TAIWANESE JOURNAL OF MATHEMATICS
Vol. 27, No. 6, pp. 1211–1225, December 2023
DOI: 10.11650/tjm/231004

On the Limiting Spectral Distributions of Stochastic Block Models

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Abstract. Erdős–Rényi graph is a random graph in which the probability of a connection between two nodes follows a Bernoulli distribution independently. The stochastic block models (SBM) are an extension of the Erdős–Rényi graph by dividing nodes into K subsets, known as blocks or communities. Let $\widetilde{A}_N = (\widetilde{A}_{ij}^{(N)})$ be an $N \times N$ normalized adjacency matrix of the SBM with K blocks of any sizes, and let $\mu_{\widetilde{A}_N}$ be the empirical spectral density of $\widetilde{A}_N$.

In this paper, we first showed that if the connecting probabilities between nodes of different blocks are zero, then $\lim_{N\to\infty} \mu_{\widetilde{A}_N} = \mu$ exists almost surely, and we gave the explicit formulas for μ and its Stieltjes transform, respectively. Second, we showed under a suitable condition on the maximum of connecting probability between nodes in different blocks, say by ζ_0, $\mu_{\widetilde{A}_N}$ converges both in probability and expectation as first $N \to \infty$ and then $\zeta_0 \to 0$.

1. Introduction

Random matrix theory (RMT) plays an important role in many fields, such as physics, chemistry, economics, statistics, data science, and social science (see, e.g., [5, 15, 25]). An interesting problem in RMT is to determine the limiting distribution of the empirical spectral distribution (ESD) of a random matrix as its size goes to infinity. In the 1950s, Wigner [26, 27] derived the semicircle law for a particular class of real symmetric random matrices, called Wigner matrices. Later, numerous results were published on the spectrum of Wigner matrices (see, e.g., [4, 11, 12, 17, 22]).

Random graph theory is an interesting branch of RMT, which can be viewed as the intersection between graph theory, probability theory, and computer science (see, e.g., [5, 7, 13, 15, 20, 28]). In 1959, Erdős–Rényi [13] considered a random graph with N nodes, in which the associated adjacency matrix $A^{ER} = (A_{ij}^{ER})_{i,j=1}^N$ has entries that represent the connection between two nodes that independently follows a Bernoulli distribution with a probability of success $p \in (0, 1)$ that depends on N. Ding and Jiang [10] showed that if $\sup_{1\le i<j\le N} E|A_{ij}^{ER} - p|^t/\sqrt{p(1-p)} < \infty$ for some $t > 0$ and $Np(1-p) \to \infty$,

Received August 11, 2022; Accepted October 23, 2023.
Communicated by Akira Sakai.
2020 *Mathematics Subject Classification.* 60B20, 15B52, 05C42.
Key words and phrases. converge in expectation, converge in probability, converge weakly, empirical spectral density, semicircle law, stochastic block models.
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then the ESD of the scaled matrix $A^{ER}/\sqrt{Np(1-p)}$ weakly converges to the semicircle law almost surely. Tran et al. [24] derived that when $Np \to \infty$, the ESD of the matrix $A^{ER}/\sqrt{Np(1-p)}$ converges in distribution to the semicircle distribution as $N \to \infty$.

Holland et al. [14] introduced stochastic block models (SBM), which generalize the Erdős–Rényi graph. A stochastic block model is an undirected random graph with N nodes divided into K blocks, denoted by $\{C_1, C_2, \ldots, C_K\}$. Without loss of generality, we label the nodes by $1, 2, \ldots, N$ satisfying

$$C_1 = \{1, 2, \ldots, N_1\}, \;\; C_2 = \{N_1+1, \ldots, N_1+N_2\}, \;\; \ldots, \;\; C_K = \left\{\sum_{m=1}^{K-1} N_m + 1, \ldots, \sum_{m=1}^{K} N_m = N\right\},$$

where $\sum_{m=1}^{K-1} N_m = 0$ if $K = 1$. Let $\alpha_m = N_m/N$ fixed, $m = 1, 2, \ldots, K$. Then $N_m = \alpha_m N$ and $\sum_{m=1}^{K} \alpha_m = 1$. For $m, n = 1, \ldots, K$, the connection between nodes i and j exists with probability $p_m = p_m(N)$ if $i, j \in C_m$ and with probability $p_{mn} = p_{mn}(N)$ if $i \in C_m$, $j \in C_n$ and $m \neq n$. All edges are distributed independently. Without loss of generality, assume hereafter that $p_1 \geq p_2 \geq \cdots \geq p_K$.

Let $A_N = (A_{ij}^{(N)})_{i,j=1}^{N}$ be $N \times N$ *adjacency matrix* of G_N with

$$A_{ij}^{(N)} = A_{ji}^{(N)} \sim \begin{cases} \mathfrak{B}(p_m) & \text{if } i, j \in C_m,\ m = 1, \ldots, K, \\ \mathfrak{B}(p_{mn}) & \text{if } i \in C_m,\ j \in C_n,\ m \neq n, \end{cases}$$

where $A_{ij}^{(N)} \sim \mathfrak{B}(p)$ means that $A_{ij}^{(N)}$ follows the Bernoulli distribution with mean p. Let also $\widetilde{A}_N = \gamma(N)[A_N - E(A_N)]$ be the *normalized matrix* of A_N, where $\gamma = \gamma(N) = 1/\sqrt{Np_1(N)(1-p_1(N))}$ and $E[A_N] = \big(E[A_{ij}^{(N)}]\big)_{i,j=1}^{N}$. Thus, the entries of $\widetilde{A}_N$ satisfy that

$$\widetilde{A}_{ij}^{(N)} = \widetilde{A}_{ji}^{(N)} \sim \begin{cases} \mathfrak{C}(p_m, \gamma) & \text{if } i, j \in C_m,\ m = 1, \ldots, K, \\ \mathfrak{C}(p_{mn}, \gamma) & \text{if } i \in C_m,\ j \in C_n,\ m \neq n,\ m, n = 1, \ldots, K. \end{cases} \tag{1.1}$$

