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Maximal Estimates for the Bilinear Riesz Means on Heisenberg Groups

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Abstract. In this article, we investigate the maximal bilinear Riesz means S_*^α associated to the sublaplacian on the Heisenberg group. We prove that the operator S_*^α is bounded from $L^{p_1} \times L^{p_2}$ into L^p for $2 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$ when α is large than a suitable smoothness index $\alpha(p_1, p_2)$. For obtaining a lower index $\alpha(p_1, p_2)$, we define two important auxiliary operators and investigate their L^p estimates, which play a key role in our proof.

1. Introduction

A classical problem in Fourier analysis is to make precise the sense in which the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$$

holds when f is a function on $\mathbb{R}^n$. A natural way of formulating this identity is in term of a summability method. For instance, one may consider the convergence of the Bochner–Riesz means defined by

$$B_R^\delta f(x) = \int_{|\xi|<R} \widehat{f}(\xi) \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta e^{2\pi i x\cdot\xi}\, d\xi$$

as $R \to \infty$ for some suitable δ. The almost everywhere convergence of the Bochner–Riesz is related to the maximal operator $B_*^\delta = \sup_{R>0} \left|B_R^\delta\right|$. When $n = 1$, Hunt showed that if $\delta = 0$ and $f \in L^p(\mathbb{R})$, $1 < p < \infty$, $B_R^\delta f$ converges to f almost everywhere as $R \to \infty$. When $n \geq 2$ and $p \geq 2$, Carbery, Rubio de Francia and Vega [1] showed that for all $f \in L^p(\mathbb{R}^n)$ with $2 \leq p \leq 2n/(n-1-2\delta)$, $B_R^\delta f$ converges to f almost everywhere as $R \to \infty$ by using the weighted L^2-estimates of the maximal operator $B_*^\delta = \sup_{R>0} \left|B_R^\delta\right|$. When $n = 2$ and $1 < p < 2$, Tao [10] proved that if $\delta > \max\left\{3/(4p) - 3/8, 7/(6p) - 2/3\right\}$, B_*^δ is bounded from L^p into $L^{p,\infty}$, i.e., $B_R^\delta f$ converges to f almost everywhere as $R \to \infty$.

The bilinear Bochner–Riesz means is defined by

$$B_R^\alpha(f,g)(x) = \int_{|\xi|<R} \widehat{f}(\xi)\widehat{g}(\eta) \left(1 - \frac{|\xi|^2 + |\eta|^2}{R^2}\right)^\alpha e^{2\pi i x\cdot(\xi+\eta)}\, d\xi d\eta.$$

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Its almost everywhere convergence is depended on the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of the maximal operator $B_*^\alpha = \sup_{R>0} \left|B_R^\alpha\right|$. Grafakos, He and Honzík [3] showed that B_*^α is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if $\alpha > (2n+3)/4$. Jeong and Lee [4] gave a comprehensive study on this problem when $n \geq 2$ and $2 \leq p_1, p_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$. Jotsaroop and Shrivastava [5] proved improved bounds for maximal bilinear Bochner–Riesz means. Inspired by their work, we shall investigated the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of the maximal bilinear Riesz means on the Heisenberg group.

Strichartz [8, 9] developed the harmonic analysis on the Heisenberg group as the spectral theory of the sublaplacian. One may define the Riesz means in terms of the spectral decomposition of the sublaplacian. Let

$$\mathcal{L}f = \int_0^\infty \lambda P_\lambda f \, d\mu(\lambda)$$

be the spectral decomposition of the sublaplacian $\mathcal{L}$. The Riesz means associated to the sublaplacian $\mathcal{L}$ is defined by

$$S_r^\delta f = \int_0^\infty (1 - r\lambda)_+^\delta P_\lambda f \, d\mu(\lambda).$$

Gorges and Müller [2] investigate the almost everywhere convergence of the Riesz means and obtained a similar result to that in [1]. They showed that $S_r^\delta f \to f$ as $r \to 0$ for $\delta > 0$ and $f \in L^p(\mathbb{H}^n)$ provided that $\frac{Q-1}{Q}\left(\frac{1}{2} - \frac{\delta}{D-1}\right) < \frac{1}{p} \leq \frac{1}{2}$.

The bilinear Riesz means associated to the sublaplacian $\mathcal{L}$ on the Heisenberg is defined by

$$S_r^\alpha(f, g) = \int_0^\infty \int_0^\infty (1 - r(\lambda_1 + \lambda_2))_+^\alpha P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1) \, d\mu(\lambda_2).$$

The corresponding maximal operator is denoted by $S_*^\alpha = \sup_{r>0} \left|S_r^\alpha\right|$. As same as the Euclidean case, we hope to obtain the smooth indices $\alpha(p_1, p_2)$ as low as possible so that S_*^α is bounded from $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ into $L^p(\mathbb{H}^n)$ when $\alpha > \alpha(p_1, p_2)$.

2. Preliminaries

First we recall some basic facts about the Heisenberg group. These facts are familiar and easy to find in many references. Let $\mathbb{H}^n$ denote the Heisenberg group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and the group law is given by

$$(z, t)(w, s) = \left(z + w, t + s + \frac{1}{2} \operatorname{Im}(z \cdot \overline{w})\right).$$

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$. A homogeneous structure on $\mathbb{H}^n$ is given by the non-isotropic dilations $\delta_r(z, t) = (rz, r^2 t)$. We define a homogeneous norm on $\mathbb{H}^n$ by

$$|x| = \left(\frac{1}{16}|z|^4 + t^2\right)^{1/4}, \quad x = (z,t) \in \mathbb{H}^n.$$

This norm satisfies the triangle inequality and leads to a left-invariant distance $d(x,y) = |x^{-1}y|$. The ball of radius r centered at x is

$$B(x,r) = \{y \in \mathbb{H}^n : |x^{-1}y| < r\}.$$

The Haar measure dx satisfies $d\delta_r(x) = r^Q\,dx$ where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$. If f and g are functions on $\mathbb{H}^n$, their convolution is defined by

$$(f * g)(x) = \int_{\mathbb{H}^n} f(xy^{-1})g(y)\,dy, \quad x, y \in \mathbb{H}^n.$$

For each $\lambda \in \mathbb{R}^*$ and $f \in \mathscr{S}(\mathbb{H}^n)$, the inverse Fourier transform of f in variable t is defined by

$$f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z,t)\,dt.$$

An easy calculation shows that

$$(f * g)^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z-\omega)g^\lambda(\omega)e^{\frac{i}{2}\lambda \operatorname{Im}(z\cdot\overline{\omega})}\,d\omega, \quad z, \omega \in \mathbb{C}^n.$$

Thus, we are led to the convolution of the form

$$f *_\lambda g = \int_{\mathbb{C}^n} f(z-\omega)g(\omega)e^{\frac{i}{2}\lambda \operatorname{Im}(z\cdot\overline{\omega})}\,d\omega,$$

which are called the λ-twisted convolution.

The sublaplacian $\mathcal{L}$ is defined by

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2)$$

where

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j\frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j\frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n,$$

are left invariant vector fields on $\mathbb{H}^n$. Up to a constant multiple, $\mathcal{L}$ is the unique left invariant, rotation invariant differential operator that is homogeneous of degree two. Therefore, it is regarded as the counterpart of the Laplacian on $\mathbb{R}^n$. The sublaplacian $\mathcal{L}$ is a positive and essentially self-adjoint operator. In the following, we state the spectral decomposition of $\mathcal{L}$ (cf. [11]).

