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Filter Regularization Method for Inverse Source Problem of the Rayleigh–Stokes Equation

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Abstract. In this paper, we consider a problem of recovering a space-dependent source term for the Rayleigh–Stokes equation, where the additional data is the observation at a final moment $t = T$, which is ill-posed in the sense of Hadamard. Firstly, the uniqueness, ill-posedness and the conditional stability of inverse source problem is given. Next, we develop a filter regularization method to overcome the ill-posedness of the problem. Under reasonable a priori bound assumption about the source function, a Hölder-type error estimate of the regularized solution is proved for a priori regularization parameter choice rule. Furthermore, a logarithmic-type error estimate between the exact solution and the regularized solution is established based on a posteriori regularization parameter choice rule.

1. Introduction

In this paper, we consider the inverse space-dependent source problem of the Rayleigh–Stokes equation for a generalized second-grade fluid model with time derivative:

$$\begin{cases} \partial_t u - (1 + \gamma \partial_t^\alpha) \Delta u = f(x) q(t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T], \\ u(x,0) = 0, & x \in \Omega, \\ u(x,T) = g(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a smooth domain with boundary $\partial\Omega$, and $T > 0$ is a given time. Here, $\gamma > 0$ is a constant, $\partial_t = \partial/\partial t$, and $\partial_t^\alpha u(x,t)$ is the Riemann–Liouville fractional derivative, which is defined by [10]

$$\partial_t^\alpha f(t) = \frac{d}{dt} \int_0^t \omega_{1-\alpha}(t-s) f(s)\, ds, \quad \omega_\alpha = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha \in (0,1).$$

An inverse source problem based on Problem (1.1) is to determine the source function $F(x,t)$ at a previous time from its value at the final time $t = T$ as follows:

$$u(x,T) = g(x), \quad x \in \Omega.$$

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Assuming that the right-hand side term, i.e., the source term is a function represented in the form of variable separation, i.e., assuming that the source form $F = F(x,t)$ can be split into a product of $f(x)q(t)$, where $q(t)$ is known in advance. In practice, the function $(g(x), q(t))$ are obtained from observation data $g^\delta(x)$, $q^\delta(t)$, such that

$$\|g^\delta(x) - g(x)\|_{L^2(\Omega)} + \|q^\delta(t) - q(t)\|_{C[0,T]} \le \delta, \tag{1.2}$$

$$0 < q_0 \le q(t), \quad q^\delta(t) \le q_1, \quad \forall\, t \in [0,T]. \tag{1.3}$$

Here, δ is a noise level. The space-dependent source term $f(x)$ is also determined from the observation data of $g(x)$, $q(t)$ at the final data $t = T$. According to the Hadamard requirements, we know that the inverse source problem mentioned above is ill-posed in general, i.e., a solution does not always exist, and in the case of existence of a solution, which does not depend continuously on the given data. In fact, form a small noise of a physical measurement, the corresponding solutions may have a large error. Hence it is impossible to solve the problem (1.1) by using classical method. For stable reconstruction, we require some regularization techniques.

The Rayleigh–Stokes problem (1.1) plays an important role in describing the behaviour of some non-Newtonian fluids [11]. The direct problems of the Rayleigh–Stokes problem have been studied in [2, 4, 5, 11, 13]. Nevertheless, in many practical problems, initial information, boundary information, coefficient information, source term information might not be given and then we need to recover them by extra measured information which is able to yield to some fractional Rayleigh–Stokes inverse problems [1, 3, 6, 8, 9, 12]. To the best of our knowledge, there are few results on inverse source problem for the Rayleigh–Stokes problem (1.1). Our main purpose is to provide a filter regularization method to deal with the inverse source problem of Rayleigh–Stokes equation.

The outline of this paper is as follows. In Section 2, we first introduce some preliminaries results on fractional Rayleigh–Stokes equation, and we give the uniqueness, ill-posedness and conditional stability of the inverse source problem. In Section 3, we present a filter regularization method and prove the convergence estimate under a priori regularization parameter choice rule. Next, we show the convergence estimate under a posteriori regularization parameter choice rule in Section 4. Finally, some comments are given in Section 5.

2. Preliminaries

Throughout this article, we use the following definitions and lemmas.