Here, the random variables $\mathfrak{C}(p, \gamma)$ with $0 \leq p \leq 1$ are distributed as follows:

$$\mathfrak{C}(p, \gamma) = \begin{cases} \gamma(1-p) & \text{with probability } p, \\ -\gamma p & \text{with probability } 1-p. \end{cases}$$

The stochastic block models have been widely studied in the statistical analysis of graphs and networks (see, e.g., [1,8,19]). Athreya et al. [2] derived the weak convergence of the joint limiting distribution of the largest eigenvalues of the SBM's adjacency matrix. Avrachenkov et al. [3] analyzed the asymptotic ESD of the normalized adjacency matrix of the SBM with each community having the same size. Recently, there has been significant interest in community detection in SBM (see, e.g., [1,6]). The aim of community detection is to partition nodes into groups with a higher probability of connection within the same group than between different groups. Therefore, in this paper, we consider the SBM with the assumption $p_m \geq p_{mn}$, $m, n = 1, \ldots, K$ and $m \neq n$. Throughout this paper, we mainly employ tools developed in RMT to study the asymptotic ESD of the adjacency matrix of the SBM in which communities can be of different sizes.

The organization of this paper is as follows. First, in Theorem 1.5, we show that if the connecting probabilities between the nodes of different blocks are zero, then the limiting ESD of the stochastic block models exist almost surely. We provide explicit formulas for the limiting ESD and its Stieltjes transform. In particular, the semicircle law for the limiting ESD of the Erdős–Rényi graph is a special case of our results. In Corollary 1.6, we give all the moments of limiting ESD. Finally, in Theorem 1.7, we derive that under a suitable condition on the probabilities of connection between nodes in different blocks, the limiting ESD converges in both probability and expectation. All proofs are presented in Section 2.

Before presenting the main results, we review some definitions.

Definition 1.1. [21, 22] Let $M_N = (\xi_{ij})$ be an $N \times N$ symmetric random matrix with independent random entries $\xi_{ij} = \xi_{ji}$. Define the *empirical spectral distribution* (ESD) of M_N by

$$\mu_{M_N}(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}(x)$$

and the *empirical cumulative distribution function* (abbreviated to ECDF) of M_N by

$$F_{M_N}(x) = \frac{1}{N}\sum_{i=1}^{N} \chi\{\lambda_i \leq x\} = \int_{-\infty}^{x} d\mu_{M_N}(t),$$

where $\lambda_i = \lambda_i(M_N)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ are the eigenvalues of M_N, χ is the indicator function and δ_p is the Dirac's delta function at the point p.

Let $C_c(\mathbb{R})$ be the set of continuous functions with compact support. Next, we will examine three distinct notions of weak convergence for μ_{M_N}. For brevity, we will use the symbol "$\xrightarrow{w}$" to denote weak convergence from this point forward.

Definition 1.2. [21–23] Let μ_{M_N} be the ESD of a random matrix M_N which are random probability measures, and let μ be a deterministic probability measure.

(i) $\mu_{M_N} \xrightarrow{w} \mu$ in the almost sure sense (a.s.) if $\Pr\big(\lim_{N\to\infty} \int_{\mathbb{R}} \varphi\, d\mu_{M_N} = \int_{\mathbb{R}} \varphi\, d\mu\big) = 1$ for all $\varphi \in C_c(\mathbb{R})$.

(ii) $\mu_{M_N} \xrightarrow{w} \mu$ in probability (in p) if for any $\epsilon > 0$, $\lim_{N\to\infty} \Pr\big(\big|\int_{\mathbb{R}} \varphi\, d\mu_{M_N} - \int_{\mathbb{R}} \varphi\, d\mu\big| > \epsilon\big) = 0$ for all $\varphi \in C_c(\mathbb{R})$.

(iii) $\mu_{M_N} \xrightarrow{w} \mu$ in expectation if $\lim_{N\to\infty} E\big[\int_{\mathbb{R}} \varphi \, d\mu_{M_N}\big] = \int_{\mathbb{R}} \varphi \, d\mu$ for all $\varphi \in C_c(\mathbb{R})$.

Remark 1.3. From Definition 1.2 and probability theory, it is not hard to see that if $\mu_{M_N} \xrightarrow{w} \mu$ a.s., then $\mu_{M_N} \xrightarrow{w} \mu$ in p.

Next, we review the Stieltjes transform, which is a useful tool in random matrix theory.

Definition 1.4. [22] The Stieltjes transform of a given cumulative distribution function F (or a probability measure μ, respectively) is defined by

$$s_F(z) = \int_{\mathbb{R}} \frac{1}{x-z}\, dF(x) \quad \left(\text{or by } s_\mu(z) = \int_{\mathbb{R}} \frac{1}{x-z}\, \mu(dx), \text{ resp.}\right)$$

for $z \in \mathbb{C}^+ = \{x + iy : x \in \mathbb{R}, y > 0\}$.

Notice that the density μ can be recovered from the Stieltjes inversion formula:

$$\mu(x) = \lim_{y \downarrow 0^+} \frac{1}{\pi} \Im(s(z)), \quad z = x + iy \in \mathbb{C}^+,\ x, y \in \mathbb{R}, \tag{1.2}$$

where $\Im(z)$ is the imaginary part of the complex number z.

