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Singular Limit Solutions for a 4-dimensional Emden–Fowler System of Liouville Type in Some General Case

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Abstract. We prove the existence of singular limit solutions for a nonlinear elliptic Emden–Fowler system with Navier boundary conditions by using the nonlinear domain decomposition method and the Pohozaev identity.

1. Introduction and statement of the results

In recent years, nonlinear system have received a very important attention in the domain of mathematics and physics since several phenomena in these domains are described through nonlinear differential system such as thermionic emissions, isothermal gas sphere, gas combustion and gauge theory [28]. The main purpose of studying nonlinear initial boundary value problems involving partial differential equations is to designate whether solutions to a given equation develop a singularity. The blow up problem can have an impact on the physical relevance and the validity of the underlying model, therefore, it is interesting to solve and to characterize this type of problem.

In geometry, the semilinear elliptic equations with exponential nonlinearities play a fundamental role, precisely, in the prescription of the Q-curvature on 4-dimensional Riemannian manifolds [12, 13]:

$$Q_g = \frac{1}{12}(-\Delta_g S_g + S_g^2 - 3|\operatorname{Ric}_g|^2),$$

where Ric_g denotes the Ricci tensor and S_g is the scalar curvature of the metric g. Remember that the Q-curvature changes under a conformal change of metric $g_w = e^{2w}g$ according to

$$P_g w + 2Q_g = 2Q_{g_w} e^{4w}, \tag{1.1}$$

where

$$P_g := \Delta_g^2 + \delta\left(\frac{2}{3}S_g I - 2\operatorname{Ric}_g\right)d$$

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is the Paneitz operator, which is an elliptic 4th order partial differential operator and it transforms according to

$$e^{4w}P_{e^{2w}g} = P_g,$$

under a conformal change of metric $g_w := e^{2w}g$. In particular case where the manifold is the Euclidean space, the Paneitz operator is given by

$$P_{g_{\text{eucl}}} = \Delta^2.$$

In that case, the equation (1.1) is now transformed to a partial differential equation with an exponential type differential nonlinearity modulo the Q-curvature:

$$\Delta^2 w = Q_{g_w} e^{4w}.$$

For more details and recent developments of this problem, see [12, 18].

Let $\Omega \subset \mathbb{R}^4$ be a regular bounded open domain, we consider the following nonlinear elliptic system of Emden–Fowler:

$$\begin{cases} \Delta(a(x)\Delta u_1) - V(x)\operatorname{div}(a(x)\nabla u_1) = \rho^4 a(x) e^{\gamma u_1 + (1-\gamma)u_2} & \text{in } \Omega, \\ \Delta(a(x)\Delta u_2) - V(x)\operatorname{div}(a(x)\nabla u_2) = \rho^4 a(x) e^{\xi u_2 + (1-\xi)u_1} & \text{in } \Omega, \\ \Delta u_1 = \Delta u_2 = u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here ρ, γ, ξ are constants such that $\gamma, \xi \in (0,1)$ and $\gamma + \xi > 1$, so in the following we have naturally $\frac{1-\gamma}{\xi}, \frac{1-\xi}{\gamma} \in (0,1)$. The potential $V(x)$ belongs to $L^\infty_{\text{loc}}(\mathbb{R}^4)$ which is smooth and bounded, the function $a = a(x)$ is a given smooth function over $\overline{\Omega}$, called the Schrödinger wave function, solution of the linear form of stationary smooth nonhomogenous Schrödinger problem

$$\begin{cases} -\Delta a(x) + V(x)a(x) = \lambda f(x,a) & \text{in } \overline{\Omega}, \\ \|\nabla a\|_\infty \le \beta \end{cases} \tag{1.3}$$

satisfying

$$0 \le c_1 \le a(x) \le c_2 \le +\infty, \tag{H}$$

λ and β are small parameters and f is a smooth bounded function over $\overline{\Omega}$. Furthermore,

$$|f(x,a(x))| \le c(1 + |a(x)|^3), \quad x \in \mathbb{R}^4,\ a \in \mathbb{R} \text{ and } c > 0.$$

For more details about asymptotical behaviors of the solution $a(x)$ of problem (1.3), see [26] and some references therein, see also [9, 10].

We are interested to the study of the existence of positive solutions for the problem (1.2) with singular limits as the parameter ρ tends to 0 and when the singular sets intercept each other.

In [19], the authors considered the system (1.2), precisely they studied the existence of a singular limits solution for the system in the case where the singular sets are disjoint. Several researchers are interested in the study of problem (1.2) without the term $\Delta(a(x)\Delta u)$, see [15, 30]. In dimension 2 and 4, the researchers in [1, 7, 29], are interested exactly in the study of system (1.2) without the term $V(x)\,\mathrm{div}(a(x)\nabla u)$ and the function $a(x)$, in fact, they proved the existence of solutions with singular limits when parameter ρ tends to 0.