Definition 2.1. Let $\{\lambda_p, \phi_p\}$ be the Dirichlet eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω. The family of eigenvalues $\{\lambda_p\}_{p=1}^\infty$ satisfy $0 < \lambda_1 \le$

$\lambda_2 \le \cdots \le \lambda_p \le \cdots$, where $\lambda_p \to \infty$ as $p \to \infty$:

$$\begin{cases} \Delta\phi_p(x) = -\lambda_p\phi_p(x), & x \in \Omega, \\ \phi_p(x) = 0, & x \in \partial\Omega. \end{cases}$$

Definition 2.2. Let $(\cdot\,,\cdot)$ be an inner product in $L^2(\Omega)$. The notation $\|\cdot\|_X$ stands for in the norm in the Banach space. For $k > 0$, we define the Hilbert space

$$H^k(\Omega) := \left\{ f \in L^2(\Omega) \,\middle|\, \sum_{k=1}^{\infty} \lambda_p^{2k} |(f, \phi_p)|^2 < \infty \right\},$$

equipped with the norm

$$\|f\|_{H^k(\Omega)} = \left(\sum_{k=1}^{\infty} \lambda_p^{2k} |(f, \phi_p)|^2 \right)^{1/2}.$$

2.1. The formula of source term f

In this subsection, we introduce the mild solution of the following initial value problem:

$$\begin{cases} \partial_t u - (1 + \gamma\partial_t^\alpha)\Delta u = F(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,T) = g(x), & x \in \Omega. \end{cases}$$

Here, $F(x,t) = f(x)q(t)$.

According to the eigenfunction expansion, we can obtain the solution of the Rayleigh–Stokes problem as follows:

$$u(x,t) = \sum_{p=1}^{\infty} \mathcal{R}_p(\alpha,t)(u_0,\phi_p)\phi_p(x) + \sum_{p=1}^{\infty} \left(\int_0^t \mathcal{R}_p(\alpha, t-s) q(s)\, ds\, f_p \right) \phi_p(x),$$

where $f_p = (f, \phi_p)$, and $\mathcal{R}_p(\alpha,t)$ satisfies the following equation

$$\begin{cases} \frac{d}{dt}\mathcal{R}_p(\alpha,t) + \lambda_p(1 + \gamma\partial_t^\alpha)\mathcal{R}_p(\alpha,t) = 0, & t \in (0,T), \\ \mathcal{R}_p(\alpha,0) = 1. \end{cases}$$

Taking $t = T$ and applying $u(x,0) = 0$, we have

$$g(x) = \sum_{p=1}^{\infty} \left(\int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds \right) f_p \phi_p(x) := Kf(x).$$

Hence the source function f is given by the Fourier series

$$f(x) = \sum_{p=1}^{\infty} f_p \phi_p(x) = \sum_{p=1}^{\infty} \frac{(g, \phi_p)\phi_p(x)}{\int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds}. \tag{2.1}$$

Using [2], we obtain the Laplace transform of $\mathcal{R}_p(\alpha, t)$ as follows

$$\mathcal{L}(\mathcal{R}_p(\alpha, t)) = \frac{1}{t + \gamma \lambda_p t^{\alpha} + \lambda_p}.$$

Lemma 2.3. [3] *The functions $\mathcal{R}_p(\alpha, t)$, $p = 1, 2, \ldots$, are equal to*

$$\mathcal{R}_p(\alpha, t) = \int_0^{\infty} e^{-rt} Y_p(\alpha, r)\, dr,$$

where

$$Y_p(\alpha, r) = \frac{\gamma}{\pi} \frac{\lambda_p r^{\alpha} \sin \alpha\pi}{(-r + \lambda_p \gamma r^{\alpha} \cos \alpha\pi + \lambda_p)^2 + (\lambda_p \gamma r^{\alpha} \sin \alpha\pi)^2}.$$

Lemma 2.4. [8] *Let $\alpha \in (1/2, 1)$, we have the following estimate for all $t \in [0, T]$:*

$$\mathcal{R}_p(\alpha, t) \geq \frac{C(\gamma, \alpha, \lambda_1)}{\lambda_p},$$

and there exists D such that

$$\int_0^T |\mathcal{R}_p(\alpha, t)|^2\, dt \leq \frac{D^2}{\lambda_p^2} \frac{T^{2\alpha-1}}{2\alpha - 1},$$

where

$$C(\gamma, \alpha, \lambda_1) = \gamma \sin \alpha\pi \int_0^{\infty} \frac{e^{-rT} r^{\alpha}}{\gamma^2 r^{2\alpha} + \frac{r^2}{\lambda_1^2} + 1}\, dr, \quad 0 < \alpha < 1.$$

Moreover, we present a useful estimate

Lemma 2.5. [3] *From Lemma 2.4, we can get*

$$\int_0^T \mathcal{R}_p(\alpha, T-s)\, ds \geq \int_0^T \frac{C(\gamma, \alpha, \lambda_1)}{\lambda_p}\, ds \geq \frac{T C(\gamma, \alpha, \lambda_1)}{\lambda_p}. \tag{2.2}$$