Let φ_k be the Laguerre functions on $\mathbb{C}^n$ given by

$$\varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right)e^{-\frac{1}{4}|z|^2},$$

where L_k^{n-1} are the Laguerre polynomials of type $n-1$ defined on $\mathbb{R}$ by

$$L_k^{n-1}(t)e^{-t}t^{n-1} = \frac{1}{k!}\left(\frac{d}{dt}\right)^k (e^{-t}t^{k+n-1}).$$

Define functions

$$e_k^\lambda(z,t) = e^{-i\lambda t}\varphi_k^\lambda(z) = e^{-i\lambda t}\varphi_k\big(\sqrt{|\lambda|}z\big), \quad \lambda \in \mathbb{R}^*.$$

For $f \in L^2(\mathbb{H}^n)$, we have the expansion

$$f(z,t) = \sum_{k=0}^{\infty}\int_{-\infty}^{\infty} f * e_k^\lambda(z,t)\, d\mu(\lambda) \tag{2.1}$$

where $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n\, d\lambda$ is the Plancherel measure for $\mathbb{H}^n$. Each $f * e_k^\lambda$ is the eigenfunction of $\mathcal{L}$ with eigenvalue $(2k+n)|\lambda|$. We also have the Plancherel formula

$$\|f\|_2^2 = (2\pi)^{-2n-1}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{C}^n}\big|f^\lambda *_\lambda \varphi_k^\lambda(z)\big|^2\lambda^{2n}\, dz d\lambda.$$

Defining

$$\widetilde{e}_k^\lambda(z,t) = e_k^{\lambda/(2k+n)}(z,t),$$

we can rewrite the decomposition (2.1) as

$$f(z,t) = \int_{-\infty}^{\infty}\sum_{k=0}^{\infty}(2k+n)^{-n-1} f * \widetilde{e}_k^\lambda(z,t)\, d\mu(\lambda).$$

Let

$$P_\lambda f(z,t) = \sum_{k=0}^{\infty}(2k+n)^{-n-1} f * (\widetilde{e}_k^\lambda + \widetilde{e}_k^{-\lambda})(z,t).$$

Then (2.1) can be written as

$$f(z,t) = \int_0^\infty P_\lambda f(z,t)\, d\mu(\lambda).$$

It is clear that $P_\lambda f$ is an eigenfunction of the $\mathcal{L}$ with eigenvalue λ and we have the spectral decomposition

$$\mathcal{L}f = \int_0^\infty \lambda P_\lambda f\, d\mu(\lambda).$$

Define the bilinear Riesz means associated to the sublaplacian $\mathcal{L}$ for $f, g \in \mathscr{S}(\mathbb{H}^n)$ by

$$S_r^\alpha(f,g) = \int_0^\infty\int_0^\infty (1 - r(\lambda_1+\lambda_2))_+^\alpha P_{\lambda_1} f P_{\lambda_2} g\, d\mu(\lambda_1)\, d\mu(\lambda_2).$$

The corresponding maximal operator is defined by

$$S_*^\alpha(f,g)(x) = \sup_{r>0}\big|S_r^\alpha(f,g)(x)\big|.$$

3. L^p-estimate for auxiliary multiplier operator

In this section, we shall define two auxiliary operators and investigate their $L^p(l^\infty)$ estimates, which play a key role in the proof of our main Theorem 4.1.

Let $I = [-1, 1]$ and consider a class of smooth function

$$C^N(I) = \left\{ \varphi : \operatorname{supp} \varphi \subset I, \|\varphi\|_{C^N(\mathbb{R})} = \max_{0 \le n \le N} \left\| \frac{d^n}{dt^n} \varphi \right\|_{L^\infty(\mathbb{R})} \le 1 \right\}.$$

For $\varphi \in C^N(I)$ and $\rho, \delta, r > 0$, we define the multiplier operator

$$F^\varphi_{\rho,\delta,r} f(x) = \int_0^\infty \varphi\left(\frac{\rho - r\lambda}{\delta}\right) P_\lambda f \, d\mu(\lambda), \quad f \in \mathscr{S}(\mathbb{H}^n),$$

and define its corresponding operator for $k \in \mathbb{Z}$ by

$$D^\varphi_{\delta,k} f(x) = \left(\sum_{\rho \in \delta\mathbb{Z} \cap [0,2]} \int_1^2 \left| F^\varphi_{\rho,\delta,2^k r} f(x) \right|^2 dr \right)^{1/2}, \quad f \in \mathscr{S}(\mathbb{H}^n).$$

Write $F^\varphi_{\rho,\delta} = F^\varphi_{\rho,\delta,1}$ for simplicity. Since P_λ is a convolution operator, we have

$$F^\varphi_{\rho,\delta,r} f(x) = \int_{\mathbb{H}^n} f(x\omega^{-1}) K^\varphi_{\rho,\delta,r}(\omega) \, d\omega$$

where the kernel $K^\varphi_{\rho,\delta,r}$ is given by

$$K^\varphi_{\rho,\delta,r}(\omega) = \sum_{k=0}^\infty (2k+n)^{-n-1} \int_{\mathbb{R}} \varphi\left(\frac{\rho - r|\lambda|}{\delta}\right) \widetilde{e}^\lambda_k(\omega) \, d\mu(\lambda).$$

Notice that for any $t > 0$,

$$K^\varphi_{\rho,\delta,r/t}(\omega) = t^{Q/2} K^\varphi_{\rho,\delta,r}\big(\delta_{\sqrt{t}}\omega\big).$$

It is easy to verify that

$$F^\varphi_{t\rho,t\delta,r} f(x) = F^\varphi_{\rho,\delta,r/t} f(x) = F^\varphi_{\rho,\delta,r} f_{1/\sqrt{t}}\big(\delta_{\sqrt{t}} x\big), \tag{3.1}$$

where $f_s = f(\delta_s \,\cdot\,)$ for any $s \in \mathbb{R}$, $s \ne 0$. Especially, if $r = 1$ and $t = 1/r$, we have

$$K^\varphi_{\rho,\delta,r}(\omega) = \left(\frac{1}{r}\right)^{Q/2} K^\varphi_{\rho,\delta}\big(\delta_{1/\sqrt{r}}\omega\big) \quad \text{and} \quad F^\varphi_{\rho,\delta,r} f(x) = F^\varphi_{\rho,\delta} f_{\sqrt{r}}\big(\delta_{1/\sqrt{r}} x\big). \tag{3.2}$$

Proposition 3.1. *Let $2 \le p \le \infty$ and $0 < \delta \le 1/4$. Suppose that $b > \frac{1}{2}(D-1)$ where $D = 2n+1$ is the topological dimension of $\mathbb{H}^n$. Then, we have that*

$$\left\| \left(\int_{1/2}^1 \left| F^\varphi_{\rho,\delta} f \right|^2 d\rho \right)^{1/2} \right\|_{L^p(\mathbb{H}^n)} \le C\delta^{-(b-1/2)} \|f\|_{L^p(\mathbb{H}^n)}. \tag{3.3}$$