The author in [25], investigated the case of an Emden–Fowler equation with exponential nonlinearity, he proved the existence of singular limits for solution of this type of equation given by

$$\begin{cases} \Delta(a(x)\Delta u) - V(x)\,\mathrm{div}(a(x)\nabla u) = \rho^4 a(x)e^u & \text{in } \Omega \subset \mathbb{R}^4, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Indeed, he looks for solutions which concentrate at the points $x^j \in \Omega$, $j = 1, \dots, m$ as the parameter ρ tends to 0. He gives sufficient conditions under which, as ρ tends to 0, there exists an explicit class of solutions which admit a concentration behavior with a prescribed bubble profile around some given m-points in Ω, for any given integer m.

In dimension 2, Chanillo and Keissling [14] established a strict isoperimetric inequality and a Pohozaev–Rellich identity for the system

$$-\Delta u_i = \exp\left(\sum_{j\in\mathcal{J}} \gamma^{i,j} u_j\right) \ \text{ in } \mathbb{R}^2, \quad i \in \mathcal{J} = \{1, \dots, N\}, \tag{1.4}$$

under the finite mass conditions

$$\int_{\mathbb{R}^2} e^{u_i}\,dx < \infty, \quad i \in \mathcal{J}. \tag{1.5}$$

Here $\{\gamma^{i,j}\} \equiv \gamma \in \mathrm{GL}_N(\mathbb{R})$ is a symmetric matrix such that $\gamma^{i,j} \geq 0$ and $\gamma^{i,i} > 0$, satisfying

$$\sum_{j\in\mathcal{J}} \gamma^{i,j} = 1, \quad i \in \mathcal{J}. \tag{1.6}$$

They prove that all solutions u_i are radially symmetric and decreasing about some point. This system of nonlinear elliptic PDEs of Liouville type (called "L-systems") is a natural generalization of Liouville's equation, see [23]:

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^2. \tag{1.7}$$

In another way, (1.7) is the simplest special case of L-system. In [16], Chen and Li proved the following important classification result.

Theorem 1.1. [16] *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ be a weak solution of* (1.7)*, satisfying the finite-mass condition*

$$\int_{\mathbb{R}^2} e^u\,dx < \infty. \tag{1.8}$$

Then u is radial symmetric and decreasing about some point in $\mathbb{R}^2$.

This result is definitive for solving completely (1.7) under (1.8), since it reduces the problem to a simple ODE problem. Then Chen and Li conclude that all solutions of (1.7) and (1.8) are given by

$$u(x) = -2\log\frac{1+\lambda^2|x-x_0|}{2\sqrt{2}\lambda},$$

where $\lambda > 0$ and $x_0 \in \mathbb{R}^2$.

The system (1.4), under slightly more general conditions which include (1.6) as a special case, is applied precisely in the physics of charged particle beams. For more details, see [11, 21, 22]. Moreover, as the Liouville's equation, the system (1.4) has an obvious geometrical significance. A solution N-tuple u_i of (1.4)–(1.6) defines a set of N metrics, all of which are conformally equivalent to the Euclidean metric on $\mathbb{R}^2$.

If we consider the corresponding Dirichlet problem on a bounded domain in $\mathbb{R}^2$,

$$-\Delta u = \rho^2 e^u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{1.9}$$

when the parameter ρ tends to 0, the blow-up analysis and the asymptotic behavior of nontrivial branches of solution of equation (1.9) are well-understood thanks to the result of Suzuki [27] and Nagasaki–Suzuki [24]. Moreover, this result allows to localize the blow-up set of singular limit solutions (up to subsequence) as critical point of functions given by the Green's functions. In contrast, the blow-up analysis for Liouville system (1.2) is almost open.

Consider the following problem

$$\begin{cases} -\Delta u = \rho^2 f(u) & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

when ρ tends to 0 and

$$|f(t)| \le ce^t.$$

Theorem 1.2. [27] *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. Assume that u_ρ is a solution of* (1.10) *which converges to some nontrivial function u^* as ρ tends to* 0*. Then the limit function u^* is a solution of problem*

$$\begin{cases} -\Delta u^* = \sum_{j=1}^n 8\pi\delta_{x_j} & \text{in } \Omega, \\ u^* = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$ *is a critical point of the functional*

$$W\colon (x_1, \dots, x_n) \in (\mathbb{R}^2)^n \mapsto \sum_{j=1}^{n} H(x_j, x_j) + \sum_{j \neq l} G(x_j, x_l).$$

Moreover, the general result has been obtained by Baraket–Paccard [6] and Baraket–Ye [8] for $e^u + e^{\gamma u}$, $\gamma \in (0,1)$ instead of e^u.

