setminus (\Lambda_1)_*}$. If both $\Lambda_1^* \cap \Lambda_2^* = \emptyset$ and $(\Lambda_1)_* \cap (\Lambda_2)_* = \emptyset$, we define $\Lambda_1 \cup \Lambda_2 = \binom{\Lambda_1^* \cup \Lambda_2^*}{(\Lambda_1)_* \cup (\Lambda_2)_*}$.

For a symbol Λ given in (2.2), its *rank* and *defect* are defined by

$$\operatorname{rk}(\Lambda) = \sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j - \left\lfloor \left(\frac{|A| + |B| - 1}{2} \right)^2 \right\rfloor \quad \text{and} \quad \operatorname{def}(\Lambda) = |A| - |B|.$$

From the definition, it is not difficult to check that

$$\operatorname{rk}(\Lambda) \geq \left\lfloor \left(\frac{\operatorname{def}(\Lambda)}{2} \right)^2 \right\rfloor. \tag{2.3}$$

A symbol Λ is called *degenerate* if $\Lambda^{\mathrm{t}} = \Lambda$. If Λ is degenerate, then $\operatorname{rk}(\Lambda)$ is even and $\operatorname{def}(\Lambda) = 0$.

We define an equivalence relation "$\sim$" generated by

$$\binom{a_1, a_2, \ldots, a_{m_1}}{b_1, b_2, \ldots, b_{m_2}} \sim \binom{a_1 + 1, a_2 + 1, \ldots, a_{m_1} + 1, 0}{b_1 + 1, b_2 + 1, \ldots, b_{m_2} + 1, 0}$$

on the set of symbols. If $\Lambda_1 \sim \Lambda_2$, two symbols Λ_1, Λ_2 are called *similar*. It is not difficult to see that two symbols in the same similarity class have the same rank and the same defect, and each similarity class contains a unique reduced symbol. Let $\mathscr{S}$ denote the set of similarity classes of symbols, and let $\mathscr{S}_{n,\delta} \subset \mathscr{S}$ denote the subset of similarity classes of symbols of rank n and defect δ.

A symbol Λ is called *cuspidal* if (2.3) is an equality. It is not difficult to see that a symbol is cuspidal if and only if it is similar to a symbol of the forms $\binom{k, k-1, \ldots, 0}{-}$ or $\binom{-}{k, k-1, \ldots, 0}$ for some nonnegative integer k. Note that $\binom{A}{-}$ means that the second row of the symbol is the empty set.

A mapping Υ from symbols to bi-partitions is defined by

$$\Upsilon\colon \binom{a_1, a_2, \ldots, a_{m_1}}{b_1, b_2, \ldots, b_{m_2}} \mapsto \begin{bmatrix} a_1 - (m_1 - 1), a_2 - (m_1 - 2), \ldots, a_{m_1 - 1} - 1, a_{m_1} \\ b_1 - (m_2 - 1), b_2 - (m_2 - 2), \ldots, b_{m_2} - 1, b_{m_2} \end{bmatrix}.$$

If $\Upsilon(\Lambda) = \left[\begin{smallmatrix}\mu \\ \nu\end{smallmatrix}\right]$, we write $\Upsilon(\Lambda)^* = \mu$ and $\Upsilon(\Lambda)_* = \nu$ to denote the first row and the second row of the bi-partition $\Upsilon(\Lambda)$. We can check that $\Upsilon(\Lambda_1) = \Upsilon(\Lambda_2)$ if $\Lambda_1 \sim \Lambda_2$, and then Υ gives a bijection

$$\mathscr{S}_{n,\delta} \to \mathscr{P}_2\big(n - \big\lfloor \big(\tfrac{\delta}{2}\big)^2 \big\rfloor\big). \tag{2.4}$$

Modified from Lusztig, we define

$$
\begin{aligned}
\mathscr{S}_{\mathrm{O}^+_{2n}} &= \{\Lambda\in\mathscr{S}\mid \mathrm{rk}(\Lambda)=n,\ \mathrm{def}(\Lambda)\equiv 0\pmod 4\},\\
\mathscr{S}_{\mathrm{Sp}_{2n}} &= \{\Lambda\in\mathscr{S}\mid \mathrm{rk}(\Lambda)=n,\ \mathrm{def}(\Lambda)\equiv 1\pmod 4\},\\
\mathscr{S}_{\mathrm{O}^-_{2n}} &= \{\Lambda\in\mathscr{S}\mid \mathrm{rk}(\Lambda)=n,\ \mathrm{def}(\Lambda)\equiv 2\pmod 4\},\\
\mathscr{S}_{\mathrm{SO}_{2n+1}} &= \{\Lambda\in\mathscr{S}\mid \mathrm{rk}(\Lambda)=n,\ \mathrm{def}(\Lambda)\equiv 3\pmod 4\}.
\end{aligned}\tag{2.5}
$$

Note that $\Lambda\in\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}$ if and only if $\Lambda^\mathrm{t}\in\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}$ where $\epsilon=+$ or $-$. Then we define

$$\mathscr{S}_{\mathrm{SO}^\epsilon_{2n}}=\{\Lambda\in\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}\mid\Lambda\neq\Lambda^\mathrm{t}\}/\{\Lambda,\Lambda^\mathrm{t}\}\cup\{\Lambda^\mathrm{I},\Lambda^\mathrm{II}\mid\Lambda\in\mathscr{S}_{\mathrm{O}^\epsilon_{2n}},\Lambda=\Lambda^\mathrm{t}\},$$

i.e., in $\mathscr{S}_{\mathrm{SO}^\epsilon_{2n}}$ a non-degenerate symbol is identified with its transpose, and a degenerated symbol Λ occurs with multiplicity 2 and the two copies are denoted by Λ^I, Λ^II respectively. Note that $\mathscr{S}_{\mathrm{O}^-_{2n}}$ does not contain any degenerate symbols and so $\mathscr{S}_{\mathrm{SO}^-_{2n}}=\mathscr{S}_{\mathrm{O}^-_{2n}}/\{\Lambda,\Lambda^\mathrm{t}\}$.

2.3. Special symbols

Let $\mathbf{G}=\mathrm{Sp}_{2n}$, SO_{2n+1}, O^ϵ_{2n} or $\mathrm{SO}^\epsilon_{2n}$ where $\epsilon=+$ or $-$. A symbol $Z=\binom{a_1,a_2,\ldots,a_{m+1}}{b_1,b_2,\ldots,b_m}$ of defect 1 is called *special* if $a_1\geq b_1\geq a_2\geq b_2\geq\cdots\geq a_m\geq b_m\geq a_{m+1}$; similarly, a symbol $Z=\binom{a_1,a_2,\ldots,a_m}{b_1,b_2,\ldots,b_m}$ of defect 0 is called *special* if $a_1\geq b_1\geq a_2\geq b_2\geq\cdots\geq a_m\geq b_m$. Define

$$\delta_0=\begin{cases}1 & \text{if }\mathbf{G}=\mathrm{Sp}_{2n}\text{ or }\mathrm{SO}_{2n+1},\\ 0 & \text{if }\mathbf{G}=\mathrm{SO}^\epsilon_{2n}\text{ or }\mathrm{O}^\epsilon_{2n}.\end{cases}$$

For a special symbol Z of defect δ_0, we define

$$
\begin{aligned}
\mathscr{S}_Z &= \{\Lambda\in\mathscr{S}\mid\Lambda^*\cup\Lambda_*=Z^*\cup Z_*,\Lambda^*\cap\Lambda_*=Z^*\cap Z_*\},\\
\mathscr{S}_{Z,\delta_0} &= \{\Lambda\in\mathscr{S}_Z\mid\mathrm{def}(\Lambda)=\delta_0\},\\
\mathscr{S}_Z^\mathbf{G} &= \mathscr{S}_Z\cap\mathscr{S}_\mathbf{G},
\end{aligned}
$$

i.e., $\mathscr{S}_Z$ is the subset of $\mathscr{S}$ consisting of the symbols of the exactly same entries of Z. Note that in the above definition of $\mathscr{S}_Z^\mathbf{G}$, the special symbol Z is not required to be in $\mathscr{S}_\mathbf{G}$. It is clear that

$$\mathscr{S}_\mathbf{G}=\bigcup_Z\mathscr{S}_Z^\mathbf{G}\tag{2.6}$$

where $\bigcup_Z$ runs over

- special symbols of rank n and defect 1 if $\mathbf{G}=\mathrm{Sp}_{2n}$ or SO_{2n+1};
- special symbols of rank n and defect 0 if $\mathbf{G}=\mathrm{SO}^+_{2n}$ or O^+_{2n};

- non-degenerate special symbols of rank n and defect 0 if $\mathbf{G} = \mathrm{SO}^-_{2n}$ or O^-_{2n}.

Example 2.1. (1) Suppose that $Z = \binom{1}{0} \in \mathscr{S}_{1,0}$. Then we have

$$\mathscr{S}_Z^{\mathrm{O}_2^+} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad \mathscr{S}_Z^{\mathrm{O}_2^-} = \left\{ \begin{pmatrix} - \\ 1,0 \end{pmatrix}, \begin{pmatrix} 1,0 \\ - \end{pmatrix} \right\}.$$

(2) Suppose that $Z = \binom{2,0}{1} \in \mathscr{S}_{2,1}$. Then we have

$$\begin{aligned}
\mathscr{S}_Z^{\mathrm{Sp}_4} &= \left\{ \begin{pmatrix} 2,0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2,1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1,0 \\ 2 \end{pmatrix}, \begin{pmatrix} - \\ 2,1,0 \end{pmatrix} \right\}, \\
\mathscr{S}_Z^{\mathrm{SO}_5} &= \left\{ \begin{pmatrix} 1 \\ 2,0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2,1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1,0 \end{pmatrix}, \begin{pmatrix} 2,1,0 \\ - \end{pmatrix} \right\}.
\end{aligned}$$

For a special symbol Z, let Z_{I} denote the subsymbol consisting of "*singles*", i.e.,

$$Z_{\mathrm{I}} = Z \setminus \begin{pmatrix} Z^* \cap Z_* \\ Z^* \cap Z_* \end{pmatrix},$$

and we define the *degree* of Z by

$$\deg(Z) = \frac{1}{2}(|Z_{\mathrm{I}}| - \operatorname{def}(Z))$$

where $|Z_{\mathrm{I}}|$ denotes the number of entries in Z_{I}, i.e., $|Z_{\mathrm{I}}| = |(Z_{\mathrm{I}})^*| + |(Z_{\mathrm{I}})_*|$. Note that $\deg(Z)$ is always a nonnegative integer. For a subsymbol $M \subset Z_{\mathrm{I}}$, we define a symbol $\Lambda_M \in \mathscr{S}_Z$ by

$$\Lambda_M = (Z \setminus M) \cup M^{\mathrm{t}}.$$

It is not difficult to see that

$$\mathscr{S}_Z^{\mathbf{G}} = \begin{cases} \{\Lambda_M \mid M \subset Z_{\mathrm{I}}, |M| \text{ even}\} & \text{if } \mathbf{G} = \mathrm{Sp}_{2n} \text{ and } \operatorname{def}(Z) = 1, \\ \{\Lambda_M \mid M \subset Z_{\mathrm{I}}, |M| \text{ odd}\} & \text{if } \mathbf{G} = \mathrm{SO}_{2n+1} \text{ and } \operatorname{def}(Z) = 1, \\ \{\Lambda_M \mid M \subset Z_{\mathrm{I}}, |M| \text{ even}\} & \text{if } \mathbf{G} = \mathrm{O}^+_{2n} \text{ and } \operatorname{def}(Z) = 0, \\ \{\Lambda_M \mid M \subset Z_{\mathrm{I}}, |M| \text{ odd}\} & \text{if } \mathbf{G} = \mathrm{O}^-_{2n} \text{ and } \operatorname{def}(Z) = 0. \end{cases}$$

Note that

$$\mathscr{S}_Z^{\mathrm{SO}^\epsilon_{2n}} = \begin{cases} \{Z^{\mathrm{I}}, Z^{\mathrm{II}}\} & \text{if } Z \text{ is degenerate (and } \epsilon = +), \\ \mathscr{S}_Z^{\mathrm{O}^\epsilon_{2n}}/\{\Lambda, \Lambda^{\mathrm{t}}\} & \text{if } Z \text{ is non-degenerate,} \end{cases}$$

where Z^{I}, Z^{II} are both equal to Z but are regarded as two elements. Then we have

$$|\mathscr{S}_Z^{\mathbf{G}}| = \begin{cases} 2^{2\deg(Z)} & \text{if } \mathbf{G} = \mathrm{Sp}_{2n} \text{ or } \mathrm{SO}_{2n+1}, \\ 2^{2\deg(Z)-1} & \text{if } \mathbf{G} = \mathrm{O}^\epsilon_{2n} \text{ and } \deg(Z) > 0, \\ 2^{2\deg(Z)-2} & \text{if } \mathbf{G} = \mathrm{SO}^\epsilon_{2n} \text{ and } \deg(Z) > 0, \\ 1 & \text{if } \mathbf{G} = \mathrm{O}^+_{2n} \text{ and } \deg(Z) = 0, \\ 2 & \text{if } \mathbf{G} = \mathrm{SO}^+_{2n} \text{ and } \deg(Z) = 0. \end{cases}$$

Note that the family $\mathscr{S}_Z^{\mathbf{G}}$ is empty if $\mathbf{G} = \mathrm{O}_{2n}^-$, SO_{2n}^- and $\deg(Z) = 0$.

Finally we define a pairing $\langle\,\cdot\,,\cdot\,\rangle\colon \mathscr{S}_Z^{\mathbf{G}} \times \mathscr{S}_{Z,\delta_0} \to \mathbf{F}_2$ by

$$\langle \Lambda_{M_1}, \Lambda_{M_2}\rangle \equiv |M_1 \cap M_2| \pmod 2. \tag{2.7}$$

Lemma 2.2. *Let Z be a special symbol of defect δ_0. Then for any $\Lambda \in \mathscr{S}_Z^{\mathbf{G}}$ and $\Sigma \in \mathscr{S}_{Z,\delta_0}$, we have*

$$\langle \Lambda, \Sigma\rangle = \langle \Lambda^{\mathrm{t}}, \Sigma\rangle \quad \textit{and} \quad \langle \Lambda, \Sigma\rangle \begin{cases} = \langle \Lambda, \Sigma^{\mathrm{t}}\rangle & \textit{if } \mathbf{G} = \mathrm{SO}_{2n}^+,\ \mathrm{O}_{2n}^+, \\ \neq \langle \Lambda, \Sigma^{\mathrm{t}}\rangle & \textit{if } \mathbf{G} = \mathrm{SO}_{2n}^-,\ \mathrm{O}_{2n}^-. \end{cases}$$

Proof. Suppose that $\Lambda = \Lambda_{M_1}$ and $\Sigma = \Lambda_{M_2}$ for some $M_1, M_2 \subset Z_{\mathrm{I}}$. It is clear that $\Lambda^{\mathrm{t}} = \Lambda_{Z_{\mathrm{I}} \setminus M_1}$. The assumption $\Sigma \in \mathscr{S}_{Z,\delta_0}$ implies that $|M_2|$ is even. Then

$$|M_1 \cap M_2| + |(Z_{\mathrm{I}} \setminus M_1) \cap M_2| = |M_2| \equiv 0 \pmod 2.$$

Hence the first equality is obtained.

Now suppose that $\mathbf{G} = \mathrm{SO}_{2n}^\epsilon$ or O_{2n}^ϵ. Note that $|M_1|$ is even if $\epsilon = +$, and $|M_1|$ is odd if $\epsilon = -$. Then the remaining assertion is obtained by the analogous argument. $\square$

3. Lusztig parametrization of unipotent characters

3.1. Unipotent characters of GL_n or U_n

In this subsection, let $\mathbf{G}$ be a general linear group GL_n or a unitary group U_n. It is well-known that the Weyl group $W_{\mathbf{G}} = S_n$, and $\mathscr{E}(S_n)$ is parametrized by $\mathscr{P}(n)$. For $\lambda \in \mathscr{P}(n)$, we define

$$R_\lambda^{\mathbf{G}} = \begin{cases} \frac{1}{|S_n|} \sum_{w \in S_n} \varphi_\lambda(w) R_{\mathbf{T}_w,1}^{\mathbf{G}} & \text{if } \mathbf{G} = \mathrm{GL}_n, \\ \frac{1}{|S_n|} \sum_{w \in S_n} \varphi_\lambda(ww_0) R_{\mathbf{T}_w,1}^{\mathbf{G}} & \text{if } \mathbf{G} = \mathrm{U}_n, \end{cases}$$

where φ_λ denotes the irreducible character of S_n corresponding to λ, w_0 is the longest element in S_n. It is known that $R_\lambda^{\mathbf{G}}$ is up to sign an irreducible unipotent character of G (cf. [6, §11.7]). Let $\mathscr{S}_{\mathrm{GL}_n} = \mathscr{S}_{\mathrm{U}_n} = \mathscr{P}(n)$, and let ρ_λ be $R_\lambda^{\mathbf{G}}$ or $-R_\lambda^{\mathbf{G}}$ so that ρ_λ is an irreducible character. Then the mapping

$$\mathscr{L}_1\colon \mathscr{S}_{\mathbf{G}} \to \mathscr{E}(\mathbf{G}, 1) \quad \text{given by } \lambda \mapsto \rho_\lambda$$

is a bijection.

It is known that the parametrization $\mathscr{L}_1$ above is compatible with the parabolic induction on unipotent characters. More precisely, let $\mathbf{G}_n = \mathrm{GL}_n$, U_{2n} or U_{2n+1}. For $\rho \in \mathscr{E}(\mathbf{G}_n, 1)$, define

$$\Omega(\rho) = \left\{ \rho' \in \mathscr{E}(\mathbf{G}_{n+1}, 1) \;\middle|\; \left\langle \rho', R_{\mathbf{G}_n \times \mathrm{GL}_1^\dagger}^{\mathbf{G}_{n+1}}(\rho \otimes \mathbf{1}) \right\rangle_{\mathbf{G}_{n+1}} \neq 0 \right\} \tag{3.1}$$

where $R^{\mathbf{G}_{n+1}}_{\mathbf{G}_n\times\mathrm{GL}_1^\dagger}$ is the standard parabolic induction, $\mathrm{GL}_1^\dagger=\mathrm{GL}_1$ defined over $\mathbf{F}_q$ if $\mathbf{G}_n=\mathrm{GL}_n$; and $\mathrm{GL}_1^\dagger$ is the restriction to $\mathbf{F}_q$ of GL_1 defined over a quadratic extension of $\mathbf{F}_q$ if $\mathbf{G}_n=\mathrm{U}_{2n}$ or U_{2n+1}. For $\lambda\in\mathscr{S}_{\mathbf{G}_n}$, we define $\Omega(\lambda)$ to be a subset of $\mathscr{S}_{\mathbf{G}_{n+1}}$ consisting of partitions of the following types:

- If $\mathbf{G}_n=\mathrm{GL}_n$ and $\lambda\in\mathscr{S}_{\mathbf{G}_n}$, then $\Omega(\lambda)$ consists of all partitions $\lambda'\in\mathscr{S}_{\mathbf{G}_{n+1}}$ whose Young diagrams are obtained by adding a box to the Young diagram of λ.
- If $\mathbf{G}_n=\mathrm{U}_{2n}$ or U_{2n+1} and $\lambda\in\mathscr{S}_{\mathbf{G}_n}$, then $\Omega(\lambda)$ consists of all partitions $\lambda'\in\mathscr{S}_{\mathbf{G}_{n+1}}$ whose Young diagrams are obtained by adding two boxes in the same row or in the same column to the Young diagram of λ.

Example 3.1. (1) Suppose that $\lambda=[3,1,1]\in\mathscr{S}_{\mathrm{GL}_5}$. Then

$$\Omega(\lambda)=\{[4,1,1],[3,2,1],[3,1,1,1]\}\subset\mathscr{S}_{\mathrm{GL}_6}.$$

(2) Suppose that $\lambda=[3,1,1]\in\mathscr{S}_{\mathrm{U}_5}$. Then

$$\Omega(\lambda)=\{[5,1,1],[3,3,1],[3,2,2],[3,1,1,1,1]\}\subset\mathscr{S}_{\mathrm{U}_7}.$$

Now the parametrization $\mathscr{L}_1\colon\mathscr{S}_{\mathbf{G}}\to\mathscr{E}(\mathbf{G},1)$ by $\lambda\mapsto\rho_\lambda$ is said to be *compatible with parabolic induction* if the following diagram

$$\begin{tikzcd} \mathscr{S}_{\mathbf{G}_n} \arrow[r, "\Omega"] \arrow[d, "\mathscr{L}_1"'] & \mathscr{S}_{\mathbf{G}_{n+1}} \arrow[d, "\mathscr{L}_1"] \\ \mathscr{E}(\mathbf{G}_n,1) \arrow[r, "\Omega"] & \mathscr{E}(\mathbf{G}_{n+1},1) \end{tikzcd} \tag{3.2}$$

commutes, i.e., $\Omega(\rho_\lambda)=\{\rho_{\lambda'}\mid\lambda'\in\Omega(\lambda)\}$ for any $\lambda\in\mathscr{S}_{\mathbf{G}_n}$.

3.2. Uniform almost characters

First suppose $\mathbf{G}$ is a connected classical group SO_{2n+1}, Sp_{2n}, or $\mathrm{SO}^\epsilon_{2n}$. For a rational maximal torus $\mathbf{T}^*$ in the dual group $\mathbf{G}^*$ and a rational element $s\in T^*$, the *Deligne–Lusztig virtual characters* $R^{\mathbf{G}}_{\mathbf{T}^*,s}$ is defined in [4] (see also [3]). For the disconnected group O^ϵ_{2n}, we define

$$R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T}^*,s}=\mathrm{Ind}^{\mathrm{O}^\epsilon_{2n}}_{\mathrm{SO}^\epsilon_{2n}}R^{\mathrm{SO}^\epsilon_{2n}}_{\mathbf{T}^*,s}. \tag{3.3}$$

For rational maximal tori in $\mathbf{G}$, the following is well known (cf. [21, §3]):

- If $\mathbf{G}=\mathrm{Sp}_{2n}$ or SO_{2n+1}, it is known that any rational maximal torus in $\mathbf{G}$ is conjugate (under G) to $\mathbf{T}_w$ for some $w\in W_n$. Moreover, $\mathbf{T}_w$ and $\mathbf{T}_{w'}$ are conjugate if and only if w, w' are conjugate under W_n.

- If $\mathbf{G}=\mathrm{SO}^\epsilon_{2n}$ or O^ϵ_{2n}, it is known that any rational maximal torus is conjugate to $\mathbf{T}_w$ for some $w\in W^\epsilon_n$ (cf. Subsection 2.1). Moreover, $\mathbf{T}_w$ and $\mathbf{T}_{w'}$ are conjugate in O^ϵ_{2n} (resp. $\mathrm{SO}^\epsilon_{2n}$) if and only if w, w' are conjugate under W_n (resp. W^+_n).

We recall some definitions from [12, §3.17 and §3.19] (see also [18, §2.3]). For $\Sigma\in\mathscr{S}_{n,\delta_0}$ (cf. Subsection 2.3), we define a uniform unipotent class function $R^{\mathbf{G}}_\Sigma\in\mathscr{V}(\mathbf{G},1)^\sharp$ by

$$
\tag{3.4}
\begin{aligned}
R^{\mathbf{G}}_\Sigma &= \begin{cases}
\frac{1}{\sqrt{2}|W^+_n|}\sum_{w\in W^+_n}\varphi_{\Upsilon(\Sigma)}(w)R^{\mathbf{G}}_{\mathbf{T}_w,1} & \text{if } \mathbf{G}=\mathrm{SO}^+_{2n} \text{ and } \Sigma \text{ degenerate},\\
\frac{1}{|W^\epsilon_n|}\sum_{w\in W^\epsilon_n}\varphi_{\Upsilon(\Sigma)}(w)R^{\mathbf{G}}_{\mathbf{T}_w,1} & \text{if } \mathbf{G}=\mathrm{SO}^\epsilon_{2n} \text{ and } \Sigma \text{ non-degenerate},\\
\frac{1}{|W_n|}\sum_{w\in W_n}\varphi_{\Upsilon(\Sigma)}(w)R^{\mathbf{G}}_{\mathbf{T}_w,1} & \text{if } \mathbf{G}=\mathrm{Sp}_{2n},\\
\frac{1}{|W_n|}\sum_{w\in W_n}\varphi_{\Upsilon(\Sigma^{\mathrm{t}})}(w)R^{\mathbf{G}}_{\mathbf{T}_w,1} & \text{if } \mathbf{G}=\mathrm{SO}_{2n+1},
\end{cases}\\
R^{\mathrm{O}^\epsilon_{2n}}_\Sigma &= \frac{1}{\sqrt{2}}\mathrm{Ind}^{\mathrm{O}^\epsilon_{2n}}_{\mathrm{SO}^\epsilon_{2n}}(R^{\mathrm{SO}^\epsilon_{2n}}_\Sigma).
\end{aligned}
$$

Note that $\varphi_{\Upsilon(\Sigma)}$ is the irreducible character of W_n given in (2.1) and Υ is the bijection $\mathscr{S}_{n,\delta_0}\to\mathscr{P}_2(n)$ given in (2.4). The class function $R^{\mathbf{G}}_\Sigma$ is called an *almost character* of G.