Next, from the above formula by putting $\inf_{t \in [0,T]} |q^{\delta}(t)| = q_0 > 0$, we have

$$\frac{1}{\int_0^T \mathcal{R}_p(\alpha, T-s) q^{\delta}(s)\, ds} \leq \frac{1}{q_0} \frac{1}{\int_0^T \mathcal{R}_p(\alpha, T-s)\, ds} \leq \frac{\lambda_p}{q_0 T C(\gamma, \alpha, \lambda_1)}.$$

2.2. The uniqueness, ill-posedness and conditional stability for the inverse source problem

From Lemma 2.5, we know

$$\int_0^T \mathcal{R}_p(\alpha, T-s)q(s)\,ds \geq q_0 \int_0^T \mathcal{R}_p(\alpha, T-s)\,ds \geq \frac{q_0 T C(\gamma, \alpha, \lambda_1)}{\lambda_p}. \tag{2.3}$$

Let f_1 and f_2 be the source functions corresponding to the final values g_1 and g_2 respectively, and combining (2.1) and (2.3), we get

$$f_1(x) - f_2(x) = \sum_{p=1}^{\infty} \frac{(g_1 - g_2, \phi_p)\phi_p(x)}{\int_0^T \mathcal{R}_p(\alpha, T-s)q(s)\,ds}.$$

Suppose that $g_1 = g_2$, then we can conclude that $f_1 = f_2$. This yields the uniqueness of the inverse source problem. Moreover, according to the analysis of Theorem 2.6 in [3], we know that inverse source problem is ill-posed. Therefore, a regularization method is needed.

Before solving the inverse source problem by using regularization method, we review a conditional stability result of the inverse source problem.

Theorem 2.6. [3] *Let $f \in H^k(\Omega)$ be such that $\|f\|_{H^k(\Omega)} \leq E$ for some $E > 0$. Then we have the estimate*

$$\|f\|_{L^2(\Omega)} \leq \mathcal{B}(k, E)\|g\|_{L^2(\Omega)}^{k/(k+1)},$$

where

$$\mathcal{B}(k, E) = \frac{1}{q_0^{k/(k+1)} T^{k/(k+1)} C^{k/(k+1)}(\gamma, \alpha, \lambda_1)} E^{1/(k+1)}.$$

3. A priori filter regularization method and convergence estimate

In this section, we propose a filter regularization method to solve the inverse source problem. To regularize the problem, our aim to replace the term $\frac{1}{\int_0^T \mathcal{R}_p(\alpha, T-s)q(s)\,ds}$ by another term. For this purpose, a regularized solution with the measured data $\{g^\delta(x), q^\delta(t)\}$ is defined as follows:

$$f_\beta^\delta(x) = \sum_{p=1}^{\infty} \frac{(g^\delta, \phi_p)\phi_p(x)}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s)q^\delta(s)\,ds}, \tag{3.1}$$

where $\beta > 0$ is a regularization parameter, and a regularization solution with the exact data $\{g(x), q(t)\}$ is defined as follows:

$$f_\beta(x) = \sum_{p=1}^{\infty} \frac{(g, \phi_p)\phi_p(x)}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s)q(s)\,ds}. \tag{3.2}$$

From (3.1) and (3.2), we can find two common properties of a new filter term $\frac{1}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds}$.

(a) If the parameter β is small, the filter term $\frac{1}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds}$ is close to $\frac{1}{\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds}$;

(b) If the parameter β is fixed, the filter term $\frac{1}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds}$ is bounded.

Property (a) indicates that the constructed regularized solution is an approximation of the exact solution. Property (b) indicates that the constructed regularized solution is continuously dependent on the data. Both properties (a) and (b) are also the basic requirement of the general regularization principle [7].

In the following, we prove the convergence estimate for $\|f_\beta^\delta - f\|$ by using a priori choice rule for the regularization parameter. We first give the following lemmas.