In fact, they prove the inverse problem of the result of Suzuki [27], more precisely they prove

Theorem 1.3. [6] *Let* Ω *be a smooth open subset of* $\mathbb{R}^2$*,* $\beta \in (0,1)$ *and* $x_1, \dots, x_m \in \Omega$*. Assume that* $(x_1, \dots, x_m)$ *is a nondegenerate critical point of the function*

$$E\colon (x_1, \dots, x_m) \in \mathbb{C}^m \to \sum_{j} H(x_j, x_j) + \sum_{j \neq l} G(x_j, x_l),$$

then there exists a constant $\rho_0 > 0$ *and* $(u_\rho)_{\rho \in (0,\rho_0)}$ *is a one parameter family of solutions of* (1.9) *such that*

$$\lim_{\rho \to 0} u_\rho = \sum_{i=1}^{m} G(\,\cdot\,, x_i) \quad \text{in } \mathcal{C}^{2,\beta}_{\mathrm{loc}}(\Omega \setminus \{x_1, \dots, x_m\}),$$

where $G(x,x')$ *is the Green's function defined on* $\Omega \times \Omega$*, the solution of*

$$-\Delta G(x,x') = 8\pi \delta_{x=x'} \ \text{ in } \Omega, \quad G(x,x') = 0 \ \text{ on } \partial\Omega$$

and $H(x,x') := G(x,x') + 4 \log |x - x'|$ *its smooth part.*

Some generalizations can be found in [3, 5, 8, 17, 20].

To describe our result, let us denote by

$$\begin{aligned}
\Delta_a^2 u_1 - \Delta_a u_1 &= \frac{1}{a}\Delta(a\Delta u_1) - \frac{V(x)}{a}\operatorname{div}(a\nabla u_1) = \Delta^2 u_1 + \Sigma_a^1 u_1 + \Sigma_a^2 u_1, \\
\Delta_a^2 u_2 - \Delta_a u_2 &= \frac{1}{a}\Delta(a\Delta u_2) - \frac{V(x)}{a}\operatorname{div}(a\nabla u_2) = \Delta^2 u_2 + \Sigma_a^1 u_2 + \Sigma_a^2 u_2,
\end{aligned}$$

where

$$\Sigma_a^1 u_i = 2\frac{\nabla a}{a} \cdot \nabla(\Delta u_i) - V(x)\nabla \log a \cdot \nabla u_i \quad \text{and} \quad \Sigma_a^2 u_i = \left(\frac{\Delta a}{a} - V(x)\right)\Delta u_i, \quad i = 1, 2.$$

Then to solve (1.2) is equivalent to solve the following system

$$\begin{cases}
\Delta_a^2 u_1 - \Delta_a u_1 = \rho^4 e^{\gamma u_1 + (1-\gamma) u_2} & \text{in } \Omega, \\
\Delta_a^2 u_2 - \Delta_a u_2 = \rho^4 e^{\xi u_2 + (1-\xi) u_1} & \text{in } \Omega, \\
\Delta u_1 = \Delta u_2 = u_1 = u_2 = 0 & \text{on } \partial\Omega
\end{cases} \tag{1.11}$$

when the parameter ρ tends to 0. Let $G_a(x,y)$ defined over $\Omega\times\Omega$ be the Green function associated to the bi-Laplacian operator with a Navier boundary conditions, which is the solution of

$$\begin{cases} \Delta^2 G_a(x,y) = 64\pi^2\delta_{x=y} & \text{in } \Omega, \\ \Delta G_a(x,y) = G_a(x,y) = 0 & \text{on } \partial\Omega \end{cases}$$

and denote by $H_a(x,y) := G_a(x,y) + 8\ln|x-y|$ its smooth part.

In this paper, we prove the following results.

Theorem 1.4. *Let Ω be a regular open subset of $\mathbb{R}^4$ and $x^1,x^2,x^3\in\Omega$ be given distinct points. Suppose that (u_1^ρ,u_2^ρ) is a one parameter family of solutions of* (1.11) *such that*

$$\lim_{\rho\to 0} u_1^\rho = \frac{1}{\gamma}G_a(\,\cdot\,,x^1) + G_a(\,\cdot\,,x^3) = u_1^* \quad \text{in } \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega\setminus\{x^1,x^3\})$$

and

$$\lim_{\rho\to 0} u_2^\rho = \frac{1}{\xi}G_a(\,\cdot\,,x^2) + G_a(\,\cdot\,,x^3) = u_2^* \quad \text{in } \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega\setminus\{x^2,x^3\}).$$

Then (x^1,x^2,x^3) is a critical point of the functional

$$\begin{aligned}\mathcal{E}(x^1,x^2,x^3) = {}& \frac{1-\xi}{2\gamma}H_a(x^1,x^1) + \frac{1-\gamma}{2\xi}H_a(x^2,x^2) + \frac{2-\gamma-\xi}{2}H_a(x^3,x^3) \\ &+ \frac{1-\gamma}{\xi}\frac{1-\xi}{\gamma}G_a(x^1,x^2) + \frac{1-\xi}{\gamma}G_a(x^1,x^3) + \frac{1-\gamma}{\xi}G_a(x^2,x^3).\end{aligned}$$

A natural question that arises: can one find a solution u^1 respectively u^2 that concentrates in x^1, x^3 respectively in x^2, x^3. Before giving a partial answer of this question, we define an auxiliary function φ which is a truncation function in $\mathcal{C}_0^\infty(\Omega)$ such that

$$\varphi \equiv \begin{cases} 1 & \text{in } B(x^1,r_0), \\ 1 & \text{in } B(x^2,r_0), \\ 0 & \text{in } \Omega\setminus(B(x^1,r_0)\cup B(x^2,r_0)), \end{cases}$$

where $r_0>0$ and such that $B(x^i,2r_0)\subset\Omega$ for $i=1,2$ and $B(x^1,2r_0)\cap B(x^2,2r_0)=\emptyset$.