It follows that for any $\varepsilon > 0$,

$$\left\|\sup_{k\in\mathbb{Z}}\left|D^{\varphi}_{\delta,k}f\right|\right\|_{L^p(\mathbb{H}^n)} = \left\|\sup_{k\in\mathbb{Z}}\left(\sum_{\rho\in\delta\mathbb{Z}\cap[0,2]}\int_1^2\left|F^{\varphi}_{\rho,\delta,2^k r}f\right|^2 dr\right)^{1/2}\right\|_{L^p(\mathbb{H}^n)} \le C\delta^{-(b-1/2)-\varepsilon}\|f\|_{L^p(\mathbb{H}^n)}. \tag{3.4}$$

Proof. It is easy to prove (3.3). By Corollary 2.6 in [7], we know that the kernel $K^{\varphi}_{\rho,\delta}$ of multiplier operator $F^{\varphi}_{\rho,\delta}$ satisfies

$$\int_{\mathbb{H}^n}\left|K^{\varphi}_{\rho,\delta}(\omega)\right| d\omega \le \left\|\varphi\left(\frac{\rho-\cdot}{\delta}\right)\right\|_{L^2_{b+1/2}} \le C\delta^{-b}$$

for any $b > \frac{1}{2}(D-1)$, where the Sobolev norm is defined by

$$\|f\|_{L^2_\alpha} = \left(\int_{\mathbb{R}}\left||x|^\alpha \widehat{f}(x)\right|^2 dx\right)^{1/2}.$$

By Young's inequality, we get that for any $1 \le p \le \infty$,

$$\left\|F^{\varphi}_{\rho,\delta}f\right\|_{L^p(\mathbb{H}^n)} \le \left\|K^{\varphi}_{\rho,\delta}\right\|_{L^1(\mathbb{H}^n)}\|f\|_{L^p(\mathbb{H}^n)} \le C\delta^{-b}\|f\|_{L^p(\mathbb{H}^n)}.$$

Then, using Minkowski's inequality for $p \ge 2$, it follows that

$$\begin{aligned}\left\|\left(\int_{1/2}^1\left|F^{\varphi}_{\rho,\delta}f\right|^2 d\rho\right)^{1/2}\right\|_{L^p(\mathbb{H}^n)} &\le \left(\int_{1/2}^1\left(\int_{\mathbb{H}^n}\left|F^{\varphi}_{\rho,\delta}f\right|^{2\cdot\frac{p}{2}} dx\right)^{2/p} d\rho\right)^{1/2} \\ &\le C\delta^{1/2}\left\|F^{\varphi}_{\rho,\delta}f\right\|_{L^p(\mathbb{H}^n)} \le C\delta^{-(b-1/2)}\|f\|_{L^p(\mathbb{H}^n)}.\end{aligned}$$

To obtain (3.4), we decompose interval $[0,2]$ into dyadic subintervals as follows:

$$[0,2] = [0,4\delta]\cup[4\delta,2], \quad [4\delta,2] = \bigcup_{j=-1}^{j_0} I_j = \bigcup_{j=-1}^{j_0}[4\delta,2]\cap[2^{-j-1},2^{-j}],$$

where j_0 is the smallest integer satisfying $2^{-j_0-1} \le 4\delta$. Then, by the triangle inequality, we have that

$$\begin{aligned}&\left\|\sup_{k\in\mathbb{Z}}\left|D^{\varphi}_{\delta,k}f(x)\right|\right\|_{L^p(\mathbb{H}^n)} \\ &= \left\|\sup_{k\in\mathbb{Z}}\left(\sum_{\rho\in\delta\mathbb{Z}\cap[0,2]}\int_1^2\left|F^{\varphi}_{\rho,\delta,2^k r}f(x)\right|^2 dr\right)^{1/2}\right\|_{L^p(\mathbb{H}^n)} \\ &\le \left\|\left(\sup_{k\in\mathbb{Z}}\sum_{\rho\in\delta\mathbb{Z}\cap[0,4\delta]}\int_1^2\left|F^{\varphi}_{\rho,\delta,2^k r}f(x)\right|^2 dr + \sum_{j=-1}^{j_0}\sup_{k\in\mathbb{Z}}\sum_{\rho\in\delta\mathbb{Z}\cap I_j}\int_1^2\left|F^{\varphi}_{\rho,\delta,2^k r}f(x)\right|^2 dr\right)^{1/2}\right\|_{L^p(\mathbb{H}^n)} \\ &\le \left\|\left(\sup_{k\in\mathbb{Z}}\int_1^2\sum_{\rho\in\delta\mathbb{Z}\cap[0,4\delta]}\left|F^{\varphi}_{\rho,\delta,2^k r}f(x)\right|^2 dr\right)^{1/2}\right\|_{L^p(\mathbb{H}^n)}\end{aligned}$$

$$+ \sum_{j=-1}^{j_0} \left\| \left(\sup_{k\in\mathbb{Z}} \int_1^2 \sum_{\rho\in\delta\mathbb{Z}\cap I_j} \left| F^{\varphi}_{\rho,\delta,2^k r} f(x) \right|^2 dr \right)^{1/2} \right\|_{L^p(\mathbb{H}^n)}.$$

Setting

$$I_j = \left\| \left(\sup_{k\in\mathbb{Z}} \sum_{\rho\in\delta\mathbb{Z}\cap I_j} \int_1^2 \left| F^{\varphi}_{\rho,\delta,2^k r} f(x) \right|^2 dr \right)^{1/2} \right\|_{L^p(\mathbb{H}^n)} \quad \text{for } -1 \le j \le j_0$$

and

$$II = \left\| \left(\sup_{k\in\mathbb{Z}} \sum_{\rho\in\delta\mathbb{Z}\cap[0,4\delta]} \int_1^2 \left| F^{\varphi}_{\rho,\delta,2^k r} f(x) \right|^2 dr \right)^{1/2} \right\|_{L^p(\mathbb{H}^n)},$$

it follows that

$$\left\| \sup_{k\in\mathbb{Z}} \left| D^{\varphi}_{\delta,k} f(x) \right| \right\|_{L^p(\mathbb{H}^n)} \le \sum_{j=-1}^{j_0} I_j + II. \tag{3.5}$$

By the first equality relation in (3.1), we notice that for any $-1 \le j \le j_0$, $j \ne 0$,

$$\begin{aligned} &\left\| \left(\sup_{k\in\mathbb{Z}} \sum_{\rho\in\delta\mathbb{Z}\cap I_j} \int_1^2 \left| F^{\varphi}_{\rho,\delta,2^k r} f(x) \right|^2 dr \right)^{1/2} \right\|_{L^p(\mathbb{H}^n)} \\ &= \left\| \left(\sup_{k\in\mathbb{Z}} \sum_{2^j\rho\in 2^j\rho\delta\mathbb{Z}\cap I_0} \int_1^2 \left| F^{\varphi}_{2^j\rho,2^j\delta,2^{k-j} r} f(x) \right|^2 dr \right)^{1/2} \right\|_{L^p(\mathbb{H}^n)}, \end{aligned}$$

and $2^{-j} \ge 2^{-j_0} > 4\delta$ such that $2^j\delta < 1/4$ for any $0 < \delta \le 1/4$. These imply that once $I_0 \le \delta^{-(b-1/2)} \|f\|_{L^p(\mathbb{H}^n)}$, then $I_j \le (2^j\delta)^{-(b-1/2)}$ and