Lemma 3.2. *Suppose that $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$ where $\epsilon=+$ or $-$. Then $R^{\mathbf{G}}_{\Sigma^{\mathrm{t}}}=\epsilon R^{\mathbf{G}}_\Sigma$ for any $\Sigma\in\mathscr{S}_{n,0}$.*

Proof. From (3.4), we know that

$$
R^{\mathrm{O}^\epsilon_{2n}}_{\Sigma^{\mathrm{t}}}=\begin{cases}
\frac{1}{2|W^+_n|}\sum_{w\in W^+_n}\varphi_{\Upsilon(\Sigma^{\mathrm{t}})}(w)R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T}_w,1} & \text{if } \epsilon=+ \text{ and } \Sigma \text{ degenerate},\\
\frac{1}{\sqrt{2}|W^\epsilon_n|}\sum_{w\in W^\epsilon_n}\varphi_{\Upsilon(\Sigma^{\mathrm{t}})}(w)R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T}_w,1} & \text{if } \Sigma \text{ non-degenerate}.
\end{cases}
$$

Moreover, from Subsection 2.1, we have $\varphi_{\Upsilon(\Sigma^{\mathrm{t}})}(w)=\epsilon\varphi_{\Upsilon(\Sigma)}(w)$ for $w\in W^\epsilon_n$. □

Now we define $\mathscr{S}^\sharp_{\mathbf{G}}$ by

$$
\tag{3.5}
\mathscr{S}^\sharp_{\mathbf{G}}=\begin{cases}
\mathscr{S}_{n,1} & \text{if } \mathbf{G}=\mathrm{Sp}_{2n},\ \mathrm{SO}_{2n+1},\\
\mathscr{S}_{n,0}/\{\Sigma,\Sigma^{\mathrm{t}}\} & \text{if } \mathbf{G}=\mathrm{SO}^+_{2n},\ \mathrm{O}^+_{2n},\\
\{\Sigma\in\mathscr{S}_{n,0}\mid\Sigma\neq\Sigma^{\mathrm{t}}\}/\{\Sigma,\Sigma^{\mathrm{t}}\} & \text{if } \mathbf{G}=\mathrm{SO}^-_{2n},\ \mathrm{O}^-_{2n}.
\end{cases}
$$

Here $\mathscr{S}_{n,0}/\{\Sigma,\Sigma^{\mathrm{t}}\}$ means that we choose a representative for each subset $\{\Sigma,\Sigma^{\mathrm{t}}\}$ in $\mathscr{S}_{n,0}$.

Example 3.3. We know that $\mathscr{S}_{2,0}=\left\{\binom{2}{0},\binom{0}{2},\binom{2,1}{1,0},\binom{1,0}{2,1},\binom{1}{1}\right\}$. The following are all four possible choices of $\mathscr{S}^\sharp_{\mathrm{SO}^-_4}$ (or $\mathscr{S}^\sharp_{\mathrm{O}^-_4}$):

$$
\left\{\binom{2}{0},\binom{2,1}{1,0}\right\},\quad\left\{\binom{2}{0},\binom{1,0}{2,1}\right\},\quad\left\{\binom{0}{2},\binom{2,1}{1,0}\right\},\quad\left\{\binom{0}{2},\binom{1,0}{2,1}\right\}.
$$

Moreover, for each above choice of $\mathscr{S}^\sharp_{\mathrm{SO}^-_4}$, we see that $\mathscr{S}_{2,0}\setminus\mathscr{S}^\sharp_{\mathrm{SO}^-_4}$ is a choice of $\mathscr{S}^\sharp_{\mathrm{SO}^+_4}$ (or $\mathscr{S}^\sharp_{\mathrm{O}^+_4}$).

Similar to (2.6), we have the decomposition

$$\mathscr{S}^\sharp_{\mathbf{G}}=\bigcup_Z \mathscr{S}_Z^{\mathbf{G}\sharp}$$

where the union $\bigcup_Z$ is as in (2.6) and

$$\mathscr{S}_Z^{\mathbf{G}\sharp}=\begin{cases}\mathscr{S}_Z\cap\mathscr{S}_{n,1} & \text{if } \mathbf{G}=\mathrm{Sp}_{2n},\ \mathrm{SO}_{2n+1},\\ (\mathscr{S}_Z\cap\mathscr{S}_{n,0})/\{\Sigma,\Sigma^{\mathrm{t}}\} & \text{if } \mathbf{G}=\mathrm{SO}^\epsilon_{2n},\ \mathrm{O}^\epsilon_{2n}.\end{cases}$$

Lemma 3.4. *Let $\mathbf{G}=\mathrm{Sp}_{2n}$, SO_{2n+1}, $\mathrm{SO}^\epsilon_{2n}$ or O^ϵ_{2n}. Then the set $\{R^{\mathbf{G}}_\Sigma\mid\Sigma\in\mathscr{S}^\sharp_{\mathbf{G}}\}$ forms an orthonormal basis for $\mathscr{V}(\mathbf{G},1)^\sharp$.*

Proof. If $\mathbf{G}$ is connected, this is [15, Corollary 4.25]. Now we assume that $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$. From (3.4), we know that

$$R^{\mathrm{O}^\epsilon_{2n}}_\Sigma(g)=\begin{cases}\sqrt{2}R^{\mathrm{SO}^\epsilon_{2n}}_\Sigma(g) & \text{if } g\in\mathrm{SO}^\epsilon_{2n}(q),\\ 0 & \text{if } g\in\mathrm{O}^\epsilon_{2n}(q)\setminus\mathrm{SO}^\epsilon_{2n}(q)\end{cases}$$

for any $\Sigma\in\mathscr{S}^\sharp_{\mathrm{O}^\epsilon_{2n}}$. Because $|\mathrm{O}^\epsilon_{2n}(q)|=2|\mathrm{SO}^\epsilon_{2n}(q)|$, we have

$$\langle R^{\mathrm{O}^\epsilon_{2n}}_\Sigma,R^{\mathrm{O}^\epsilon_{2n}}_{\Sigma'}\rangle_{\mathrm{O}^\epsilon_{2n}}=\langle R^{\mathrm{SO}^\epsilon_{2n}}_\Sigma,R^{\mathrm{SO}^\epsilon_{2n}}_{\Sigma'}\rangle_{\mathrm{SO}^\epsilon_{2n}},$$

i.e., the set $\{R^{\mathrm{O}^\epsilon_{2n}}_\Sigma\mid\Sigma\in\mathscr{S}^\sharp_{\mathrm{O}^\epsilon_{2n}}\}$ is orthonormal. On the other hand, for $w\in W^\epsilon_n$, it is not difficult to see that

$$\sum_{\substack{\Sigma\in\mathscr{S}^\sharp_{\mathrm{O}^\epsilon_{2n}}\\ \Sigma\text{ non-degenerate}}}\sqrt{2}\,\overline{\varphi_{\Upsilon(\Sigma)}(w)}R^{\mathrm{O}^\epsilon_{2n}}_\Sigma+\sum_{\substack{\Sigma\in\mathscr{S}^\sharp_{\mathrm{O}^\epsilon_{2n}}\\ \Sigma\text{ degenerate}}}\overline{\varphi_{\Upsilon(\Sigma)}(w)}R^{\mathrm{O}^\epsilon_{2n}}_\Sigma=R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T}_w,1}.$$

Hence the lemma is proved. □

3.3. Lusztig's parametrization of unipotent characters

Proposition 3.5 (Lusztig)**.** *Let $\mathbf{G}=\mathrm{Sp}_{2n}$, SO_{2n+1}, $\mathrm{SO}^\epsilon_{2n}$, or O^ϵ_{2n} where $\epsilon=+$ or $-$. There exists a bijection $\mathscr{L}_1=\mathscr{L}_1^{\mathbf{G}}\colon\mathscr{S}_{\mathbf{G}}\to\mathscr{E}(\mathbf{G},1)$ denoted by $\Lambda\mapsto\rho_\Lambda$ satisfying*

$$\langle\rho_\Lambda,R^{\mathbf{G}}_\Sigma\rangle_{\mathbf{G}}=\begin{cases}\frac{1}{c_Z}(-1)^{\langle\Lambda,\Sigma\rangle} & \text{if } \Lambda\in\mathscr{S}_Z^{\mathbf{G}},\ \Sigma\in\mathscr{S}_Z^{\mathbf{G}\sharp} \text{ for some special } Z,\\ 0 & \text{otherwise,}\end{cases}\tag{3.6}$$

where

$$c_Z=\begin{cases}2^{\deg(Z)} & \text{if } \mathbf{G}=\mathrm{Sp}_{2n},\ \mathrm{SO}_{2n+1},\\ 2^{\deg(Z)} & \text{if } \mathbf{G}=\mathrm{O}^+_{2n} \text{ and } Z \text{ degenerate},\\ 2^{\deg(Z)+1/2} & \text{if } \mathbf{G}=\mathrm{SO}^+_{2n} \text{ and } Z \text{ degenerate},\\ 2^{\deg(Z)-1/2} & \text{if } \mathbf{G}=\mathrm{SO}^\epsilon_{2n},\ \mathrm{O}^\epsilon_{2n} \text{ and } Z \text{ non-degenerate}.\end{cases}$$

Proof. If $\mathbf{G}=\mathrm{Sp}_{2n}$, SO_{2n+1} or $\mathrm{SO}^\epsilon_{2n}$, the result is from [13, Theorem 5.8] and [14, Theorem 3.15]. If $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$, a proof can be found in [19, Proposition 3.6]. Note that the definition of $R^{\mathbf{G}}_\Sigma$ here is slightly different from that in [19, §3.4]. □

A bijective mapping $\mathscr{S}_{\mathbf{G}}\to\mathscr{E}(\mathbf{G},1)$ satisfying (3.6) is called a *Lusztig parametrization of unipotent characters* for $\mathbf{G}$.

Remark 3.6. For $\mathbf{G}=\mathrm{Sp}_{2n}$, our definition $\mathscr{S}_{\mathbf{G}}$ in (2.5) is slightly different from the original definition by Lusztig in [11, p. 134]. Lusztig considers the set Φ_n of similarity classes of unordered pairs (A,B) of rank n and odd defects. It is easy to see that the mapping $\mathscr{S}_{\mathrm{Sp}_{2n}}\to\Phi_n$ by $\binom{A}{B}\mapsto(A,B)$ is a bijection (cf. [19, (3.5)]). Our definition here is more convenient for the description of the theta correspondence on unipotent characters.

Lemma 3.7. *Suppose that* $\mathbf{G}=\mathrm{Sp}_{2n}$, SO_{2n+1}, $\mathrm{SO}^\epsilon_{2n}$, *or* O^ϵ_{2n} *where* $\epsilon=+$ *or* $-$. *Let* $\mathscr{L}_1,\mathscr{L}_1'\colon\mathscr{S}_{\mathbf{G}}\to\mathscr{E}(\mathbf{G},1)$ *be two Lusztig parametrizations of unipotent characters for* $\mathbf{G}$. *Then* $\mathscr{L}_1(\Lambda)^\sharp=\mathscr{L}_1'(\Lambda)^\sharp$.

Proof. The lemma follows directly from the definition of the Lusztig parametrizations. □

Suppose that $\mathbf{G}=\prod_{i=1}^k\mathbf{G}_i$ where each $\mathbf{G}_i$ is a classical group. It is clear that

$$\mathscr{E}(\mathbf{G},1)=\{\rho_1\otimes\cdots\otimes\rho_k\mid\rho_i\in\mathscr{E}(\mathbf{G}_i,1)\text{ for }i=1,\ldots,k\},$$
$$\mathscr{V}(\mathbf{G},1)=\bigotimes_{i=1}^k\mathscr{V}(\mathbf{G}_i,1)\quad\text{and}\quad\mathscr{V}(\mathbf{G},1)^\sharp=\bigotimes_{i=1}^k\mathscr{V}(\mathbf{G}_i,1)^\sharp.$$

Then we define

$$\mathscr{S}_{\mathbf{G}}=\prod_{i=1}^k\mathscr{S}_{\mathbf{G}_i},\quad\mathscr{S}^\sharp_{\mathbf{G}}=\prod_{i=1}^k\mathscr{S}^\sharp_{\mathbf{G}_i},$$

and then any Lusztig parametrization $\mathscr{L}_1\colon\mathscr{S}_{\mathbf{G}}\to\mathscr{E}(\mathbf{G},1)$ is of the form

$$\begin{array}{rccc}\mathscr{L}_1\colon & \prod_{i=1}^k\mathscr{S}_{\mathbf{G}_i} & \to & \mathscr{E}\left(\prod_{i=1}^k\mathbf{G}_i,1\right)\\ & (\Lambda_1,\ldots,\Lambda_k) & \mapsto & \rho_{\Lambda_1}\otimes\cdots\otimes\rho_{\Lambda_k}\end{array}$$

where $\Lambda_i\mapsto\rho_{\Lambda_i}$ is given by a Lusztig parametrization $\mathscr{S}_{\mathbf{G}_i}\to\mathscr{E}(\mathbf{G}_i,1)$.

The parametrization $\mathscr{L}_1$ given by Lusztig in Proposition 3.5 is compatible with parabolic induction as follows. Let $\mathbf{G}_n=\mathrm{Sp}_{2n}$, SO_{2n+1}, $\mathrm{SO}^\epsilon_{2n}$ or O^ϵ_{2n}. For $\rho\in\mathscr{E}(\mathbf{G}_n,1)$, let $\Omega(\rho)\subset\mathscr{E}(\mathbf{G}_{n+1},1)$ be defined similarly as in (3.1). For $\Lambda\in\mathscr{S}_{\mathbf{G}_n}$, then $\Omega(\Lambda)$ consists of all symbols $\Lambda'\in\mathscr{S}_{\mathbf{G}_{n+1}}$ such that

- $\mathrm{def}(\Lambda')=\mathrm{def}(\Lambda)$, and
- $\Upsilon(\Lambda')$ is obtained from $\Upsilon(\Lambda)$ by adding a box to the Young diagram of $\Upsilon(\Lambda)^*$ or the Young diagram of $\Upsilon(\Lambda)_*$.

Example 3.8. Suppose that $\mathbf{G}_n = \mathrm{Sp}_4$ and $\Lambda = \binom{2,0}{1}$, then $\Upsilon(\Lambda) = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, and so $\Upsilon(\Lambda')$ is equal to $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1,1\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, or $\left[\begin{smallmatrix}1\\1,1\end{smallmatrix}\right]$. Therefore

$$\Omega(\Lambda) = \left\{ \begin{pmatrix}3,0\\1\end{pmatrix}, \begin{pmatrix}2,1\\1\end{pmatrix}, \begin{pmatrix}2,0\\2\end{pmatrix}, \begin{pmatrix}3,1,0\\2,1\end{pmatrix} \right\} \subset \mathscr{S}_{\mathrm{Sp}_6}.$$

For $\Sigma \in \mathscr{S}^\sharp_{\mathbf{G}_n}$, it is known that (cf. [8, §6.1.9])

$$\mathrm{Ind}^{W_{n+1}}_{W_n\times S_1}(\varphi_{\Upsilon(\Sigma)} \otimes \mathbf{1}) = \sum_{\Sigma'\in\Omega(\Sigma)} \varphi_{\Upsilon(\Sigma')}.$$

By direct computation (cf. [13, (4.6.3)]), we have

$$R^{\mathbf{G}_{n+1}}_{\mathbf{G}_n\times\mathrm{GL}_1}(R^{\mathbf{G}_n}_\Sigma) = \sum_{\Sigma'\in\Omega(\Sigma)} R^{\mathbf{G}_{n+1}}_{\Sigma'}.$$

We say that the parametrization $\mathscr{L}_1\colon \mathscr{S}_\mathbf{G} \to \mathscr{E}(\mathbf{G}, 1)$ is *compatible with parabolic induction* if the diagram analogous to (3.2) commutes, i.e.,

$$\Omega(\rho_\Lambda) = \{\rho_{\Lambda'} \mid \Lambda' \in \Omega(\Lambda)\}.$$

Note that $\mathrm{def}(\Lambda') = \mathrm{def}(\Lambda)$ for any $\Lambda' \in \Omega(\Lambda)$. This means that under the parametrization by Lusztig the defects of symbols are preserved by parabolic induction on unipotent characters. Therefore, if $\mathrm{def}(\Lambda') \neq 0$ and $\Lambda' \in \Omega(\Lambda)$, then $\Lambda'^{\mathrm{t}} \notin \Omega(\Lambda)$.

Lemma 3.9. *Let* $\Lambda \in \mathscr{S}_{\mathrm{O}^+_{2n}}$ *such that* $\mathrm{def}(\Lambda) = 0$ *and* $\Lambda \neq \Lambda^{\mathrm{t}}$. *Suppose that* $n \geq 2$. *Then there exists* $\Lambda_1 \in \mathscr{S}_{\mathrm{O}^+_{2(n-1)}}$ *such that* $\Lambda \in \Omega(\Lambda_1)$ *and* $\Lambda^{\mathrm{t}} \notin \Omega(\Lambda_1)$.

Proof. Write $\Lambda = \binom{a_1,\dots,a_m}{b_1,\dots,b_m}$ where a_m, b_m are not both zero. Let i be the largest index such that $a_i \neq b_i$. Such an index i exists because we assume that $\Lambda \neq \Lambda^{\mathrm{t}}$. Now we consider the following cases:

- Suppose that $i = m = 1$. We know that $\Lambda \neq \binom{1}{0}, \binom{0}{1}$ because $\binom{1}{0}, \binom{0}{1} \in \mathscr{S}_{\mathrm{O}^+_2}$ which contradicts to the assumption $n \geq 2$.
  - If either $a_1 \geq 2$ and $b_1 = 0$; or $b_1 > a_1 \geq 1$, let $\Lambda_1 = \binom{a_1-1}{b_1}$.
  - If either $a_1 = 0$ and $b_1 \geq 2$; or $a_1 > b_1 \geq 1$, let $\Lambda_1 = \binom{a_1}{b_1-1}$.
- Suppose that $i = m > 1$.
  - If $a_m > b_m$ and $b_{m-1} \geq a_{m-1}$, then we have $b_{m-1} > b_m + 1$ and let $\Lambda_1 = \binom{a_1,\dots,a_m}{b_1,\dots,b_{m-2},b_{m-1}-1,b_m}$.
  - If $a_m > b_m$ and $a_{m-1} > b_{m-1}$, let $\Lambda_1 = \binom{a_1,\dots,a_{m-1},a_m-1}{b_1,\dots,b_m}$.
  - If $b_m > a_m$ and $a_{m-1} \geq b_{m-1}$, let $\Lambda_1 = \binom{a_1,\dots,a_{m-2},a_{m-1}-1,a_m}{b_1,\dots,b_m}$.

  - If $b_m > a_m$ and $b_{m-1} > a_{m-1}$, let $\Lambda_1 = \binom{a_1,\ldots,a_m}{b_1,\ldots,b_{m-1},b_m-1}$.

- Suppose that $i < m$.
  - If $a_i > b_i$, then $a_i > b_i > b_{i+1} = a_{i+1}$ and let $\Lambda_1 = \binom{a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_m}{b_1,\ldots,b_m}$.
  - If $b_i > a_i$, then $b_i > a_i > a_{i+1} = b_{i+1}$ and let $\Lambda_1 = \binom{a_1,\ldots,a_m}{b_1,\ldots,b_{i-1},b_i-1,b_{i+1},\ldots,b_m}$.

For all cases, it is not difficult to check that $\Lambda_1 \in \mathscr{S}_{\mathrm{O}^+_{2(n-1)}}$, $\Lambda \in \Omega(\Lambda_1)$ and $\Lambda^{\mathrm{t}} \notin \Omega(\Lambda_1)$. $\square$

Remark 3.10. It is obvious that the statement in above lemma is not true without the assumption $n \geq 2$. Note that $\Lambda = \binom{1}{0} \in \mathscr{S}_{\mathrm{O}^+_2}$, $\Lambda \neq \Lambda^{\mathrm{t}}$. However, $\mathscr{S}_{\mathrm{O}^+_0} = \{\binom{-}{-}\}$ and $\Omega(\binom{-}{-}) = \{\binom{1}{0}, \binom{0}{1}\}$.

3.4. Cells in a family of unipotent characters

In this subsection, we recall some result on *cells* by Lusztig (cf. [13,14]). Some details and examples can be found in [17, §4]. Let Z be a special symbol of rank n. An *arrangement* Φ of Z is defined as follows:

- if $\operatorname{def}(Z) = 1$, Φ is a partition of Z_{I} into $\deg(Z)$ pairs and one isolated element such that each pair contains one element in $(Z_{\mathrm{I}})^*$ and one element in $(Z_{\mathrm{I}})_*$;
- if $\operatorname{def}(Z) = 0$, Φ is a partition of Z_{I} into $\deg(Z)$ pairs such that each pair contains one element in $(Z_{\mathrm{I}})^*$ and one element in $(Z_{\mathrm{I}})_*$.

A subset of pairs Ψ in an arrangement Φ is denoted by $\Psi \leq \Phi$. For such an arrangement Φ and a subset of pairs Ψ, we define a subset $C^{\mathbf{G}}_{\Phi,\Psi} = C_{\Phi,\Psi}$ of $\mathscr{S}_{\mathbf{G}}$ as follows:

- if $\operatorname{def}(Z) = 1$, we define
$$C_{\Phi,\Psi} = \{\Lambda_M \mid M \subset Z_{\mathrm{I}},\ |M| \text{ even},\ |M\cap\Psi'| \equiv |(\Phi\setminus\Psi)\cap\Psi'^*| \pmod 2 \text{ for all } \Psi' \leq \Phi\};$$
- if $\operatorname{def}(Z) = 0$, we define
$$C_{\Phi,\Psi} = \{\Lambda_M \mid M \subset Z_{\mathrm{I}},\ |M \cap \Psi'| \equiv |(\Phi \setminus \Psi) \cap \Psi'^*| \pmod 2 \text{ for all } \Psi' \leq \Phi\}.$$

It is not difficult to see that
$$C_{\Phi,\Psi} \subset \begin{cases} \mathscr{S}_Z^{\mathrm{Sp}_{2n}} & \text{if } \operatorname{def}(Z) = 1, \\ \mathscr{S}_Z^{\mathrm{O}^+_{2n}} & \text{if } \operatorname{def}(Z) = 0 \text{ and } \#(\Phi \setminus \Psi) \text{ even}, \\ \mathscr{S}_Z^{\mathrm{O}^-_{2n}} & \text{if } \operatorname{def}(Z) = 0 \text{ and } \#(\Phi \setminus \Psi) \text{ odd}. \end{cases}$$

Here $\#(\Phi \setminus \Psi)$ denotes the number of pairs in $\Phi \setminus \Psi$. For $\mathbf{G} = \mathrm{O}^{\epsilon}_{2n}$, a special symbol Z of rank n and defect 0, and an arrangement Φ of Z, a subset of pairs Ψ is called *admissible* if $\#(\Phi \setminus \Psi)$ is even when $\epsilon = +$; and $\#(\Phi \setminus \Psi)$ is odd when $\epsilon = -$.

The following lemmas are from [17, Lemma 4.17, Proposition 4.18, Lemma 4.34 and Proposition 4.35].

Lemma 3.11. *Let $\mathbf{G} = \mathrm{Sp}_{2n}$, and let $\Lambda \mapsto \rho_\Lambda$ be a Lusztig parametrization of unipotent characters. Let Z be a special symbol of rank n and defect 1, Φ an arrangement of Z, Ψ a subset of pairs of Φ.*

(i) *The class function $\sum_{\Lambda \in C_{\Phi,\Psi}} \rho_\Lambda$ is uniform.*

(ii) *For any two distinct symbols $\Lambda_1, \Lambda_2 \in \mathscr{S}_Z^{\mathbf{G}}$, there exists an arrangement Φ of Z with two subsets of pairs Ψ_1, Ψ_2 such that $\Lambda_i \in C_{\Phi,\Psi_i}$ for $i = 1, 2$ and $C_{\Phi,\Psi_1} \cap C_{\Phi,\Psi_2} = \emptyset$.*

Lemma 3.12. *Let $\mathbf{G} = \mathrm{O}^{\epsilon}_{2n}$ where $\epsilon = +$ or $-$, and let $\Lambda \mapsto \rho_\Lambda$ be a Lusztig parametrization of unipotent characters. Let Z be a special symbol of rank n and defect 0, Φ an arrangement of Z, Ψ an admissible subset of pairs of Φ.*

(i) *The class function $\sum_{\Lambda \in C_{\Phi,\Psi}} \rho_\Lambda$ is uniform.*

(ii) *$\Lambda \in C_{\Phi,\Psi}$ if and only if $\Lambda^{\mathrm{t}} \in C_{\Phi,\Psi}$.*

(iii) *For any two symbols $\Lambda_1, \Lambda_2 \in \mathscr{S}_Z^{\mathbf{G}}$ such that $\Lambda_1 \neq \Lambda_2, \Lambda_2^{\mathrm{t}}$, there exists an arrangement Φ of Z with subsets of pairs Ψ_1, Ψ_2 such that $\Lambda_i \in C_{\Phi,\Psi_i}$ for $i = 1, 2$ and $C_{\Phi,\Psi_1} \cap C_{\Phi,\Psi_2} = \emptyset$.*

4. Uniqueness of the Lusztig parametrizations

4.1. Unipotent characters of Sp_{2n}

In this subsection, let $\mathbf{G} = \mathrm{Sp}_{2n}$. The following lemma is [18, Proposition 3.3].

Lemma 4.1. *Suppose that $\mathbf{G} = \mathrm{Sp}_{2n}$, and let $\rho_1, \rho_2 \in \mathscr{E}(\mathbf{G}, 1)$. If $\rho_1^\sharp = \rho_2^\sharp$, then $\rho_1 = \rho_2$.*

Proposition 4.2. *Let $\mathbf{G}$ be Sp_{2n}. Then there is a unique bijection $\mathscr{L}_1 \colon \mathscr{S}_{\mathbf{G}} \to \mathscr{E}(\mathbf{G}, 1)$ satisfying (3.6).*

Proof. Suppose that we have two parametrizations $\Lambda \mapsto \rho_\Lambda$ and $\Lambda \mapsto \rho'_\Lambda$ from $\mathscr{S}_{\mathbf{G}}$ to $\mathscr{E}(\mathbf{G}, 1)$ satisfying (3.6). From Lemma 3.4, we see that condition (3.6) implies that $(\rho_\Lambda)^\sharp = (\rho'_\Lambda)^\sharp$. Then by Lemma 4.1, we conclude that $\rho_\Lambda = \rho'_\Lambda$, i.e., two parametrizations coincide. $\square$

Corollary 4.3. *Let* $\mathbf{G}_n = \mathrm{Sp}_{2n}$*. Then the bijection* $\mathscr{L}_1\colon \mathscr{S}_{\mathbf{G}_n} \to \mathscr{E}(\mathbf{G}_n, 1)$ *given in Proposition* 4.2 *is compatible with the parabolic induction, i.e., the diagram analogous to* (3.2) *commutes.*

Proof. The original construction of the bijection $\mathscr{S}_{\mathbf{G}_n} \to \mathscr{E}(\mathbf{G}_n, 1)$ by Lusztig is compatible with the parabolic induction (cf. [11]). By Proposition 4.2, Lusztig's original construction is the only bijection satisfying (3.6) and hence the corollary is obtained. □

For a nonnegative integer k, we define the symbol

$$\Lambda_k^{\mathrm{Sp}} = \begin{cases} \binom{2k,2k-1,\dots,0}{-} & \text{if } k \text{ is even,} \\ \binom{-}{2k,2k-1,\dots,0} & \text{if } k \text{ is odd.} \end{cases} \tag{4.1}$$

The following are easy to check:

- $\mathrm{rk}(\Lambda_k^{\mathrm{Sp}}) = k(k+1)$ and $\mathrm{def}(\Lambda_k^{\mathrm{Sp}}) \equiv 1 \pmod 4$, i.e., $\Lambda_k^{\mathrm{Sp}} \in \mathscr{S}_{\mathrm{Sp}_{2k(k+1)}}$,
- if $\Lambda \in \mathscr{S}_{\mathrm{Sp}_{2n}}$ with $n < k(k+1)$, then $|\,\mathrm{def}(\Lambda)| < |\,\mathrm{def}(\Lambda_k^{\mathrm{Sp}})|$,
- if $\Lambda \in \mathscr{S}_{\mathrm{Sp}_{2k(k+1)}}$ and $\Lambda \neq \Lambda_k^{\mathrm{Sp}}$, then $|\,\mathrm{def}(\Lambda)| < |\,\mathrm{def}(\Lambda_k^{\mathrm{Sp}})|$.