Lemma 3.1. *Let $\alpha \in (1/2, 1)$, and assume that (1.2) holds, then we have the following estimate*

$$\|f_\beta - f_\beta^\delta\| \le \frac{\delta}{q_0}\|f\|_{L^2(\Omega)} + \frac{\delta}{\beta}.$$

Proof. From (3.1) and (3.2), we have

$$\begin{aligned}
f_\beta - f_\beta^\delta &= \sum_{p=1}^{\infty}\left(\frac{(g,\phi_p)\phi_p(x)}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds} - \frac{(g^\delta,\phi_p)\phi_p(x)}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds}\right)\\
&= \sum_{p=1}^{\infty}\left(\frac{(g,\phi_p)\phi_p(x)}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds} - \frac{(g,\phi_p)\phi_p(x)}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds}\right)\\
&\quad + \sum_{p=1}^{\infty}\left(\frac{(g,\phi_p)\phi_p(x)}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds} - \frac{(g^\delta,\phi_p)\phi_p(x)}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds}\right)\\
&= \sum_{p=1}^{\infty}\left(\frac{(g,\phi_p)\phi_p(x)\int_0^T \mathcal{R}_p(\alpha,T-s)(q^\delta(s)-q(s))\,ds}{\left(\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds\right)\left(\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds\right)}\right)\\
&\quad + \sum_{p=1}^{\infty}\left(\frac{(g-g^\delta,\phi_p)\phi_p(x)}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds}\right)\\
&= Q_1 + Q_2.
\end{aligned} \tag{3.3}$$

We continue to estimate the error by two steps as follows:

Step 1. We estimate $\|Q_1\|_{L^2(\Omega)}$, and by using (1.3) and (2.1), we have

$$\|Q_1\|^2_{L^2(\Omega)} = \sum_{p=1}^{\infty}\left(\frac{(g,\phi_p)\int_0^T \mathcal{R}_p(\alpha,T-s)(q^\delta(s)-q(s))\,ds}{\left(\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds\right)\left(\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds\right)}\right)^2$$

$$\leq \sum_{p=1}^{\infty} \left| \frac{\int_0^T \mathcal{R}_p(\alpha, T-s)(q^\delta(s) - q(s))\, ds}{\int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds} \right|^2 \left| \frac{(g, \phi_p)}{\int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds} \right|^2$$
$$\leq \frac{\|q - q^\delta\|^2_{C[0,T]}}{q_0^2} \sum_{p=1}^{\infty} |(f, \phi_p)|^2 = \frac{\|q - q^\delta\|^2_{C[0,T]}}{q_0^2} \|f\|^2_{L^2(\Omega)}.$$

So, we get

$$\|Q_1\|_{L^2(\Omega)} \leq \frac{\|q - q^\delta\|_{C[0,T]}}{q_0} \|f\|_{L^2(\Omega)} \leq \frac{\delta}{q_0} \|f\|_{L^2(\Omega)}. \tag{3.4}$$

Step 2. We estimate $\|Q_2\|_{L^2(\Omega)}$, we obtain

$$\|Q_2\|^2_{L^2(\Omega)} = \sum_{p=1}^{\infty} \left(\frac{(g - g^\delta, \phi_p)}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds} \right)^2 \leq \frac{1}{\beta^2} \sum_{p=1}^{\infty} (g - g^\delta, \phi_p)^2 \leq \frac{\delta^2}{\beta^2}.$$

Then, we can have

$$\|Q_2\|_{L^2(\Omega)} \leq \frac{\delta}{\beta}. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we obtain

$$\|f_\beta - f_\beta^\delta\| \leq \frac{\delta}{q_0} \|f\|_{L^2(\Omega)} + \frac{\delta}{\beta}.$$

The proof of Lemma 3.1 is completed. □

Lemma 3.2. *For $\alpha \in (1/2, 1)$ and assume that $f \in H^k(\Omega)$ with $\|f\|_{H^k(\Omega)} \leq E$ hold, then we obtain the following estimate*

$$\|f - f_\beta\|_{L^2(\Omega)} \leq \begin{cases} \beta^{1/2} E \dfrac{\lambda_1^{1/2-k}}{2\sqrt{q_0 T C(\gamma, \alpha, \lambda_1)}}, & k \geq 1/2, \\ \beta^k E \sqrt{\left(\dfrac{1}{2\sqrt{q_0 T C(\gamma, \alpha, \lambda_1)}} \right)^2 + 1}, & 0 < k < 1/2. \end{cases}$$

Proof. From (2.1) and (3.2), using the Parseval identity, we get

$$\|f - f_\beta\|^2_{L^2(\Omega)} = \sum_{p=1}^{\infty} \left(\frac{(g, \phi_p)}{\int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds} - \frac{(g, \phi_p)}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds} \right)^2$$
$$= \sum_{p=1}^{\infty} \left(\frac{\beta (g, \phi_p)}{\int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds \big(\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds\big)} \right)^2$$
$$= \sum_{p=1}^{\infty} \frac{\beta^2 (g, \phi_p)^2}{\big(\int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds\big)^2 \big(\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds\big)^2} \tag{3.6}$$

$$= \sum_{p=1}^{\infty} \frac{\beta^2 \lambda_p^{-2k} \lambda_p^{2k} (g,\phi_p)^2}{\left(\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds\right)^2 \left(\beta + \int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds\right)^2}$$
$$= \sum_{p=1}^{\infty} G(p)^2 \frac{\lambda_p^{2k}(g,\phi_p)^2}{\left(\int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds\right)^2},$$

where

$$G(p) = \frac{\beta \lambda_p^{-k}}{\beta + \int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds}.$$