To simplify the notation, throughout this paper, let

$$\zeta_m(N) = \frac{p_m(N)(1-p_m(N))}{p_1(N)(1-p_1(N))}, \quad \zeta_0(N) = \max_{\substack{m,n=1,\dots,K \\ m \neq n}} \frac{p_{mn}(N)(1-p_{mn}(N))}{p_1(N)(1-p_1(N))}$$

for $m, n = 1, 2, \dots, K$ and $m \neq n$, and let

$$\zeta_m = \lim_{N\to\infty} \zeta_m(N), \quad m = 0, 1, 2, \dots, K.$$

Note that $\zeta_m(N)$ and ζ_m are deterministic for all m. The main contribution of this paper is the derivation of Theorems 1.5 and 1.7. All proofs are in Section 2.

Let $\widetilde{B}_N = (\widetilde{B}_{ij}^{(N)})_{i,j=1}^N$ be the normalized matrix of the SBM given by (1.1) with $p_{mn} = 0$ for all $m \neq n$. Then we have the following theorem. Moreover, in the proof of Theorem 1.7, the matrix $\widetilde{B}_N$ will depend on the matrix $\widetilde{A}_N$ given in Theorem 1.7.

Theorem 1.5. *Let $\widetilde{B}_N$ be the normalized matrix of the SBM given by* (1.1) *with $p_{mn} = 0$ for all $m \neq n$, and let $\mu_{\widetilde{B}_N}$ be the ESD of $\widetilde{B}_N$. If $\lim_{N\to\infty} \gamma = 0$ and $0 < \zeta_m < \infty$ for $m = 1, 2, \dots, K$, then there exists a deterministic probability measure μ such that for all x,*

$$\mu_{\widetilde{B}_N} \xrightarrow{w} \mu \quad \textit{almost surely as } N \to \infty,$$

where

$$\mu(x) = \sum_{m=1}^{K} \frac{\sqrt{4\alpha_m\zeta_m - x^2}}{2\pi\zeta_m} \chi\{|x| < \sqrt{4\alpha_m\zeta_m}\}$$

and the Stieltjes transform of μ is given by

$$s_\mu(z) = \sum_{m=1}^{K} \frac{-z + \sqrt{z^2 - 4\alpha_m\zeta_m}}{2\zeta_m}. \tag{1.3}$$

Corollary 1.6. *Assume μ is as defined in Theorem 1.5. Then the k-th moment of μ is*

$$\int_{\mathbb{R}} x^k \, d\mu(x) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ \frac{1}{j+1}\binom{2j}{j} \sum_{m=1}^{K} \alpha_m^{j+1} \zeta_m^j & \text{if } k = 2j \text{ is even}, \end{cases}$$

where $\binom{m}{n} = \frac{m!}{n!(m-n)!}$.

In Theorem 1.7, we consider the adjacency matrix determined by (1.1) with assuming $p_m \geq p_{mn}$, $m, n = 1, \ldots, K$ and $m \neq n$.

Theorem 1.7. *Let $\widetilde{A}_N$ be the normalized matrix of the SBM given by* (1.1)*. Assume $p_m \geq p_{mn}$, $m, n = 1, \ldots, K$ and $m \neq n$. If $\lim_{N\to\infty} \gamma = 0$ and $0 < \zeta_m < \infty$ for $m = 0, 1, 2, \ldots, K$, then, for almost all $x \in \mathbb{R}$, $\lim_{\zeta_0 \to 0} \lim_{N\to\infty} \mu_{\widetilde{A}_N} \xrightarrow{w} \mu$ in p and in expectation, where $\mu(x)$ is as defined in Theorem* 1.5.

2. Proofs

2.1. Proof of Theorem 1.5

In our model, the distributions of all A_{ij} depend on N, so for any fixed i and j in the same block and for different values of N, the distributions of $\{A_{ij}^{(N)}\}_{\{N \in \mathbb{N}\}}$ are not identical. Therefore, to prove Theorem 1.5, we will use Theorem 2.9 in [4], which is stated as follows:

Theorem 2.1. [4, Theorem 2.9] *Suppose that $M_N = (M_{ij}^{(N)})_{i,j\geq 1}$ is an $N \times N$ Wigner matrix whose entries above or on the diagonal are independent (not necessarily identically distributed) random variables with mean zero and unit variance. If, for any constant $\tau > 0$,*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{ij} E\left[|M_{ij}^{(N)}|^2 \chi\{|M_{ij}^{(N)}| \geq \tau\sqrt{N}\} \right] = 0,$$

then the ESD of $M_N^{(N)}/\sqrt{N}$ converges weakly to the semicircle distribution μ_{sc} a.s. as $N \to \infty$, where

$$\mu_{sc}(x) = \frac{\sqrt{4-x^2}}{2\pi} \chi\{|x| \leq 2\}.$$

We also need the following lemma for the proof of Theorem 1.5. Recall that $\gamma = \gamma(N) = 1/\sqrt{Np_1(1-p_1)}$, $\mathbb{C}^+ = \{x + iy : x \in \mathbb{R}, y > 0\}$ and $\{\zeta_m\}_{m=1}^K$ are defined in Section 1.