Because the defects are preserved by the parabolic induction, we have the following corollary.

Corollary 4.4. *Let* $\mathscr{L}_1\colon \mathscr{S}_{\mathrm{Sp}_{2n}} \to \mathscr{E}(\mathrm{Sp}_{2n}, 1)$*,* $\Lambda \mapsto \rho_\Lambda$*, be the parametrization in Proposition* 4.2*. Then the unique cuspidal unipotent character* ζ_k^{Sp} *of* $\mathrm{Sp}_{2k(k+1)}(q)$ *is parametrized by the symbol* Λ_k^{Sp}*, i.e.,* $\zeta_k^{\mathrm{Sp}} = \rho_{\Lambda_k^{\mathrm{Sp}}}$*.*

Lemma 4.5. *Let* $\mathscr{L}_1\colon \mathscr{S}_{\mathrm{Sp}_{2n}} \to \mathscr{E}(\mathrm{Sp}_{2n}, 1)$*,* $\Lambda \mapsto \rho_\Lambda$*, be the parametrization in Proposition* 4.2*. Then*

(i) $\mathbf{1}_{\mathrm{Sp}_{2n}} = \rho_{\binom{n}{-}}$,

(ii) $\mathrm{St}_{\mathrm{Sp}_{2n}} = \rho_\Lambda$ *where* $\Lambda = \binom{n,n-1,\dots,1,0}{n,n-1,\dots,1}$.

Proof. From [3, Corollary 7.6.5], we know that $R^{\mathrm{Sp}_{2n}}_{\binom{n}{-}} = \mathbf{1}_{\mathrm{Sp}_{2n}}$. Because now $\binom{n}{-}$ is a special symbol of degree 0, we have $\mathscr{S}^{\mathrm{Sp}_{2n}}_{\binom{n}{-}} = \{\binom{n}{-}\}$, $\rho_{\binom{n}{-}} = R^{\mathrm{Sp}_{2n}}_{\binom{n}{-}}$ by (3.6), and so (i) is proved.

Write $\Lambda = \binom{n,n-1,\dots,1,0}{n,n-1,\dots,1}$. From [3, Corollary 7.6.6], we see that $R^{\mathrm{Sp}_{2n}}_\Lambda = \mathrm{St}_{\mathrm{Sp}_{2n}}$. Again, now Λ is a special symbol of degree 0, we have $\rho_\Lambda = R^{\mathrm{Sp}_{2n}}_\Lambda$, and (ii) is proved. □

Example 4.6. Let $\mathbf{G}=\mathrm{Sp}_4$. We know that

$$\mathscr{S}_{\mathrm{Sp}_4}=\left\{\binom{2}{-}\right\}\cup\left\{\binom{2,1,0}{2,1}\right\}\cup\left\{\binom{2,1}{0},\binom{2,0}{1},\binom{1,0}{2},\binom{-}{2,1,0}\right\},$$
$$\mathscr{S}^\sharp_{\mathrm{Sp}_4}=\left\{\binom{2}{-}\right\}\cup\left\{\binom{2,1,0}{2,1}\right\}\cup\left\{\binom{2,1}{0},\binom{2,0}{1},\binom{1,0}{2}\right\}.$$

The character values of irreducible characters in $\mathscr{E}(W_2)=\left\{\varphi_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}\mid \left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]\in\mathscr{P}_2(2)\right\}$ are given by the following table:

(w)	$\{1\}$	$\{\sigma_2, s_1\sigma_2s_1\}$	$\{s_1\sigma_2s_1\sigma_2\}$	$\{s_1, \sigma_2s_1\sigma_2\}$	$\{s_1\sigma_2, \sigma_2s_1\}$
$\varphi_{\left[\begin{smallmatrix}2\\ 0\end{smallmatrix}\right]}$	1	1	1	1	1
$\varphi_{\left[\begin{smallmatrix}1,1\\ 0\end{smallmatrix}\right]}$	1	1	1	-1	-1
$\varphi_{\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]}$	2	0	-2	0	0
$\varphi_{\left[\begin{smallmatrix}0\\ 2\end{smallmatrix}\right]}$	1	-1	1	1	-1
$\varphi_{\left[\begin{smallmatrix}0\\ 1,1\end{smallmatrix}\right]}$	1	-1	1	-1	1
$\mathbf{T}_w$	$\mathbf{T}_1$	$\mathbf{T}_2$	$\mathbf{T}_3$	$\mathbf{T}_4$	$\mathbf{T}_5$

where s_1, σ_2 are defined in Subsection 2.1. We can check that

$$\begin{aligned}
R_{\mathbf{T}_1,1}&=2\theta_9+\theta_{11}+\theta_{12}+\theta_{13}+\theta_0,\\
R_{\mathbf{T}_2,1}&=\theta_{11}-\theta_{12}-\theta_{13}+\theta_0,\\
R_{\mathbf{T}_3,1}&=-2\theta_{10}-\theta_{11}-\theta_{12}+\theta_{13}+\theta_0,\\
R_{\mathbf{T}_4,1}&=-\theta_{11}+\theta_{12}-\theta_{13}+\theta_0,\\
R_{\mathbf{T}_5,1}&=-\theta_9+\theta_{10}+\theta_{13}+\theta_0,
\end{aligned}$$

where the θ_i's are the notions from [20]. Therefore by (3.4), we have

$$\begin{aligned}
R^{\mathbf{G}}_{\binom{2}{-}}&=\frac{1}{8}\big[R_{\mathbf{T}_1,1}+2R_{\mathbf{T}_2,1}+R_{\mathbf{T}_3,1}+2R_{\mathbf{T}_4,1}+2R_{\mathbf{T}_5,1}\big]=\theta_0,\\
R^{\mathbf{G}}_{\binom{2,0}{1}}&=\frac{1}{4}\big[R_{\mathbf{T}_1,1}-R_{\mathbf{T}_3,1}\big]=\frac{1}{2}(\theta_9+\theta_{10}+\theta_{11}+\theta_{12}),\\
R^{\mathbf{G}}_{\binom{2,1}{0}}&=\frac{1}{8}\big[R_{\mathbf{T}_1,1}+2R_{\mathbf{T}_2,1}+R_{\mathbf{T}_3,1}-2R_{\mathbf{T}_4,1}-2R_{\mathbf{T}_5,1}\big]=\frac{1}{2}(\theta_9-\theta_{10}+\theta_{11}-\theta_{12}),\\
R^{\mathbf{G}}_{\binom{1,0}{2}}&=\frac{1}{8}\big[R_{\mathbf{T}_1,1}-2R_{\mathbf{T}_2,1}+R_{\mathbf{T}_3,1}+2R_{\mathbf{T}_4,1}-2R_{\mathbf{T}_5,1}\big]=\frac{1}{2}(\theta_9-\theta_{10}-\theta_{11}+\theta_{12}),\\
R^{\mathbf{G}}_{\binom{2,1,0}{2,1}}&=\frac{1}{8}\big[R_{\mathbf{T}_1,1}-2R_{\mathbf{T}_2,1}+R_{\mathbf{T}_3,1}-2R_{\mathbf{T}_4,1}+2R_{\mathbf{T}_5,1}\big]=\theta_{13}.
\end{aligned}$$

It is known that $\theta_0=\mathbf{1}_{\mathrm{Sp}_4}=\rho_{\binom{2}{-}}$ and $\theta_{13}=\mathrm{St}_{\mathrm{Sp}_4}=\rho_{\binom{2,1,0}{2,1}}$ by Lemma 4.5. Let $Z=\binom{2,0}{1}$. The table for $(-1)^{\langle\Sigma,\Lambda\rangle}$ for $\Sigma\in\mathscr{S}_{Z,1}$ and $\Lambda\in\mathscr{S}_Z^{\mathbf{G}}$ is

	$\binom{2,0}{1}$	$\binom{2,1}{0}$	$\binom{1,0}{2}$	$\binom{-}{2,1,0}$
$\binom{2,0}{1}$	1	1	1	1
$\binom{2,1}{0}$	1	1	-1	-1
$\binom{1,0}{2}$	1	-1	1	-1

Then we have

$$\rho_{\binom{2,0}{1}}^\sharp = \frac{1}{2}\left[R_{\binom{2,0}{1}} + R_{\binom{2,1}{0}} + R_{\binom{1,0}{2}}\right], \qquad \rho_{\binom{2,1}{0}}^\sharp = \frac{1}{2}\left[R_{\binom{2,0}{1}} + R_{\binom{2,1}{0}} - R_{\binom{1,0}{2}}\right],$$
$$\rho_{\binom{1,0}{2}}^\sharp = \frac{1}{2}\left[R_{\binom{2,0}{1}} - R_{\binom{2,1}{0}} + R_{\binom{1,0}{2}}\right], \qquad \rho_{\binom{-}{2,1,0}}^\sharp = \frac{1}{2}\left[R_{\binom{2,0}{1}} - R_{\binom{2,1}{0}} - R_{\binom{1,0}{2}}\right].$$

We know that

$$R^{\mathrm{Sp}_4}_{\mathrm{Sp}_2\times\mathrm{GL}_1}(\rho_{\binom{1}{-}}\otimes\mathbf{1}) = \mathbf{1}_{\mathrm{Sp}_4} + \theta_{11} + \theta_9, \quad R^{\mathrm{Sp}_4}_{\mathrm{Sp}_2\times\mathrm{GL}_1}(\rho_{\binom{1,0}{1}}\otimes\mathbf{1}) = \theta_9 + \theta_{12} + \mathrm{St}_{\mathrm{Sp}_4},$$
$$\Omega\left(\binom{1}{-}\right) = \left\{\binom{2}{-}, \binom{2,1}{0}, \binom{2,0}{1}\right\}, \quad \Omega\left(\binom{1,0}{1}\right) = \left\{\binom{2,0}{1}, \binom{1,0}{2}, \binom{2,1,0}{2,1}\right\}.$$

Then by Corollary 4.3, we conclude that $\theta_9 = \rho_{\binom{2,0}{1}}$, $\theta_{10} = \rho_{\binom{-}{2,1,0}}$, $\theta_{11} = \rho_{\binom{2,1}{0}}$, and $\theta_{12} = \rho_{\binom{1,0}{2}}$.

4.2. Unipotent characters of SO_{2n+1}

The following lemma is an analogue of Lemma 4.1.

Lemma 4.7. *Suppose that* $\mathbf{G} = \mathrm{SO}_{2n+1}$*, and let* $\rho_1, \rho_2 \in \mathscr{E}(\mathbf{G}, 1)$*. If* $\rho_1^\sharp = \rho_2^\sharp$*, then* $\rho_1 = \rho_2$*.*

Proposition 4.8. *Suppose that* $\mathbf{G} = \mathrm{SO}_{2n+1}$*. Then there is a unique bijection* $\mathscr{S}_{\mathbf{G}} \to \mathscr{E}(\mathbf{G}, 1)$ *satisfying* (3.6)*.*

Proof. The proof is similar to that of Proposition 4.2. □

Corollary 4.9. *Let* $\mathscr{L}_1\colon \mathscr{S}_{\mathrm{SO}_{2n+1}} \to \mathscr{E}(\mathrm{SO}_{2n+1}, 1)$ *be the parametrization given in Proposition* 4.8*. Then*

(i) $\mathbf{1}_{\mathrm{SO}_{2n+1}} = \rho_{\binom{-}{n}}$;

(ii) $\mathrm{St}_{\mathrm{SO}_{2n+1}} = \rho_\Lambda$ *where* $\Lambda = \binom{n,n-1,\ldots,1}{n,n-1,\ldots,1,0}$;

(iii) *if* $n = k(k+1)$ *for some nonnegative integer* k*, then the unique cuspidal unipotent character* $\zeta_k^{\mathrm{SO}_{\mathrm{odd}}}$ *of* $\mathrm{SO}_{2n+1}(q)$ *is parametrized by the symbol*

$$\Lambda_k^{\mathrm{SO}_{\mathrm{odd}}} = \begin{cases} \binom{-}{2k,2k-1,\ldots,0} & \text{if } k \text{ is even,} \\ \binom{2k,2k-1,\ldots,0}{-} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. For (i) and (ii), the proofs are analogous to that of Corollary 4.4, for (iii) the proof is analogous to that of Lemma 4.5. $\square$

4.3. Unipotent characters of O^ϵ_{2n}

From (3.3), we know that $R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T},1}\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}}=R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T},1}$ and then

$$\langle\rho,R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T},1}\rangle_{\mathrm{O}^\epsilon_{2n}}=\langle\rho\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}},R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T},1}\rangle_{\mathrm{O}^\epsilon_{2n}}$$

for any $\rho\in\mathscr{E}(\mathrm{O}^\epsilon_{2n},1)$. Therefore,

$$\rho^\sharp=(\rho\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}})^\sharp,\tag{4.2}$$

i.e., two irreducible characters ρ, $\rho\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}}$ are not distinguishable by their uniform projections. The following lemma is [18, Proposition 3.5].

Lemma 4.10. *Let $\mathscr{L}_1\colon\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}\to\mathscr{E}(\mathrm{O}^\epsilon_{2n},1)$ by $\Lambda\mapsto\rho_\Lambda$ be a Lusztig parametrization of unipotent characters. Then $(\rho_{\Lambda_1})^\sharp=(\rho_{\Lambda_2})^\sharp$ if and only if $\Lambda_1=\Lambda_2$ or $\Lambda_1=\Lambda_2^{\mathrm{t}}$.*

Corollary 4.11. *Let $\mathscr{L}_1\colon\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}\to\mathscr{E}(\mathrm{O}^\epsilon_{2n},1)$ by $\Lambda\mapsto\rho_\Lambda$ be a Lusztig parametrization of unipotent characters. Then $\rho_{\Lambda^{\mathrm{t}}}=\rho_\Lambda\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}}$.*

Proof. If Λ is degenerate, then clearly $\rho_{\Lambda^{\mathrm{t}}}=\rho_\Lambda=R^{\mathrm{O}^\epsilon_{2n}}_\Lambda=R^{\mathrm{O}^\epsilon_{2n}}_\Lambda\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}}=\rho_\Lambda\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}}$. If Λ is non-degenerate, from (4.2) we know that $(\rho_\Lambda\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}})^\sharp=(\rho_\Lambda)^\sharp$, and by Lemma 4.10 we conclude that $\rho_\Lambda\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}}=\rho_{\Lambda^{\mathrm{t}}}$. $\square$

Corollary 4.12. *Let $\rho_1,\rho_2\in\mathscr{E}(\mathrm{O}^\epsilon_{2n},1)$. If $\rho_1^\sharp=\rho_2^\sharp$, then either $\rho_1=\rho_2$ or $\rho_1=\rho_2\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}}$.*

Corollary 4.13. *Let $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$, and let $\Lambda\mapsto\rho_\Lambda$ be a Lusztig parametrization of unipotent characters. Then any bijective mapping $\mathscr{S}_{\mathbf{G}}\mapsto\mathscr{E}(\mathbf{G},1)$ such that $\{\Lambda,\Lambda^{\mathrm{t}}\}\to\{\rho_\Lambda,\rho_{\Lambda^{\mathrm{t}}}\}$ is also a Lusztig parametrization of unipotent characters.*

Proof. Suppose that $\mathscr{S}_{\mathbf{G}}\mapsto\mathscr{E}(\mathbf{G},1)$ given by $\Lambda\mapsto\rho'_\Lambda$ is a bijection such that $\{\Lambda,\Lambda^{\mathrm{t}}\}\to\{\rho_\Lambda,\rho_{\Lambda^{\mathrm{t}}}\}$, i.e., $\{\rho'_\Lambda,\rho'_{\Lambda^{\mathrm{t}}}\}=\{\rho_\Lambda,\rho_{\Lambda^{\mathrm{t}}}\}$. This implies that $(\rho'_\Lambda)^\sharp=(\rho'_{\Lambda^{\mathrm{t}}})^\sharp=(\rho_\Lambda)^\sharp=(\rho_{\Lambda^{\mathrm{t}}})^\sharp$ and hence the mapping $\Lambda\mapsto\rho'_\Lambda$ satisfies (3.6), i.e., $\Lambda\mapsto\rho'_\Lambda$ is also a Lusztig parametrization of unipotent characters for O^ϵ_{2n}. $\square$

Corollary 4.14. *Let $\mathscr{L}_1\colon\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}\to\mathscr{E}(\mathrm{O}^\epsilon_{2n},1)$ be a Lusztig parametrization of unipotent characters.*

(i) *If $\epsilon=+$, then*

  (a) $\mathscr{L}_1\colon\left\{\binom{n}{0},\binom{0}{n}\right\}\to\left\{\mathbf{1}_{\mathrm{O}^+_{2n}},\mathrm{sgn}_{\mathrm{O}^+_{2n}}\right\}$,

  (b) *$\mathscr{L}_1$ maps $\left\{\binom{n,n-1,\ldots,1}{n-1,n-2,\ldots,0},\binom{n-1,n-2,\ldots,0}{n,n-1,\ldots,1}\right\}$ to the two Steinberg characters of O^+_{2n}.*

(ii) *If* $\epsilon=-$, *then*

(a) $\mathscr{L}_1\colon \{\binom{-}{n,0},\binom{n,0}{-}\}\to\{\mathbf{1}_{\mathrm{O}^-_{2n}},\mathrm{sgn}_{\mathrm{O}^-_{2n}}\}$,

(b) $\mathscr{L}_1$ *maps* $\{\binom{n,n-1,\dots,1,0}{n-1,n-2,\dots,1},\binom{n-1,n-2,\dots,1}{n,n-1,\dots,1,0}\}$ *to the two Steinberg characters of* O^-_{2n}.

Proof. First suppose that $\epsilon=+$. From [3, Corollary 7.6.5], we know that $R^{\mathrm{SO}^+_{2n}}_{\binom{n}{0}}=\mathbf{1}_{\mathrm{SO}^+_{2n}}$. Therefore,

$$R^{\mathrm{O}^+_{2n}}_{\binom{n}{0}}=R^{\mathrm{O}^+_{2n}}_{\binom{0}{n}}=\frac{1}{\sqrt{2}}(\mathbf{1}_{\mathrm{O}^+_{2n}}+\mathrm{sgn}_{\mathrm{O}^+_{2n}})$$

and (i.a) is proved from (3.6). Write $\Lambda=\binom{n-1,n-2,\dots,0}{n,n-1,\dots,1}$. From [3, Corollary 7.6.6], we see that $R^{\mathrm{SO}^+_{2n}}_\Lambda=\mathrm{St}_{\mathrm{SO}^+_{2n}}$. Then $\sqrt{2}R^{\mathrm{O}^+_{2n}}_\Lambda=\sqrt{2}R^{\mathrm{O}^+_{2n}}_{\Lambda^{\mathrm{t}}}$ is the sum of two Steinberg characters of O^+_{2n} and (i.b) is proved.

The proof of (ii) is similar. □

It is known that $\mathrm{O}^{\epsilon_k}_{2k^2}$ where $k\geq 1$ and $\epsilon_k=(-1)^k$ has two cuspidal unipotent characters, denoted by ζ^{I}_k and ζ^{II}_k. Then from above we see that any Lusztig parametrization $\mathscr{L}_1\colon\mathscr{S}_{\mathrm{O}^{\epsilon_k}_{2k^2}}\to\mathscr{E}(\mathrm{O}^{\epsilon_k}_{2k^2},1)$ maps

$$\left\{\binom{2k-1,2k-2,\dots,1,0}{-},\binom{-}{2k-1,2k-2,\dots,1,0}\right\}\to\{\zeta^{\mathrm{I}}_k,\zeta^{\mathrm{II}}_k\}$$

bijectively.

Example 4.15. Let $\mathbf{T}_i$ for $i=1,\dots,5$ be parametrized as in Example 4.6. It is know that $\mathbf{T}_1$, $\mathbf{T}_3$, $\mathbf{T}_4$ are maximal tori in O^+_4, and $\mathbf{T}_2$, $\mathbf{T}_5$ are maximal tori in O^-_4. It is know that

$$\begin{aligned}
R^{\mathrm{O}^+_4}_{\mathbf{T}_1,1}&=\mathbf{1}_{\mathrm{O}^+_4}+\mathrm{sgn}_{\mathrm{O}^+_4}+2\chi^+_{2q}+\chi^+_{q^2}+\chi^+_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}^+_4},\\
R^{\mathrm{O}^+_4}_{\mathbf{T}_3,1}&=\mathbf{1}_{\mathrm{O}^+_4}+\mathrm{sgn}_{\mathrm{O}^+_4}-2\chi^+_{2q}+\chi^+_{q^2}+\chi^+_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}^+_4},\\
R^{\mathrm{O}^+_4}_{\mathbf{T}_4,1}&=\mathbf{1}_{\mathrm{O}^+_4}+\mathrm{sgn}_{\mathrm{O}^+_4}-\chi^+_{q^2}-\chi^+_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}^+_4},\\
R^{\mathrm{O}^-_4}_{\mathbf{T}_2,1}&=\mathbf{1}_{\mathrm{O}^-_4}+\mathrm{sgn}_{\mathrm{O}^-_4}+\chi^-_{q^2}+\chi^-_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}^-_4},\\
R^{\mathrm{O}^-_4}_{\mathbf{T}_5,1}&=\mathbf{1}_{\mathrm{O}^-_4}+\mathrm{sgn}_{\mathrm{O}^-_4}-\chi^-_{q^2}-\chi^-_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}^-_4},
\end{aligned}$$

where $\chi^+_{2q},\chi^+_{q^2}$ are irreducible characters of $\mathrm{O}^+_4(q)$ of degrees $2q,q^2$ respectively; similarly $\chi^-_{q^2}$ is an irreducible character of $\mathrm{O}^-_4(q)$ of degree q^2. And so we have

$$\begin{aligned}
\mathscr{E}(\mathrm{O}^+_4,1)&=\{\mathbf{1}_{\mathrm{O}^+_4},\mathrm{sgn}_{\mathrm{O}^+_4},\chi^+_{2q},\chi^+_{q^2},\chi^+_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}^+_4}\},\\
\mathscr{E}(\mathrm{O}^-_4,1)&=\{\mathbf{1}_{\mathrm{O}^-_4},\mathrm{sgn}_{\mathrm{O}^-_4},\chi^-_{q^2},\chi^-_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}^-_4}\}.
\end{aligned}$$

We know that

$$\mathscr{S}_{\mathrm{O}_4^+}=\left\{\binom{2}{0},\binom{0}{2},\binom{2,1}{1,0},\binom{1,0}{2,1},\binom{1}{1}\right\},$$
$$\mathscr{S}_{\mathrm{O}_4^-}=\left\{\binom{-}{2,0},\binom{2,0}{-},\binom{1}{2,1,0},\binom{2,1,0}{1}\right\}.$$

Suppose we choose $\mathscr{S}^\sharp_{\mathrm{O}_4^+}=\{\binom{2}{0},\binom{2,1}{1,0},\binom{1}{1}\}$, and $\mathscr{S}^\sharp_{\mathrm{O}_4^-}=\{\binom{0}{2},\binom{1,0}{2,1}\}$. Then we have

$$\begin{aligned}
R^{\mathrm{O}_4^+}_{\binom{2}{0}}&=\frac{1}{4\sqrt{2}}\big[R^{\mathrm{O}_4^+}_{\mathbf{T}_1,1}+R^{\mathrm{O}_4^+}_{\mathbf{T}_3,1}+2R^{\mathrm{O}_4^+}_{\mathbf{T}_4,1}\big]=\frac{1}{\sqrt{2}}\big[\mathbf{1}_{\mathrm{O}_4^+}+\mathrm{sgn}_{\mathrm{O}_4^+}\big],\\
R^{\mathrm{O}_4^+}_{\binom{1}{1}}&=\frac{1}{4}\big[R^{\mathrm{O}_4^+}_{\mathbf{T}_1,1}-R^{\mathrm{O}_4^+}_{\mathbf{T}_3,1}\big]=\chi^+_{2q},\\
R^{\mathrm{O}_4^+}_{\binom{2,1}{1,0}}&=\frac{1}{4\sqrt{2}}\big[R^{\mathrm{O}_4^+}_{\mathbf{T}_1,1}+R^{\mathrm{O}_4^+}_{\mathbf{T}_3,1}-2R^{\mathrm{O}_4^+}_{\mathbf{T}_4,1}\big]=\frac{1}{\sqrt{2}}\big[\chi^+_{q^2}+\chi^+_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}_4^+}\big],\\
R^{\mathrm{O}_4^-}_{\binom{0}{2}}&=\frac{1}{2\sqrt{2}}\big[-R^{\mathrm{O}_4^-}_{\mathbf{T}_2,1}-R^{\mathrm{O}_4^-}_{\mathbf{T}_5,1}\big]=\frac{1}{\sqrt{2}}\big[-\mathbf{1}_{\mathrm{O}_4^-}-\mathrm{sgn}_{\mathrm{O}_4^-}\big],\\
R^{\mathrm{O}_4^-}_{\binom{1,0}{2,1}}&=\frac{1}{2\sqrt{2}}\big[-R^{\mathrm{O}_4^-}_{\mathbf{T}_2,1}+R^{\mathrm{O}_4^-}_{\mathbf{T}_5,1}\big]=\frac{1}{\sqrt{2}}\big[-\chi^-_{q^2}-\chi^-_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}_4^-}\big].
\end{aligned}$$

From (2.7), we know that

$$\left\langle\binom{2}{0},\binom{2}{0}\right\rangle=\left\langle\binom{0}{2},\binom{2}{0}\right\rangle=\left\langle\binom{2,1}{1,0},\binom{2,1}{1,0}\right\rangle=\left\langle\binom{1,0}{2,1},\binom{2,1}{1,0}\right\rangle\equiv 0\pmod 2,$$
$$\left\langle\binom{-}{2,0},\binom{0}{2}\right\rangle=\left\langle\binom{2,0}{-},\binom{0}{2}\right\rangle=\left\langle\binom{1}{2,1,0},\binom{1,0}{2,1}\right\rangle=\left\langle\binom{2,1,0}{1},\binom{1,0}{2,1}\right\rangle\equiv 1\pmod 2.$$

By (3.6), we have

$$\begin{aligned}
(\mathbf{1}_{\mathrm{O}_4^+})^\sharp=(\mathrm{sgn}_{\mathrm{O}_4^+})^\sharp&=\frac{1}{\sqrt{2}}R^{\mathrm{O}_4^+}_{\binom{2}{0}}=\frac12(\mathbf{1}_{\mathrm{O}_4^+}+\mathrm{sgn}_{\mathrm{O}_4^+}),\\
(\chi^+_{q^2})^\sharp=(\chi^+_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}_4^+})^\sharp&=\frac{1}{\sqrt{2}}R^{\mathrm{O}_4^+}_{\binom{2,1}{1,0}}=\frac12(\chi^+_{q^2}+\chi^+_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}_4^+}),\\
\chi^+_{2q}&=R^{\mathrm{O}_4^+}_{\binom{1}{1}},\\
(\mathbf{1}_{\mathrm{O}_4^-})^\sharp=(\mathrm{sgn}_{\mathrm{O}_4^-})^\sharp&=-\frac{1}{\sqrt{2}}R^{\mathrm{O}_4^-}_{\binom{0}{2}}=\frac12(\mathbf{1}_{\mathrm{O}_4^-}+\mathrm{sgn}_{\mathrm{O}_4^-}),\\
(\chi^-_{q^2})^\sharp=(\chi^-_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}_4^-})^\sharp&=-\frac{1}{\sqrt{2}}R^{\mathrm{O}_4^-}_{\binom{1,0}{2,1}}=\frac12(\chi^-_{q^2}+\chi^-_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}_4^-}).
\end{aligned}$$

Therefore any bijection $\mathscr{S}_{\mathrm{O}_4^+}\to\mathscr{E}(\mathrm{O}_4^+,1)$ such that

$$\left\{\binom{2}{0},\binom{0}{2}\right\}\mapsto\{\mathbf{1}_{\mathrm{O}_4^+},\mathrm{sgn}_{\mathrm{O}_4^+}\},\quad\binom{1}{1}\mapsto\chi^+_{2q},\quad\left\{\binom{2,1}{1,0},\binom{1,0}{2,1}\right\}\mapsto\{\chi^+_{q^2},\chi^+_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}_4^+}\}$$

is a Lusztig parametrization of unipotent characters for O_4^+, and any bijection $\mathscr{S}_{\mathrm{O}_4^-}\to\mathscr{E}(\mathrm{O}_4^-,1)$ such that

$$\left\{\binom{-}{2,0},\binom{2,0}{-}\right\}\mapsto\{\mathbf{1}_{\mathrm{O}_4^-},\mathrm{sgn}_{\mathrm{O}_4^-}\},\quad\left\{\binom{1}{2,1,0},\binom{2,1,0}{1}\right\}\mapsto\{\chi^-_{q^2},\chi^-_{q^2}\cdot\mathrm{sgn}_{\mathrm{O}_4^-}\}$$

is a Lusztig parametrization of unipotent characters for O_4^-. In particular, the parametrizations for O_4^+ or O_4^- are not unique.