Now, we estimate $G(p)$,

$$G(p) \le \frac{\beta \lambda_p^{-k}}{2\sqrt{\beta \int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds}} \le \frac{\beta \lambda_p^{-k}}{2\sqrt{\beta q_0 \int_0^T \mathcal{R}_p(\alpha,T-s)\,ds}}. \tag{3.7}$$

Hence, combining (2.2) and (3.7), we get

$$G(p) \le \frac{\sqrt{\beta}\lambda_p^{1/2-k}}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}}. \tag{3.8}$$

We divide into the two following cases:

Case 1. If $k \ge 1/2$, we note

$$\lambda_p^{1/2-k} = \frac{1}{\lambda_p^{k-1/2}} \le \frac{1}{\lambda_1^{k-1/2}} = \lambda_1^{1/2-k}. \tag{3.9}$$

Combining (3.6), (3.8) and (3.9), we obtain

$$\|f - f_\beta\|_{L^2(\Omega)} \le \frac{\sqrt{\beta}\lambda_1^{1/2-k}}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}} \|f\|_{H^k(\Omega)} \le \beta^{1/2} E \frac{\lambda_1^{1/2-k}}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}}.$$

Case 2. If $0 < k < 1/2$, we choose any $\eta \in (0,1/2)$ and rewrite $\mathbb{N} = \mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\mathcal{A}_1 = \left\{p \in \mathbb{N} \mid \lambda_p^{1/2-k} \le \beta^{-\eta}\right\}, \quad \mathcal{A}_2 = \left\{p \in \mathbb{N} \mid \lambda_p^{1/2-k} > \beta^{-\eta}\right\}. \tag{3.10}$$

Combining (3.6), (3.8) and (3.10), we have

$$\begin{aligned}
&\|f - f_\beta\|^2_{L^2(\Omega)} \\
&\le \sup_{p\in\mathcal{A}_1} \left(\frac{\sqrt{\beta}\lambda_p^{1/2-k}}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}}\right)^2 \sum_{p\in\mathcal{A}_1} \lambda_p^{2k}(f,\phi_p)^2 \\
&\quad + \sum_{p\in\mathcal{A}_2} \left(\frac{\beta\lambda_p^{-k}}{\beta + \int_0^T \mathcal{R}_p(\alpha,T-s)q(s)\,ds}\right)^2 \lambda_p^{2k}(f,\phi_p)^2 \\
&\le \left(\frac{1}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}}\right)^2 \beta^{1-2\eta}\|f\|^2_{H^k(\Omega)} + \sup_{p\in\mathcal{A}_2} \lambda_p^{-2k} \sum_{p\in\mathcal{A}_2} \lambda_p^{2k}(f,\phi_p)^2 \\
&\le \left(\frac{1}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}}\right)^2 \beta^{1-2\eta}\|f\|^2_{H^k(\Omega)} + \beta^{\frac{2\eta k}{1/2-k}}\|f\|^2_{H^k(\Omega)}.
\end{aligned} \tag{3.11}$$

Since $\|f\|_{H^k(\Omega)} \le E$, by choosing $\eta = 1/2 - k$ in (3.11), we have

$$\|f - f_\beta\|_{L^2(\Omega)} \le \beta^k E \sqrt{\left(\frac{1}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}}\right)^2 + 1}.$$

The proof of Lemma 3.2 is completed. □

By modifying the ideas in [3], we are now able to prove the following theorem, which is the first main result of this paper.