Theorem 1.5. *Let Ω be a regular open subset of $\mathbb{R}^4$ and $x^1,x^2,x^3\in\Omega$ be given distinct points. Suppose that (x^1,x^2,x^3) is a nondegenerate critical point of the functional*

$$\begin{aligned}\mathcal{E}(x^1,x^2,x^3) = {}& \frac{1-\xi}{2\gamma}H_a(x^1,x^1) + \frac{1-\gamma}{2\xi}H_a(x^2,x^2) + \frac{2-\gamma-\xi}{2}H_a(x^3,x^3) \\ &+ \frac{1-\gamma}{\xi}\frac{1-\xi}{\gamma}G_a(x^1,x^2) + \frac{1-\xi}{\gamma}G_a(x^1,x^3) + \frac{1-\gamma}{\xi}G_a(x^2,x^3).\end{aligned}$$

Then, there exists a constant $\rho_0 > 0$ *and* $(u_1^\rho, u_2^\rho)_{\rho \le \rho_0}$ *is a one parameter family of solutions of* (1.11) *such that*

$$\lim_{\rho\to 0} \varphi u_1^\rho = \frac{\varphi}{\gamma} G_a(\,\cdot\,, x^1) \quad \text{in } \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega \setminus \{x^1\}),$$
$$\lim_{\rho\to 0} \varphi u_2^\rho = \frac{\varphi}{\xi} G_a(\,\cdot\,, x^2) \quad \text{in } \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega \setminus \{x^2\}),$$

and

$$\begin{aligned}\lim_{\rho\to 0} \big((1-\xi)u_1^\rho + (1-\gamma)u_2^\rho\big) &= \frac{1-\xi}{\gamma} G_a(\,\cdot\,, x^1) + \frac{1-\gamma}{\xi} G_a(\,\cdot\,, x^2) \\ &\quad + (2-\gamma-\xi) G_a(\,\cdot\,, x^3) \quad \text{in } \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega \setminus \{x^1, x^2, x^3\}).\end{aligned}$$

Under an additional assumption on the points x^1, x^2 and x^3, we can give the asymptotic behavior of u_1^ρ and u_2^ρ separately.

Theorem 1.6. *Let* Ω *be a regular open subset of* $\mathbb{R}^4$ *and* $x^1, x^2, x^3 \in \Omega$ *be given distinct points. Suppose that* (x^1, x^2, x^3) *is a nondegenerate critical point of the functional*

$$\begin{aligned}\mathcal{E}(x^1, x^2, x^3) &= \frac{1-\xi}{2\gamma} H_a(x^1, x^1) + \frac{1-\gamma}{2\xi} H_a(x^2, x^2) + \frac{2-\gamma-\xi}{2} H_a(x^3, x^3) \\ &\quad + \frac{1-\gamma}{\xi}\frac{1-\xi}{\gamma} G_a(x^1, x^2) + \frac{1-\xi}{\gamma} G_a(x^1, x^3) + \frac{1-\gamma}{\xi} G_a(x^2, x^3)\end{aligned}$$

such that

$$\frac{1}{\gamma} G_a(x^3, x^1) = \frac{1}{\xi} G_a(x^3, x^2) \quad \text{and} \quad \frac{1}{\gamma} \nabla G_a(\,\cdot\,, x^1)(x^3) = \frac{1}{\xi} \nabla G_a(\,\cdot\,, x^2)(x^3). \tag{1.12}$$

Then, there exists a constant $\rho_0 > 0$ *and* $(u_1^\rho, u_2^\rho)_{\rho \le \rho_0}$ *is a one parameter family of solutions of* (1.11) *such that*

$$\lim_{\rho\to 0} u_1^\rho = \frac{1}{\gamma} G_a(\,\cdot\,, x^1) + G_a(\,\cdot\,, x^3) \quad \text{in } \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega \setminus \{x^1, x^3\}),$$
$$\lim_{\rho\to 0} u_2^\rho = \frac{1}{\xi} G_a(\,\cdot\,, x^2) + G_a(\,\cdot\,, x^3) \quad \text{in } \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega \setminus \{x^2, x^3\}).$$

Our paper is organized as follows: In Section 2, we give necessary condition about the position of the points (x^1, x^2, x^3) thanks to the Pohozaev identity and by using the techniques inspired by the work of Suzuki [27], we determine the appropriate functional, which proves Theorem 1.4. Next, we want to show the inverse result of Theorem 1.4 without adding any condition which is a priori impossible, so we thought of adding an auxiliary function which is proved in Section 3, where we prove Theorem 1.5 motivated by the technics of Baraket et al. [4]. In fact, we introduce some crucial results and definitions about the weighted Hölder spaces, linearized operators and the bi-harmonic extensions, we recall some studies about the analysis of the bi-Laplace operator in weighted spaces. We construct our approximate solution of (1.2) by using the appropriate transformation in a large balls. Next, we consider a nonlinear interior problem, where the existence of a family of solutions of (1.2), which are close to the approximate solution is proven. Then, we prove the existence of a family of solutions to (1.2), which are defined on Ω with small balls removed. Finally, we show how parameters of these families can be connected to produce solutions of (1.2). Indeed, we patch these pieces together via a nonlinear version of Cauchy data matching. In Section 4, we prove Theorem 1.6 by considering the topological condition (1.12) and by using the same techniques as in the previous section.