5. Finite theta correspondence of unipotent characters

In this section we want to purpose several conditions to enforce the parametrization $\mathscr{S}_{\mathbf{G}} \to \mathscr{E}(\mathbf{G},1)$ for $\mathbf{G} = \mathrm{O}_{2n}^\epsilon$ to be unique.

5.1. Finite theta correspondence on unipotent characters

For a nontrivial additive character ψ of $\mathbf{F}_q$, let $\omega^\psi_{\mathrm{Sp}(W)}$ denote the (character of the) *Weil representation* of the finite symplectic group $\mathrm{Sp}(W)$ with respect to ψ (cf. [9]).

Let $(\mathbf{G},\mathbf{G}')$ be a *reductive dual pair* of the form $(\mathrm{Sp}_{2n},\mathrm{SO}_{2n'+1})$ or $(\mathrm{O}_{2n}^\epsilon,\mathrm{Sp}_{2n'})$ where $\epsilon = +$ or $-$ (cf. [2, 10]). The restriction $\omega^\psi_{\mathbf{G},\mathbf{G}'}$ of the Weil character to $G\times G'$ gives a decomposition

$$\omega^\psi_{\mathbf{G},\mathbf{G}'} = \sum_{\substack{\rho\in\mathscr{E}(\mathbf{G})\\ \rho'\in\mathscr{E}(\mathbf{G}')}} m_{\rho,\rho'}\,\rho\otimes\rho' \tag{5.1}$$

where the multiplicity $m_{\rho,\rho'}$ is either 1 or 0. Then we have a relation

$$\Theta^\psi_{\mathbf{G},\mathbf{G}'} = \{(\rho,\rho')\in\mathscr{E}(\mathbf{G})\times\mathscr{E}(\mathbf{G}') \mid m_{\rho,\rho'}\neq 0\}$$

between $\mathscr{E}(\mathbf{G})$ and $\mathscr{E}(\mathbf{G}')$ called the *finite theta correspondence* (or *Howe duality*) for the dual pair $(\mathbf{G},\mathbf{G}')$. We say that an irreducible character $\rho\in\mathscr{E}(\mathbf{G})$ *occurs in* $\Theta^\psi_{\mathbf{G},\mathbf{G}'}$ if there exists $\rho'\in\mathscr{E}(\mathbf{G}')$ such that $(\rho,\rho')\in\Theta^\psi_{\mathbf{G},\mathbf{G}'}$.

For a symplectic space V over $\mathbf{F}_q$, we have the *symplectic similitude group*

$$\mathrm{GSp}(V) = \{g\in\mathrm{GL}(V) \mid \langle gv,gw\rangle = k_g\langle v,w\rangle \text{ for some } k_g\in\mathbf{F}_q^\times \text{ and any } v,w\in V\}.$$

Note that $\mathrm{GSp}(V)$ normalizes the symplectic group $\mathrm{Sp}(V)$. Choose an element $h\in\mathrm{GSp}_{2n}(q)$ such that k_h is a non-square element in $\mathbf{F}_q^\times$. For $\rho\in\mathscr{E}(\mathrm{Sp}_{2n})$, we define the *conjugate character* $\rho^c\in\mathscr{E}(\mathrm{Sp}_{2n})$ by $\rho^c(g) = \rho(hgh^{-1})$ for any $g\in\mathrm{Sp}_{2n}(q)$.

Lemma 5.1. *Suppose that $(\mathbf{G},\mathbf{G}') = (\mathrm{Sp}_{2n},\mathrm{SO}_{2n'+1})$. If $(\rho,\rho')\in\Theta^\psi_{\mathbf{G},\mathbf{G}'}$, then $(\rho^c,\rho')\in\Theta^{\psi_a}_{\mathbf{G},\mathbf{G}'}$ where ψ_a is another additive character of $\mathbf{F}_q$ given by $\psi_a(x) := \psi(ax)$ and a is a non-square element in $\mathbf{F}_q^\times$.*

Proof. Suppose that $G = \mathrm{Sp}(V)$ and $G' = \mathrm{SO}(V')$ for a $2n$-dimensional symplectic space V and a $(2n+1)$-dimensional orthogonal space V', and write $\omega^\psi_{\mathbf{G},\mathbf{G}'}$ as in (5.1). Choose $h\in\mathrm{GSp}(V)$ such that k_h is a non-square element in $\mathbf{F}_q^\times$, and let $\widetilde{h} = \iota(h,1)\in\mathrm{GSp}(V\otimes V')$ where

$$\iota\colon\ \mathrm{GSp}(V)\times\mathrm{SO}(V')\to\mathrm{GSp}(V\otimes V').$$

Now clearly $k_{\widetilde{h}}=k_h$, and then by [22, Proposition 11] we have $\omega^{\psi}_{\mathrm{Sp}(V\otimes V')}\circ\mathrm{Ad}_{\widetilde{h}}=\omega^{\psi_a}_{\mathrm{Sp}(V\otimes V')}$. Therefore,

$$\omega^{\psi_a}_{\mathbf{G},\mathbf{G}'}=\sum_{\substack{\rho\in\mathscr{E}(\mathbf{G})\\ \rho'\in\mathscr{E}(\mathbf{G}')}} m_{\rho,\rho'}(\rho\circ\mathrm{Ad}_h)\otimes\rho'=\sum_{\substack{\rho\in\mathscr{E}(\mathbf{G})\\ \rho'\in\mathscr{E}(\mathbf{G}')}} m_{\rho,\rho'}\rho^c\otimes\rho'.$$

Thus the lemma is proved. □

For a quadratic space V over $\mathbf{F}_q$, we have the *orthogonal similitude group*

$$\mathrm{GO}(V)=\{g\in\mathrm{GL}(V)\mid\langle gv,gw\rangle=k_g\langle v,w\rangle\text{ for some }k_g\in\mathbf{F}_q^\times\text{ and any }v,w\in V\}.$$

For $\rho\in\mathscr{E}(\mathrm{O}^\epsilon_{2n})$, we can define the *conjugate character* $\rho^c\in\mathscr{E}(\mathrm{O}^\epsilon_{2n})$ as we did for a symplectic group above.

Lemma 5.2. *Suppose that* $(\mathbf{G},\mathbf{G}')=(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2n'})$. *If* $(\rho,\rho')\in\Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$, *then* $(\rho^c,\rho'^c)\in\Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$, *and* $(\rho^c,\rho'),(\rho,\rho'^c)\in\Theta^{\psi_a}_{\mathbf{G},\mathbf{G}'}$ *where* ψ_a *is given as in Lemma* 5.1.

Proof. Suppose that $G=\mathrm{O}(V)$ and $G'=\mathrm{Sp}(V')$ for a $2n$-dimensional orthogonal space V and a $2n'$-dimensional symplectic space V', and write $\omega^{\psi}_{\mathbf{G},\mathbf{G}'}$ as in (5.1). Choose $h\in\mathrm{GO}(V)$ and $h'\in\mathrm{GSp}(V')$ such that both k_h, $k_{h'}$ are non-square elements in $\mathbf{F}_q^\times$ and let $\widetilde{h}=\iota(h,h')\in\mathrm{GSp}(V\otimes V')$ where

$$\iota\colon\ \mathrm{GO}(V)\times\mathrm{GSp}(V')\to\mathrm{GSp}(V\otimes V').$$

Now $k_{\widetilde{h}}=k_hk_{h'}$ becomes a square element in $\mathbf{F}_q^\times$ and therefore

$$\omega^{\psi}_{\mathbf{G},\mathbf{G}'}=\omega^{\psi}_{\mathbf{G},\mathbf{G}'}\circ\mathrm{Ad}_{\widetilde{h}}=\sum_{\substack{\rho\in\mathscr{E}(\mathbf{G})\\ \rho'\in\mathscr{E}(\mathbf{G}')}} m_{\rho,\rho'}(\rho\circ\mathrm{Ad}_h)\otimes(\rho'\circ\mathrm{Ad}_{h'})=\sum_{\substack{\rho\in\mathscr{E}(\mathbf{G})\\ \rho'\in\mathscr{E}(\mathbf{G}')}} m_{\rho,\rho'}\rho^c\otimes\rho'^c.$$

So we have shown that $(\rho,\rho')\in\Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$ implies that $(\rho^c,\rho'^c)\in\Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$. The other assertions can be proved by an analogous argument in the proof of Lemma 5.1. □

Let $\mathbf{G}'_{n'}$ denote $\mathrm{SO}_{2n'+1}$, $\mathrm{Sp}_{2n'}$, or $\mathrm{O}^\epsilon_{2n'}$. For $\rho\in\mathscr{E}(\mathbf{G})$, it is well-known that if ρ occurs in $\Theta^{\psi}_{\mathbf{G},\mathbf{G}'_{n'}}$, then it also occurs in $\Theta^{\psi}_{\mathbf{G},\mathbf{G}'_{n''}}$ for any $n''\geq n'$ (cf. [16, Chapter 3]). We say that ρ *first occurs in* $\Theta^{\psi}_{\mathbf{G},\mathbf{G}'_{n'}}$ if it occurs in $\Theta^{\psi}_{\mathbf{G},\mathbf{G}'_{n'}}$ and does not occur in $\Theta^{\psi}_{\mathbf{G},\mathbf{G}'_{n'-1}}$.

5.2. Finite theta correspondence on unipotent characters

If $(\mathbf{G},\mathbf{G}')=(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2n'})$, then the unipotent characters are preserved by $\Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$ (cf. [1, Theorem 3.5]), i.e., we can write

$$\omega^{\psi}_{\mathbf{G},\mathbf{G}',1}=\sum_{\substack{\rho\in\mathscr{E}(\mathbf{G},1)\\ \rho'\in\mathscr{E}(\mathbf{G}',1)}} m_{\rho,\rho'}\rho\otimes\rho',\quad\Theta^{\psi}_{\mathbf{G},\mathbf{G}',1}=\Theta^{\psi}_{\mathbf{G},\mathbf{G}'}\cap(\mathscr{E}(\mathbf{G},1)\times\mathscr{E}(\mathbf{G}',1)),$$

where $\omega^\psi_{\mathbf{G},\mathbf{G}',1}$ denotes the unipotent part of $\omega^\psi_{\mathbf{G},\mathbf{G}'}$. Recall that $\mathrm{O}^{\epsilon_k}_{2k^2}(q)$ where $\epsilon_k=(-1)^k$ and $k\geq 1$ has two irreducible cuspidal unipotent characters. It is well-known that (cf. [1, Theorem 5.2]) there is a unique labelling ζ^{I}_k, ζ^{II}_k of these two cuspidal unipotent characters such that ζ^{I}_k (resp. ζ^{II}_k) first occurs in the correspondence for the pair $(\mathrm{O}^{\epsilon_k}_{2k^2},\mathrm{Sp}_{2k(k-1)})$ (resp. $(\mathrm{O}^{\epsilon_k}_{2k^2},\mathrm{Sp}_{2k(k+1)})$).

Now we recall some results on $\Theta^\psi_{\mathbf{G},\mathbf{G}',1}$ from [17]. For two partitions $\lambda=[\lambda_1,\lambda_2,\ldots]$ (with $\lambda_1\geq\lambda_2\geq\cdots$), $\mu=[\mu_1,\mu_2,\ldots]$ (with $\mu_1\geq\mu_2\geq\cdots$), we denote

$$\lambda\preccurlyeq\mu\quad\text{if } \mu_1\geq\lambda_1\geq\mu_2\geq\lambda_2\geq\mu_3\geq\lambda_3\geq\cdots.$$

And then we define a relation $\mathscr{B}_{\mathbf{G},\mathbf{G}'}$ between $\mathscr{S}_\mathbf{G}$ and $\mathscr{S}_{\mathbf{G}'}$ for $(\mathbf{G},\mathbf{G}')=(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2n'})$ as follows:

- If $\epsilon=+$, let $\mathscr{B}_{\mathbf{G},\mathbf{G}'}$ be the set consisting of pairs $(\Lambda,\Lambda')\in\mathscr{S}_\mathbf{G}\times\mathscr{S}_{\mathbf{G}'}$ such that
  (1) $\Upsilon(\Lambda)_*\preccurlyeq\Upsilon(\Lambda')^*$ and $\Upsilon(\Lambda')_*\preccurlyeq\Upsilon(\Lambda)^*$,
  (2) $\mathrm{def}(\Lambda')=-\mathrm{def}(\Lambda)+1$;
- if $\epsilon=-$, let $\mathscr{B}_{\mathbf{G},\mathbf{G}'}$ be the set consisting of pairs $(\Lambda,\Lambda')\in\mathscr{S}_\mathbf{G}\times\mathscr{S}_{\mathbf{G}'}$ such that
  (1) $\Upsilon(\Lambda)^*\preccurlyeq\Upsilon(\Lambda')_*$ and $\Upsilon(\Lambda')^*\preccurlyeq\Upsilon(\Lambda)_*$,
  (2) $\mathrm{def}(\Lambda')=-\mathrm{def}(\Lambda)-1$.

We say that a symbol $\Lambda\in\mathscr{S}_\mathbf{G}$ *occurs in* $\mathscr{B}_{\mathbf{G},\mathbf{G}'}$ if there is $\Lambda'\in\mathscr{S}_{\mathbf{G}'}$ such that $(\Lambda,\Lambda')\in\mathscr{B}_{\mathbf{G},\mathbf{G}'}$. For $\Lambda\in\mathscr{S}_\mathbf{G}$, it is not difficult to see that if Λ occurs in $\mathscr{B}_{\mathbf{G},\mathbf{G}'_{n'}}$, then it also occurs in $\mathscr{B}_{\mathbf{G},\mathbf{G}'_{n''}}$ for any $n''\geq n'$. We say that Λ *first occurs in* $\mathscr{B}_{\mathbf{G},\mathbf{G}'_{n'}}$ if it occurs in $\mathscr{B}_{\mathbf{G},\mathbf{G}'_{n'}}$ and does not occur in $\mathscr{B}_{\mathbf{G},\mathbf{G}'_{n'-1}}$.

Example 5.3. (1) We have $\binom{1}{0},\binom{0}{1}\in\mathscr{S}_{\mathrm{O}^+_2}$, $\binom{0}{-}\in\mathscr{S}_{\mathrm{Sp}_0}$, and $\binom{1}{-}\in\mathscr{S}_{\mathrm{Sp}_2}$. Now $\Upsilon(\binom{1}{0})=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\Upsilon(\binom{0}{1})=\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $\Upsilon(\binom{0}{-})=\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, and $\Upsilon(\binom{1}{-})=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, and so $\binom{1}{0}$ first occurs in $\mathscr{B}_{\mathrm{O}^+_2,\mathrm{Sp}_0}$ and $\binom{0}{1}$ first occurs in $\mathscr{B}_{\mathrm{O}^+_2,\mathrm{Sp}_2}$.

(2) Suppose that k is even, and let $\Lambda^{\mathrm{I}}_k=\binom{2k-1,2k-2,\ldots,0}{-}$. Then $\Lambda^{\mathrm{I}}_k\in\mathscr{S}_{\mathrm{O}^+_{2k^2}}$, $\mathrm{def}(\Lambda^{\mathrm{I}}_k)=2k$ and $\Upsilon(\Lambda^{\mathrm{I}}_k)=\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. Therefore, if Λ^{I}_k occurs in $\mathscr{B}_{\mathrm{O}^+_{2k^2},\mathrm{Sp}_{2n'}}$, then $\mathscr{S}_{\mathrm{Sp}_{2n'}}$ contains a symbol of defect $-2k+1$. By (2.3), we have $n'\geq k(k-1)$, i.e., Λ^{I}_k first occurs in $\mathscr{B}_{\mathrm{O}^+_{2k^2},\mathrm{Sp}_{2k(k-1)}}$.

(3) Suppose that k is odd, and let $\Lambda^{\mathrm{I}}_k=\binom{-}{2k-1,2k-2,\ldots,0}$. Then $\Lambda^{\mathrm{I}}_k\in\mathscr{S}_{\mathrm{O}^-_{2k^2}}$, $\mathrm{def}(\Lambda^{\mathrm{I}}_k)=-2k$ and $\Upsilon(\Lambda^{\mathrm{I}}_k)=\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. This means that if Λ^{I}_k occurs in $\mathscr{B}_{\mathrm{O}^-_{2k^2},\mathrm{Sp}_{2n'}}$, then $\mathscr{S}_{\mathrm{Sp}_{2n'}}$ contains a symbol of defect $2k-1$. By (2.3), we have $n'\geq k(k-1)$, i.e., Λ^{I}_k first occurs in $\mathscr{B}_{\mathrm{O}^-_{2k^2},\mathrm{Sp}_{2k(k-1)}}$.

The following proposition is from [17, Corollary 5.36].

Proposition 5.4. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2n'})$ *where* $\epsilon=+$ *or* $-$. *Let* $\mathscr{L}_1\colon\mathscr{S}_{\mathbf{G}}\to\mathscr{E}(\mathbf{G},1)$ *by* $\Lambda\mapsto\rho_\Lambda$ *and* $\mathscr{L}'_1\colon\mathscr{S}_{\mathbf{G}'}\to\mathscr{E}(\mathbf{G}',1)$ *by* $\Lambda'\mapsto\rho_{\Lambda'}$ *be any Lusztig parametrizations for* $\mathbf{G}$, $\mathbf{G}'$ *respectively. Then* $(\rho_\Lambda,\rho_{\Lambda'})$ *or* $(\rho_{\Lambda^{\mathrm{t}}},\rho_{\Lambda'})$ *occurs in* $\Theta^\psi_{\mathbf{G},\mathbf{G}',1}$ *if and only if* (Λ,Λ') *or* $(\Lambda^{\mathrm{t}},\Lambda')$ *occurs in* $\mathscr{B}_{\mathbf{G},\mathbf{G}'}$.

Remark 5.5. Note that the parametrization $\mathscr{L}'_1$ is unique by Corollary 4.4 but $\mathscr{L}_1$ is not. So we want to enforce more conditions on $\mathscr{L}_1$ so that $\mathscr{L}_1$ is unique and eliminate the ambiguity in the above proposition.

5.3. On the uniqueness of Lusztig parametrization for even orthogonal groups

Now we want to enforce extra conditions on the Lusztig parametrization $\mathscr{L}_1\colon\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}\to\mathscr{E}(\mathrm{O}^\epsilon_{2n},1)$ to make it be uniquely determined.

(I) We require that $\mathscr{L}_1$ by $\Lambda\mapsto\rho_\Lambda$ is compatible with the parabolic induction on unipotent characters, i.e., we require that $\Omega(\rho_\Lambda)=\{\rho_{\Lambda'}\mid\Lambda'\in\Omega(\Lambda)\}$ where $\Omega(\rho_\Lambda)$ and $\Omega(\Lambda)$ are defined as in Subsection 3.3.

(II) We require that

- for $k\geq 1$, $\mathscr{L}_1(\Lambda^{\mathrm{I}}_k)=\zeta^{\mathrm{I}}_k$ and $\mathscr{L}_1(\Lambda^{\mathrm{II}}_k)=\zeta^{\mathrm{II}}_k$, i.e., $\zeta^{\mathrm{I}}_k=\rho_{\Lambda^{\mathrm{I}}_k}$ and $\zeta^{\mathrm{II}}_k=\rho_{\Lambda^{\mathrm{II}}_k}$ where

$$\Lambda^{\mathrm{I}}_k=\begin{cases}\binom{2k-1,2k-2,\dots,1,0}{-} & \text{if } k \text{ is even,}\\ \binom{-}{2k-1,2k-2,\dots,1,0} & \text{if } k \text{ is odd,}\end{cases}\qquad \Lambda^{\mathrm{II}}_k=(\Lambda^{\mathrm{I}}_k)^{\mathrm{t}} \tag{5.2}$$

  and ζ^{I}_k, ζ^{II}_k the two cuspidal unipotent characters of $\mathrm{O}^{\epsilon_k}_{2k^2}$ given in the previous subsection;

- $\mathscr{L}_1(\binom{1}{0})=\mathbf{1}_{\mathrm{O}^+_2}$ and $\mathscr{L}_1(\binom{0}{1})=\mathrm{sgn}_{\mathrm{O}^+_2}$, i.e., $\rho_{\binom{1}{0}}=\mathbf{1}_{\mathrm{O}^+_2}$ and $\rho_{\binom{0}{1}}=\mathrm{sgn}_{\mathrm{O}^+_2}$.

Note that in addition to the specification of $\mathscr{L}_1$ on cuspidal symbols, due to Remark 3.10 we also need to assign the image of $\mathscr{L}_1$ at $\binom{1}{0}$ or $\binom{0}{1}$.

Proposition 5.6. *There is a unique bijective parametrization* $\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}\to\mathscr{E}(\mathrm{O}^\epsilon_{2n},1)$ *where* $\epsilon=+$ *or* $-$ *satisfying* (3.6), *and (I), (II) above.*

Proof. By Corollary 4.13, the existence of such a bijection $\mathscr{L}_1$ is obvious, so now we consider the uniqueness. Let $\mathscr{L}_1,\mathscr{L}'_1\colon\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}\to\mathscr{E}(\mathrm{O}^\epsilon_{2n},1)$ be two Lusztig parametrizations of unipotent characters for O^ϵ_{2n}. Moreover, suppose that $\mathscr{L}_1$, $\mathscr{L}'_1$ both satisfy (I) and (II) above. For $\Lambda\in\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}$, we know that $\mathscr{L}_1(\Lambda)^\sharp=\mathscr{L}'_1(\Lambda)^\sharp$ by Lemma 3.7, and hence by Corollary 4.12 either

$$\mathscr{L}_1(\Lambda)=\mathscr{L}'_1(\Lambda)\quad\text{or}\quad\mathscr{L}_1(\Lambda)=\mathscr{L}'_1(\Lambda)\cdot\mathrm{sgn}_{\mathrm{O}^\epsilon_{2n}}=\mathscr{L}'_1(\Lambda^{\mathrm{t}}),$$

i.e., if $\mathscr{L}_1(\Lambda)=\mathscr{L}'_1(\Lambda')$, then either $\Lambda'=\Lambda$ or $\Lambda'=\Lambda^{\rm t}$. Now we suppose that $\mathscr{L}_1(\Lambda)=\mathscr{L}'_1(\Lambda')$ and consider the following three cases:

(1) Suppose that Λ is degenerate, i.e., $\Lambda=\Lambda^{\rm t}$. Then $\mathscr{L}_1(\Lambda)=\mathscr{L}'_1(\Lambda')$ implies that $\mathscr{L}_1(\Lambda)=\mathscr{L}'_1(\Lambda)$ immediately.