Theorem 3.3. *Let* $\alpha \in (1/2, 1)$*. Let* f *be the exact solution* (2.1) *with the exact data* g*, and let* f_β^δ *be the regularized solution* (3.1) *with the noisy data* g^δ*. Let the noise assumption* (1.2), (1.3) *and a priori condition*

$$\|f\|_{H^k(\Omega)} \le E \tag{3.12}$$

hold. The error estimate between the exact solution and the regularized solution is as follows:

(a) *If* $0 < k < 1/2$*, and we choose the parameter*

$$\beta = \left(\frac{\delta}{E}\right)^{1/(k+1)}, \tag{3.13}$$

we have the following convergence estimate

$$\begin{aligned}\|f_\beta^\delta - f\|_{L^2(\Omega)} \le &\left(1 + \frac{\delta^{1/(k+1)} E^{k/(k+1)}}{q_0 \lambda_1^k} + \sqrt{\left(\frac{1}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}}\right)^2 + 1}\right) \\ &\times \delta^{k/(k+1)} E^{1/(k+1)}.\end{aligned} \tag{3.14}$$

(b) *If* $k \ge 1/2$*, and we choose the parameter*

$$\beta = \left(\frac{\delta}{E}\right)^{2/3}, \tag{3.15}$$

we have the following convergence estimate

$$\|f_\beta^\delta - f\|_{L^2(\Omega)} \le \left(1 + \frac{\delta^{2/3} E^{1/3}}{q_0 \lambda_1^k} + \frac{\lambda_1^{1/2-k}}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}}\right) \delta^{1/3} E^{2/3}. \tag{3.16}$$

Proof. (a) If $0 < k < 1/2$, using the triangle inequality, and from Lemmas 3.1 and 3.2, we get

$$\begin{aligned}\|f_\beta^\delta - f\|_{L^2(\Omega)} &\le \|f_\beta^\delta - f_\beta\|_{L^2(\Omega)} + \|f_\beta - f\|_{L^2(\Omega)} \\ &\le \frac{\delta}{q_0}\|f\|_{L^2(\Omega)} + \frac{\delta}{\beta} + \beta^k E \sqrt{\left(\frac{1}{2\sqrt{q_0 T C(\gamma,\alpha,\lambda_1)}}\right)^2 + 1},\end{aligned}$$

and we easily know

$$\|f\|_{L^2(\Omega)} \le \frac{1}{\lambda_1^k}\|f\|_{H^k(\Omega)} \le \frac{E}{\lambda_1^k}, \tag{3.17}$$

using (3.13), we can obtain the convergence estimate (3.14).

(b) If $k \ge 1/2$, using the triangle inequality, and from Lemmas 3.1 and 3.2, we get

$$\begin{aligned}\|f_\beta^\delta - f\|_{L^2(\Omega)} &\le \|f_\beta^\delta - f_\beta\|_{L^2(\Omega)} + \|f_\beta - f\|_{L^2(\Omega)}\\ &\le \frac{\delta}{q_0}\|f\|_{L^2(\Omega)} + \frac{\delta}{\beta} + \beta^{1/2}E\frac{\lambda_1^{1/2-k}}{2\sqrt{q_0TC(\gamma,\alpha,\lambda_1)}},\end{aligned}$$

using (3.15) and (3.17), we can obtain the convergence estimate (3.16). Hence, the proof of Theorem 3.3 is completed. □

4. A posteriori filter regularization method and convergence estimate

In this section, we prove the convergence estimate between the exact solution and the regularized solution by using a posteriori choice rule for a regularization parameter, i.e., Morozov's discrepancy principle.

According to the Morozov's discrepancy principle [7], we choose the regularization parameter β as the solution of the following equation

$$\left\|\sum_{p=1}^{+\infty}\frac{\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds}(g^\delta,\phi_p)\phi_p(x) - g^\delta(x)\right\| = \delta + \tau\left(\log\left(\log\left(\frac{E}{\delta}\right)\right)\right)^{-1}, \tag{4.1}$$

where $\tau > 1$ is a constant.

Lemma 4.1. *Let*

$$\rho(\beta) = \left\|\sum_{p=1}^{+\infty}\frac{\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds}{\beta+\int_0^T \mathcal{R}_p(\alpha,T-s)q^\delta(s)\,ds}(g^\delta,\phi_p)\phi_p(x) - g^\delta(x)\right\|.$$

Let $\alpha \in (1/2, 1)$. *If* $0 < \delta + \tau(\log(\log(\frac{E}{\delta})))^{-1} < \|g^\delta\|$, *then the following results hold:*

(a) $\rho(\beta)$ *is a continuous function;*

(b) $\lim_{\beta\to 0}\rho(\beta) = 0$*;*

(c) $\lim_{\beta\to+\infty}\rho(\beta) = \|g^\delta\|$*;*

(d) $\rho(\beta)$ *is a strictly increasing function over* $(0, +\infty)$.

The proof is obvious, and we omit it here.

Remark 4.2. According to Lemma 4.1, we know there exists a unique solution for (4.1) if $0 < \delta + \tau(\log(\log(\frac{E}{\delta})))^{-1} < \|g^\delta\|$.