$$\sum_{j=-1}^{j_0} I_j \le \sum_{j=-1}^{j_0} (2^j\delta)^{-(b-1/2)} \le C\delta^{-(b-1/2)-\varepsilon} \|f\|_{L^p(\mathbb{H}^n)} \tag{3.6}$$

for any $\varepsilon > 0$ since $j_0 = O(\log(1/\delta))$. Thus, to obtain (3.4), it suffices to show that

$$\max\{I_0, II\} \le \delta^{-(b-1/2)}.$$

To estimate I_0, we consider the Littlewood–Paley projection operator P_m, $m \in \mathbb{Z}$, defined by

$$P_m f = \int_0^\infty \beta(2^{-m}\lambda) P_\lambda f \, d\mu(\lambda),$$

where $\beta \in C_0^\infty[1/2, 2]$ satisfying $0 \le \beta \le 1$ and $\sum_{m\in\mathbb{Z}} \beta(2^{-m}t) = 1$ for each $t > 0$. Then, we have that

$$f = \int_0^\infty P_\lambda f \, d\mu(\lambda) = \sum_{m\in\mathbb{Z}} \int_0^\infty \beta(2^{-m}\lambda) P_\lambda f \, d\mu(\lambda) = \sum_{m\in\mathbb{Z}} P_m f.$$

Since $\operatorname{supp}\varphi \subset [-1,1]$, $\rho \in [1/2,1]$, $r \in [1,2]$ and $0 < \delta \le 1/4$, we see that

$$\varphi\left(\frac{\rho - 2^k r\lambda}{\delta}\right)\beta(2^{-m}\lambda) \equiv 0 \quad \text{expect } -3 \le k+m \le 2.$$

Using this, we can get that for any $k \in \mathbb{Z}$,

(3.7)

$$\begin{aligned} F^{\varphi}_{\rho,\delta,2^k r} f &= F^{\varphi}_{\rho,\delta,2^k r}\left(\sum_{m\in\mathbb{Z}} P_m f\right) = \sum_{m\in\mathbb{Z}} \int_0^\infty \varphi\left(\frac{\rho - 2^k r\lambda}{\delta}\right)\beta(2^{-m}\lambda) P_\lambda f \, d\mu(\lambda) \\ &= \sum_{\substack{m\in\mathbb{Z}\\ -3\le k+m\le 2}} \int_0^\infty \varphi\left(\frac{\rho - 2^k r\lambda}{\delta}\right)\beta(2^{-m}\lambda) P_\lambda f \, d\mu(\lambda) = \sum_{\substack{m\in\mathbb{Z}\\ -3\le k+m\le 2}} F^{\varphi}_{\rho,\delta,2^k r}(P_m f). \end{aligned}$$

Since $p \ge 2$, applying (3.7), (3.2) and Mikowski's inequality, it follows that

(3.8)

$$\begin{aligned} I_0 &\le \left(\int_{\mathbb{H}^n}\left(\sup_{k\in\mathbb{Z}} \sum_{\rho\in\delta\mathbb{Z}\cap I_0} \int_1^2 \left|F^{\varphi}_{\rho,\delta,2^k r} f(x)\right|^2 dr\right)^{p/2} dx\right)^{1/p} \\ &= \left(\int_{\mathbb{H}^n}\left(\sup_{k\in\mathbb{Z}} \sum_{\rho\in\delta\mathbb{Z}\cap I_0} \int_1^2 \left|\sum_{\substack{m\in\mathbb{Z}\\ -3\le k+m\le 2}} F^{\varphi}_{\rho,\delta,2^k r}(P_m f)(x)\right|^2 dr\right)^{p/2} dx\right)^{1/p} \\ &= \left(\int_{\mathbb{H}^n}\left(\sup_{k\in\mathbb{Z}} \sum_{\rho\in\delta\mathbb{Z}\cap I_0} \int_1^2 \left|\sum_{\substack{m\in\mathbb{Z}\\ -3\le k+m\le 2}} F^{\varphi}_{\rho,\delta}(P_m f)_{\sqrt{2^k r}}\big(\delta_{1/\sqrt{2^k r}} x\big)\right|^2 dr\right)^{p/2} dx\right)^{1/p} \\ &\le \sum_{k\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\ -3\le k+m\le 2}} \left(\int_{\mathbb{H}^n}\left(\int_1^2 \sum_{\rho\in\delta\mathbb{Z}\cap I_0} \left|F^{\varphi}_{\rho,\delta}(P_m f)_{\sqrt{2^k r}}\big(\delta_{1/\sqrt{2^k r}} x\big)\right|^2 dr\right)^{p/2} dx\right)^{2/p\cdot 1/2} \\ &\le \sum_{k\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\ -3\le k+m\le 2}} \left(\int_1^2 \big(\sqrt{2^k r}\big)^{2Q/p} \left\|\sum_{\rho\in\delta\mathbb{Z}\cap I_0} \left|F^{\varphi}_{\rho,\delta}(P_m f)_{\sqrt{2^k r}}\right|^2\right\|_{L^{p/2}(\mathbb{H}^n)} dr\right)^{1/2} \\ &= \sum_{k\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\ -3\le k+m\le 2}} \left(\int_1^2 \big(\sqrt{2^k r}\big)^{2Q/p} \left\|\left(\sum_{\rho\in\delta\mathbb{Z}\cap I_0} \left|F^{\varphi}_{\rho,\delta}(P_m f)_{\sqrt{2^k r}}\right|^2\right)^{1/2}\right\|^2_{L^{p}(\mathbb{H}^n)} dr\right)^{1/2}. \end{aligned}$$

Notice that the L^p-boundedness properties of the square function in (3.3) and the discretize square function in the above are essentially equivalent. Hence, we have that

$$\begin{aligned} \left\|\left(\sum_{\rho\in\delta\mathbb{Z}\cap I_0} \left|F^{\varphi}_{\rho,\delta}(P_m f)_{\sqrt{2^k r}}\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{H}^n)} &\le C\delta^{-(b-1/2)} \big\|(P_m f)_{\sqrt{2^k r}}\big\|_{L^p(\mathbb{H}^n)} \\ &\le C\delta^{-(b-1/2)} \big(\sqrt{2^k r}\big)^{-Q/p} \|P_m f\|_{L^p(\mathbb{H}^n)}. \end{aligned}$$

Inserting this into (3.8) and using the Littlewood–Paley theorem, we can obtain that

(3.9)

$$\begin{aligned}I_0 &\leq \sum_{k\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\ -3\leq k+m\leq 2}} \left(\int_1^2 (\sqrt{2^k r})^{2Q/p} \left\| \left(\sum_{\rho\in\delta\mathbb{Z}\cap I_0} \left| F^{\varphi}_{\rho,\delta}(P_m f)_{\sqrt{2^k r}} \right|^2 \right)^{1/2} \right\|^2_{L^p(\mathbb{H}^n)} dr \right)^{1/2} \\ &\leq C\delta^{-(b-1/2)} \left(\sum_{k\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\ -3\leq k+m\leq 2}} \|P_m f\|^2_{L^p(\mathbb{H}^n)} \right)^{1/2} \\ &\leq C\delta^{-(b-1/2)} \left\| \left(\sum_{m\in\mathbb{Z}} |P_m f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{H}^n)} \leq C\delta^{-(b-1/2)} \|f\|_{L^p(\mathbb{H}^n)}.\end{aligned}$$