Lemma 2.2. [22, pages 171–172] *Let $\{\mu_N\}$ be a sequence of random probability measures on the real line, μ be a deterministic probability measure, and $s_{\mu_N}(z)$ (or $s_\mu(z)$) be the Stieltjes transform of μ_N (or μ, respectively).*

(i) $\mu_N \xrightarrow{w} \mu$ *a.s. if and only if $s_{\mu_N}(z)$ converges to s_μ a.s. for every $z \in \mathbb{C}^+$.*

(ii) $\mu_N \xrightarrow{w} \mu$ *in* p *if and only if* $s_{\mu_N}(z)$ *converges to* s_μ *in* p *for every* $z \in \mathbb{C}^+$.

(iii) $\mu_N \xrightarrow{w} \mu$ *in expectation if and only if* $Es_{\mu_N}(z)$ *converges to* s_μ *for every* $z \in \mathbb{C}^+$.

Proof of Theorem 1.5. From the definition of $\widetilde{B}_N$, denote

$$\widetilde{B}_N = \begin{pmatrix} W_1 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_K \end{pmatrix}, \tag{2.1}$$

where W_m is an $N_m \times N_m$ symmetric matrix whose entries follow the distribution $\mathfrak{C}(p_m, \gamma)$ with mean zero and variance $\zeta_m(N)/N$, $m = 1, \ldots, K$. Since $0 < \zeta_m = \lim_{N\to\infty} \zeta_m(N) < \infty$ for all $m = 1, 2, \ldots, K$, it follows that for any $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\epsilon < \zeta_m(N) < \infty$ for all $m = 1, 2, \ldots, K$ and all $N \geq N_0$. Thus, for any $m = 1, \ldots, K$ and all $N \geq N_0$, all entries of the matrix $X_m^{(N_m)} = W_m\sqrt{N/\zeta_m(N)}$ are mean zero and unit variance. Moreover,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N_m^2} \sum_{i,j\in C_m} E\left[|(X_m^{(N_m)})_{ij}|^2 \chi\big\{|(X_m^{(N_m)})_{ij}| \geq \tau\sqrt{N_m}\big\}\right] \\
&= \lim_{N\to\infty} \frac{1}{N_m^2} \sum_{i,j\in C_m} \left[\frac{(1-p_m)^2}{p_m(1-p_m)} p_m \chi\left(\frac{1-p_m}{\sqrt{p_m(1-p_m)}} > \tau\sqrt{N_m}\right)\right. \\
&\qquad\qquad \left. + \frac{p_m^2}{p_m(1-p_m)}(1-p_m)\chi\left(\frac{p_m}{\sqrt{p_m(1-p_m)}} > \tau\sqrt{N_m}\right)\right] \\
&= \lim_{N\to\infty} \frac{1}{N_m^2} \sum_{i,j\in C_m} \left[(1-p_m)\chi\left(\frac{1-p_m}{\sqrt{p_1(1-p_1)}} > \tau\sqrt{\frac{\alpha_m N p_m(1-p_m)}{p_1(1-p_1)}}\right)\right. \\
&\qquad\qquad \left. + p_m \chi\left(\frac{p_m}{\sqrt{p_1(1-p_1)}} > \tau\sqrt{\frac{\alpha_m N p_m(1-p_m)}{p_1(1-p_1)}}\right)\right] \\
&= \lim_{N\to\infty} \left[(1-p_m)\chi\big(\gamma(1-p_m) > \tau\sqrt{\alpha_m\zeta_m(N)}\big) + p_m\chi\big(\gamma p_m > \tau\sqrt{\alpha_m\zeta_m(N)}\big)\right] \\
&\leq \lim_{N\to\infty} \left[\chi\big(\gamma(1-p_m) > \tau\sqrt{\alpha_m\zeta_m(N)}\big) + \chi\big(\gamma p_m > \tau\sqrt{\alpha_m\zeta_m(N)}\big)\right].
\end{aligned}$$

Since $\lim_{N\to\infty} \gamma = 0$ and $0 < \zeta_m = \lim_{N\to\infty} \zeta_m(N) < \infty$ for all $m = 1, \ldots, K$, it follows that for any $m = 1, \ldots, K$ and $\tau > 0$,

$$\lim_{N\to\infty} \gamma(1-p_m) = \lim_{N\to\infty} \gamma p_m = 0 < \tau\sqrt{\alpha_m\zeta_m} = \lim_{N\to\infty} \tau\sqrt{\alpha_m\zeta_m(N)},$$

which implies that

$$\lim_{N\to\infty} \chi\big(\gamma(1-p_m) > \tau\sqrt{\alpha_m\zeta_m(N)}\big) = 0 = \lim_{N\to\infty} \chi\big(\gamma p_m > \tau\sqrt{\alpha_m\zeta_m(N)}\big).$$

Therefore,

$$\lim_{N\to\infty}\frac{1}{N_m^2}\sum_{i,j\in C_m}E\left[|(X_m^{(N_m)})_{ij}|^2\chi\{|(X_m^{(N_m)})_{ij}|\geq\tau\sqrt{N_m}\}\right]=0.$$