Lemma 4.3. *Let $\alpha \in (1/2, 1)$, and assume that $\|f\|_{H^k(\Omega)} \le E$ holds. If β is the solution of* (4.1), *we can obtain the following inequality*

$$\frac{1}{\beta} \le \frac{q_1 E}{\tau q_0 \lambda_1^k}\left(\log\left(\log\left(\frac{E}{\delta}\right)\right)\right). \tag{4.2}$$

Proof. From (4.1), there holds

$$\begin{aligned}
\delta + \tau\left(\log\left(\log\left(\frac{E}{\delta}\right)\right)\right)^{-1} &= \left\|\sum_{p=1}^{+\infty} \frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds}(g^\delta, \phi_p)\phi_p(x)\right\| \\
&\le \left\|\sum_{p=1}^{+\infty} \frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds}(g^\delta - g, \phi_p)\phi_p(x)\right\| \\
&\quad + \left\|\sum_{p=1}^{+\infty} \frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds}(g, \phi_p)\phi_p(x)\right\| \\
&\le \delta + \left\|\sum_{p=1}^{+\infty} \frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds}(g, \phi_p)\phi_p(x)\right\|.
\end{aligned}$$

Then, we know

$$\tau\left(\log\left(\log\left(\frac{E}{\delta}\right)\right)\right)^{-1} \le \left\|\sum_{p=1}^{+\infty} \frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds}(g, \phi_p)\phi_p(x)\right\|. \tag{4.3}$$

Since $\|f\|_{H^k(\Omega)} \le E$ and (2.1), then

$$\begin{aligned}
&\left\|\sum_{p=1}^{+\infty} \frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds}(g, \phi_p)\phi_p(x)\right\| \\
&= \left\|\sum_{p=1}^{+\infty} \frac{\beta \int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds \cdot \lambda_p^{-k}}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds} \frac{\lambda_p^k (g, \phi_p)\phi_p(x)}{\int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds}\right\| \\
&\le \left\{\sum_{p=1}^{+\infty} \left(\frac{\beta \int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds \cdot \lambda_p^{-k}}{\int_0^T \mathcal{R}_p(\alpha, T-s) q^\delta(s)\, ds}\right)^2 \left(\frac{\lambda_p^k (g, \phi_p)}{\int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds}\right)^2\right\}^{1/2} \qquad (4.4) \\
&\le \frac{\beta q_1 \lambda_1^{-k}}{q_0}\left(\sum_{p=1}^{+\infty} \frac{\lambda_p^{2k}(g, \phi_p)^2}{\left(\int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds\right)^2}\right)^{1/2} \\
&\le \frac{\beta q_1}{q_0 \lambda_1^k}\|f\|_{H^k(\Omega)} \le \frac{\beta q_1}{q_0 \lambda_1^k} E.
\end{aligned}$$

Combining (4.3) and (4.4), we can obtain

$$\tau\left(\log\left(\log\left(\frac{E}{\delta}\right)\right)\right)^{-1} \leq \frac{\beta q_1}{q_0\lambda_1^k}E.$$

Then, we get the estimate (4.2). Hence, the proof of Lemma 4.3 is completed. □

Theorem 4.4. *Let* $\alpha \in (1/2, 1)$. *Suppose a priori conditions* (1.2) *and* (3.12) *hold and taking the solution of* (4.1) *as the regularization parameter, then we obtain the following error estimate*

$$\begin{aligned} \|f_\beta^\delta - f\|_{L^2(\Omega)} &\leq \left(\frac{1}{q_0\lambda_1^k} + P(k)\right)\delta E \\ &\quad + M(k)\left(2\delta + \tau\left(\log\left(\log\left(\frac{E}{\delta}\right)\right)\right)^{-1}\right)^{k/(k+1)} E^{1/(k+1)}, \end{aligned} \tag{4.5}$$

where

$$P(k) = \frac{q_1}{\tau q_0\lambda_1^k}\left(\log\left(\log\left(\frac{E}{\delta}\right)\right)\right), \quad M(k) = \frac{1}{q_0^{k/(k+1)}T^{k/(k+1)}C^{k/(k+1)}(\gamma,\alpha,\lambda_1)}.$$

Proof. Due to the triangle inequality, we have

$$\|f_\beta^\delta - f\|_{L^2(\Omega)} \leq \|f_\beta^\delta - f_\beta\|_{L^2(\Omega)} + \|f_\beta - f\|_{L^2(\Omega)} := I_1 + I_2. \tag{4.6}$$