Next, we consider the estimate of II. Notice that

$$F^{\varphi}_{\rho,\delta} f(x) = \int_0^\infty \varphi\left(\frac{\rho-\lambda}{\delta}\right) P_\lambda f\, d\mu(\lambda) = \int_{\mathbb{H}^n} f(x\omega^{-1}) K_{\rho,\delta}(\omega)\, d\omega.$$

Setting $R^l_t(\omega)$ to be the kernel of the Riesz means $\int_0^t (1-\lambda/t)^l P_\lambda f\, d\mu(\lambda)$, we see that

$$t \to R^0_t(\omega)$$

is a function of bounded variation. Then, the kernel $K^{\varphi}_{\rho,\delta}$ can be written as

$$K_{\rho,\delta}(\omega) = \int_0^\infty \varphi\left(\frac{\rho-\lambda}{\delta}\right) \frac{\partial}{\partial\lambda} R^0_\lambda(\omega)\, d\mu(\lambda).$$

Integration by parts and using the identity

$$\frac{\partial}{\partial t}(t^m R^m_t(\omega)) = m t^{m-1} R^{m-1}_t(\omega),$$

where m is a positive integer, we get that

$$K_{\rho,\delta}(\omega) = c_m \int_0^\infty \left(\partial_\lambda^{2m+2} \varphi\left(\frac{\rho-\lambda}{\delta}\right)\right) \lambda^{2m+1} R^{2m+1}_\lambda(\omega)\, d\mu(\lambda).$$

It is known that (see Theorem 2.5.3 in [11])

$$\left| R^{2m+1}_\lambda(\omega) \right| \leq C\lambda^{Q/2} (1+\lambda^{1/2}|\omega|)^{-2m}. \tag{3.10}$$

We let $m = Q/2+1$ and $\varphi \in C^N(I)$ with $N = 2m+2$. Then, we have that

$$\left| \partial_\lambda^{2m+2} \varphi\left(\frac{\rho-\lambda}{\delta}\right) \right| \leq \delta^{-(2m+2)}.$$

This together with (3.10) yields that for any $\rho \in \delta\mathbb{Z}\cap[0,4\delta]$,

$$\begin{aligned}\left| K_{\rho,\delta}(\omega) \right| &\leq c_m (1+|\omega|)^{-2m} \delta^{-(2m+2)} \int_0^{5\delta} \lambda^{2m+1-m+Q/2+n}\, d\lambda \\ &\leq c_m \delta^{-m+n+Q/2} (1+|\omega|)^{-2m} \leq c_m (1+|\omega|)^{-2m}.\end{aligned}$$

Using (3.2) and Young's inequality, it follows that for any $r \in [1,2]$, $k \in \mathbb{Z}$ and $\rho \in \delta\mathbb{Z} \cap [0, 4\delta]$,

$$\left\|F^{\varphi}_{\rho,\delta,2^k r} f\right\|_{L^p(\mathbb{H}^n)} = \left\|F^{\varphi}_{\rho,\delta} f\right\|_{L^p(\mathbb{H}^n)} \leq c_m \|f\|_{L^p(\mathbb{H}^n)} \int_{\mathbb{H}^n} (1+|\omega|)^{-2m}\, d\omega \leq C\|f\|_{L^p(\mathbb{H}^n)}.$$

Hence, by Minkowski's inequality, we can get that

$$\begin{aligned} II &= \left\| \left(\sup_{k\in\mathbb{Z}} \sum_{\rho\in\delta\mathbb{Z}\cap[0,4\delta]} \int_1^2 \left|F^{\varphi}_{\rho,\delta,2^k r} f(x)\right|^2 dr \right)^{1/2} \right\|_{L^p(\mathbb{H}^n)} \\ &\leq \sup_{k\in\mathbb{Z}} \left(\int_1^2 \left\| \sum_{\rho\in\delta\mathbb{Z}\cap[0,4\delta]} \left|F^{\varphi}_{\rho,\delta,2^k r} f\right|^2 \right\|_{L^{p/2}(\mathbb{H}^n)} dr \right)^{1/2} \\ &\leq \sup_{k\in\mathbb{Z}} \left(\int_1^2 \sum_{\rho\in\delta\mathbb{Z}\cap[0,4\delta]} \left\|F^{\varphi}_{\rho,\delta,2^k r} f\right\|^2_{L^p(\mathbb{H}^n)} dr \right)^{1/2} \leq C\|f\|_{L^p}. \end{aligned} \tag{3.11}$$

Applying (3.5), (3.6) and the above estimates (3.9), (3.11), we can conclude that for any $\varepsilon > 0$,

$$\left\| \sup_{k\in\mathbb{Z}} \left|D^{\varphi}_{\delta,k} f(x)\right| \right\|_{L^p(\mathbb{H}^n)} \leq C\delta^{-(b-1/2)} \|f\|_{L^p(\mathbb{H}^n)}.$$

The proof of (3.4) is complete. □

In [6], we proved that for any function $m \in L^\infty(\mathbb{R})$ and $0 \leq a < b$, the multiplier operator $T_m f = \int_a^b m(\lambda) P_\lambda f\, d\mu(\lambda)$ is bounded from $L^p(\mathbb{H}^n)$ into $L^2(\mathbb{H}^n)$ for any $1 \leq p \leq 2$, i.e.,

$$\|T_m f\|_2 \leq C\|m\|_\infty \big((b-a)b^n\big)^{(1/p-1/2)} \|f\|_p.$$

Using this estimate, it is easy to check that

$$\left\| \left(\int_{1/2}^1 \left|F^{\varphi}_{\rho,\delta} f(x)\right|^2 d\rho \right)^{1/2} \right\|_{L^2(\mathbb{H}^n)} \leq C\delta^{1/2} \|f\|_{L^2(\mathbb{H}^n)}.$$

By the same argument of Proposition 3.1, we have that

$$\left\| \sup_{k\in\mathbb{Z}} \left|D^{\varphi}_{\delta,k} f\right| \right\|_{L^2(\mathbb{H}^n)} \leq C\delta^{1/2-\varepsilon} \|f\|_{L^2(\mathbb{H}^n)} \tag{3.12}$$

for any $\varepsilon > 0$. Then, by interpolation between the estimates in (3.12) and (3.4) for $p = \infty$, we can get the following result.

Corollary 3.2. *Let $2 \leq p \leq \infty$ and $0 < \delta \leq 1/4$. Suppose that $b > \frac{1}{2}(D-1)$ where $D = 2n+1$ is the topological dimension of $\mathbb{H}^n$. Then,*

$$\left\| \sup_{k\in\mathbb{Z}} \left|D^{\varphi}_{\delta,k} f\right| \right\|_{L^p(\mathbb{H}^n)} \leq C\delta^{-[(b-1/2)(1-2/p)-1/p]-\varepsilon} \|f\|_{L^p(\mathbb{H}^n)}.$$

4. Boundedness of the maximal operator S_*^α

Theorem 4.1. *Let* $2 \le p_1, p_2 \le \infty$ *and* $1/p = 1/p_1 + 1/p_2$. *If* $\alpha > D(1-1/p) + 1/p$, *then* S_*^α *is bounded from* $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ *into* $L^p(\mathbb{H}^n)$.