(2) Suppose that $\operatorname{def}(\Lambda)\neq 0$. Suppose that the unipotent character $\mathscr{L}_1(\Lambda)=\mathscr{L}'_1(\Lambda')$ where $\Lambda'=\Lambda$ or $\Lambda^{\rm t}$ is in the Harish–Chandra series initiated by some unipotent cuspidal character ζ. Because $\operatorname{def}(\Lambda)\neq 0$, we have $\zeta\neq\zeta\cdot\operatorname{sgn}$. By the requirement in (II), we have $\mathscr{L}_1(\Lambda_0)=\zeta=\mathscr{L}'_1(\Lambda_0)$ for some cuspidal symbol Λ_0 such that $\operatorname{def}(\Lambda_0)\neq 0$. By (I), we must have $\operatorname{def}(\Lambda)=\operatorname{def}(\Lambda')$, and then we conclude that $\Lambda'=\Lambda$, i.e., $\mathscr{L}_1(\Lambda)=\mathscr{L}'_1(\Lambda)$.

(3) Suppose that Λ is non-degenerate and $\operatorname{def}(\Lambda)=0$, i.e., $\Lambda\in\mathscr{S}_{\mathrm{O}^+_{2n}}$ for some n. Now we are going to prove this case by induction on n. For $n=1$, the equality $\mathscr{L}_1(\Lambda)=\mathscr{L}'_1(\Lambda)$ is enforced by (II) above. Now suppose that $n\geq 2$. Because now $\Lambda^{\rm t}\neq\Lambda$, by Lemma 3.9, there exists $\Lambda_1\in\mathscr{S}_{\mathrm{O}^+_{2(n-1)}}$ such that $\Lambda\in\Omega(\Lambda_1)$ and $\Lambda^{\rm t}\notin\Omega(\Lambda_1)$. By (I) and the induction hypothesis, we have

$$\mathscr{L}_1(\Lambda)\in\Omega(\mathscr{L}_1(\Lambda_1))=\Omega(\mathscr{L}'_1(\Lambda_1))\not\ni\mathscr{L}'_1(\Lambda^{\rm t}).$$

Now $\mathscr{L}'_1(\Lambda^{\rm t})\neq\mathscr{L}_1(\Lambda)$ implies that $\mathscr{L}'_1(\Lambda)=\mathscr{L}_1(\Lambda)$.

Hence the proposition is proved. □

Corollary 5.7. *Let* $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$*, and let* $\Lambda\mapsto\rho_\Lambda$ *be the Lusztig parametrization in Proposition* 5.6*. Then*

(i) $\mathbf{1}_{\mathrm{O}^-_{2n}}=\rho_{\binom{-}{n,0}}$ *and* $\operatorname{sgn}_{\mathrm{O}^-_{2n}}=\rho_{\binom{n,0}{-}}$;

(ii) $\mathbf{1}_{\mathrm{O}^+_{2n}}=\rho_{\binom{n}{0}}$ *and* $\operatorname{sgn}_{\mathrm{O}^+_{2n}}=\rho_{\binom{0}{n}}$.

Proof. Let $\Lambda\mapsto\rho_\Lambda$ be the parametrization for O^ϵ_{2n} satisfying (3.6) and (I), (II) above. We know that $\mathbf{1}_{\mathrm{O}^-_2}$ (resp. $\operatorname{sgn}_{\mathrm{O}^-_2}$) first occurs in the correspondence for the pair $(\mathrm{O}^-_2,\mathrm{Sp}_0)$ (resp. $(\mathrm{O}^-_2,\mathrm{Sp}_4)$). Therefore by the requirement in (II) above, we have $\mathbf{1}_{\mathrm{O}^-_2}=\zeta^{\rm I}_1=\rho_{\binom{-}{1,0}}$ (resp. $\operatorname{sgn}_{\mathrm{O}^-_2}=\zeta^{\rm II}_1=\rho_{\binom{1,0}{-}}$). Write $\mathbf{1}_{\mathrm{O}^-_{2n}}=\rho_\Lambda$ for some $\Lambda\in\mathscr{S}_{\mathrm{O}^-_{2n}}$. By Corollary 4.14 we know that Λ is either $\binom{-}{n,0}$ or $\binom{n,0}{-}$. Because $\mathbf{1}_{\mathrm{O}^-_{2n}}$ is an irreducible constituent of $R^{\mathrm{O}^-_{2n}}_{\mathrm{O}^-_2\times\mathrm{GL}_1^{n-1}}(\rho_{\binom{-}{1,0}}\otimes\mathbf{1})$. By (I) above, we must have $\operatorname{def}(\Lambda)=\operatorname{def}\big(\binom{-}{1,0}\big)=-2$, and so we conclude that $\mathbf{1}_{\mathrm{O}^-_{2n}}=\rho_{\binom{-}{n,0}}$ and hence $\operatorname{sgn}_{\mathrm{O}^-_{2n}}=\rho_{\binom{n,0}{-}}$.

Now we are going to prove case (ii) by induction on n. By (II) above, we have $\rho_{\binom{1}{0}}=\mathbf{1}_{\mathrm{O}^+_2}$ and $\rho_{\binom{0}{1}}=\operatorname{sgn}_{\mathrm{O}^+_2}$. Now by the induction hypothesis, for $n\geq 2$, we assume that $\mathbf{1}_{\mathrm{O}^+_{2(n-1)}}=\rho_{\binom{n-1}{0}}$ and $\operatorname{sgn}_{\mathrm{O}^+_{2(n-1)}}=\rho_{\binom{0}{n-1}}$. Suppose that $\mathbf{1}_{\mathrm{O}^+_{2n}}=\rho_\Lambda$ for some $\Lambda\in\mathscr{S}_{\mathrm{O}^+_{2n}}$. Then we know that either $\Lambda=\binom{n}{0}$ or $\Lambda=\binom{0}{n}$ by Corollary 4.14. Because $\mathbf{1}_{\mathrm{O}^+_{2n}}\in\Omega(\mathbf{1}_{\mathrm{O}^+_{2(n-1)}})$, by (II) we see that $\Lambda\in\Omega\big(\binom{n-1}{0}\big)$ and therefore Λ must be $\binom{n}{0}$, i.e., we conclude that $\mathbf{1}_{\mathrm{O}^+_{2n}}=\rho_{\binom{n}{0}}$ and $\operatorname{sgn}_{\mathrm{O}^+_{2n}}=\rho_{\binom{0}{n}}$. □

Example 5.8. Keep the notations in Example 4.15. Suppose that $\rho_1,\rho_2\in\mathscr{E}(\mathrm{O}_4^-,1)$ are the two irreducible characters of degree q^2 satisfying

$$R^{\mathrm{O}_4^-}_{\mathrm{O}_2^-\times\mathrm{GL}_1}(\mathbf{1}_{\mathrm{O}_2^-}\otimes\mathbf{1})=\mathbf{1}_{\mathrm{O}_4^-}+\rho_1,\quad R^{\mathrm{O}_4^-}_{\mathrm{O}_2^-\times\mathrm{GL}_1}(\mathrm{sgn}_{\mathrm{O}_2^-}\otimes\mathbf{1})=\mathrm{sgn}_{\mathrm{O}_4^-}+\rho_2.$$

Because $\mathbf{1}_{\mathrm{O}_2^-}=\rho_{\binom{-}{1,0}}$ and $\mathrm{sgn}_{\mathrm{O}_2^-}=\rho_{\binom{1,0}{-}}$, the parametrization $\mathscr{S}_{\mathrm{O}_4^-}\to\mathscr{E}(\mathrm{O}_4^-,1)$ satisfying (I), (II) above must be

$$\rho_{\binom{-}{2,0}}=\mathbf{1}_{\mathrm{O}_4^-},\quad \rho_{\binom{2,0}{-}}=\mathrm{sgn}_{\mathrm{O}_4^-},\quad \rho_{\binom{1}{2,1,0}}=\rho_1,\quad \rho_{\binom{2,1,0}{1}}=\rho_2.$$

The following proposition which justifies our choice of $\mathscr{L}_1$ in Proposition 5.6 is from [17, Theorem 1.8].

Proposition 5.9. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{O}^\epsilon_{2n'},\mathrm{Sp}_{2n})$ *where* $\epsilon=+$ *or* $-$. *Let* $\mathscr{L}_1\colon\mathscr{S}_{\mathbf{G}}\to\mathscr{E}(\mathbf{G},1)$ *and* $\mathscr{L}_1'\colon\mathscr{S}_{\mathbf{G}'}\to\mathscr{E}(\mathbf{G}',1)$ *be the unique Lusztig parametrizations given in Propositions* 5.6 *and* 4.2 *respectively. Then the diagram*

$$\begin{tikzcd}
\mathscr{S}_{\mathbf{G}} \arrow[r, "\mathscr{B}_{\mathbf{G},\mathbf{G}'}"] \arrow[d, "\mathscr{L}_1"'] & \mathscr{S}_{\mathbf{G}'} \arrow[d, "\mathscr{L}_1'"] \\
\mathscr{E}(\mathbf{G},1) \arrow[r, "\Theta^\psi_{\mathbf{G},\mathbf{G}',1}"] & \mathscr{E}(\mathbf{G}',1)
\end{tikzcd}$$

commutes, i.e., $(\rho_\Lambda,\rho_{\Lambda'})$ *occurs in* $\Theta^\psi_{\mathbf{G},\mathbf{G}',1}$ *if and only if* $(\Lambda,\Lambda')\in\mathscr{B}_{\mathbf{G},\mathbf{G}'}$.

When both Λ, Λ' are cuspidal, the commutativity of the above diagram can be seen by the requirement (II) (cf. Example 5.3). For general Λ, Λ', the commutativity follows from the fact that both the correspondence $\Theta^\psi_{\mathbf{G},\mathbf{G}'}$ and the parametrizations $\mathscr{L}_1$, $\mathscr{L}_1'$ are compatible with the parabolic induction. Details of the proof can be found in [17].

Example 5.10. Let $\mathscr{L}_1\colon\mathscr{S}_{\mathrm{O}^\epsilon_{2n}}\to\mathscr{E}(\mathrm{O}^\epsilon_{2n},1)$ by $\Lambda\mapsto\rho_\Lambda$ be the parametrization in Proposition 5.6. Then by Corollary 4.14 the two Steinberg characters of O^ϵ_{2n} are parametrized by the symbols

$$\begin{cases}\binom{n,n-1,\dots,1}{n-1,n-2,\dots,0},\ \binom{n-1,n-2,\dots,0}{n,n-1,\dots,1} & \text{if } \epsilon=+,\\ \binom{n,n-1,\dots,0}{n-1,n-2,\dots,1},\ \binom{n-1,n-2,\dots,1}{n,n-1,\dots,0} & \text{if } \epsilon=-.\end{cases}$$

(1) If $\epsilon=+$, then one of the Steinberg characters of O^+_{2n} first occurs in the correspondence $\Theta^\psi_{\mathrm{O}^+_{2n},\mathrm{Sp}_{2(n-1)},1}$ and is paired with $\mathrm{St}_{\mathrm{Sp}_{2(n-1)}}$, and this Steinberg character is parametrized by the symbol $\binom{n,n-1,\dots,1}{n-1,n-2,\dots,0}$.

(2) If $\epsilon=-$, then one of the Steinberg characters of O^-_{2n} first occurs in the correspondence $\Theta^\psi_{\mathrm{O}^-_{2n},\mathrm{Sp}_{2(n-1)},1}$ and is paired with $\mathrm{St}_{\mathrm{Sp}_{2(n-1)}}$, and this Steinberg character is parametrized by the symbol $\binom{n-1,n-2,\dots,1}{n,n-1,\dots,0}$.

6. Lusztig correspondence and finite theta correspondence

6.1. Lusztig correspondences

Let $\mathbf{G}$ be a classical group, and let s be a semisimple element in the connected component $(G^*)^0$ of G^*. A rational maximal torus $\mathbf{T}^*$ in $\mathbf{G}^*$ contains s if and only if it is a rational maximal torus in $C_{\mathbf{G}^*}(s)$. From [3, Theorem 7.3.4], it is known that

$$\langle R^{\mathbf{G}}_{\mathbf{T}^*,s}, R^{\mathbf{G}}_{\mathbf{T}'^*,s}\rangle_{\mathbf{G}} = \langle R^{C_{\mathbf{G}^*}(s)}_{\mathbf{T}^*,1}, R^{C_{\mathbf{G}^*}(s)}_{\mathbf{T}'^*,1}\rangle_{C_{\mathbf{G}^*}(s)}$$

for any rational maximal tori $\mathbf{T}^*$, $\mathbf{T}'^*$ of $\mathbf{G}^*$ containing s. Then the mapping

$$\epsilon_{\mathbf{G}} R^{\mathbf{G}}_{\mathbf{T}^*,s} \mapsto \epsilon_{C_{\mathbf{G}^*}(s)} R^{C_{\mathbf{G}^*}(s)}_{\mathbf{T}^*,1}$$

can be extended uniquely to an isometry from $\mathscr{V}(\mathbf{G},s)^\sharp$ onto $\mathscr{V}(C_{\mathbf{G}^*}(s),1)^\sharp$. Now a Lusztig correspondence $\mathfrak{L}_s\colon \mathscr{E}(\mathbf{G},s) \to \mathscr{E}(C_{\mathbf{G}^*}(s),1)$, i.e., a bijective mapping satisfying (1.2), can be extended linearly to be an isometry, still denoted by $\mathfrak{L}_s$, of inner product spaces

$$\mathfrak{L}_s\colon \mathscr{V}(\mathbf{G},s) \to \mathscr{V}(C_{\mathbf{G}^*}(s),1)$$

whose restriction to $\mathscr{V}(\mathbf{G},s)^\sharp$ is uniquely determined.

Suppose that $\mathbf{G} = \prod_{i=1}^k \mathbf{G}_k$. Then $\mathbf{G}^* = \prod_{i=1}^k \mathbf{G}_i^*$ where $\mathbf{G}_i^*$ is the dual group of $\mathbf{G}_i$ for each i. For a semisimple element $s \in G^*$, we write $s = (s_1,\ldots,s_k)$ where each $s_i \in G_i^*$ is semisimple, and then $C_{\mathbf{G}^*}(s) = \prod_{i=1}^k C_{\mathbf{G}_i^*}(s_i)$. Now a rational maximal torus $\mathbf{T}^*$ containing s can be written as $\mathbf{T}^* = \prod_{i=1}^k \mathbf{T}_i^*$ where $\mathbf{T}_i^*$ is a rational maximal torus in $\mathbf{G}_i^*$. Therefore, we have $R^{\mathbf{G}}_{\mathbf{T}^*,s} = \bigotimes_{i=1}^k R^{\mathbf{G}_i}_{\mathbf{T}_i^*,s_i}$.

Corollary 6.1. *Then a bijection $\mathfrak{L}_s\colon \mathscr{E}(\mathbf{G},s) \to \mathscr{E}(C_{\mathbf{G}^*}(s),1)$ is a Lusztig correspondence if and only if $\mathfrak{L}_s = \prod_{i=1}^k \mathfrak{L}_{s_i}$ where each $\mathfrak{L}_{s_i}\colon \mathscr{E}(\mathbf{G}_i,s_i) \to \mathscr{E}(C_{\mathbf{G}_i^*}(s_i),1)$ is a Lusztig correspondence.*

Proof. This is obvious. □

For a semisimple element $s \in (G^*)^0$ with eigenvalues $\lambda_1,\ldots,\lambda_n$, we define

$$\begin{aligned}
\mathbf{G}^{(0)} &= \mathbf{G}^{(0)}(s) = \prod_{\langle\lambda\rangle\subset\{\lambda_1,\ldots,\lambda_n\},\,\lambda\neq\pm1} \mathbf{G}_{[\lambda]}(s),\\
\mathbf{G}^{(1)} &= \mathbf{G}^{(1)}(s) = \mathbf{G}_{[-1]}(s),\\
\mathbf{G}^{(2)} &= \mathbf{G}^{(2)}(s) = \mathbf{G}_{[1]}(s),
\end{aligned}$$

where $\mathbf{G}_{[\lambda]}(s)$ is given in [2, Subsection 1.B] (see also [18, Subsection 2.2]). We know that

$$C_{\mathbf{G}^*}(s) \simeq \mathbf{G}^{(0)} \times \mathbf{G}^{(1)} \times \mathbf{G}^{(2)},$$

and $\mathbf{G}^{(0)}$ is a product of general linear groups or unitary group, and

$$(\mathbf{G}^{(1)},\mathbf{G}^{(2)})=\begin{cases}(\mathrm{Sp}_{2n^{(1)}},\mathrm{Sp}_{2n^{(2)}}) & \text{if } \mathbf{G}=\mathrm{SO}_{2n+1},\\ (\mathrm{O}^{\epsilon^{(1)}}_{2n^{(1)}},\mathrm{SO}_{2n^{(2)}+1}) & \text{if } \mathbf{G}=\mathrm{Sp}_{2n},\\ (\mathrm{O}^{\epsilon^{(1)}}_{2n^{(1)}},\mathrm{O}^{\epsilon^{(2)}}_{2n^{(2)}}) & \text{if } \mathbf{G}=\mathrm{O}^{\epsilon}_{2n}\end{cases}$$

for some nonnegative integers $n^{(1)}$, $n^{(2)}$ depending on s, and some $\epsilon^{(1)}$, $\epsilon^{(2)}$. Note that if $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$, then $\epsilon^{(1)}$, $\epsilon^{(2)}$ also depend on s (and ϵ), if $\mathbf{G}=\mathrm{Sp}_{2n}$, then $\epsilon^{(1)}$ can be $+$ or $-$ for each s such that $n^{(1)}\geq 1$. The element s can be written as

$$s=s^{(0)}\times s^{(1)}\times s^{(2)}$$

where $s^{(1)}$ (resp. $s^{(2)}$) is the part whose eigenvalues are all equal to -1 (resp. 1), and $s^{(0)}$ is the part whose eigenvalues do not contain 1 or -1. In particular, $s^{(j)}$ is in the center of $\mathbf{G}^{(j)}$. Then a Lusztig correspondence

$$\mathfrak{L}_s\colon \mathscr{E}(\mathbf{G},s)\rightarrow\mathscr{E}(\mathbf{G}^{(0)}\times\mathbf{G}^{(1)}\times\mathbf{G}^{(2)},1) \tag{6.1}$$

can be written as

$$\mathfrak{L}_s(\rho)=\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)} \tag{6.2}$$

where $\rho^{(j)}\in\mathscr{E}(\mathbf{G}^{(j)},1)$ for $j=0,1,2$. It is known that ρ is cuspidal if and only if $\mathbf{G}^{(0)}$ is a product of unitary groups (i.e., no general linear groups) and each $\rho^{(j)}$ is cuspidal.

Now $\mathscr{S}^\sharp_{C_{\mathbf{G}^*}(s)}=\mathscr{S}^\sharp_{\mathbf{G}^{(0)}}\times\mathscr{S}^\sharp_{\mathbf{G}^{(1)}}\times\mathscr{S}^\sharp_{\mathbf{G}^{(2)}}$ and

$$R^{C_{\mathbf{G}^*}(s)}_\Sigma=R^{\mathbf{G}^{(0)}}_x\otimes R^{\mathbf{G}^{(1)}}_{\Sigma^{(1)}}\otimes R^{\mathbf{G}^{(2)}}_{\Sigma^{(2)}}\in\mathscr{V}(C_{\mathbf{G}^*}(s),1)^\sharp$$

for $\Sigma=(x,\Sigma^{(1)},\Sigma^{(2)})\in\mathscr{S}^\sharp_{C_{\mathbf{G}^*}(s)}$. We define

$$R^{\mathbf{G}}_\Sigma=\mathfrak{L}_s^{-1}(R^{C_{\mathbf{G}^*}(s)}_\Sigma)\in\mathscr{V}(\mathbf{G},s)^\sharp. \tag{6.3}$$

Because $\{R^{C_{\mathbf{G}^*}(s)}_\Sigma\mid\Sigma\in\mathscr{S}^\sharp_{C_{\mathbf{G}^*}(s)}\}$ is an orthonormal basis for $\mathscr{V}(C_{\mathbf{G}^*}(s),1)^\sharp$ and

$$\mathfrak{L}_s\colon\mathscr{V}(\mathbf{G},s)\rightarrow\mathscr{V}(C_{\mathbf{G}^*}(s),1)$$

is an isometry which maps $\mathscr{V}(\mathbf{G},s)^\sharp$ onto $\mathscr{V}(C_{\mathbf{G}^*}(s),1)^\sharp$, we see that $\{R^{\mathbf{G}}_\Sigma\mid\Sigma\in\mathscr{S}^\sharp_{C_{\mathbf{G}^*}(s)}\}$ forms an orthonormal basis for the space $\mathscr{V}(\mathbf{G},s)^\sharp$. For $\rho\in\mathscr{E}(\mathbf{G},s)$, we have

$$\rho^\sharp=\sum_{\Sigma\in\mathscr{S}^\sharp_{C_{\mathbf{G}^*}(s)}}\langle\rho,R^{\mathbf{G}}_\Sigma\rangle_{\mathbf{G}}R^{\mathbf{G}}_\Sigma,\quad \mathfrak{L}_s(\rho)^\sharp=\sum_{\Sigma\in\mathscr{S}^\sharp_{C_{\mathbf{G}^*}(s)}}\langle\mathfrak{L}_s(\rho),R^{C_{\mathbf{G}^*}(s)}_\Sigma\rangle_{C_{\mathbf{G}^*}(s)}R^{C_{\mathbf{G}^*}(s)}_\Sigma. \tag{6.4}$$

Therefore we have $\mathfrak{L}_s(\rho^\sharp)=\mathfrak{L}_s(\rho)^\sharp$ for any $\rho\in\mathscr{E}(\mathbf{G},s)$.

Lemma 6.2. *Let $\mathfrak{L}_s, \mathfrak{L}'_s\colon \mathscr{E}(\mathbf{G},s)\to\mathscr{E}(C_{\mathbf{G}^*}(s),1)$ be two Lusztig correspondences, and write $\mathfrak{L}_s(\rho)=\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}$, $\mathfrak{L}'_s(\rho)=\rho'^{(0)}\otimes\rho'^{(1)}\otimes\rho'^{(2)}$. Then*

$$\rho^{(0)}=\rho'^{(0)},\quad (\rho^{(1)})^\sharp=(\rho'^{(1)})^\sharp,\quad (\rho^{(2)})^\sharp=(\rho'^{(2)})^\sharp.$$

Proof. Because $\mathfrak{L}_s(\rho^\sharp)=\mathfrak{L}_s(\rho)^\sharp$, we have

$$\mathfrak{L}_s(\rho^\sharp)=(\rho^{(0)})^\sharp\otimes(\rho^{(1)})^\sharp\otimes(\rho^{(2)})^\sharp,\quad \mathfrak{L}'_s(\rho^\sharp)=(\rho'^{(0)})^\sharp\otimes(\rho'^{(1)})^\sharp\otimes(\rho'^{(2)})^\sharp.$$

Because the restrictions of $\mathfrak{L}_s$ and $\mathfrak{L}'_s$ to $\mathscr{V}(\mathbf{G},s)^\sharp$ are the same, i.e., $\mathfrak{L}_s(\rho^\sharp)=\mathfrak{L}'_s(\rho^\sharp)$, we have $(\rho^{(0)})^\sharp=(\rho'^{(0)})^\sharp$, $(\rho^{(1)})^\sharp=(\rho'^{(1)})^\sharp$, $(\rho^{(2)})^\sharp=(\rho'^{(2)})^\sharp$. Now $\mathbf{G}^{(0)}$ is a product of general linear groups or unitary groups, so we have $\rho^{(0)}=(\rho^{(0)})^\sharp=(\rho'^{(0)})^\sharp=\rho'^{(0)}$. $\square$

6.2. Lusztig correspondence and parabolic induction

Let $\mathbf{G}_n=\mathrm{SO}_{2n+1}$, Sp_{2n} or O^ϵ_{2n} where $\epsilon=+$ or $-$. The group $\mathbf{G}_n\times\mathrm{GL}_l$ is the Levi factor of a parabolic subgroup of $\mathbf{G}_{n+l}$. Let σ be an irreducible cuspidal character of GL_l, and so $\sigma\in\mathscr{E}(\mathrm{GL}_l,t)$ for some semisimple element $t\in\mathrm{GL}_l(q)$. For $\rho\in\mathscr{E}(\mathbf{G}_n,s)$, an irreducible constituent of $R^{\mathbf{G}_{n+l}}_{\mathbf{G}_n\times\mathrm{GL}_l}(\rho\otimes\sigma)$ is in $\mathscr{E}(\mathbf{G}_{n+l},s')$ where $s'=(s,t)$ is regarded as an element in $(G^*_{n+l})^0$. Then we define a relation $\Omega_t\colon\mathscr{E}(\mathbf{G}_n,s)\to\mathscr{E}(\mathbf{G}_{n+l},s')$ by

$$\Omega_t(\rho)=\left\{\rho'\in\mathscr{E}(\mathbf{G}_{n+l},s')\mid \langle\rho',R^{\mathbf{G}_{n+l}}_{\mathbf{G}_n\times\mathrm{GL}_l}(\rho\otimes\sigma)\rangle_{\mathbf{G}_{n+l}}\neq 0\right\}.$$

Because we assume that σ is cuspidal, one can see that $C_{\mathbf{G}^*_n}(s)\times\mathrm{GL}^\dagger_1$ is a Levi subgroup of $C_{\mathbf{G}^*_{n+l}}(s')$ where $\mathrm{GL}^\dagger_1$ denotes the restriction to $\mathbf{F}_q$ of GL_1 defined over a finite extension (depending on t) of $\mathbf{F}_q$. Suppose that $\rho'\in\Omega_t(\rho)$, it is clear that $\rho'^c\in\Omega_t(\rho^c)$ for $\mathbf{G}_n=\mathrm{Sp}_{2n}$, O^ϵ_{2n} (cf. [23, §4.4]), and $\rho'\cdot\mathrm{sgn}_{\mathbf{G}_{n+l}}\in\Omega_t(\rho\cdot\mathrm{sgn}_{\mathbf{G}_n})$ for $\mathbf{G}_n=\mathrm{O}^\epsilon_{2n}$.