Next, from Theorem 2.1, for each m and all x, as $N\to\infty$, the ESD μ_m of $X_m^{(N_m)}/\sqrt{N_m}$ $= W_m/\sqrt{\alpha_m\zeta_m(N)}$ converges weakly to the semicircle law μ_{sc} a.s. That is, for each m and all x, as $N\to\infty$,

$$\mu_m(x)\xrightarrow{w}\mu_{sc}(x)\quad\text{a.s.}$$

Let $s_{\mu_{\widetilde{B}_N}}$ and s_{μ_m} be the Stieltjes transforms of $\mu_{\widetilde{B}_N}$ and μ_m, respectively, where $m=1,\ldots,K$. According to Lemma 2.2, we see that for all $z\in\mathbb{C}^+$,

$$s_{\mu_m}(z)\to s_{sc}(z)\quad\text{a.s. as } N\to\infty. \tag{2.2}$$

Refer to [22, page 172], the Stieltjes transform for the ESD μ_M of an $N\times N$ symmetric matrix M is defined as

$$s_{\mu_M}=\frac{1}{N}\operatorname{tr}(M-zI_N)^{-1},$$

where I_N is the $N\times N$ identity matrix. Consequently, this implies that for every $z\in\mathbb{C}^+$,

$$\begin{aligned}
s_{\mu_{\widetilde{B}_N}}(z)&=\frac{1}{N}\operatorname{tr}[(\widetilde{B}_N-zI_N)^{-1}]\\
&=\frac{1}{N}\sum_{m=1}^K\operatorname{tr}[(W_m-zI_{N_m})^{-1}]\\
&=\sum_{m=1}^K\frac{N_m}{N}\frac{1}{N_m}\sqrt{\frac{1}{\alpha_m\zeta_m(N)}}\operatorname{tr}\left[\left(\frac{W_m}{\sqrt{\alpha_m\zeta_m(N)}}-\frac{z}{\sqrt{\alpha_m\zeta_m(N)}}I_{N_m}\right)^{-1}\right]\\
&=\sum_{m=1}^K\sqrt{\frac{\alpha_m}{\zeta_m(N)}}\frac{1}{N_m}\operatorname{tr}\left[\left(\frac{W_m}{\sqrt{\alpha_m\zeta_m(N)}}-\frac{z}{\sqrt{\alpha_m\zeta_m(N)}}I_{N_m}\right)^{-1}\right]\\
&=\sum_{m=1}^K\sqrt{\frac{\alpha_m}{\zeta_m(N)}}s_{\mu_m}\left(\frac{z}{\sqrt{\alpha_m\zeta_m(N)}}\right),
\end{aligned}$$

which combining with (2.2) implies that

$$s_{\mu_{\widetilde{B}_N}}(z)=\sum_{m=1}^K\sqrt{\frac{\alpha_m}{\zeta_m(N)}}s_{\mu_m}\left(\frac{z}{\sqrt{\alpha_m\zeta_m(N)}}\right)\xrightarrow{a.s.}\sum_{m=1}^K\sqrt{\frac{\alpha_m}{\zeta_m}}s_{\mu_{sc}}\left(\frac{z}{\sqrt{\alpha_m\zeta_m}}\right) \tag{2.3}$$

as $N\to\infty$.

Further, since the Stieltjes transform of μ_{sc} is

$$s_{sc}(z)=\frac{-z+\sqrt{z^2-4}}{2},\quad z\in\mathbb{C}^+,$$

it follows that by (2.3), for every $z \in \mathbb{C}^+$, as $N \to \infty$,

$$
\begin{aligned}
s_{\mu_{\widetilde{B}_N}}(z) \to \sum_{m=1}^{K} \sqrt{\frac{\alpha_m}{\zeta_m}} s_{\mu_{sc}}\left(\frac{z}{\sqrt{\alpha_m\zeta_m}}\right) &= \sum_{m=1}^{K} \sqrt{\frac{\alpha_m}{\zeta_m}} \left(\frac{-z/\sqrt{\alpha_m\zeta_m} + \sqrt{z^2/\alpha_m\zeta_m - 4}}{2}\right) \\
&= \sum_{m=1}^{K} \frac{-z + \sqrt{z^2 - 4\alpha_m\zeta_m}}{2\zeta_m} \\
&= \sum_{m=1}^{K} c_m(z),
\end{aligned}
$$

where $c_m(z) = (-z + \sqrt{z^2 - 4\alpha_m\zeta_m})/(2\zeta_m)$, $m = 1, \dots, K$.

Using the formula on the square root of a complex number (see [9, page 72]), it follows that for each $m = 1, 2, \dots, K$ and any $z = x + iy \in \mathbb{C}^+$, the imaginary part of $c_m(z)$ is

$$
\Im(c_m(z)) = \frac{1}{2\zeta_m}\left(\sqrt{\frac{\sqrt{(x^2 - y^2 - 4\alpha_m\zeta_m)^2 + 4x^2y^2} - (x^2 - y^2 - 4\alpha_m\zeta_m)}{2}} - y\right),
$$

and then

$$
\begin{aligned}
\lim_{y \downarrow 0^+} \frac{1}{\pi} \Im\left(\sum_{m=1}^{K} c_m(z)\right) &= \frac{1}{\pi} \sum_{m=1}^{K} \lim_{y \downarrow 0^+} \Im(c_m(z)) \\
&= \sum_{m=1}^{K} \frac{1}{2\pi\zeta_m} \sqrt{\frac{|x^2 - 4\alpha_m\zeta_m| - (x^2 - 4\alpha_m\zeta_m)}{2}} \\
&= \sum_{m=1}^{K} \frac{\sqrt{4\alpha_m\zeta_m - x^2}}{2\pi\zeta_m} \chi\{|x| < \sqrt{4\alpha_m\zeta_m}\}.
\end{aligned}
\tag{2.4}
$$

Since $\sum_{m=1}^{K} \frac{\sqrt{4\alpha_m\zeta_m - x^2}}{2\pi\zeta_m} \chi\{|x| < \sqrt{4\alpha_m\zeta_m}\}$ is a deterministic probability measure, denote

$$
\mu(x) = \sum_{m=1}^{K} \frac{\sqrt{4\alpha_m\zeta_m - x^2}}{2\pi\zeta_m} \chi\{|x| < \sqrt{4\alpha_m\zeta_m}\}.
$$