2. Proof of Theorem 1.4

Let ω be a subset of Ω. We multiply the equation $\Delta_a^2 u_1 - \Delta_a u_1 = \rho^4 e^{\gamma u_1+(1-\gamma)u_2}$ by $\nabla(\gamma u_1 + (1-\gamma)u_2)$ and then integrating over ω, we obtain a Pohozaev type identity

(2.1)
$$\gamma\int_\omega(\Delta_a^2 u_1 - \Delta_a u_1)\nabla u_1 + (1-\gamma)\int_\omega(\Delta_a^2 u_1 - \Delta_a u_1)\nabla u_2 = \rho^4\int_{\partial\omega}(e^{\gamma u_1+(1-\gamma)u_2}-1)\nu\, d\sigma.$$

We have

$$\begin{aligned}\int_\omega(\Delta_a^2 u_1)\nabla u_1 &= \int_\omega \frac{1}{a}\Delta(a\Delta u_1)\nabla u_1\\ &= \int_\omega(\Delta^2 u_1)\nabla u_1 + \int_\omega\frac{1}{a}\Delta a\Delta u_1\cdot\nabla u_1 + \int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_1)\cdot\nabla u_1.\end{aligned}$$

Using the Green's formula we obtain

$$\begin{aligned}\int_\omega(\Delta_a^2 u_1)\nabla u_1 = &-\frac{1}{2}\int_{\partial\omega}(\Delta u_1)^2\nu\, d\sigma - \int_{\partial\omega}(\nabla(\Delta u_1)\cdot\nabla u_1)\nu\, d\sigma + \int_{\partial\omega}\nabla u_1\cdot\nu\nabla(\Delta u_1)\, d\sigma\\ &+\int_{\partial\omega}\nabla(\Delta u_1)\cdot\nu\nabla u_1\, d\sigma + \int_\omega\frac{1}{a}\Delta a\Delta u_1\cdot\nabla u_1 + \int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_1)\cdot\nabla u_1.\end{aligned}$$

On the other hand,

$$\int_\omega\Delta_a u_1\nabla u_1 = \int_\omega\frac{V(x)}{a}\operatorname{div}(a\nabla u_1)\nabla u_1 = \int_\omega\frac{V(x)}{a}\nabla a\Delta u_1 + \int_\omega V(x)\Delta u_1\nabla u_1.$$

Then

$$\begin{aligned}&\int_\omega(\Delta_a^2 u_1 - \Delta_a u_1)\nabla u_1\\ &= -\frac{1}{2}\int_{\partial\omega}(\Delta u_1)^2\nu\, d\sigma - \int_{\partial\omega}(\nabla(\Delta u_1)\cdot\nabla u_1)\nu\, d\sigma + \int_{\partial\omega}\nabla u_1\cdot\nu\nabla(\Delta u_1)\, d\sigma\\ &\quad+\int_{\partial\omega}\nabla(\Delta u_1)\cdot\nu\nabla u_1\, d\sigma + \int_\omega\frac{1}{a}\Delta a\Delta u_1\cdot\nabla u_1 + \int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_1)\cdot\nabla u_1\\ &\quad-\int_\omega\frac{V(x)}{a}\nabla a\Delta u_1 - \int_\omega V(x)\Delta u_1\cdot\nabla u_1.\end{aligned}$$

Similarly we multiply the equation $\Delta_a^2 u_2 - \Delta_a u_2 = \rho^4 e^{\xi u_2 + (1-\xi)u_1}$ by $\nabla(\xi u_2 + (1-\xi)u_1)$ and then integrating over ω we obtain a Pohozaev type identity

$$(2.2)\quad \xi \int_\omega (\Delta_a^2 u_2 - \Delta_a u_2)\nabla u_2 + (1-\xi)\int_\omega (\Delta_a^2 u_2 - \Delta_a u_2)\nabla u_1 = \rho^4 \int_{\partial\omega} (e^{\xi u_2 + (1-\xi)u_1} - 1)\nu\, d\sigma.$$

Using the Green's formula we obtain

$$\begin{aligned}
&\int_\omega (\Delta_a^2 u_2 - \Delta_a u_2)\nabla u_2 \\
&= -\frac{1}{2}\int_{\partial\omega} (\Delta u_2)^2 \nu\, d\sigma - \int_{\partial\omega} (\nabla(\Delta u_2)\cdot\nabla u_2)\nu\, d\sigma + \int_{\partial\omega} \nabla u_2 \cdot \nu \nabla(\Delta u_2)\, d\sigma \\
&\quad + \int_{\partial\omega} \nabla(\Delta u_2)\cdot \nu \nabla u_2\, d\sigma + \int_\omega \frac{1}{a}\Delta a \Delta u_2 \cdot \nabla u_2 + \int_\omega \frac{2}{a}\nabla a \nabla(\Delta u_2)\cdot\nabla u_2 \\
&\quad - \int_\omega \frac{V(x)}{a}\nabla a \Delta u_2 - \int_\omega V(x)\Delta u_2 \cdot \nabla u_2.
\end{aligned}$$