We define the relation $\Omega\colon\mathscr{E}(C_{\mathbf{G}^*_n}(s),1)\to\mathscr{E}(C_{\mathbf{G}^*_{n+l}}(s'),1)$ as in (3.1), i.e.,

$$\Omega(\rho)=\left\{\rho'\in\mathscr{E}(C_{\mathbf{G}^*_{n+l}}(s'),1)\mid \langle\rho',R^{C_{\mathbf{G}^*_{n+l}}(s')}_{C_{\mathbf{G}^*_n}(s)\times\mathrm{GL}^\dagger_1}(\rho\otimes\mathbf{1})\rangle_{C_{\mathbf{G}^*_{n+l}}(s')}\neq 0\right\}$$

for $\rho\in\mathscr{E}(C_{\mathbf{G}^*_n}(s),1)$. Then we say that a Lusztig correspondence

$$\mathfrak{L}_s\colon\mathscr{E}(\mathbf{G},s)\to\mathscr{E}(C_{\mathbf{G}^*}(s),1)$$

is *compatible with the parabolic induction* if the following diagram

$$\begin{tikzcd}
\mathscr{E}(\mathbf{G}_n,s) \arrow[r, "\Omega_t"] \arrow[d, "\mathfrak{L}_s"'] & \mathscr{E}(\mathbf{G}_{n+l},s') \arrow[d, "\mathfrak{L}_{s'}"] \\
\mathscr{E}(C_{\mathbf{G}^*_n}(s),1) \arrow[r, "\Omega"] & \mathscr{E}(C_{\mathbf{G}^*_{n+l}}(s'),1)
\end{tikzcd} \tag{6.5}$$

commutes for any s and $s'=(s,t)$ given as above, i.e., for any $\rho\in\mathscr{E}(\mathbf{G}_n,s)$, the two subsets $\mathfrak{L}_{s'}(\Omega_t(\rho))$ and $\Omega(\mathfrak{L}_s(\rho))$ of $\mathscr{E}(C_{\mathbf{G}^*_{n+l}}(s'),1)$ are equal (cf. (3.2)).

Now write

$$C_{\mathbf{G}^*_n}(s)=\mathbf{G}^{(0)}(s)\times\mathbf{G}^{(1)}(s)\times\mathbf{G}^{(2)}(s),\quad C_{\mathbf{G}^*_{n+l}}(s')=\mathbf{G}^{(0)}(s')\times\mathbf{G}^{(1)}(s')\times\mathbf{G}^{(2)}(s')$$

and $\mathfrak{L}_s(\rho)=\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}$ as in (6.2). Then the diagram (6.5) can be described more precisely according to the following three cases:

(1) If $t=1$ and so $l=1$, then $\mathbf{G}^{(0)}(s)=\mathbf{G}^{(0)}(s')$ and $\mathbf{G}^{(1)}(s)=\mathbf{G}^{(1)}(s')$, and then the relation Ω is given by

$$\Omega(\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)})=\rho^{(0)}\otimes\rho^{(1)}\otimes\Omega(\rho^{(2)})$$

where $\Omega(\rho^{(2)})$ is defined as in (3.1).

(2) If $t=-1$ and so $l=1$, then $\mathbf{G}^{(0)}(s)=\mathbf{G}^{(0)}(s')$ and $\mathbf{G}^{(2)}(s)=\mathbf{G}^{(2)}(s')$, and then

$$\Omega(\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)})=\rho^{(0)}\otimes\Omega(\rho^{(1)})\otimes\rho^{(2)}$$

where $\Omega(\rho^{(1)})$ is defined as in (3.1).

(3) If $t\neq\pm1$, then $\mathbf{G}^{(1)}(s)=\mathbf{G}^{(1)}(s')$ and $\mathbf{G}^{(2)}(s)=\mathbf{G}^{(2)}(s')$, and then

$$\Omega(\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)})=\Omega(\rho^{(0)})\otimes\rho^{(1)}\otimes\rho^{(2)}$$

where $\Omega(\rho^{(0)})$ is defined as in (3.1).

6.3. Modified Lusztig correspondence

For a semisimple element $s\in G^*$, we define

$$\mathbf{G}^{(-)}=\mathbf{G}^{(1)},\quad \mathbf{G}^{(+)}=\begin{cases}\mathbf{G}^{(2)} & \text{if } \mathbf{G}=\mathrm{SO}_{2n+1}\text{ or }\mathrm{O}^\epsilon_{2n},\\ (\mathbf{G}^{(2)})^* & \text{if }\mathbf{G}=\mathrm{Sp}_{2n}.\end{cases}\tag{6.6}$$

Combining $\mathscr{L}_1$ in Proposition 3.5 (for $\mathbf{G}^{(0)}\times\mathbf{G}^{(-)}\times\mathbf{G}^{(+)}$) and the inverse of $\mathfrak{L}_s$ in (6.1), we obtain a bijection

$$\begin{aligned}\mathscr{L}_s\colon\ \mathscr{S}_{\mathbf{G}^{(0)}(s)}\times\mathscr{S}_{\mathbf{G}^{(-)}(s)}\times\mathscr{S}_{\mathbf{G}^{(+)}(s)} &\ \to\ \mathscr{E}(\mathbf{G},s)\\ (x,\Lambda_1,\Lambda_2) &\ \mapsto\ \rho_{x,\Lambda_1,\Lambda_2}=\rho^{\mathbf{G}}_{x,\Lambda_1,\Lambda_2}.\end{aligned}$$

Note that from Proposition 4.2, Lemmas 4.10 and 6.2, we have

(1) $(\rho^{\mathrm{Sp}_{2n}}_{x,\Lambda_1,\Lambda_2})^\sharp=(\rho^{\mathrm{Sp}_{2n}}_{x',\Lambda'_1,\Lambda'_2})^\sharp$ if and only if

- $x'=x$,

- $\Lambda_1' = \Lambda_1, \Lambda_1^{\rm t}$,
- $\Lambda_2' = \Lambda_2$;

(2) $(\rho^{\mathrm{O}^\epsilon_{2n}}_{x,\Lambda_1,\Lambda_2})^\sharp = (\rho^{\mathrm{O}^\epsilon_{2n}}_{x',\Lambda_1',\Lambda_2'})^\sharp$ if and only if

- $x' = x$,
- $\Lambda_1' = \Lambda_1, \Lambda_1^{\rm t}$,
- $\Lambda_2' = \Lambda_2, \Lambda_2^{\rm t}$.

Moreover, diagram (6.5) becomes

$$\begin{tikzcd} \mathscr{S}_{\mathbf{G}^{(0)}(s)} \times \mathscr{S}_{\mathbf{G}^{(-)}(s)} \times \mathscr{S}_{\mathbf{G}^{(+)}(s)} \arrow[r, "\Omega"] \arrow[d, "\mathscr{L}_s"'] & \mathscr{S}_{\mathbf{G}^{(0)}(s')} \times \mathscr{S}_{\mathbf{G}^{(-)}(s')} \times \mathscr{S}_{\mathbf{G}^{(+)}(s')} \arrow[d, "\mathscr{L}_{s'}"] \\ \mathscr{E}(\mathbf{G}_n, s) \arrow[r, "\Omega_t"] & \mathscr{E}(\mathbf{G}_{n+l}, s') \end{tikzcd} \tag{6.7}$$

where the relation Ω is given as in Subsections 3.1 or 3.3.

Remark 6.3. If s is a semisimple element in $(G^*)^0$ such that $\mathbf{G}^{(0)}(s)$ is trivial, then an irreducible character $\rho \in \mathscr{E}(\mathbf{G}, s)$ is called *quadratic unipotent*. A Lusztig correspondence $\mathscr{L}_s\colon (\Lambda_1, \Lambda_2) \mapsto \rho_{\Lambda_1,\Lambda_2}$ of quadratic unipotent characters for $\mathbf{G} = \mathrm{Sp}_{2n}$, SO_{2n+1} or O^ϵ_{2n} is given in [23, §4.11, §4.8, §4.4] respectively.

6.4. Theta correspondence and modified Lusztig correspondence

First suppose that $(\mathbf{G}, \mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{SO}_{2n'+1})$, and $s \in G^*$, $s' \in G'^*$ semisimple. Then $\mathbf{G}^{(-)} = \mathrm{O}^{\epsilon^{(-)}}_{2n^{(-)}}$, $\mathbf{G}^{(+)} = \mathrm{Sp}_{2n^{(+)}}$, $\mathbf{G}'^{(-)} = \mathrm{Sp}_{2n'^{(-)}}$, and $\mathbf{G}'^{(+)} = \mathrm{Sp}_{2n'^{(+)}}$, for some $\epsilon^{(-)}$ and some $n^{(-)}$, $n^{(+)}$, $n'^{(-)}$, $n'^{(+)}$. The following proposition is from [18, Proposition 8.3].

Proposition 6.4. *Let* $(\mathbf{G}, \mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{SO}_{2n'+1})$, *and* $s \in G^*$, $s' \in G'^*$ *semisimple. Let*

$$\mathscr{L}_s\colon \mathscr{S}_{\mathbf{G}^{(0)}(s)} \times \mathscr{S}_{\mathbf{G}^{(-)}(s)} \times \mathscr{S}_{\mathbf{G}^{(+)}(s)} \to \mathscr{E}(\mathbf{G}, s),$$
$$\mathscr{L}_{s'}\colon \mathscr{S}_{\mathbf{G}'^{(0)}(s')} \times \mathscr{S}_{\mathbf{G}'^{(-)}(s')} \times \mathscr{S}_{\mathbf{G}'^{(+)}(s')} \to \mathscr{E}(\mathbf{G}', s')$$

be any modified Lusztig correspondences for $\mathbf{G}$ *and* $\mathbf{G}'$ *respectively. Then one of*

$$(\rho_{x,\Lambda_1,\Lambda_2}, \rho_{x',\Lambda_1',\Lambda_2'}), \quad (\rho_{x,\Lambda_1^{\rm t},\Lambda_2}, \rho_{x',\Lambda_1',\Lambda_2'})$$

occurs in $\Theta^\psi_{\mathbf{G},\mathbf{G}'}$ *if and only if*

- $s^{(0)} = -s'^{(0)}$ *(up to conjugation) and* $x = x'$,
- $\Lambda_2 = \Lambda_1'$, *and*

- (Λ_1,Λ'_2) *or* $(\Lambda_1^{\rm t},\Lambda'_2)$ *is in* $\mathscr{B}_{\mathbf{G}^{(-)}(s),\mathbf{G}'^{(+)}(s')}$.

Remark 6.5. We shall see in Theorem 7.1 that the modified Lusztig correspondence $\mathscr{L}_{s'}$ for SO_{2n+1} is unique.

Next suppose that $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ where $\epsilon=+$ or $-$, and $s\in G^*$, $s'\in(G'^*)^0$ semisimple. Then $\mathbf{G}^{(-)}=\mathrm{O}^{\epsilon^{(-)}}_{2n^{(-)}}$, and $\mathbf{G}^{(+)}=\mathrm{Sp}_{2n^{(+)}}$, $\mathbf{G}'^{(-)}=\mathrm{O}^{\epsilon'^{(-)}}_{2n'^{(-)}}$, $\mathbf{G}'^{(+)}=\mathrm{O}^{\epsilon'^{(+)}}_{2n'^{(+)}}$, for some $\epsilon^{(-)}$, $\epsilon'^{(-)}$, $\epsilon'^{(+)}$, and some $n^{(-)}$, $n^{(+)}$, $n'^{(-)}$, $n'^{(+)}$. The following proposition is from [18, Proposition 8.1].

Proposition 6.6. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ *where* $\epsilon=+$ *or* $-$, *and* $s\in G^*$, $s'\in(G'^*)^0$ *semisimple. Let*

$$\mathscr{L}_s\colon \mathscr{S}_{\mathbf{G}^{(0)}(s)}\times\mathscr{S}_{\mathbf{G}^{(-)}(s)}\times\mathscr{S}_{\mathbf{G}^{(+)}(s)}\to\mathscr{E}(\mathbf{G},s),$$
$$\mathscr{L}_{s'}\colon \mathscr{S}_{\mathbf{G}'^{(0)}(s')}\times\mathscr{S}_{\mathbf{G}'^{(-)}(s')}\times\mathscr{S}_{\mathbf{G}'^{(+)}(s')}\to\mathscr{E}(\mathbf{G}',s')$$

be any modified Lusztig correspondences for $\mathbf{G}$ *and* $\mathbf{G}'$ *respectively. Then one of*

$$(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'_2}),\quad(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'^{\rm t}_1,\Lambda'_2}),\quad(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'^{\rm t}_2}),\quad(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'^{\rm t}_1,\Lambda'^{\rm t}_2})$$

occurs in $\Theta^\psi_{\mathbf{G},\mathbf{G}'}$ *if and only if*

- $s^{(0)}=s'^{(0)}$ *(up to conjugation) and* $x=x'$,
- $\Lambda_1=\Lambda'_1,\ \Lambda'^{\rm t}_1$, *and*
- (Λ_2,Λ'_2) *or* $(\Lambda_2,\Lambda'^{\rm t}_2)$ *is in* $\mathscr{B}_{\mathbf{G}^{(+)}(s),\mathbf{G}'^{(+)}(s')}$.

7. Lusztig correspondences for SO_{2n+1}

7.1. Lusztig correspondence for SO_{2n+1}

Let $\mathbf{G}=\mathrm{SO}_{2n+1}$. For a semisimple element $s\in G^*$, recall that (cf. (6.1), (6.2)) we have

$$\begin{aligned}\mathfrak{L}_s\colon\ \mathscr{E}(\mathbf{G},s)&\to\mathscr{E}(\mathbf{G}^{(0)}(s)\times\mathbf{G}^{(1)}(s)\times\mathbf{G}^{(2)}(s),1)\\ \rho&\mapsto\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}.\end{aligned}$$

Now we know that $\mathbf{G}^{(1)}(s)=\mathrm{Sp}_{2n^{(1)}}$ and $\mathbf{G}^{(2)}(s)=\mathrm{Sp}_{2n^{(2)}}$ for some nonnegative integers $n^{(1)}$, $n^{(2)}$ depending on s. The group SO_{2n+1} is connected with connected center, so the following result is considered in [5, Theorem 7.1].

Theorem 7.1. *Let* $\mathbf{G}=\mathrm{SO}_{2n+1}$ *and* $s\in G^*$. *There exists a unique bijection* $\mathfrak{L}_s\colon\mathscr{E}(\mathbf{G},s)\to\mathscr{E}(C_{\mathbf{G}^*}(s),1)$ *satisfying (1.2), i.e., the Lusztig correspondence is unique.*

Remark 7.2. The Lusztig correspondence $\mathfrak{L}_s$ given in the theorem satisfies the commutativity of the diagram in (6.5). This is a special case of [7, Theorem 4.7.5].

Corollary 7.3. *Let* $\mathbf{G} = \mathrm{SO}_{2n+1}$ *and* $s \in G^*$. *Then there is a unique modified Lusztig correspondence* $\mathscr{L}_s\colon \mathscr{S}_{\mathbf{G}^{(0)}(s)} \times \mathscr{S}_{\mathbf{G}^{(-)}(s)} \times \mathscr{S}_{\mathbf{G}^{(+)}(s)} \to \mathscr{E}(\mathbf{G}, s)$.

Corollary 7.4. *For* $\rho, \rho' \in \mathscr{E}(\mathrm{SO}_{2n+1})$, *then* $\rho'^\sharp = \rho^\sharp$ *if and only if* $\rho' = \rho$.

Corollary 7.5. *The bijection* $\mathfrak{L}_1\colon \mathscr{E}(\mathrm{SO}_{2n+1}, 1) \to \mathscr{E}(\mathrm{Sp}_{2n}, 1)$ *is given by* $\rho_\Lambda \mapsto \rho_{\Lambda^{\mathrm{t}}}$ *for* $\Lambda \in \mathscr{S}_{\mathrm{SO}_{2n+1}}$, *i.e., the diagram*

$$\begin{tikzcd}
\mathscr{S}_{\mathrm{SO}_{2n+1}} \arrow[r] \arrow[d, "\mathscr{L}_1"'] & \mathscr{S}_{\mathrm{Sp}_{2n}} \arrow[d, "\mathscr{L}_1"] \\
\mathscr{E}(\mathrm{SO}_{2n+1}, 1) \arrow[r, "\mathfrak{L}_1"] & \mathscr{E}(\mathrm{Sp}_{2n}, 1)
\end{tikzcd}$$

commutes where the mapping on the top is given by $\Lambda \mapsto \Lambda^{\mathrm{t}}$.

Proof. Recall that $W_n = W_{\mathrm{SO}_{2n+1}} = W_{\mathrm{Sp}_{2n}}$ and $\mathfrak{L}_1\colon R^{\mathrm{SO}_{2n+1}}_{\mathbf{T}^*_w,1} \mapsto R^{\mathrm{Sp}_{2n}}_{\mathbf{T}^*_w,1}$ for $w \in W_n$. From (3.4), we see that the isometry $\mathfrak{L}_1\colon \mathscr{V}(\mathrm{SO}_{2n+1}, 1) \to \mathscr{V}(\mathrm{Sp}_{2n}, 1)$ maps $R^{\mathrm{SO}_{2n+1}}_\Sigma \mapsto R^{\mathrm{Sp}_{2n}}_{\Sigma^{\mathrm{t}}}$ for $\Sigma \in \mathscr{S}^\sharp_{\mathrm{SO}_{2n+1}}$. Then we see that $\mathfrak{L}_1(\rho_\Lambda)^\sharp = (\rho_{\Lambda^{\mathrm{t}}})^\sharp$ for any $\Lambda \in \mathscr{S}_{\mathrm{SO}_{2n+1}}$. By Theorem 7.1, we conclude that $\mathfrak{L}_1(\rho_\Lambda) = \rho_{\Lambda^{\mathrm{t}}}$. ☐

Lemma 7.6. *Let* $\mathbf{G} = \mathrm{SO}_{2n+1}$, *and let* $\mathscr{L}_s$ *be the Lusztig correspondence given in Corollary* 7.3. *Then*

$$\chi_{\mathbf{G}}\rho_{x,\Lambda_1,\Lambda_2} = \rho_{x,\Lambda_2,\Lambda_1} \tag{7.1}$$

where $\chi_{\mathbf{G}}$ *denotes the character (of order two) derived from the spinor norm of* $\mathbf{G}$.

Proof. It is known that $\chi_{\mathbf{G}} R^{\mathbf{G}}_{\mathbf{T}^*,s} = R^{\mathbf{G}}_{\mathbf{T}^*,-s}$ for each pair $(\mathbf{T}^*, s)$, and $\mathbf{G}^{(0)}(-s) = \mathbf{G}^{(0)}(s)$, $\mathbf{G}^{(-)}(-s) = \mathbf{G}^{(+)}(s)$ and $\mathbf{G}^{(+)}(-s) = \mathbf{G}^{(-)}(s)$. Then the mapping $R^{\mathbf{G}}_{\mathbf{T}^*,s} \mapsto \chi_{\mathbf{G}} R^{\mathbf{G}}_{\mathbf{T}^*,s}$ induces an isometry $\mathscr{V}(\mathbf{G}, s)^\sharp \to \mathscr{V}(\mathbf{G}, -s)^\sharp$ such that $R^{\mathbf{G}}_{x,\Sigma_1,\Sigma_2} \mapsto R^{\mathbf{G}}_{x,\Sigma_2,\Sigma_1}$ where $(x, \Sigma_1, \Sigma_2) \in \mathscr{S}^\sharp_{\mathbf{G}^*(s)}$ (cf. (6.3)). This means that $(\chi_{\mathbf{G}}\rho_{x,\Lambda_1,\Lambda_2})^\sharp = (\rho_{x,\Lambda_2,\Lambda_1})^\sharp$ for any $x \in \mathscr{S}_{\mathbf{G}^{(0)}(s)}$, $\Lambda_1 \in \mathscr{S}_{\mathbf{G}^{(-)}(s)}$ and $\Lambda_2 \in \mathscr{S}_{\mathbf{G}^{(+)}(s)}$. Then we conclude that $\chi_{\mathbf{G}}\rho_{x,\Lambda_1,\Lambda_2} = \rho_{x,\Lambda_2,\Lambda_1}$ by Corollary 7.4. ☐

Remark 7.7. For the case that $\mathbf{G}^{(0)}(s)$ is trivial, (7.1) is also given in [23, Proposition 4.8].

8. Lusztig correspondence for O^ϵ_{2n}

8.1. Lusztig correspondence for O^ϵ_{2n}

Let $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$ where $\epsilon=+$ or $-$. For a semisimple element $s\in(G^*)^0$, recall that (cf. (6.1), (6.2)) we have

$$\begin{array}{rccc}\mathfrak{L}_s\colon & \mathscr{E}(\mathbf{G},s) & \rightarrow & \mathscr{E}(\mathbf{G}^{(0)}(s)\times\mathbf{G}^{(1)}(s)\times\mathbf{G}^{(2)}(s),1)\\ & \rho & \mapsto & \rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}.\end{array}$$

Now we know that $\mathbf{G}^{(1)}(s)=\mathrm{O}^{\epsilon^{(1)}}_{2n^{(1)}}$ and $\mathbf{G}^{(2)}(s)=\mathrm{O}^{\epsilon^{(2)}}_{2n^{(2)}}$ for some nonnegative integers $n^{(1)}$, $n^{(2)}$ and some $\epsilon^{(1)}$, $\epsilon^{(2)}$ depending on s such that $\epsilon^{(1)}\epsilon^{(2)}=\epsilon$.

Lemma 8.1. *Let $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$ where $\epsilon=+$ or $-$, and let $s\in(G^*)^0$ be semisimple. Let $\mathfrak{L}_s\colon\mathscr{E}(\mathbf{G},s)\rightarrow\mathscr{E}(C_{\mathbf{G}^*}(s),1)$ be a Lusztig correspondence and write $\mathfrak{L}_s(\rho)=\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}$. Moreover, let $\mathfrak{L}'_s\colon\mathscr{E}(\mathbf{G},s)\rightarrow\mathscr{E}(C_{\mathbf{G}^*}(s),1)$ be a bijective mapping and write $\mathfrak{L}'_s(\rho)=\rho'^{(0)}\otimes\rho'^{(1)}\otimes\rho'^{(2)}$. Then $\mathfrak{L}'_s$ is a Lusztig correspondence if and only if*

- $\rho'^{(0)}=\rho^{(0)}$;
- $\rho'^{(1)}=\rho^{(1)},\rho^{(1)}\cdot\mathrm{sgn}$;
- $\rho'^{(2)}=\rho^{(2)},\rho^{(2)}\cdot\mathrm{sgn}$.

Proof. First suppose that $\mathfrak{L}'_s$ is a Lusztig correspondence. By Lemma 6.2, we know that $\mathfrak{L}_s(\rho)^\sharp=\mathfrak{L}'_s(\rho)^\sharp$, i.e., $\rho^{(0)}=\rho'^{(0)}$, $(\rho^{(1)})^\sharp=(\rho'^{(1)})^\sharp$, $(\rho^{(2)})^\sharp=(\rho'^{(2)})^\sharp$. Now $\mathbf{G}^{(0)}$ is a product of general linear groups or unitary groups; $\mathbf{G}^{(1)}$ and $\mathbf{G}^{(2)}$ are even orthogonal groups. Then by Corollary 4.12, we have $\rho'^{(0)}=\rho^{(0)}$, and $\rho'^{(i)}=\rho^{(i)}$ or $\rho'^{(i)}=\rho^{(i)}\cdot\mathrm{sgn}$ for $i=1,2$.

Next suppose that $\rho'^{(0)}=\rho^{(0)}$, and $\rho'^{(i)}=\rho^{(i)}$ or $\rho'^{(i)}=\rho^{(i)}\cdot\mathrm{sgn}$ for $i=1,2$. Then we have $(\rho^{(1)})^\sharp=(\rho'^{(1)})^\sharp$, $(\rho^{(2)})^\sharp=(\rho'^{(2)})^\sharp$, i.e., $\mathfrak{L}_s(\rho)^\sharp=\mathfrak{L}'_s(\rho)^\sharp$ for any $\rho\in\mathscr{E}(\mathbf{G},s)$. This means that $\mathfrak{L}'_s$ also satisfies (1.2), i.e., $\mathfrak{L}'_s$ is also a Lusztig correspondence. $\square$

Recall that for $\rho\in\mathscr{E}(\mathbf{G})$ the character ρ^c is defined in Subsection 5.1.

Lemma 8.2. *Let $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$, $s\in(G^*)^0$ semisimple, and let $\mathfrak{L}_s\colon\mathscr{E}(\mathbf{G},s)\rightarrow\mathscr{E}(C_{\mathbf{G}^*}(s),1)$ be a Lusztig correspondence. Suppose that $\rho\in\mathscr{E}(\mathbf{G},s)$ and write $\mathfrak{L}_s(\rho)=\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}$. Then we have $\rho^c\in\mathscr{E}(\mathbf{G},s)$ and*

$$\mathfrak{L}_s(\rho^c)=\rho^{(0)}\otimes(\rho^{(1)}\cdot\mathrm{sgn})\otimes\rho^{(2)}.$$

Proof. When $\mathbf{G}^{(0)}$ is trivial, i.e., when ρ is quadratic unipotent, the result is proved in [23, §4.4]. The same argument still works here. $\square$

Corollary 8.3. *Let* $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$*,* $s\in(G^*)^0$ *semisimple. Suppose that* $\rho\in\mathscr{E}(\mathbf{G},s)$ *and* $\rho=\rho_{x,\Lambda_1,\Lambda_2}$ *under a modified Lusztig correspondence* $\mathscr{L}_s$*. Then* $\rho^c=\rho_{x,\Lambda_1^{\mathrm{t}},\Lambda_2}$*.*

Proof. This follows from Lemma 8.2 and Corollary 4.11 immediately. □

Corollary 8.4. *For* $\rho,\rho'\in\mathscr{E}(\mathrm{O}^\epsilon_{2n})$*, then* $\rho'^\sharp=\rho^\sharp$ *if and only if* $\rho'=\rho,\ \rho^c,\ \rho\cdot\mathrm{sgn},\ \rho^c\cdot\mathrm{sgn}$.

Proof. Let $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$, and let $\rho,\rho'\in\mathscr{E}(\mathbf{G})$. By Lemmas 8.2, 6.2 and (4.2), we have $(\rho^c)^\sharp=\rho^\sharp$. Moreover, because of (3.3), we have $R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T}^*,s}\cdot\mathrm{sgn}=R^{\mathrm{O}^\epsilon_{2n}}_{\mathbf{T}^*,s}$ and then $(\rho\cdot\mathrm{sgn})^\sharp=\rho^\sharp$. Therefore, if $\rho'=\rho$, ρ^c, $\rho\cdot\mathrm{sgn}$, or $\rho^c\cdot\mathrm{sgn}$, we have $\rho'^\sharp=\rho^\sharp$.