Proof. Fix $\alpha > 0$. Let us choose $\psi \in C_0^\infty([1/2, 2])$ and $\psi_0 \in C_0^\infty([-3/4, 3/4])$ such that

$$(1-t)_+^\alpha = \sum_{\delta \in D} \delta^\alpha \psi\left(\frac{1-t}{\delta}\right) + \psi_0(t), \quad 0 \le t \le 1,$$

where $D = \{2^k : k \in \mathbb{Z} \text{ and } k \le -2\}$. Using this, we can decompose

$$S_r^\alpha = \sum_{\delta \in D}^{\infty} \delta^\alpha S_r^\delta + S_r^0$$

where

$$S_r^\delta(f,g) = \int_0^\infty \int_0^\infty \psi\left(\frac{1 - r\lambda_1 - r\lambda_2}{\delta}\right) P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1) \, d\mu(\lambda_2)$$

and

$$S_r^0(f,g) = \int_0^\infty \int_0^\infty \psi_0(r\lambda_1 + r\lambda_2) P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1) \, d\mu(\lambda_2).$$

It follows that

$$S_*^\alpha(f,g)(x) \le \sum_{\delta \in D} \delta^\alpha \sup_{r>0} \left|S_r^\delta(f,g)(x)\right| + \sup_{r>0} \left|S_r^0(f,g)(x)\right|.$$

Since $\psi_0 \in C_0^\infty([-3/4, 3/4])$, using Holder's inequality, it is easy to see that for any $2 \le p_1, p_2 \le \infty$ and $1/p = 1/p_1 + 1/p_2$,

$$\left\|S_*^0(f,g)\right\|_{L^p(\mathbb{H}^n)} = \left\|\sup_{r>0} \left|S_r^0(f,g)\right|\right\|_{L^p(\mathbb{H}^n)} \le \|fg\|_{L^p(\mathbb{H}^n)} \le \|f\|_{L^{p_1}(\mathbb{H}^n)} \|g\|_{L^{p_2}(\mathbb{H}^n)}.$$

Therefore, to obtain Theorem 4.1, we have to focus on obtaining estimates for the maximal operator

$$S_*^\delta(f,g)(x) = \sup_{r>0} \left|S_r^\delta(f,g)(x)\right| \quad \text{for } 0 < \delta \le 1/4.$$

By the fundamental theorem of calculus, we see that $|F(t)| \le |F(s)| + \int_1^2 |F'(\tau)| \, d\tau$ for any $t, s \in [1,2]$. This implies that

$$\begin{aligned} S_*^\delta(f,g)(x) &= \sup_{k \in \mathbb{Z}} \sup_{1 \le r \le 2} \left|S_{2^k r}^\delta(f,g)(x)\right| \\ &\le \sup_{k \in \mathbb{Z}} \int_1^2 \left|S_{2^k r}^\delta(f,g)(x)\right| dr + \sup_{k \in \mathbb{Z}} \int_1^2 \left|\frac{\partial}{\partial r} S_{2^k r}^\delta(f,g)(x)\right| dr. \end{aligned}$$

Since

$$\frac{\partial}{\partial r} S_{2^k r}^\delta(f,g) = \frac{-2^k}{\delta} \int_0^\infty \int_0^\infty (\lambda_1 + \lambda_2) \psi'\left(\frac{1 - 2^k r(\lambda_1 + \lambda_2)}{\delta}\right) P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1) \, d\mu(\lambda_2),$$

we can conclude that $\frac{\partial}{\partial r} S^\delta_{2^k r}(f,g)$ satisfies the same quantitative properties as $\frac{1}{\delta} S^\delta_{2^k r}(f,g)$ when $1 \le r \le 2$. Hence, to estimate $S^\delta_*(f,g)$, it suffices to consider the operator

$$(f,g) \to \sup_{k\in\mathbb{Z}} \int_1^2 \left| S^\delta_{2^k r}(f,g)(x) \right| dr.$$

To estimate this operator, we choose $\varphi \in C_0^\infty(I)$ satisfying $\sum_{l\in\mathbb{Z}} \varphi(t+l) = 1$ for all $t \in \mathbb{R}$. Fix $0 < \delta \le 1/4$ and set $\widetilde{\delta} = \delta^{1+\kappa}$ with some $\kappa > 0$. Then, for any $r > 0$, we can write

$$\psi\left(\frac{1 - r\lambda_1 - r\lambda_2}{\delta}\right) = \sum_{\sigma\in\widetilde{\delta}\mathbb{Z}} \sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]} \varphi\left(\frac{\rho - r\lambda_1}{\widetilde{\delta}}\right) \varphi\left(\frac{\sigma - \rho - r\lambda_2}{\widetilde{\delta}}\right) \psi\left(\frac{1 - r\lambda_1 - r\lambda_2}{\delta}\right).$$

Since $\operatorname{supp}\varphi \subset [-1,1]$ and $\operatorname{supp}\psi \subset [1/2, 2]$, then

$$\varphi\left(\frac{\rho - r\lambda_1}{\widetilde{\delta}}\right) \psi\left(\frac{1 - r\lambda_1 - r\lambda_2}{\delta}\right) = 0 \quad \text{except } 1 - 3\delta \le r\lambda_2 + \rho \le 1 + \delta.$$

It follows that

$$\varphi\left(\frac{\rho - r\lambda_1}{\widetilde{\delta}}\right) \varphi\left(\frac{\sigma - \rho - r\lambda_2}{\widetilde{\delta}}\right) \psi\left(\frac{1 - r\lambda_1 - r\lambda_2}{\delta}\right) = 0 \quad \text{except } \sigma \in [1 - 4\delta, 1 + 2\delta].$$

Thus,

$$\begin{aligned} S^\delta_r(f,g) = & \sum_{\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]} \sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]} \int_0^\infty \int_0^\infty \varphi\left(\frac{\rho - r\lambda_1}{\widetilde{\delta}}\right) \varphi\left(\frac{\sigma - \rho - r\lambda_2}{\widetilde{\delta}}\right) \\ & \times \psi\left(\frac{1 - r\lambda_1 - r\lambda_2}{\delta}\right) P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1)\, d\mu(\lambda_2). \end{aligned} \tag{4.1}$$

At the same time, by the Fourier inversion formula, we have

$$\begin{aligned} \psi\left(\frac{1 - r\lambda_1 - r\lambda_2}{\delta}\right) &= \int_{\mathbb{R}} \widehat{\psi}(\tau) e^{2\pi i\tau\left(\frac{1 - r\lambda_1 - r\lambda_2}{\delta}\right)} \, d\tau \\ &= \int_{\mathbb{R}} \widehat{\psi}(\tau) e^{2\pi i\tau\left(\frac{\sigma - r\lambda_1 - r\lambda_2}{\delta}\right)} e^{2\pi i\tau\left(\frac{1-\sigma}{\delta}\right)} \, d\tau. \end{aligned} \tag{4.2}$$