Then from the Stieltjes inversion formula (1.2) and (2.4),

$$
s_\mu(z) = \sum_{m=1}^{K} c_m(z) = \sum_{m=1}^{K} \frac{-z + \sqrt{z^2 - 4\alpha_m\zeta_m}}{2\zeta_m}. \tag{2.5}
$$

By (2.3) and (2.5), as $N \to \infty$, $s_{\mu_{\widetilde{B}_N}}(z) \to s_\mu(z)$ a.s. for every $z \in \mathbb{C}^+$. Therefore, using Lemma 2.2, as $N \to \infty$, $\mu_{\widetilde{B}_N} \xrightarrow{w} \mu$ a.s. Hence the proof is completed. □

Remark 2.3. The deterministic probability measure μ can also be obtained from (1.3) and we show it as follows:

From the definition of the Stieltjes transform and by $\mu_{sc}(x) = \frac{\sqrt{4-x^2}}{2\pi}\chi\{|x| \le 2\}$, it follows that for every $z \in \mathbb{C}^+$,

(2.6)
$$\begin{aligned}
\sum_{m=1}^{K} \sqrt{\frac{\alpha_m}{\zeta_m}} s_{\mu_{sc}}\left(\frac{z}{\sqrt{\alpha_m\zeta_m}}\right) &= \frac{1}{2\pi}\sum_{m=1}^{K}\sqrt{\frac{\alpha_m}{\zeta_m}}\int_{\mathbb{R}}\frac{\mu_{sc}(dx)}{x-(z/\sqrt{\alpha_m\zeta_m})} \\
&= \frac{1}{2\pi}\sum_{m=1}^{K}\sqrt{\frac{\alpha_m}{\zeta_m}}\int_{\{|x|\le 2\}}\frac{\sqrt{4-x^2}}{x-(z/\sqrt{\alpha_m\zeta_m})}\,dx \\
&= \sum_{m=1}^{K}\frac{1}{2\pi\zeta_m}\int_{\{|x|\le 2\}}\frac{\sqrt{4\alpha_m\zeta_m-(\sqrt{\alpha_m\zeta_m}x)^2}}{\sqrt{\alpha_m\zeta_m}x-z}\,d\sqrt{\alpha_m\zeta_m}x \\
&= \int_{\mathbb{R}}\frac{1}{y-z}\left(\sum_{m=1}^{K}\frac{\sqrt{4\alpha_m\zeta_m-y^2}}{2\pi\zeta_m}\chi\{|y|\le\sqrt{4\alpha_m\zeta_m}\}\right)dy \\
&= \int_{\mathbb{R}}\frac{1}{y-z}\,\mu(dy) \\
&= s_\mu(z).
\end{aligned}$$

Then by (2.3) and (2.6), as $N \to \infty$, $s_{\mu_{\widetilde{B}_N}}(z) \to s_\mu(z)$ a.s. for every $z \in \mathbb{C}^+$, so by Lemma 2.2, as $N \to \infty$, $\mu_{\widetilde{B}_N} \xrightarrow{w} \mu$ a.s.

2.2. Proof of Corollary 1.6

Proof of Corollary 1.6. Notice that

$$\begin{aligned}
\int_{\mathbb{R}} x^k\,d\mu(x) &= \sum_{m=1}^{K}\int_{\mathbb{R}} x^k\frac{\sqrt{4\alpha_m\zeta_m-x^2}}{2\pi\zeta_m}\chi\{|x|<\sqrt{4\alpha_m\zeta_m}\}\,dx \\
&= \sum_{m=1}^{K}\int_{-\sqrt{4\alpha_m\zeta_m}}^{\sqrt{4\alpha_m\zeta_m}}\frac{x^k\sqrt{4\alpha_m\zeta_m-x^2}}{2\pi\zeta_m}\,dx.
\end{aligned}$$

If k is odd, then $\frac{x^k\sqrt{4\alpha_m\zeta_m-x^2}}{2\pi\zeta_m}$ is an odd function for all $m = 1, \ldots, K$ and so

$$\int_{\mathbb{R}} x^k\,d\mu(x) = \sum_{m=1}^{K}\int_{-\sqrt{4\alpha_m\zeta_m}}^{\sqrt{4\alpha_m\zeta_m}}\frac{x^k\sqrt{4\alpha_m\zeta_m-x^2}}{2\pi\zeta_m}\,dx = 0.$$

If $k = 2j$ is even, where j is a positive integer, then

$$\int_{\mathbb{R}} x^k\,d\mu(x) = \int_{\mathbb{R}} x^{2j}\,d\mu(x) = \sum_{m=1}^{K}\int_{-\sqrt{4\alpha_m\zeta_m}}^{\sqrt{4\alpha_m\zeta_m}}\frac{x^{2j}\sqrt{4\alpha_m\zeta_m-x^2}}{2\pi\zeta_m}\,dx.$$

Let $x = \left(\sqrt{4\alpha_m\zeta_m}\right)\sin\theta$, $-\pi/2 \le \theta \le \pi/2$, and so

$$\begin{aligned}
\int_{\mathbb{R}} x^k\, d\mu(x) &= \sum_{m=1}^{K} \frac{(4\alpha_m\zeta_m)^{j+1}}{2\zeta_m}\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin^{2j}\theta\cos^2\theta\, d\theta \\
&= \sum_{m=1}^{K} 2^{2j+2}\alpha_m^{j+1}\zeta_m^j\left[\frac{1}{\pi}\int_0^{\pi/2}(\sin^{2j}\theta - \sin^{2j+2}\theta)\, d\theta\right] \\
&= \sum_{m=1}^{K} 2^{2j+2}\alpha_m^{j+1}\zeta_m^j\left[\binom{2j}{j}\frac{1}{2^{2j+1}} - \binom{2j+2}{j+1}\frac{1}{2^{2j+3}}\right] \\
&= \frac{1}{j+1}\binom{2j}{j}\sum_{m=1}^{K}\alpha_m^{j+1}\zeta_m^j.
\end{aligned}$$