Next we suppose that $\rho^\sharp=\rho'^\sharp$ and ρ is in $\mathscr{E}(\mathbf{G},s)$ for some semisimple $s\in(G^*)^0$. As in the proof of Corollary 7.4, we also have $\rho'\in\mathscr{E}(\mathbf{G},s)$. Write $\mathfrak{L}_s(\rho)=\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}$ and $\mathfrak{L}_s(\rho')=\rho'^{(0)}\otimes\rho'^{(1)}\otimes\rho'^{(2)}$. By (6.4), we have

$$(\rho^{(0)})^\sharp\otimes(\rho^{(1)})^\sharp\otimes(\rho^{(2)})^\sharp=\mathfrak{L}_s(\rho^\sharp)=\mathfrak{L}_s(\rho'^\sharp)=(\rho'^{(0)})^\sharp\otimes(\rho'^{(1)})^\sharp\otimes(\rho'^{(2)})^\sharp,$$

i.e., $\rho'^{(0)}=\rho^{(0)}$, $\rho'^{(1)}=\rho^{(1)}$, $\rho^{(1)}\cdot\mathrm{sgn}$, and $\rho'^{(2)}=\rho^{(2)}$, $\rho^{(2)}\cdot\mathrm{sgn}$. Now two sets $\{\rho,\rho^c,\rho\cdot\mathrm{sgn},\rho^c\cdot\mathrm{sgn}\}$ and

$$\begin{aligned}\big\{&\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)},\rho^{(0)}\otimes(\rho^{(1)}\cdot\mathrm{sgn})\otimes\rho^{(2)},\\&\rho^{(0)}\otimes\rho^{(1)}\otimes(\rho^{(2)}\cdot\mathrm{sgn}),\rho^{(0)}\otimes(\rho^{(1)}\cdot\mathrm{sgn})\otimes(\rho^{(2)}\cdot\mathrm{sgn})\big\}\end{aligned}$$

have the same cardinality. We see that if $\rho'^\sharp=\rho^\sharp$, then ρ' must be one of ρ, ρ^c, $\rho\cdot\mathrm{sgn}$, or $\rho^c\cdot\mathrm{sgn}$. □

8.2. Basic characters of O^ϵ_{2n}

Let $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$. For a semisimple element $s\in(G^*)^0$, let $\rho\in\mathscr{E}(\mathbf{G},s)$ and write $\mathfrak{L}_s(\rho)=\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}$ for some Lusztig correspondence $\mathfrak{L}_s$. Assume that both $\mathbf{G}^{(1)}(s)$, $\mathbf{G}^{(2)}(s)$ are not trivial, by Lemma 8.1 and Corollary 8.4 we know that any Lusztig correspondence gives a bijection between $\{\rho,\rho^c,\rho\cdot\mathrm{sgn},\rho^c\cdot\mathrm{sgn}\}$ and

$$\begin{aligned}\big\{&\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)},\rho^{(0)}\otimes(\rho^{(1)}\cdot\mathrm{sgn})\otimes\rho^{(2)},\\&\rho^{(0)}\otimes\rho^{(1)}\otimes(\rho^{(2)}\cdot\mathrm{sgn}),\rho^{(0)}\otimes(\rho^{(1)}\cdot\mathrm{sgn})\otimes(\rho^{(2)}\cdot\mathrm{sgn})\big\}.\end{aligned}$$

Now we consider the situation where

- $\mathbf{G}^{(0)}(s)$ is a product of unitary groups, and $\rho^{(0)}$ is cuspidal; and
- each of $\rho^{(1)}$, $\rho^{(2)}$ is either cuspidal (i.e., is ζ^{I}_k, ζ^{II}_k for some k), or is $\mathbf{1}_{\mathrm{O}^+_2},\mathrm{sgn}_{\mathrm{O}^+_2}$.

An irreducible character ρ satisfying the above conditions is called a *basic* character. Note that the class of basic characters is slightly larger than the class of cuspidal characters.

Now we denote the set $\{\rho,\rho^c,\rho\cdot\mathrm{sgn},\rho^c\cdot\mathrm{sgn}\}$ by $\{\rho_1,\rho_2,\rho_3,\rho_4\}$. From Proposition 6.6 and the result in Subsection 5.2, we know that exactly two elements (says ρ_1, ρ_2) in $\{\rho_1,\rho_2,\rho_3,\rho_4\}$ first occur in the correspondence for the pair $(\mathbf{G},\mathbf{G}')=(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-k')})$ where

$$k'=\begin{cases} k & \text{if } \rho^{(2)}=\zeta^{\mathrm{I}}_k,\,\zeta^{\mathrm{II}}_k,\\ 1 & \text{if } \rho^{(2)}=\mathbf{1}_{\mathrm{O}^+_2},\,\mathrm{sgn}_{\mathrm{O}^+_2}.\end{cases}\tag{8.1}$$

Then we know that $\{\rho_3,\rho_4\}=\{\rho_1\cdot\mathrm{sgn},\rho_2\cdot\mathrm{sgn}\}$ and $\rho_2=\rho_1^c$.

We know that $\rho_i\chi_{\mathbf{G}}\in\mathscr{E}(\mathbf{G},-s)$ for $i=1,2,3,4$ where $\chi_{\mathbf{G}}$ denotes the spinor character (cf. Lemma 7.6), and any Lusztig correspondence

$$\mathfrak{L}_{-s}\colon\mathscr{E}(\mathbf{G},-s)\to\mathscr{E}(C_{\mathbf{G}^*}(-s),1)=\mathscr{E}(\mathbf{G}^{(0)}(s)\times\mathbf{G}^{(2)}(s)\times\mathbf{G}^{(1)}(s),1)$$

gives a bijection between $\{\rho_1\chi_{\mathbf{G}},\rho_2\chi_{\mathbf{G}},\rho_3\chi_{\mathbf{G}},\rho_4\chi_{\mathbf{G}}\}$ and

$$\begin{aligned}\big\{&\rho^{(0)}\otimes\rho^{(2)}\otimes\rho^{(1)},\rho^{(0)}\otimes(\rho^{(2)}\cdot\mathrm{sgn})\otimes\rho^{(1)},\\ &\rho^{(0)}\otimes\rho^{(2)}\otimes(\rho^{(1)}\cdot\mathrm{sgn}),\rho^{(0)}\otimes(\rho^{(2)}\cdot\mathrm{sgn})\otimes(\rho^{(1)}\cdot\mathrm{sgn})\big\}.\end{aligned}$$

Again, we know that there are exactly two elements in $\{\rho_1\chi_{\mathbf{G}},\rho_2\chi_{\mathbf{G}},\rho_3\chi_{\mathbf{G}},\rho_4\chi_{\mathbf{G}}\}$ first occur in the correspondence for the pair $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-h')})$ where

$$h'=\begin{cases} h & \text{if } \rho^{(1)}=\zeta^{\mathrm{I}}_h,\,\zeta^{\mathrm{II}}_h,\\ 1 & \text{if } \rho^{(1)}=\mathbf{1}_{\mathrm{O}^+_2},\,\mathrm{sgn}_{\mathrm{O}^+_2}.\end{cases}\tag{8.2}$$

Lemma 8.5. *Keep the above settings. There exists a unique character ρ in $\{\rho_1,\rho_2,\rho_3,\rho_4\}$ above such that*

- *ρ first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-k')})$, and*
- *$\rho\chi_{\mathbf{G}}$ first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-h')})$*

where k', h' are given as in (8.1) *and* (8.2)*.*

Proof. We know that there exists ρ', ρ'' in $\{\rho_1,\rho_2,\rho_3,\rho_4\}$ such that

- ρ', ρ'^c first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-k')})$, and
- $\rho''\chi_{\mathbf{G}}$, $(\rho''\chi_{\mathbf{G}})^c$ first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-h')})$.

Moreover, we have

$$\{\rho_1,\rho_2,\rho_3,\rho_4\}=\{\rho',\rho'^c,\rho'\cdot\mathrm{sgn},\rho'^c\cdot\mathrm{sgn}\},$$
$$\{\rho_1\chi_{\mathbf{G}},\rho_2\chi_{\mathbf{G}},\rho_3\chi_{\mathbf{G}},\rho_4\chi_{\mathbf{G}}\}=\{\rho''\chi_{\mathbf{G}},(\rho''\chi_{\mathbf{G}})^c,\rho''\chi_{\mathbf{G}}\cdot\mathrm{sgn},(\rho''\chi_{\mathbf{G}})^c\cdot\mathrm{sgn}\}.$$

By [23, (1) in §4.3], we know that $(\rho''\chi_{\mathbf{G}})^c=\rho''^c\chi_{\mathbf{G}}\cdot\mathrm{sgn}$, and so the intersection $\{\rho',\rho'^c\}\cap\{\rho'',\rho''^c\cdot\mathrm{sgn}\}$ clearly contains exactly one element. □

Remark 8.6. Let ρ be the character given in Lemma 8.5. Then

- ρ^c first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-k')})$,
- $\rho^c\chi_{\mathbf{G}}$ first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n+h')})$,
- $\rho^c\cdot\mathrm{sgn}$ first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n+k')})$,
- $(\rho^c\cdot\mathrm{sgn})\chi_{\mathbf{G}}$ first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-h')})$,
- $\rho\cdot\mathrm{sgn}$ first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n+k')})$,
- $(\rho\cdot\mathrm{sgn})\chi_{\mathbf{G}}$ first occurs in the correspondence for $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n+h')})$,

where k', h' are given in (8.1) and (8.2) respectively.

Remark 8.7. If $\mathbf{G}^{(1)}(s)$ or $\mathbf{G}^{(2)}(s)$ is trivial, the situation is easier as follows.

(1) If $\mathbf{G}^{(1)}(s)$ is trivial and $\mathbf{G}^{(2)}(s)$ is not, then $\rho=\rho^c$, and ρ, $\rho\cdot\mathrm{sgn}$ are the only two irreducible characters whose uniform projection is equal to $\rho^\sharp$. And it is clear that exactly one of them first occurs in the correspondence for the pair $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-k')})$.

(2) If $\mathbf{G}^{(2)}(s)$ is trivial and $\mathbf{G}^{(1)}(s)$ is not, then $\rho^c=\rho\cdot\mathrm{sgn}$, and ρ, ρ^c are the only two irreducible characters whose uniform projection is equal to $\rho^\sharp$. Moreover, there is a unique element ρ_1 in $\{\rho,\rho^c\}$ such that $\rho_1\chi_{\mathbf{G}}$ first occurs in the correspondence for the pair $(\mathrm{O}^\epsilon_{2n},\mathrm{Sp}_{2(n-h')})$.

(3) If both $\mathbf{G}^{(1)}(s)$, $\mathbf{G}^{(2)}(s)$ are trivial, then $\rho=\rho^c=\rho\cdot\mathrm{sgn}$, and so ρ is uniquely determined by its uniform projection.

8.3. A uniqueness choice of $\mathscr{L}_s$

To make a modified Lusztig correspondence

$$\mathscr{L}_s\colon\mathscr{S}_{\mathbf{G}^{(0)}(s)}\times\mathscr{S}_{\mathbf{G}^{(-)}(s)}\times\mathscr{S}_{\mathbf{G}^{(+)}(s)}\to\mathscr{E}(\mathbf{G},s)$$

uniquely determined for $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$, we follow the same idea used in Subsection 5.3 and consider the following requirements:

(I) $\mathscr{L}_s$ is compatible with the parabolic induction as in (6.7).

(II) Suppose $\rho \in \mathscr{E}(\mathbf{G}, s)$ is the unique basic character given in Lemma 8.5 (or in Remark 8.7), then $\mathscr{L}_s$ is required that

$$\text{(8.3)} \qquad \rho = \rho_{x,\Lambda_1,\Lambda_2}, \quad \rho^c = \rho_{x,\Lambda_1^{\mathrm{t}},\Lambda_2}, \quad \rho^c \cdot \mathrm{sgn} = \rho_{x,\Lambda_1,\Lambda_2^{\mathrm{t}}}, \quad \rho \cdot \mathrm{sgn} = \rho_{x,\Lambda_1^{\mathrm{t}},\Lambda_2^{\mathrm{t}}}$$

where

- $x \in \mathscr{S}_{\mathbf{G}^{(0)}(s)}$ is determined by $\rho^{(0)}$ (cf. Subsection 3.1),
- $\Lambda_1 \in \mathscr{S}_{\mathbf{G}^{(-)}(s)}$ is given by

$$\Lambda_1 = \begin{cases} \Lambda_h^{\mathrm{I}} & \text{if } \rho^{(1)} = \zeta_h^{\mathrm{I}}, \zeta_h^{\mathrm{II}}, \\ \binom{1}{0} & \text{if } \rho^{(1)} = \mathbf{1}_{\mathrm{O}_2^+}, \mathrm{sgn}_{\mathrm{O}_2^+}, \end{cases}$$

- $\Lambda_2 \in \mathscr{S}_{\mathbf{G}^{(+)}(s)}$ is given by

$$\Lambda_2 = \begin{cases} \Lambda_k^{\mathrm{I}} & \text{if } \rho^{(2)} = \zeta_k^{\mathrm{I}}, \zeta_k^{\mathrm{II}}, \\ \binom{1}{0} & \text{if } \rho^{(2)} = \mathbf{1}_{\mathrm{O}_2^+}, \mathrm{sgn}_{\mathrm{O}_2^+}. \end{cases}$$

Theorem 8.8. *Let $\mathbf{G} = \mathrm{O}_{2n}^\epsilon$ where $\epsilon = +$ or $-$, and let $s \in (G^*)^0$ be semisimple. There exists a unique bijection $\mathfrak{L}_s \colon \mathscr{E}(\mathbf{G}, s) \to \mathscr{E}(C_{\mathbf{G}^*}(s), 1)$ satisfying (1.2) and (I), (II) above.*

Proof. For $s \in (G^*)^0$, let $\mathfrak{L}_s, \mathfrak{L}'_s \colon \mathscr{E}(\mathbf{G}, s) \to \mathscr{E}(C_{\mathbf{G}^*}(s), 1)$ be two Lusztig correspondences satisfying (I) and (II) above. Let

$$\mathscr{L}_s, \mathscr{L}'_s \colon \mathscr{S}_{\mathbf{G}^{(0)}(s)} \times \mathscr{S}_{\mathbf{G}^{(-)}(s)} \times \mathscr{S}_{\mathbf{G}^{(+)}(s)} \to \mathscr{E}(\mathbf{G}, s)$$

be the corresponding modified Lusztig correspondences. Suppose that $\mathscr{L}_s(x, \Lambda_1, \Lambda_2) = \mathscr{L}'_s(x', \Lambda'_1, \Lambda'_2)$ for some $x, x' \in \mathscr{S}_{\mathbf{G}^{(0)}(s)}$, $\Lambda_1, \Lambda'_1 \in \mathscr{S}_{\mathbf{G}^{(-)}(s)}$, $\Lambda_2, \Lambda'_2 \in \mathscr{S}_{\mathbf{G}^{(+)}(s)}$. Then by Lemma 8.1 we know that $x = x'$, $\Lambda_1 = \Lambda'_1, \Lambda'^{\mathrm{t}}_1$, and $\Lambda_2 = \Lambda'_2, \Lambda'^{\mathrm{t}}_2$. So we need to show that $\Lambda_1 = \Lambda'_1$ and $\Lambda_2 = \Lambda'_2$.

For Λ_1, we consider the following two situations:

(1) Suppose that Λ_1 is degenerate, i.e., $\Lambda_1 = \Lambda_1^{\mathrm{t}}$. This implies that $\Lambda_1 = \Lambda'_1$ immediately.

(2) Next we consider the case that Λ_1 is non-degenerate. Suppose that ρ is an irreducible constituent of the parabolic induced character $R^{\mathrm{O}_{2n}^\epsilon}_{\mathrm{O}_{2n_0}^\epsilon \times \mathbf{L}}(\zeta \otimes \sigma)$ where ζ is a cuspidal character of $\mathrm{O}_{2n_0}^\epsilon(q)$ for some $n_0 \leq n$, $\mathbf{L}$ is a product of general linear groups, and σ is a cuspidal character of $\mathbf{L}$. Suppose that $\zeta \in \mathscr{E}(\mathrm{O}_{2n_0}^\epsilon, s_0)$ and write

$$\zeta = \mathscr{L}_{s_0}(x_0, \Lambda_{0,1}, \Lambda_{0,2}) = \mathscr{L}'_{s_0}(x'_0, \Lambda'_{0,1}, \Lambda'_{0,2})$$

for some $x_0, x'_0 \in \mathscr{S}_{\mathbf{G}^{(0)}(s_0)}$, $\Lambda_{0,1}, \Lambda'_{0,1} \in \mathscr{S}_{\mathbf{G}^{(-)}(s_0)}$ and $\Lambda_{0,2}, \Lambda'_{0,2} \in \mathscr{S}_{\mathbf{G}^{(+)}(s_0)}$. By condition (II) above, we know that $x_0 = x'_0$, $\Lambda_{0,1} = \Lambda'_{0,1}$, and $\Lambda_{0,2} = \Lambda'_{0,2}$.

(a) Suppose that $\mathrm{def}(\Lambda_1)\neq 0$. By condition (I), we have

$$\mathrm{def}(\Lambda_1)=\mathrm{def}(\Lambda_{0,1})=\mathrm{def}(\Lambda'_{0,1})=\mathrm{def}(\Lambda'_1)\neq\mathrm{def}(\Lambda'^{\mathrm{t}}_1).$$

This means that $\Lambda_1=\Lambda'_1$.

(b) Suppose that Λ_1 is non-degenerate and $\mathrm{def}(\Lambda_1)=0$, i.e., $\Lambda_1,\Lambda'_1\in\mathscr{S}_{\mathrm{O}^+_{2n^{(-)}}}$ for some $n^{(-)}$. Note that for this case, $\Lambda_{0,1}=\Lambda'_{0,1}=\binom{-}{-}$. Now we are going to prove this case by induction on $n^{(-)}$. For $n^{(-)}=1$, the equality $\Lambda_1=\Lambda'_1$ is enforced by (II) above. Next suppose that $n^{(-)}\geq 2$. Because now $\Lambda^{\mathrm{t}}_1\neq\Lambda_1$, by Lemma 3.9, there exists $\Lambda_{1,1}\in\mathscr{S}_{\mathrm{O}^+_{2(n^{(-)}-1)}}$ such that $\Lambda_1\in\Omega(\Lambda_{1,1})$ and $\Lambda^{\mathrm{t}}_1\notin\Omega(\Lambda_{1,1})$. By the induction hypothesis and condition (I) above, we have

$$\mathscr{L}_s(x,\Lambda_1,\Lambda_2)\in\Omega(\mathscr{L}_{s_1}(x,\Lambda_{1,1},\Lambda_2))=\Omega(\mathscr{L}'_{s_1}(x,\Lambda_{1,1},\Lambda_2))\not\ni\mathscr{L}'_s(x,\Lambda^{\mathrm{t}}_1,\Lambda_2).$$

Now $\mathscr{L}'_s(x,\Lambda^{\mathrm{t}}_1,\Lambda_2)\neq\mathscr{L}_s(x,\Lambda_1,\Lambda_2)$ implies that $\Lambda_1=\Lambda'_1$.

By the same argument we can also show that $\Lambda_2=\Lambda'_2$. And then we conclude that $\mathscr{L}_s$ (and hence $\mathfrak{L}_s$) is uniquely determined by (1.2) and (I), (II). □

Corollary 8.9. *Let* $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$ *where* $\epsilon=+$ *or* $-$, *and let* $\mathscr{L}_s$ *be the modified Lusztig correspondence given in Theorem* 8.8. *Then* $\rho_{x,\Lambda_1,\Lambda_2}\cdot\mathrm{sgn}=\rho_{x,\Lambda^{\mathrm{t}}_1,\Lambda^{\mathrm{t}}_2}$.

Proof. From (8.3) we see that the assertion is true if ρ is basic, i.e., if

- $\mathbf{G}^{(0)}$ is a product of unitary groups, and x is cuspidal; and
- each of Λ_1, Λ_2 is either cuspidal, or is $\binom{1}{0}$, $\binom{0}{1}$.

In general, suppose that $\rho'\in\Omega_t(\rho)$ for some t corresponding a cuspidal character of a general linear group (cf. Subsection 6.2). Then we have $\rho'\cdot\mathrm{sgn}\in\Omega_t(\rho\cdot\mathrm{sgn})$. Then the corollary can be proved by induction on the rank of $\mathbf{G}$ via the similar argument in the proof of Theorem 8.8. □

Corollary 8.10. *Let* $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$ *where* $\epsilon=+$ *or* $-$, *and let* $\mathscr{L}_s$ *be the modified Lusztig correspondence given in Theorem* 8.8. *Then* $\chi_{\mathbf{G}}\rho_{x,\Lambda_1,\Lambda_2}=\rho_{x,\Lambda_2,\Lambda_1}$ *where* $\chi_{\mathbf{G}}$ *denotes the spinor character.*

Proof. Suppose that $\rho_{x,\Lambda_1,\Lambda_2}\in\mathscr{E}(\mathbf{G},s)$. By the same argument in the proof of Lemma 7.6, we know that $\chi_{\mathbf{G}}\rho_{x,\Lambda_1,\Lambda_2}\in\mathscr{E}(\mathbf{G},-s)$ and $(\chi_{\mathbf{G}}\rho_{x,\Lambda_1,\Lambda_2})^\sharp=(\rho_{x,\Lambda_2,\Lambda_1})^\sharp$. From Lemma 8.5, Remark 8.6 and (8.3), we see that the assertion is true if ρ is basic. Then the remaining proof is similar to that of Corollary 8.9. □

9. Lusztig Correspondences for Sp_{2n}

9.1. Lusztig correspondence for Sp_{2n}

Let $\mathbf{G}=\mathrm{Sp}_{2n}$. For a semisimple element $s\in G^*=\mathrm{SO}_{2n+1}(q)$, recall that (cf. (6.1), (6.2)) we have

$$\begin{aligned}\mathfrak{L}_s\colon\ \mathscr{E}(\mathbf{G},s) &\rightarrow \mathscr{E}(\mathbf{G}^{(0)}(s)\times\mathbf{G}^{(1)}(s)\times\mathbf{G}^{(2)}(s),1)\\ \rho &\mapsto \rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}.\end{aligned}$$

Now we know that $\mathbf{G}^{(1)}(s)=\mathrm{O}^{\epsilon^{(1)}}_{2n^{(1)}}$ and $\mathbf{G}^{(2)}(s)=\mathrm{SO}_{2n^{(2)}+1}$ for some $\epsilon^{(1)}=+$ or $-$, and some nonnegative integers $n^{(1)}$, $n^{(2)}$ depending on s.

Lemma 9.1. *Let $\mathbf{G}=\mathrm{Sp}_{2n}$ and $s\in G^*$ semisimple. Let $\mathfrak{L}_s\colon\mathscr{E}(\mathbf{G},s)\to\mathscr{E}(C_{\mathbf{G}^*}(s),1)$ be a Lusztig correspondence and write $\mathfrak{L}_s(\rho)=\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}$. Moreover, let $\mathfrak{L}'_s\colon\mathscr{E}(\mathbf{G},s)\to\mathscr{E}(C_{\mathbf{G}^*}(s),1)$ be a bijective mapping and write $\mathfrak{L}'_s(\rho)=\rho'^{(0)}\otimes\rho'^{(1)}\otimes\rho'^{(2)}$. Then $\mathfrak{L}'_s$ is a Lusztig correspondence if and only if*

- $\rho'^{(0)}=\rho^{(0)}$;
- $\rho'^{(1)}=\rho^{(1)},\ \rho^{(1)}\cdot\mathrm{sgn}$;
- $\rho'^{(2)}=\rho^{(2)}$.

Proof. The proof is similar to that of Lemma 8.1. □

Lemma 9.2. *Let $\mathbf{G}=\mathrm{Sp}_{2n}$, $s\in G^*$ semisimple, and let $\mathfrak{L}_s\colon\mathscr{E}(\mathbf{G},s)\to\mathscr{E}(C_{\mathbf{G}^*}(s),1)$ be a Lusztig correspondence. Suppose that $\rho\in\mathscr{E}(\mathbf{G},s)$ and write $\mathfrak{L}_s(\rho)=\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)}$. Then we have $\rho^c\in\mathscr{E}(\mathbf{G},s)$ and*

$$\mathfrak{L}_s(\rho^c)=\rho^{(0)}\otimes(\rho^{(1)}\cdot\mathrm{sgn})\otimes\rho^{(2)}.$$

Proof. When $\mathbf{G}^{(0)}$ is trivial, i.e., when ρ is quadratic unipotent, the result is proved in [23, §4.11]. The same argument still works here. □

Corollary 9.3. *Let $\mathbf{G}=\mathrm{Sp}_{2n}$, $s\in G^*$ semisimple. Suppose that $\rho\in\mathscr{E}(\mathbf{G},s)$ and $\rho=\rho_{x,\Lambda_1,\Lambda_2}$ under a modified Lusztig correspondence $\mathscr{L}_s$. Then $\rho^c=\rho_{x,\Lambda_1^{\mathrm{t}},\Lambda_2}$.*

Proof. The proof is similar to that of Corollary 8.3. □

Corollary 9.4. *For $\rho,\rho'\in\mathscr{E}(\mathrm{Sp}_{2n})$, then $\rho'^{\sharp}=\rho^{\sharp}$ if and only if $\rho'=\rho$, ρ^c.*

Proof. Let $\mathbf{G}=\mathrm{Sp}_{2n}$, and let $\rho,\rho'\in\mathscr{E}(\mathbf{G})$. By Lemma 9.2, $\rho'=\rho$, ρ^c implies that $\rho'^{\sharp}=\rho^{\sharp}$. Then remaining proof is similar to that of Corollary 8.4. □

Example 9.5. Let $\mathbf{G}=\mathrm{Sp}_4$. Now we follow the notations in [20].