Applying Taylor's theorem for $e^{2\pi i\tau\left(\frac{\sigma - r\lambda_1 - r\lambda_2}{\delta}\right)}$, we get that for any $\rho \in \widetilde{\delta}\mathbb{Z} \cap [0,2]$,

$$\begin{aligned} e^{2\pi i\tau\left(\frac{\sigma - r\lambda_1 - r\lambda_2}{\delta}\right)} &= \sum_{N=0}^\infty \frac{1}{N!} \left(\frac{\tau(\sigma - r\lambda_1 - r\lambda_2)}{\delta}\right)^N \\ &= \sum_{N=0}^\infty \frac{1}{N!} \sum_{0\le a+b\le N} c_{a,b} \left(\frac{\tau(\rho - r\lambda_1)}{\delta}\right)^a \left(\frac{\tau(\sigma - \rho - r\lambda_2)}{\delta}\right)^b. \end{aligned} \tag{4.3}$$

Putting (4.3) into the right-hand side of (4.2), it follows that

$$\begin{aligned}
&\psi\left(\frac{1-r\lambda_1-r\lambda_2}{\delta}\right)\\
&=\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{0\le a+b\le N}c_{a,b}\left(\int_{\mathbb{R}}\widehat{\psi}(\tau)\tau^{a+b}e^{2\pi i\tau\left(\frac{1-\sigma}{\delta}\right)}\,d\tau\right)\left(\frac{\rho-r\lambda_1}{\delta}\right)^a\left(\frac{\sigma-\rho-r\lambda_2}{\delta}\right)^b\\
&=\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{0\le a+b\le N}c_{a,b}\psi^{(a+b)}\left(\frac{1-\sigma}{\delta}\right)\left(\frac{\rho-r\lambda_1}{\delta}\right)^a\left(\frac{\sigma-\rho-r\lambda_2}{\delta}\right)^b.
\end{aligned}$$

Insert this into (4.1) and let $\varphi_\beta(t)=t^\beta\varphi(t)$. Notice that $\varphi_\beta\in C_0^\infty(I)$ for any $\beta\in\mathbb{N}$. Then, we can obtain that

$$\begin{aligned}
S_r^\delta(f,g)=&\sum_{\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]}\;\sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]}\int_0^\infty\!\!\int_0^\infty\varphi\left(\frac{\rho-r\lambda_1}{\widetilde{\delta}}\right)\varphi\left(\frac{\sigma-\rho-r\lambda_2}{\widetilde{\delta}}\right)\\
&\times\psi\left(\frac{1-r\lambda_1-r\lambda_2}{\delta}\right)P_{\lambda_1}fP_{\lambda_2}g\,d\mu(\lambda_1)\,d\mu(\lambda_2)\\
=&\sum_{\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]}\;\sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]}\int_0^\infty\!\!\int_0^\infty\varphi\left(\frac{\rho-r\lambda_1}{\widetilde{\delta}}\right)\varphi\left(\frac{\sigma-\rho-r\lambda_2}{\widetilde{\delta}}\right)\\
&\times\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{0\le a+b\le N}c_{a,b}\left(\psi^{(a+b)}\left(\frac{1-\sigma}{\delta}\right)\right)\left(\frac{\rho-r\lambda_1}{\delta}\right)^a\left(\frac{\sigma-\rho-r\lambda_2}{\delta}\right)^b\\
&\times P_{\lambda_1}fP_{\lambda_2}g\,d\mu(\lambda_1)\,d\mu(\lambda_2)\\
=&\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{0\le a+b\le N}c_{a,b}\delta^{\kappa(a+b)}\sum_{\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]}\left(\psi^{(a+b)}\left(\frac{1-\sigma}{\delta}\right)\right)\\
&\times\sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]}\int_0^\infty\varphi_a\left(\frac{\rho-r\lambda_1}{\widetilde{\delta}}\right)P_{\lambda_1}f\,d\mu(\lambda_1)\\
&\times\int_0^\infty\varphi_b\left(\frac{\sigma-\rho-r\lambda_2}{\widetilde{\delta}}\right)P_{\lambda_2}g\,d\mu(\lambda_2).
\end{aligned}$$

Set $r=2^kr$ for $k\in\mathbb{Z}$, $r\in[1,2]$. Applying the triangle inequality, Cauchy–Schwartz' inequality and Hölder's inequality, we get that

$$\begin{aligned}
&\left\|\sup_{k\in\mathbb{Z}}\int_1^2\left|S_{2^kr}^\delta(f,g)\right|dr\right\|_{L^p(\mathbb{H}^n)}\\
&\le\left\|\sup_{k\in\mathbb{Z}}\int_1^2\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{0\le a+b\le N}c_{a,b}\delta^{\kappa(a+b)}\sum_{\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]}\left|\psi^{(a+b)}\left(\frac{1-\sigma}{\delta}\right)\right|\right.\\
&\qquad\left.\times\sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]}\left|F^{\varphi_a}_{\rho,\widetilde{\delta},2^kr}f\right|\left|F^{\varphi_b}_{\sigma-\rho,\widetilde{\delta},2^kr}g\right|dr\right\|_{L^p(\mathbb{H}^n)}
\end{aligned}\tag{4.4}$$

$$
\begin{aligned}
&\le \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0\le a+b\le N} c_{a,b}\delta^{\kappa(a+b)} \sum_{\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]} \left|\psi^{(a+b)}\left(\frac{1-\sigma}{\delta}\right)\right| \\
&\qquad\times \left\|\sup_{k\in\mathbb{Z}} \int_1^2 \sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]} \left|F^{\varphi_a}_{\rho,\widetilde{\delta},2^k r} f\right| \left|F^{\varphi_b}_{\sigma-\rho,\widetilde{\delta},2^k r} g\right| dr\right\|_{L^p(\mathbb{H}^n)} \\
&\le \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0\le a+b\le N} c_{a,b}\delta^{\kappa(a+b)} \sum_{\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]} \left|\psi^{(a+b)}\left(\frac{1-\sigma}{\delta}\right)\right| \\
&\qquad\times \left\|\sup_{k\in\mathbb{Z}} \left(\sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]} \int_1^2 \left|F^{\varphi_a}_{\rho,\widetilde{\delta},2^k r} f\right|^2 dr\right)^{1/2}\right\|_{L^{p_1}(\mathbb{H}^n)} \\
&\qquad\times \left\|\sup_{k\in\mathbb{Z}} \left(\sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]} \int_1^2 \left|F^{\varphi_b}_{\sigma-\rho,\widetilde{\delta},2^k r} g\right|^2 dr\right)^{1/2}\right\|_{L^{p_2}(\mathbb{H}^n)}.
\end{aligned}
$$

Notice that $\sigma-\rho\in\widetilde{\delta}\mathbb{Z}\cap[-4\delta-1,1+2\delta]$ for any $\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]$, $\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]$ and $F^{\varphi_b}_{\sigma-\rho,\widetilde{\delta},2^k} g = 0$ if $\sigma-\rho\in\widetilde{\delta}\mathbb{Z}\cap[-4\delta-1,0]$. So,

$$
\sup_{k\in\mathbb{Z}} \left(\sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]} \int_1^2 \left|F^{\varphi_a}_{\rho,\widetilde{\delta},2^k r} f\right|^2 dr\right)^{1/2} = \sup_{k\in\mathbb{Z}} \left|D^{\varphi_a}_{\widetilde{\delta},k} f\right|,
$$

and

$$
\begin{aligned}
\sup_{k\in\mathbb{Z}} \left(\sum_{\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]} \int_1^2 \left|F^{\varphi_b}_{\sigma-\rho,\widetilde{\delta},2^k} g\right|^2 dr\right)^{1/2} &\le \sup_{k\in\mathbb{Z}} \left(\sum_{\sigma-\rho\in\widetilde{\delta}\mathbb{Z}\cap[0,2]} \int_1^2 \left|F^{\varphi_b}_{\sigma-\rho,\widetilde{\delta},2^k} g\right|^2 dr\right)^{1/2} \\
&= \sup_{k\in\mathbb{Z}} \left|D^{\varphi_b}_{\widetilde{\delta},k} g\right|.
\end{aligned}
$$