From Lemmas 3.1 and 4.3, and using (3.17), we estimate I_1,

$$I_1 = \|f_\beta^\delta - f_\beta\|_{L^2(\mathbb{R})} \leq \frac{\delta}{q_0}\|f\|_{L^2(\Omega)} + \frac{q_1\delta E}{\tau q_0\lambda_1^k}\left(\log\left(\log\left(\frac{E}{\delta}\right)\right)\right).$$

Then we can obtain

$$I_1 \leq \frac{\delta E}{q_0\lambda_1^k} + \frac{q_1\delta E}{\tau q_0\lambda_1^k}\left(\log\left(\log\left(\frac{E}{\delta}\right)\right)\right). \tag{4.7}$$

In the following, we estimate I_2, we firstly give the following estimate

$$\begin{aligned} \|f_\beta - f\|_{H^k(\Omega)}^2 &= \sum_{p=1}^{\infty}\left(\frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s)q(s)\,ds}\right)^2 \frac{\lambda_p^{2k}(g,\phi_p)^2}{\left(\int_0^T \mathcal{R}_p(\alpha, T-s)q(s)\,ds\right)^2} \\ &\leq \sum_{p=1}^{\infty}\frac{\lambda_p^{2k}(g,\phi_p)^2}{\left(\int_0^T \mathcal{R}_p(\alpha, T-s)q(s)\,ds\right)^2} = \|f\|_{H^k(\Omega)}^2 \leq E^2, \end{aligned}$$

and from Theorem 2.6, we know

$$I_2 \leq \mathcal{B}(k,E)\|K(f_\beta - f)\|_{L^2(\Omega)}^{k/(k+1)} = \mathcal{B}(k,E)\|Kf_\beta - Kf\|_{L^2(\Omega)}^{k/(k+1)}. \tag{4.8}$$

Now, we estimate

$$
\begin{aligned}
Kf_\beta - Kf &= \sum_{p=1}^{\infty} \frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds} (g, \phi_p)\phi_p(x) \\
&= \sum_{p=1}^{\infty} \frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds} (g - g^\delta, \phi_p)\phi_p(x) \\
&\quad + \sum_{p=1}^{\infty} \frac{\beta}{\beta + \int_0^T \mathcal{R}_p(\alpha, T-s) q(s)\, ds} (g^\delta, \phi_p)\phi_p(x).
\end{aligned}
$$

By using (1.2) and (4.1), we get

$$
\|Kf_\beta - Kf\|_{L^2(\Omega)} \le 2\delta + \tau \left(\log \left(\log \left(\frac{E}{\delta} \right) \right) \right)^{-1}. \tag{4.9}
$$

Combining (4.6), (4.7), (4.8) and (4.9), we can obtain the convergence estimate (4.5). Hence, the proof of Theorem 4.4 is completed. □

Remark 4.5. Here, in order to distinguish from the symbols in this paper, we give another symbols. In [3], we know

$$
Tf_\sigma^\epsilon = \sum_{p=1}^{+\infty} \frac{\left| \int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\, ds \right|^2}{\sigma^2 + \left| \int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\, ds \right|^2} (g^\epsilon, \phi_p)\phi_p(x).
$$

If one can choose σ satisfying

$$
\begin{aligned}
\|Tf_\sigma^\epsilon - g^\epsilon(x)\| &= \left\| \sum_{p=1}^{+\infty} \frac{\left| \int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\, ds \right|^2}{\sigma^2 + \left| \int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\, ds \right|^2} (g^\epsilon, \phi_p)\phi_p(x) - g^\epsilon(x) \right\| \\
&= \epsilon + \tau \left(\log \left(\log \left(\frac{E}{\epsilon} \right) \right) \right)^{-1},
\end{aligned}
$$

it can obtain a convergence result for Tikhonov regularization method, which is analogous to Theorem 4.4 in our paper. However, the authors in [3] choose σ satisfying

$$
\|Tf_\sigma^\epsilon - g^\epsilon(x)\| = \left\| \sum_{p=1}^{+\infty} \frac{\left| \int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\, ds \right|^2}{\sigma^2 + \left| \int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\, ds \right|^2} (g^\epsilon, \phi_p)\phi_p(x) - g^\epsilon(x) \right\| = \tau\epsilon,
$$

the result in [3] is not optimal. Obviously, we provide a more optimal convergence result in this work, and improve the convergence result (see Theorem 3.6 in [3]).

5. Concluding remarks

In this paper, we have studied an inverse source problem for the Rayleigh–Stokes equation. Due to the ill-posedness of the problem, we introduce a filter regularization method to construct an approximate solution. The error estimates are proved under a priori and a posteriori regularization parameter rules. In the future work, we will try to solve the other ill-posed problems by using the proposed method.

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