Making use of the identity

$$\begin{aligned}
&\int_\omega (\Delta_a^2 u_1 - \Delta_a u_1)\nabla u_2 + \int_\omega (\Delta_a^2 u_2 - \Delta_a u_2)\nabla u_1 \\
&= \int_\omega \Delta_a^2 u_1 \nabla u_2 + \int_\omega \Delta_a^2 u_2 \nabla u_1 - \int_\omega \Delta_a u_1 \nabla u_2 - \int_\omega \Delta_a u_2 \nabla u_1
\end{aligned}$$

and calculating

$$\begin{aligned}
I_1 &= \int_\omega \Delta_a^2 u_1 \nabla u_2 + \int_\omega \Delta_a^2 u_2 \nabla u_1 \\
&= -\int_{\partial\omega} (\Delta u_1 \cdot \Delta u_2)\nu\, d\sigma + \int_{\partial\omega} \nabla u_2 \cdot \nu \nabla(\Delta u_1)\, d\sigma + \int_{\partial\omega} \nabla u_1 \cdot \nu \nabla(\Delta u_2)\, d\sigma \\
&\quad + \int_\omega \frac{1}{a}\Delta a \Delta u_1 \cdot \nabla u_2 + \int_\omega \frac{1}{a}\Delta a \Delta u_2 \cdot \nabla u_1 + \int_\omega \frac{2}{a}\nabla a \nabla(\Delta u_1)\cdot \nabla u_2 \\
&\quad + \int_\omega \frac{2}{a}\nabla a \nabla(\Delta u_2)\cdot \nabla u_1, \\
I_2 &= \int_\omega \Delta_a u_1 \nabla u_2 + \int_\omega \Delta_a u_2 \nabla u_1 \\
&= 2\int_\omega \frac{V(x)}{a}\nabla a \nabla u_1 \nabla u_2 + \int_\omega V(x)\Delta u_1 \nabla u_2 + \int_\omega V(x)\Delta u_2 \nabla u_1,
\end{aligned}$$

we obtain

$$\begin{aligned}
I_1 - I_2 &= \int_\omega (\Delta_a^2 u_1 - \Delta_a u_1)\nabla u_2 + \int_\omega (\Delta_a^2 u_2 - \Delta_a u_2)\nabla u_1 \\
&= -\int_{\partial\omega} (\Delta u_1 \cdot \Delta u_2)\nu\, d\sigma + \int_{\partial\omega} \nabla u_2 \cdot \nu \nabla(\Delta u_1)\, d\sigma + \int_{\partial\omega} \nabla u_1 \cdot \nu \nabla(\Delta u_2)\, d\sigma \\
&\quad + \int_\omega \frac{1}{a}\Delta a \Delta u_1 \cdot \nabla u_2 + \int_\omega \frac{1}{a}\Delta a \Delta u_2 \cdot \nabla u_1 + \int_\omega \frac{2}{a}\nabla a \nabla(\Delta u_1)\cdot \nabla u_2
\end{aligned}$$

$$+\int_\omega \frac{2}{a}\nabla a\nabla(\Delta u_2)\cdot\nabla u_1 - 2\int_\omega \frac{V(x)}{a}\nabla a\nabla u_1\nabla u_2 - \int_\omega V(x)\Delta u_1\nabla u_2$$
$$-\int_\omega V(x)\Delta u_2\nabla u_1.$$

Then by combination of (2.1) and (2.2) we obtain

(2.3)

$$\begin{aligned}
&\gamma(1-\xi)\Big[\int_{\partial\omega}\Big(-\frac{1}{2}(\Delta u_1)^2\nu-(\nabla(\Delta u_1)\cdot\nabla u_1)\nu+\nabla u_1\cdot\nu\nabla(\Delta u_1)\,d\sigma+\nabla(\Delta u_1)\cdot\nu\nabla u_1\Big)\,d\sigma\\
&\quad+\int_\omega\frac{1}{a}\Delta a\Delta u_1\cdot\nabla u_1+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_1)\cdot\nabla u_1-\int_\omega\frac{V(x)}{a}\nabla a\Delta u_1-\int_\omega V(x)\Delta u_1\cdot\nabla u_1\Big]\\
&+\xi(1-\gamma)\Big[\int_{\partial\omega}\Big(-\frac{1}{2}(\Delta u_2)^2\nu-(\nabla(\Delta u_2)\cdot\nabla u_2)\nu+\nabla u_2\cdot\nu\nabla(\Delta u_2)\,d\sigma+\nabla(\Delta u_2)\cdot\nu\nabla u_2\Big)\,d\sigma\\
&\quad+\int_\omega\frac{1}{a}\Delta a\Delta u_2\cdot\nabla u_2+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_2)\cdot\nabla u_2-\int_\omega\frac{V(x)}{a}\nabla a\Delta u_2-\int_\omega V(x)\Delta u_2\cdot\nabla u_2\Big]\\
&+(1-\gamma)(1-\xi)\Big[\int_{\partial\omega}\Big(-(\Delta u_1\cdot\Delta u_2)\nu+\nabla u_2\cdot\nu\nabla(\Delta u_1)+\nabla u_1\cdot\nu\nabla(\Delta u_2)\Big)\,d\sigma\\
&\quad+\int_\omega\frac{1}{a}\Delta a\Delta u_1\cdot\nabla u_2+\int_\omega\frac{1}{a}\Delta a\Delta u_2\cdot\nabla u_1+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_1)\cdot\nabla u_2+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_2)\cdot\nabla u_1\\
&\quad-2\int_\omega\frac{V(x)}{a}\nabla a\nabla u_1\nabla u_2-\int_\omega V(x)\Delta u_1\nabla u_2-\int_\omega V(x)\Delta u_2\nabla u_1\Big]\\
&=\rho^4(1-\xi)\int_{\partial\omega}(e^{\gamma u_1+(1-\gamma)u_2}-1)\nu\,d\sigma+\rho^4(1-\gamma)\int_{\partial\omega}(e^{\xi u_2+(1-\xi)u_1}-1)\nu\,d\sigma.
\end{aligned}$$