The third identity holds by using Wallis's formula for the integrals of the powers of the sine function: $\int_0^{\pi/2}\sin^{2i}\theta\, d\theta = \pi\binom{2i}{i}/2^{2i+1}$, see [18, page 540]. □

2.3. Proof of Theorem 1.7

Recall the Frobenius norm of an $N \times N$ matrix M is $\|M\|_F^2 = \sum_{i,j=1}^N M_{ij}^2 = \mathrm{tr}(M^2)$. To prove Theorem 1.7, we need the following lemmas and for the convenience of the reader, we quote them.

Lemma 2.4. [16, page 30] *For any $z \in \mathbb{C}^+$, the respective Stieltjes transforms of the spectral density functions of two matrices A and B satisfy that*

$$|s_{\mu_A}(z) - s_{\mu_B}(z)| \le \frac{1}{\sqrt{N}\Im(z)^2}\|A - B\|_F.$$

Lemma 2.5. [22, Theorem 2.3.16] *Let M be an $N \times N$ real-valued symmetric random matrix, with the upper triangular elements ξ_{ij}, $i \le j$ jointly independent with mean zero and variance one, and bounded in magnitude by $o(\sqrt{N})$. Then for any positive integer k,*

$$E[\mathrm{tr}(M^{2k})] = \left(\frac{1}{k+1}\binom{2k}{k} + o_k(1)\right)N^{k+1}.$$

Proof of Theorem 1.7. To prove $\mu_{\widetilde{A}_N} \xrightarrow{w} \mu$ in p and in expectation as first $N \to \infty$ and then $\zeta_0 \to 0$, we only need to prove for $z \in \mathbb{C}^+$,

$$\lim_{\zeta_0\to 0}\lim_{N\to\infty}|s_{\mu_{\widetilde{A}_N}}(z) - s_\mu(z)| = 0 \quad \text{in p and in expectation,}$$

according to Lemma 2.2.

Recall that $\widetilde{B}_N$ is as defined in Theorem 1.5 with $\widetilde{B}_{ij} = \widetilde{A}_{ij}$ if i and j belong to the same block, and $\widetilde{B}_{ij} = 0$ otherwise. Since

$$\begin{aligned}
&\lim_{\zeta_0\to 0}\lim_{N\to\infty}|s_{\mu_{\widetilde{A}_N}}(z) - s_\mu(z)| \\
&\le \lim_{\zeta_0\to 0}\lim_{N\to\infty}\left(|s_{\mu_{\widetilde{A}_N}}(z) - s_{\mu_{\widetilde{B}_N}}(z)| + |s_{\mu_{\widetilde{B}_N}}(z) - s_\mu(z)|\right) \quad \text{in p and in expectation.}
\end{aligned}$$

Our goal is to prove that for $z \in \mathbb{C}^+$,

$$\lim_{\zeta_0 \to 0} \lim_{N\to\infty} |s_{\mu_{\widetilde{A}_N}}(z) - s_{\mu_{\widetilde{B}_N}}(z)| = 0 \quad \text{in p and in expectation} \tag{2.7}$$

and

$$\lim_{\zeta_0 \to 0} \lim_{N\to\infty} |s_{\mu_{\widetilde{B}_N}}(z) - s_{\mu}(z)| = 0 \quad \text{in p and in expectation.} \tag{2.8}$$

To establish (2.7), observe that for each $N \in \mathbb{N}$,

$$\|\widetilde{A}_N - \widetilde{B}_N\|_F^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \big(\widetilde{A}_{ij}^{(N)} - \widetilde{B}_{ij}^{(N)}\big)^2 = \sum_{m=1}^{K}\sum_{\substack{n=1\\ n\neq m}}^{K}\sum_{\substack{i\in C_m\\ j\in C_n}} \big(\widetilde{A}_{ij}^{(N)}\big)^2$$

and for all $i \in C_m$, $j \in C_n$, $\widetilde{A}_{ij}^{(N)}$ independently and identically follow the distribution $\mathfrak{C}^2(p_{mn}, \gamma)$. Then

$$\begin{aligned}
E\left[\frac{1}{N}\|\widetilde{A}_N - \widetilde{B}_N\|_F^2\right] &= \frac{1}{N}\sum_{m=1}^{K}\sum_{\substack{n=1\\ n\neq m}}^{K}\sum_{\substack{i\in C_m\\ j\in C_n}} E\left[\big(\widetilde{A}_{ij}^{(N)}\big)^2\right] \\
&= \frac{1}{N}\sum_{m=1}^{K}\sum_{\substack{n=1\\ n\neq m}}^{K} N_m N_n E[\mathfrak{C}^2(p_{mn},\gamma)] \\
&= N\sum_{m=1}^{K}\sum_{\substack{n=1\\ n\neq m}}^{K} \alpha_m\alpha_n\big[\gamma^2(1-p_{mn})^2 p_{mn} + \gamma^2 p_{mn}^2(1-p_{mn})\big] \\
&= \sum_{m=1}^{K}\sum_{\substack{n=1\\ n\neq m}}^{K} \alpha_m\alpha_n \frac{p_{mn}(1-p_{mn})}{p_1(1-p_1)} \qquad \left(\text{since } \gamma = \frac{1}{\sqrt{Np_1(1-p_1)}}\right) \\
&\le K^2\zeta_0(N). \qquad \left(\text{since } \alpha_m, \alpha_n \le 1 \text{ and } \frac{p_{mn}(1-p_{mn})}{p_1(1-p_1)} \le \zeta_0(N)\right)
\end{aligned}$$