Using (4.4) and Corollary 3.2, we see that

$$
\begin{aligned}
&\left\|\sup_{k\in\mathbb{Z}} \int_1^2 \left|S^{\delta}_{2^k r}(f,g)\right| dr\right\|_{L^p(\mathbb{H}^n)} \\
&\le \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0\le a+b\le N} c_{a,b}\delta^{\kappa(a+b)} \sum_{\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]} \left|\psi^{(a+b)}\left(\frac{1-\sigma}{\delta}\right)\right| \\
&\qquad\times \left\|\sup_{k\in\mathbb{Z}} \left|D^{\varphi_a}_{\widetilde{\delta},k} f\right|\right\|_{L^{p_1}(\mathbb{H}^n)} \left\|\sup_{k\in\mathbb{Z}} \left|D^{\varphi_b}_{\widetilde{\delta},k} g\right|\right\|_{L^{p_2}(\mathbb{H}^n)} \\
&\le C\delta^{-[(b-1/2)(1-2/p_1)-1/p_1]-[(b-1/2)(1-2/p_2)-1/p_2]-\varepsilon} \|f\|_{L^{p_1}(\mathbb{H}^n)} \|g\|_{L^{p_2}(\mathbb{H}^n)} \\
&\qquad\times \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0\le a+b\le N} c_{a,b}\delta^{\kappa(a+b)} \sum_{\sigma\in\widetilde{\delta}\mathbb{Z}\cap[1-4\delta,1+2\delta]} \left|\psi^{(a+b)}\left(\frac{1-\sigma}{\delta}\right)\right| \\
&\le C\delta^{-(2b-1)(1-1/p)+1/p-\varepsilon} \|f\|_{L^{p_1}(\mathbb{H}^n)} \|g\|_{L^{p_2}(\mathbb{H}^n)} \delta^{-1-\kappa} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0\le a+b\le N} c_{a,b}\delta^{(\kappa-1)(a+b)}
\end{aligned}
$$

$$\leq C\delta^{-(2b-1)(1-1/p)+1/p-2-\varepsilon}\|f\|_{L^{p_1}(\mathbb{H}^n)}\|g\|_{L^{p_2}(\mathbb{H}^n)}$$

holds for $\kappa = 1+\varepsilon$ with any $\varepsilon > 0$ and $b > \frac{1}{2}(D-1)$ with $D = 2n+1$ is the topological dimension of $\mathbb{H}^n$. It follows that

$$\begin{aligned}
&\left\|S_*^\delta(f,g)\right\|_{L^p(\mathbb{H}^n)} \\
&\leq \left\|\sup_{k\in\mathbb{Z}}\int_1^2 \left|S_{2^k r}^\delta(f,g)\right| dr\right\|_{L^p(\mathbb{H}^n)} + \left\|\sup_{k\in\mathbb{Z}}\int_1^2 \left|\frac{\partial}{\partial r}S_{2^k r}^\delta(f,g)\right| dr\right\|_{L^p(\mathbb{H}^n)} \\
&\leq C\big(\delta^{-(2b-1)(1-1/p)+1/p-2-\varepsilon} + \delta^{-(2b-1)(1-1/p)+1/p-2-\varepsilon-1}\big)\|f\|_{L^{p_1}(\mathbb{H}^n)}\|g\|_{L^{p_2}(\mathbb{H}^n)} \\
&\leq C\delta^{-(2b-1)(1-1/p)+1/p-2-\varepsilon}\|f\|_{L^{p_1}(\mathbb{H}^n)}\|g\|_{L^{p_1}(\mathbb{H}^n)}.
\end{aligned}$$

Therefore, whenever $\alpha > \alpha(p_1,p_2) = D(1-1/p)+1/p$, we can choose $b > \frac{1}{2}(D-1)$ and $\varepsilon > 0$ such that $\alpha > (2b-1)(1-1/p)-1/p+2+\varepsilon$. Then

$$\begin{aligned}
&\left\|S_*^\alpha(f,g)\right\|_{L^p(\mathbb{H}^n)} \\
&\leq \sum_{\delta\in D}\delta^\alpha\left\|S_*^\delta(f,g)\right\|_{L^p(\mathbb{H}^n)} + \left\|S_*^0(f,g)\right\|_{L^p(\mathbb{H}^n)} \\
&\leq \sum_{\delta\in D}\delta^{\alpha-((2b-1)(1-1/p)-1/p+2+\varepsilon)}\|f\|_{L^{p_1}(\mathbb{H}^n)}\|g\|_{L^{p_1}(\mathbb{H}^n)} + C\|f\|_{L^{p_1}(\mathbb{H}^n)}\|g\|_{L^{p_2}(\mathbb{H}^n)} \\
&\leq C\|f\|_{L^{p_1}(\mathbb{H}^n)}\|g\|_{L^{p_2}(\mathbb{H}^n)}.
\end{aligned}$$

The proof of Theorem 4.1 is complete. □

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References

[1] A. Carbery, J. L. Rubio de Francia and L. Vega, *Almost everywhere summability of Fourier integrals*, J. London Math. Soc. (2) **38** (1988), no. 3, 513–524.

[2] D. Gorges and D. Müller, *Almost everywhere convergence of Bochner–Riesz means on the Heisenberg group and fractional integration on the dual*, Proc. London Math. Soc. (3) **85** (2002), no. 1, 139–167.

[3] L. Grafakos, D. He and P. Honzík, *Maximal operators associated with bilinear multipliers of limited decay*, J. Anal. Math. **143** (2021), no. 1, 231–251.

[4] E. Jeong and S. Lee, *Maximal estimates for the bilinear spherical averages and the bilinear Bochner–Riesz operators*, J. Funct. Anal. **279** (2020), no. 7, 108629, 29 pp.

[5] K. Jotsaroop and S. Shrivastava, *Maximal estimates for bilinear Bochner–Riesz means*, Adv. Math. **395** (2022), Paper No. 108100, 38 pp.

[6] H. Liu and M. Wang, *Bilinear Riesz means on the Heisenberg group*, Sci. China Math. **62** (2019), no. 12, 2535–2556.

[7] D. Müller, *A restriction theorem for the Heisenberg group*, Ann. of Math. (2) **131** (1990), no. 3, 567–587.

[8] R. S. Strichartz, *Harmonic analysis as spectral theory of Laplacians*, J. Funct. Anal. **87** (1989), no. 1, 51–148.

[9] ———, *L^p harmonic analysis and Radon transforms on the Heisenberg group*, J. Funct. Anal. **96** (1991), no. 2, 350–406.

[10] T. Tao, *On the maximal Bochner–Riesz conjecture in the plane for $p < 2$*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1947–1959.

[11] S. Thangavelu, *Harmonic Analysis on the Heisenberg Group*, Progr. Math. **159**, Birkhäuser Boston, Boston, MA, 1998.

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