We insert the profile of the limits of the solutions in the identity (2.3) when ρ tends to 0 and η fixed small enough, we choose $\omega = B(x^i,\eta) = B_i$ for $i = 1, 2, 3$. We obtain, thanks to the regularity of solutions of (1.11) on $\Omega\setminus\{x^1, x^2, x^3\}$,

$$\lim_{\rho\to 0}\rho^4(1-\xi)\int_{\partial\omega}(e^{\gamma u_1+(1-\gamma)u_2}-1)\nu\,d\sigma+\rho^4(1-\gamma)\int_{\partial\omega}(e^{\xi u_2+(1-\xi)u_1}-1)\nu\,d\sigma=0.$$

Then

$$\begin{aligned}
&\gamma(1-\xi)\Big[\int_{\partial\omega}\Big(-\frac{1}{2}(\Delta u_1^*)^2\nu-(\nabla(\Delta u_1^*)\cdot\nabla u_1^*)\nu+\nabla u_1^*\cdot\nu\nabla(\Delta u_1^*)\,d\sigma+\nabla(\Delta u_1^*)\cdot\nu\nabla u_1^*\Big)\,d\sigma\\
&\quad+\int_\omega\frac{1}{a}\Delta a\Delta u_1^*\cdot\nabla u_1^*+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_1^*)\cdot\nabla u_1^*-\int_\omega\frac{V(x)}{a}\nabla a\Delta u_1^*-\int_\omega V(x)\Delta u_1^*\cdot\nabla u_1^*\Big]\\
&+\xi(1-\gamma)\Big[\int_{\partial\omega}\Big(-\frac{1}{2}(\Delta u_2^*)^2\nu-(\nabla(\Delta u_2^*)\cdot\nabla u_2^*)\nu+\nabla u_2^*\cdot\nu\nabla(\Delta u_2^*)\,d\sigma+\nabla(\Delta u_2^*)\cdot\nu\nabla u_2^*\Big)\,d\sigma\\
&\quad+\int_\omega\frac{1}{a}\Delta a\Delta u_2^*\cdot\nabla u_2^*+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_2^*)\cdot\nabla u_2^*-\int_\omega\frac{V(x)}{a}\nabla a\Delta u_2^*-\int_\omega V(x)\Delta u_2^*\cdot\nabla u_2^*\Big]\\
&+(1-\gamma)(1-\xi)\Big[\int_{\partial\omega}\Big(-(\Delta u_1^*\cdot\Delta u_2^*)\nu+\nabla u_2^*\cdot\nu\nabla(\Delta u_1^*)+\nabla u_1^*\cdot\nu\nabla(\Delta u_2^*)\Big)\,d\sigma\\
&\quad+\int_\omega\frac{1}{a}\Delta a\Delta u_1^*\cdot\nabla u_2^*+\int_\omega\frac{1}{a}\Delta a\Delta u_2^*\cdot\nabla u_1^*+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_1^*)\cdot\nabla u_2^*+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_2^*)\cdot\nabla u_1^*\\
&\quad-2\int_\omega\frac{V(x)}{a}\nabla a\nabla u_1^*\nabla u_2^*-\int_\omega V(x)\Delta u_1^*\nabla u_2^*-\int_\omega V(x)\Delta u_2^*\nabla u_1^*\Big]=0.
\end{aligned}$$

In the desire to construct solutions of the system that blow-up in the points x^1, x^2 and x^3 this means that, if ρ tends to 0, we have

$$u_1 \to u_1^* = \frac{1}{\gamma} G_a(x,x^1) + G_a(x,x^3) \quad \text{and} \quad u_2 \to u_2^* = \frac{1}{\xi} G_a(x,x^2) + G_a(x,x^3).$$