(1) Let $s_1 \in \mathrm{SO}_5(q)$ such that $C_{\mathrm{SO}_5}(s_1) \simeq \mathrm{O}_4^+$. Then we can check that

$$\mathscr{E}(\mathrm{Sp}_4, s_1) = \{\theta_1, \theta_2, \Phi_9, \theta_3, \theta_4\}, \quad \mathscr{E}(\mathrm{O}_4^+, 1) = \left\{\rho_{\binom{2}{0}}, \rho_{\binom{0}{2}}, \rho_{\binom{1}{1}}, \rho_{\binom{2,1}{1,0}}, \rho_{\binom{1,0}{2,1}}\right\}$$

where $\rho_{\binom{2}{0}} = \mathbf{1}_{\mathrm{O}_4^+}$, $\rho_{\binom{0}{2}} = \mathrm{sgn}_{\mathrm{O}_4^+}$. Now $\mathfrak{L}_{s_1}\colon \mathscr{E}(\mathrm{Sp}_4, s_1) \to \mathscr{E}(\mathrm{O}_4^+, 1)$ is a bijection such that

$$\{\theta_3, \theta_4\} \to \left\{\rho_{\binom{2}{0}}, \rho_{\binom{0}{2}}\right\}, \quad \{\Phi_9\} \to \left\{\rho_{\binom{1}{1}}\right\}, \quad \{\theta_1, \theta_2\} \to \left\{\rho_{\binom{2,1}{1,0}}, \rho_{\binom{1,0}{2,1}}\right\}.$$

(2) Let $s_2 \in \mathrm{SO}_5(q)$ such that $C_{\mathrm{SO}_5}(s_2) \simeq \mathrm{O}_4^-$. Then we can check that

$$\mathscr{E}(\mathrm{Sp}_4, s_2) = \{\theta_5, \theta_6, \theta_7, \theta_8\}, \quad \mathscr{E}(\mathrm{O}_4^-, 1) = \left\{\rho_{\binom{-}{2,0}}, \rho_{\binom{2,0}{-}}, \rho_{\binom{1}{2,1,0}}, \rho_{\binom{2,1,0}{1}}\right\}$$

where $\rho_{\binom{-}{2,0}} = \mathbf{1}_{\mathrm{O}_4^-}$, $\rho_{\binom{2,0}{-}} = \mathrm{sgn}_{\mathrm{O}_4^-}$. Now $\mathfrak{L}_{s_2}\colon \mathscr{E}(\mathrm{Sp}_4, s_2) \to \mathscr{E}(\mathrm{O}_4^-, 1)$ is a bijection such that

$$\{\theta_7, \theta_8\} \to \left\{\rho_{\binom{-}{2,0}}, \rho_{\binom{2,0}{-}}\right\}, \quad \{\theta_5, \theta_6\} \to \left\{\rho_{\binom{1}{2,1,0}}, \rho_{\binom{2,1,0}{1}}\right\}.$$

9.2. Basic characters of Sp_{2n}

Let $\mathbf{G} = \mathrm{Sp}_{2n}$. For a semisimple element s in G^*, let $\rho \in \mathscr{E}(\mathbf{G}, s)$ and write $\mathfrak{L}_s(\rho) = \rho^{(0)} \otimes \rho^{(1)} \otimes \rho^{(2)}$ for a Lusztig correspondence $\mathfrak{L}_s$. Assume that $\mathbf{G}^{(1)}(s)$ is not trivial, we know that any Lusztig correspondence gives a bijection between $\{\rho, \rho^c\}$ and

$$\left\{\rho^{(0)} \otimes \rho^{(1)} \otimes \rho^{(2)}, \rho^{(0)} \otimes (\rho^{(1)} \cdot \mathrm{sgn}) \otimes \rho^{(2)}\right\}.$$

Now we consider the situation where

- $\mathbf{G}^{(0)}$ is a product of unitary groups, and $\rho^{(0)}$ is cuspidal; and
- $\rho^{(1)}$ is either cuspidal (i.e., is ζ_k^{I}, ζ_k^{II} for some k), or is $\mathbf{1}_{\mathrm{O}_2^+}$, $\mathrm{sgn}_{\mathrm{O}_2^+}$; and
- $\rho^{(2)}$ is cuspidal.

An irreducible character ρ satisfies the above conditions is called a *basic* character. From Proposition 6.6 and the result in Subsection 5.2, we know that exactly one element in $\{\rho, \rho^c\}$ first occurs in the correspondence for the pair $(\mathrm{Sp}_{2n}, \mathrm{SO}_{2(n-k')+1})$ where

$$k' = \begin{cases} k & \text{if } \rho^{(1)} = \zeta_k^{\mathrm{I}}, \zeta_k^{\mathrm{II}}, \\ 1 & \text{if } \rho^{(1)} = \mathbf{1}_{\mathrm{O}_2^+}, \mathrm{sgn}_{\mathrm{O}_2^+}. \end{cases}$$

9.3. The choice of $\mathscr{L}_s$ with respect to $(\mathrm{Sp}_{2n}, \mathrm{SO}_{2n'+1})$

To make a unique modified Lusztig correspondence

$$\mathscr{L}_s\colon \mathscr{S}_{\mathbf{G}^{(0)}(s)} \times \mathscr{S}_{\mathbf{G}^{(-)}(s)} \times \mathscr{S}_{\mathbf{G}^{(+)}(s)} \to \mathscr{E}(\mathbf{G}, s)$$

for Sp_{2n} with respect to the dual pair $(\mathrm{Sp}_{2n}, \mathrm{SO}_{2n'+1})$ we consider the following requirements:

(I) We require that $\mathscr{L}_s$ is compatible with the parabolic induction as in (6.7).

(II) Suppose $\rho \in \mathscr{E}(\mathbf{G}, s)$ is the unique irreducible basic character which first occurs in the correspondence for $(\mathrm{Sp}_{2n}, \mathrm{SO}_{2(n-k')+1})$. Then $\mathscr{L}_s$ is required that

$$\rho = \rho_{x,\Lambda_1,\Lambda_2}, \quad \rho^c = \rho_{x,\Lambda_1^{\mathrm{t}},\Lambda_2} \tag{9.1}$$

where

- $x \in \mathscr{S}_{\mathbf{G}^{(0)}(s)}$ is uniquely determined by $\rho^{(0)}$ (cf. Subsection 3.1),
- $\Lambda_1 \in \mathscr{S}_{\mathbf{G}^{(-)}(s)}$ given by

$$\Lambda_1 = \begin{cases} \Lambda_k^{\mathrm{I}} & \text{if } \rho^{(1)} = \zeta_k^{\mathrm{I}},\, \zeta_k^{\mathrm{II}}, \\ \binom{1}{0} & \text{if } \rho^{(1)} = \mathbf{1}_{\mathrm{O}_2^+},\, \mathrm{sgn}_{\mathrm{O}_2^+}, \end{cases}$$

- $\Lambda_2 \in \mathscr{S}_{\mathbf{G}^{(+)}(s)}$ is uniquely determined by $\rho^{(2)}$, i.e., $\rho^{(2)} = \rho_{\Lambda_2^{\mathrm{t}}}$ (cf. (6.6) and Corollary 7.5).

Theorem 9.6. *Let $\mathbf{G} = \mathrm{Sp}_{2n}$ and $s \in G^*$ semisimple. There exists a unique bijection $\mathfrak{L}_s\colon \mathscr{E}(\mathbf{G}, s) \to \mathscr{E}(C_{\mathbf{G}^*}(s), 1)$ satisfying* (1.2) *and (I), (II) above.*

Proof. The proof is similar to that of Proposition 5.6. For $s \in G^*$, let $\mathfrak{L}_s, \mathfrak{L}'_s\colon \mathscr{E}(\mathbf{G}, s) \to \mathscr{E}(C_{\mathbf{G}^*}(s), 1)$ be two Lusztig correspondence satisfying (I) and (II) above. Let

$$\mathscr{L}_s, \mathscr{L}'_s\colon \mathscr{S}_{\mathbf{G}^{(0)}(s)} \times \mathscr{S}_{\mathbf{G}^{(-)}(s)} \times \mathscr{S}_{\mathbf{G}^{(+)}(s)} \to \mathscr{E}(\mathbf{G}, s)$$

be the corresponding modified Lusztig correspondences. Let $\rho \in \mathscr{E}(\mathbf{G}, s)$ and suppose that

$$\rho = \mathscr{L}_s(x, \Lambda_1, \Lambda_2) = \mathscr{L}'_s(x', \Lambda'_1, \Lambda'_2)$$

for some $x, x' \in \mathscr{S}_{\mathbf{G}^{(0)}(s)}$, $\Lambda_1, \Lambda'_1 \in \mathscr{S}_{\mathbf{G}^{(-)}(s)}$, $\Lambda_2, \Lambda'_2 \in \mathscr{S}_{\mathbf{G}^{(+)}(s)}$. Then by Lemma 9.1 we know that $x = x'$, $\Lambda_2 = \Lambda'_2$, and $\Lambda_1 = \Lambda'_1, \Lambda'^{\mathrm{t}}_1$. So our goal is to prove that $\Lambda_1 = \Lambda'_1$. Now we consider the following situations:

(1) Suppose that Λ_1 is degenerate, i.e., $\Lambda_1 = \Lambda_1^{\mathrm{t}}$. This of course implies that $\Lambda_1 = \Lambda'_1$.

(2) Next suppose that Λ_1 is non-degenerate. Suppose that ρ is an irreducible constituent of the parabolic induced character $R^{\mathrm{Sp}_{2n}}_{\mathrm{Sp}_{2n_0}\times\mathbf{L}}(\zeta\otimes\sigma)$ where ζ is a cuspidal character of $\mathrm{Sp}_{2n_0}(q)$ for some $n_0\leq n$, $\mathbf{L}$ is a product of general linear groups, and σ is a cuspidal character of $\mathbf{L}$. Suppose that $\zeta\in\mathscr{E}(\mathrm{Sp}_{2n_0},s_0)$ for some s_0 and write

$$\zeta=\mathscr{L}_{s_0}(x_0,\Lambda_{0,1},\Lambda_{0,2})=\mathscr{L}'_{s_0}(x'_0,\Lambda'_{0,1},\Lambda'_{0,2})$$

for some $x_0,x'_0\in\mathscr{S}_{\mathbf{G}^{(0)}(s_0)}$, $\Lambda_{0,1},\Lambda'_{0,1}\in\mathscr{S}_{\mathbf{G}^{(-)}(s_0)}$ and $\Lambda_{0,2},\Lambda'_{0,2}\in\mathscr{S}_{\mathbf{G}^{(+)}(s_0)}$. By (II) above, we know that $\Lambda_{0,1}=\Lambda'_{0,1}$. By the same argument as in the proof of Theorem 8.8, we conclude that $\Lambda_1=\Lambda'_1$.

Therefore the theorem is proved. □

Remark 9.7. Note that the modified Lusztig correspondence $\mathscr{L}_s$ in the theorem depends on the theta correspondence $\Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$, in particular, it depends on the choice of ψ. Let $\mathscr{L}'_s$ be the corresponding modified Lusztig correspondence with respect to another character $\psi'=\psi_a$ where $a\in\mathbf{F}_q^\times$ is a non-square element. Then by Lemmas 5.1 and 9.2 we see that

$$\mathscr{L}_s(x,\Lambda_1,\Lambda_2)=\mathscr{L}'_s(x,\Lambda_1^{\mathrm{t}},\Lambda_2)$$

for $x\in\mathscr{S}_{\mathbf{G}^{(0)}(s)}$, $\Lambda_1\in\mathscr{S}_{\mathbf{G}^{(-)}(s)}$, $\Lambda_2\in\mathscr{S}_{\mathbf{G}^{(+)}(s)}$.

To justify the choice of $\mathscr{L}_s$ in the above theorem, we have the following result which refines Proposition 6.4.

Theorem 9.8. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{SO}_{2n'+1})$, *and* $s\in G^*$, $s'\in G'^*$ *semisimple. Let*

$$\mathscr{L}_s\colon\mathscr{S}_{\mathbf{G}^{(0)}(s)}\times\mathscr{S}_{\mathbf{G}^{(-)}(s)}\times\mathscr{S}_{\mathbf{G}^{(+)}(s)}\to\mathscr{E}(\mathbf{G},s),$$
$$\mathscr{L}_{s'}\colon\mathscr{S}_{\mathbf{G}'^{(0)}(s')}\times\mathscr{S}_{\mathbf{G}'^{(-)}(s')}\times\mathscr{S}_{\mathbf{G}'^{(+)}(s')}\to\mathscr{E}(\mathbf{G}',s')$$

be the modified Lusztig correspondence for $\mathbf{G}$ *given in Theorem* 9.6, *and the Lusztig correspondence for* $\mathbf{G}'$ *given by Theorem* 7.1 *respectively. Then* $(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'_2})\in\Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$ *if and only if*

- $s^{(0)}=-s'^{(0)}$ *(up to conjugation) and* $x=x'$,
- $\Lambda_2=\Lambda'_1$, *and*
- $(\Lambda_1,\Lambda'_2)\in\mathscr{B}_{\mathbf{G}^{(-)}(s),\mathbf{G}^{(+)}(s')}$.

Proof. Suppose that $(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'_2})\in\Theta^{\psi}_{\mathbf{G},\mathbf{G}'}$. Then by Proposition 6.4, we have

- $s^{(0)}=-s'^{(0)}$ and $x=x'$,
- $\Lambda_2=\Lambda'_1$, and

- (Λ_1,Λ'_2) or $(\Lambda_1^{\rm t},\Lambda'_2)$ is in $\mathscr{B}_{\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s')}$.

Now we want to show that in fact $(\Lambda_1,\Lambda'_2)\in\mathscr{B}_{\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s')}$. Note that $(\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s'))$ $=(\mathrm{O}^{\epsilon^{(1)}}_{2n^{(1)}},\mathrm{Sp}_{2n'^{(2)}})$ for some $\epsilon^{(1)}$, and some $n^{(1)}$, $n'^{(2)}$. Now we consider the following situations:

(1) Suppose that $\mathrm{def}(\Lambda_1)\neq 0$. First we consider the case that both Λ_1, Λ'_2 are cuspidal.

   (a) Suppose that $(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'_2})$ first occurs in the correspondence for the pair $(\mathrm{Sp}_{2n},\mathrm{SO}_{2(n-k)+1})$ for some k, i.e., $(\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s'))=(\mathrm{O}^{\epsilon_k}_{2k^2},\mathrm{Sp}_{2k(k-1)})$. From our choice of $\mathscr{L}_s$ (cf. (9.1)) and $\mathscr{L}_{s'}$, we know that $\Lambda_1=\Lambda_k^{\rm I}$ (cf. (5.2)) and $\Lambda'_2=\Lambda^{\rm Sp}_{k-1}$ (cf. (4.1)), and it is clearly that $(\Lambda_1,\Lambda'_2)\in\mathscr{B}_{\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s')}$.

   (b) Suppose that $(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'_2})$ first occurs in the correspondence for the pair $(\mathrm{Sp}_{2n},\mathrm{SO}_{2(n+k)+1})$, i.e., $(\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s'))=(\mathrm{O}^{\epsilon_k}_{2k^2},\mathrm{Sp}_{2k(k+1)})$. Now we have $\Lambda_1=\Lambda_k^{\rm II}$ and $\Lambda'_2=\Lambda_k^{\rm Sp}$, and again $(\Lambda_1,\Lambda'_2)\in\mathscr{B}_{\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s')}$.

   Now if Λ_1 or Λ'_2 is not cuspidal, by the same argument in the proof of [17, Proposition 6.4] we still conclude that $(\Lambda_1,\Lambda'_2)\in\mathscr{B}_{\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s')}$.

(2) Suppose that $\mathrm{def}(\Lambda_1)=0$. Then $\mathrm{def}(\Lambda'_2)=1$.

   (a) Suppose that $\Lambda_1=\binom{-}{-}$, i.e., $(\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s'))=(\mathrm{O}^+_0,\mathrm{Sp}_{2n'^{(2)}})$. This case is obvious.

   (b) Suppose that $\Lambda_1=\binom{1}{0}$ or $\binom{0}{1}$. If $(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'_2})$ first occurs in the correspondence for the pair $(\mathrm{Sp}_{2n},\mathrm{SO}_{2n-1})$, then $\Lambda_1=\binom{1}{0}$, $(\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s'))=(\mathrm{O}^+_2,\mathrm{Sp}_0)$, and $\Lambda'_2=\binom{0}{-}$. If $(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'_2})$ first occurs in the correspondence for $(\mathrm{Sp}_{2n},\mathrm{SO}_{2n+1})$, then $\Lambda_1=\binom{0}{1}$, $(\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s'))=(\mathrm{O}^+_2,\mathrm{Sp}_2)$, and $\Lambda'_2=\binom{1}{-}$. We have $(\Lambda_1,\Lambda'_2)\in\mathscr{B}_{\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s')}$ for both situations.

   Now for general Λ_1 and Λ'_2, we can use the same argument in [17, §6] (in particular the proof of Proposition 6.20) and conclude that $(\Lambda_1,\Lambda'_2)\in\mathscr{B}_{\mathbf{G}^{(1)}(s),\mathbf{G}'^{(2)}(s')}$.

Therefore the theorem is proved. □

9.4. The choice of $\mathscr{L}_s$ with respect to $(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$

Keep the settings in Subsection 9.2. Now any Lusztig correspondence gives a bijection

$$\{\rho,\rho^c\}\rightarrow\{\rho^{(0)}\otimes\rho^{(1)}\otimes\rho^{(2)},\rho^{(0)}\otimes(\rho^{(1)}\cdot\mathrm{sgn})\otimes\rho^{(2)}\}$$

where $\rho\in\mathscr{E}(\mathbf{G},s)$ is a basic character. Now $\rho^{(2)}$ is a cuspidal character of $\mathrm{SO}_{2n^{(2)}+1}$, so we assume that $\rho^{(2)}=\zeta_k^{\mathrm{SO}_{\rm odd}}$ for some k. By Proposition 6.6, we know that both ρ, ρ^c first occur in the correspondence for the pair $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2(n-k)})$ for some ϵ. Suppose that $(\rho,\rho')\in\Theta^\psi_{\mathbf{G},\mathbf{G}'}$ for some unique $\rho'\in\mathscr{E}(\mathbf{G}',s')$ and write $\rho'=\rho_{x',\Lambda'_1,\Lambda'_2}$ where $\mathscr{L}_{s'}$ is given in Theorem 8.8. Note that (ρ^c,ρ'^c) also occurs in $\Theta^\psi_{\mathbf{G},\mathbf{G}'}$ by Lemma 5.2.

Now besides the condition (I) given in Subsection 9.3, we also consider the following requirement:

(III) Suppose $\rho\in\mathscr{E}(\mathbf{G},s)$ is a basic character given above. Then $\mathscr{L}_s$ is required that

$$\rho=\rho_{x,\Lambda_1,\Lambda_2},\quad \rho^c=\rho_{x,\Lambda_1^{\mathrm{t}},\Lambda_2} \tag{9.2}$$

where

- $x\in\mathscr{S}_{\mathbf{G}^{(0)}(s)}$ is uniquely determined by $\rho^{(0)}$,
- $\Lambda_1\in\mathscr{S}_{\mathbf{G}^{(-)}(s)}$ given by $\Lambda_1=\Lambda'_1$ where Λ'_1 is determined by ρ' as above,
- $\Lambda_2\in\mathscr{S}_{\mathbf{G}^{(+)}(s)}$ is uniquely determined by $\rho^{(2)}$, i.e., $\rho^{(2)}=\rho_{\Lambda_2^{\mathrm{t}}}$ (cf. (6.6) and Corollary 7.5).

Theorem 9.9. *Let $\mathbf{G}=\mathrm{Sp}_{2n}$ and $s\in G^*$ semisimple. There exists a unique bijection $\mathfrak{L}_s\colon\mathscr{E}(\mathbf{G},s)\to\mathscr{E}(C_{\mathbf{G}^*}(s),1)$ satisfying* (1.2), *(I) in Subsection* 9.3, *and (III) above.*

Proof. The proof is analogous to that of Theorem 9.6. □

Now we have the following result which refines Proposition 6.6.

Theorem 9.10. *Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ where $\epsilon=+$ or $-$, and $s\in G^*$, $s'\in(G'^*)^0$ semisimple. Let*

$$\mathscr{L}_s\colon\mathscr{S}_{\mathbf{G}^{(0)}(s)}\times\mathscr{S}_{\mathbf{G}^{(-)}(s)}\times\mathscr{S}_{\mathbf{G}^{(+)}(s)}\to\mathscr{E}(\mathbf{G},s),$$
$$\mathscr{L}_{s'}\colon\mathscr{S}_{\mathbf{G}^{(0)}(s')}\times\mathscr{S}_{\mathbf{G}^{(-)}(s')}\times\mathscr{S}_{\mathbf{G}^{(+)}(s')}\to\mathscr{E}(\mathbf{G}',s')$$

be the modified Lusztig correspondence for $\mathbf{G}$ given in Theorem 9.9, *and the modified Lusztig correspondence for $\mathbf{G}'$ given by Theorem* 8.8 *respectively. Then $(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'_2})\in\Theta^\psi_{\mathbf{G},\mathbf{G}'}$ if and only if*

- *$s^{(0)}=s'^{(0)}$ (up to conjugation) and $x=x'$,*
- *$\Lambda_1=\Lambda'_1$, and*
- *$(\Lambda_2,\Lambda'_2)\in\mathscr{B}_{\mathbf{G}^{(+)}(s),\mathbf{G}^{(+)}(s')}$.*

Proof. Suppose that $(\rho_{x,\Lambda_1,\Lambda_2},\rho_{x',\Lambda'_1,\Lambda'_2})\in\Theta^\psi_{\mathbf{G},\mathbf{G}'}$. Then by Proposition 6.6, we have

- $s^{(0)}=s'^{(0)}$ and $x=x'$,

- $\Lambda_1 = \Lambda'_1, \Lambda'^{\mathrm{t}}_1$, and
- (Λ_2, Λ'_2) or $(\Lambda^{\mathrm{t}}_2, \Lambda'_2)$ is in $\mathscr{B}_{\mathbf{G}^{(+)}(s),\mathbf{G}'^{(+)}(s')}$.

Note that $(\mathbf{G}^{(+)}(s), \mathbf{G}'^{(+)}(s')) = (\mathrm{Sp}_{2n^{(+)}}, \mathrm{O}^{\epsilon'^{(+)}}_{2n'^{(+)}})$ for some $n^{(+)}$, $\epsilon'^{(+)}$, $n'^{(+)}$. By the same argument in the proof of Theorem 9.8, we can conclude that $(\Lambda_2, \Lambda'_2) \in \mathscr{B}_{\mathbf{G}^{(+)}(s),\mathbf{G}'^{(+)}(s')}$.

Next we want to show that $\Lambda_1 = \Lambda'_1$. If Λ_1 is cuspidal or is equal to $\binom{1}{0}$, $\binom{0}{1}$, and Λ_2 is cuspidal, we have $\Lambda_1 = \Lambda'_1$ by the requirement of $\mathscr{L}_s$ in (9.2). Then by the same argument in the proof of Proposition 5.6, we can conclude that $\Lambda_1 = \Lambda'_1$ for general Λ_1. □

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References

[1] J. Adams and A. Moy, *Unipotent representations and reductive dual pairs over finite fields*, Trans. Amer. Math. Soc. **340** (1993), no. 1, 309–321.

[2] A.-M. Aubert, J. Michel and R. Rouquier, *Correspondance de Howe pour les groupes réductifs sur les corps finis*, Duke Math. J. **83** (1996), no. 2, 353–397.

[3] R. W. Carter, *Finite Groups of Lie Type: Conjugacy classes and complex characters*, John Wiley & Sons, New York, 1985.

[4] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.

[5] F. Digne and J. Michel, *On Lusztig's parametrization of characters of finite groups of Lie type*, Astérisque (1990), no. 181–182, 6, 113–156.

[6] ———, *Representations of Finite Groups of Lie Type*, Second edition, London Math. Soc. Stud. Texts **95**, Cambridge University Press, Cambridge, 2020.

[7] M. Geck and G. Malle, *The character theory of finite groups of Lie type: A guided tour*, Cambridge Stud. Adv. Math. **187**, Cambridge University Press, Cambridge, 2020.

[8] M. Geck and G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras*, London Math. Soc. Monogr. (N.S.) **21**, The Clarendon Press, Oxford University Press, New York, 2000.

[9] P. Gérardin, *Weil representations associated to finite fields*, J. Algebra **46** (1977), no. 1, 54–101.

[10] R. Howe, *Invariant theory and duality for classical groups over finite fields with applications to their singular representation theory*, preprint.

[11] G. Lusztig, *Irreducible representations of finite classical groups*, Invent. Math. **43** (1977), no. 2, 125–175.

[12] ______, *Representations of finite Chevalley groups*, CBMS Regional Conf. Ser. in Math. **39**, American Mathematical Society, Providence, RI, 1978.

[13] ______, *Unipotent characters of the symplectic and odd orthogonal groups over a finite field*, Invent. Math. **64** (1981), no. 2, 263–296.

[14] ______, *Unipotent characters of the even orthogonal groups over a finite field*, Trans. Amer. Math. Soc. **272** (1982), no. 2, 733–751.

[15] ______, *Characters of Reductive Groups over a Finite Field*, Ann. of Math. Stud. **107**, Princeton University Press, Princeton, NJ, 1984.

[16] C. Mœglin, M.-F. Vignéras and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Math. **1291**, Springer-Verlag, Berlin, 1987.

[17] S.-Y. Pan, *Howe correspondence of unipotent characters for a finite symplectic/even-orthogonal dual pair*, arXiv:1901.00623, to appear in Amer. J. Math.

[18] ______, *Lusztig correspondence and Howe correspondence for finite reductive dual pairs*, arXiv:1906.01158.

[19] ______, *Uniform projection of the Weil character*, Adv. Math. **381** (2021), Paper No. 107624, 80 pp.

[20] B. Srinivasan, *The characters of the finite symplectic group* $\mathrm{Sp}(4,q)$, Trans. Amer. Math. Soc. **131** (1968), 488–525.

[21] ______, *Weil representations of finite classical groups*, Invent. Math. **51** (1979), no. 2, 143–153.

[22] F. Szechtman, *Weil representations of the symplectic group*, J. Algebra **208** (1998), no. 2, 662–686.

[23] J.-L. Waldspurger, *Une conjecture de Lusztig pour les groupes classiques*, Mém. Soc. Math. Fr. (N.S.) (2004), no. 96, 166 pp.

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