Thus $\lim_{N\to\infty} E\big[\frac{1}{N}\|\widetilde{A}_N - \widetilde{B}_N\|_F^2\big] \le \lim_{N\to\infty} K^2\zeta_0(N) = K^2\zeta_0$ and so

$$\lim_{\zeta_0\to 0}\lim_{N\to\infty} E\left[\frac{1}{N}\|\widetilde{A}_N - \widetilde{B}_N\|_F^2\right] = 0.$$

By Lemma 2.4, we see that for $z \in \mathbb{C}^+$,

$$\lim_{\zeta_0\to 0}\lim_{N\to\infty} E|s_{\mu_{\widetilde{A}_N}}(z) - s_{\mu_{\widetilde{B}_N}}(z)|^2 = 0,$$

which implies that for $z \in \mathbb{C}^+$, (2.7) holds.

Next, we would prove (2.8) in p, that is, for $z \in \mathbb{C}^+$,

$$\lim_{\zeta_0\to 0}\lim_{N\to\infty} |s_{\mu_{\widetilde{B}_N}}(z) - s_{\mu}(z)| = 0 \quad \text{in p.}$$

By Theorem 1.5 and Remark 1.3, $\mu_{\widetilde{B}_N} \xrightarrow{w} \mu$ a.s. implies $\mu_{\widetilde{B}_N} \xrightarrow{w} \mu$ in p. Again, using Lemma 2.2, (2.8) holds in p.

Next, we aim to establish (2.8) in expectation, that is, for $z \in \mathbb{C}^+$, $E\big[s_{\mu_{\widetilde{B}_N}}(z) - s_\mu(z)\big] \to 0$. According to [22, page 166], we need to demonstrate the following to prove that $E\big[s_{\mu_{\widetilde{B}_N}}(z) - s_\mu(z)\big] \to 0$, that is,

$$\lim_{N\to\infty} \frac{1}{N} E\big[\operatorname{tr}(\widetilde{B}_N^k)\big] = \int_{\mathbb{R}} x^k \, d\mu(x), \quad k = 1, 2, \dots. \tag{2.9}$$

From the definition of $\widetilde{B}_N$, the expression of $\widetilde{B}_N$ is as (2.1). Since $W_m/\sqrt{\zeta_m(N)}$ is a Wigner matrix (see [4, page 20]), we can conclude from [4, page 24], for any positive integer k,

$$\lim_{N\to\infty} \frac{1}{N} E\left[\operatorname{tr}\left(\frac{W_m}{\sqrt{\zeta_m(N)}}\right)^{2k-1}\right] = 0. \tag{2.10}$$

Applying Lemma 2.5 and $N_m = \alpha_m N$, for any positive integer k,

$$\begin{aligned} \frac{1}{N} E\big[\operatorname{tr}(W_m^{2k})\big] &= \frac{1}{N}\left(\frac{\zeta_m(N)}{N}\right)^k E\left[\operatorname{tr}\left(\sqrt{\frac{N}{\zeta_m(N)}} W_m\right)^{2k}\right] \\ &= \frac{1}{N}\left(\frac{\zeta_m(N)}{N}\right)^k \left(\frac{1}{k+1}\binom{2k}{k} + o_k(1)\right) N_m^{k+1} \\ &= \left(\frac{1}{k+1}\binom{2k}{k} + o_k(1)\right) \alpha_m^{k+1} \zeta_m^k(N). \end{aligned} \tag{2.11}$$

Since $\frac{1}{N}E\big[\operatorname{tr}(\widetilde{B}_N^k)\big] = \frac{1}{N}\big(E\big[\operatorname{tr}(W_1^k)\big] + \cdots + E\big[\operatorname{tr}(W_K^k)\big]\big)$ and $\lim_{N\to\infty} \zeta_m(N) = \zeta_m$ exist, we obtain by (2.10) and (2.11), for any positive integer k,

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} E\big[\operatorname{tr}(\widetilde{B}_N^{2k-1})\big] &= \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^K E\left[\operatorname{tr}\left(\frac{W_m}{\sqrt{\zeta_m(N)}}\right)^{2k-1}\right] (\zeta_m(N))^{k-1/2} = 0, \\ \lim_{N\to\infty} \frac{1}{N} E\big[\operatorname{tr}(\widetilde{B}_N^{2k})\big] &= \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^K E\big[\operatorname{tr}(W_m^{2k})\big] = \frac{1}{k+1}\binom{2k}{k} \sum_{m=1}^K \alpha_m^{k+1}\zeta_m^k. \end{aligned}$$

Applying Corollary 1.6, we can conclude that (2.9) holds, that is, for any positive integer k,

$$\lim_{N\to\infty} E\left[\frac{1}{N}\operatorname{tr}(\widetilde{B}_N^k)\right] = \int_{\mathbb{R}} x^k \, d\mu(x).$$

Hence the proof is completed. $\square$

Acknowledgments

The authors thank the referees for careful reading of the manuscript and for valuable comments or suggestions that have led to much improved readability of this paper. This work is financially supported by the National Science and Technology Council of Taiwan, MOST 111-2118-M-110-001-MY2 and NSTC 112-2811-M-110-031, and Giap Van Su is also financially supported by the Thai Nguyen University of Education, Vietnam.

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