• In $B_3 = B(x^3,\eta)$, since we have $G_a(x,x^3) = -8\ln|x-x^3| + H_a(x,x^3)$ where H_a is a smooth function in Ω, then

$$u_1^* = \frac{1}{\gamma} G_a(x,x^1) + G_a(x,x^3) = -8\ln|x-x^3| + R(x) \ \text{ with } R(x) = H_a(x,x^3) + \frac{1}{\gamma} G_a(x,x^1)$$

and

$$u_2^* = \frac{1}{\xi} G_a(x,x^2) + G_a(x,x^3) = -8\ln|x-x^3| + M(x) \ \text{ with } M(x) = H_a(x,x^3) + \frac{1}{\xi} G_a(x,x^2).$$

We set

$$\begin{aligned}
&I_{\text{lhs}} \\
&= \gamma(1-\xi)\bigg[\int_{\partial\omega}\Big(-\frac{1}{2}(\Delta u_1^*)^2\nu - (\nabla(\Delta u_1^*)\cdot\nabla u_1^*)\nu + \nabla u_1^*\cdot\nu\nabla(\Delta u_1^*)\,d\sigma + \nabla(\Delta u_1^*)\cdot\nu\nabla u_1^*\Big)\,d\sigma \\
&\quad + \int_\omega \frac{1}{a}\Delta a\Delta u_1^*\cdot\nabla u_1^* + \int_\omega \frac{2}{a}\nabla a\nabla(\Delta u_1^*)\cdot\nabla u_1^* - \int_\omega \frac{V(x)}{a}\nabla a\Delta u_1^* - \int_\omega V(x)\Delta u_1^*\cdot\nabla u_1^*\bigg] \\
&\quad + \xi(1-\gamma)\bigg[\int_{\partial\omega}\Big(-\frac{1}{2}(\Delta u_2^*)^2\nu - (\nabla(\Delta u_2^*)\cdot\nabla u_2^*)\nu + \nabla u_2^*\cdot\nu\nabla(\Delta u_2^*)\,d\sigma + \nabla(\Delta u_2^*)\cdot\nu\nabla u_2^*\Big)\,d\sigma \\
&\quad + \int_\omega \frac{1}{a}\Delta a\Delta u_2^*\cdot\nabla u_2^* + \int_\omega \frac{2}{a}\nabla a\nabla(\Delta u_2^*)\cdot\nabla u_2^* - \int_\omega \frac{V(x)}{a}\nabla a\Delta u_2^* - \int_\omega V(x)\Delta u_2^*\cdot\nabla u_2^*\bigg] \\
&\quad + (1-\gamma)(1-\xi)\bigg[\int_{\partial\omega}\Big(-(\Delta u_1^*\cdot\Delta u_2^*)\nu + \nabla u_2^*\cdot\nu\nabla(\Delta u_1^*) + \nabla u_1^*\cdot\nu\nabla(\Delta u_2^*)\Big)\,d\sigma \\
&\quad + \int_\omega \frac{1}{a}\Delta a\Delta u_1^*\cdot\nabla u_2^* + \int_\omega \frac{1}{a}\Delta a\Delta u_2^*\cdot\nabla u_1^* + \int_\omega \frac{2}{a}\nabla a\nabla(\Delta u_1^*)\cdot\nabla u_2^* + \int_\omega \frac{2}{a}\nabla a\nabla(\Delta u_2^*)\cdot\nabla u_1^* \\
&\quad - 2\int_\omega \frac{V(x)}{a}\nabla a\nabla u_1^*\nabla u_2^* - \int_\omega V(x)\Delta u_1^*\nabla u_2^* - \int_\omega V(x)\Delta u_2^*\nabla u_1^*\bigg].
\end{aligned}$$

By computation, we prove that

$$\begin{aligned}
I_{\text{lhs}} = &-\frac{8}{\eta}\left[(1-\xi)\int_{\partial B_3}\nabla\Delta R(x)\nu\,d\sigma + (1-\gamma)\int_{\partial B_3}\nabla\Delta M(x)\nu\,d\sigma\right] \\
&+ \frac{16}{\eta^2}\left[(1-\xi)\int_{\partial B_3}\Delta R(x)\nu\,d\sigma + (1-\gamma)\int_{\partial B_3}\Delta M(x)\nu\,d\sigma\right] \\
&+ \frac{32}{\eta^3}\left[(1-\xi)\int_{\partial B_3}\nabla R(x)\nu\,d\sigma + (1-\gamma)\int_{\partial B_3}\nabla M(x)\nu\,d\sigma\right] + O(\eta)
\end{aligned}$$

with

$$R(x) = H_a(x,x^3) + \frac{1}{\gamma} G_a(x,x^1) \quad \text{and} \quad M(x) = H_a(x,x^3) + \frac{1}{\xi} G_a(x,x^2).$$

Then

$$\begin{aligned}
&(1-\xi)\int_{\partial B_3}\nabla R(x)\nu\,d\sigma+(1-\gamma)\int_{\partial B_3}\nabla M(x)\nu\,d\sigma\\
&=(2-\gamma-\xi)\int_{\partial B_3}\nabla H_a(x,x^3)\,d\sigma+\frac{1-\xi}{\gamma}\int_{\partial B_3}\nabla G_a(x,x^1)\,d\sigma+\frac{1-\gamma}{\xi}\int_{\partial B_3}\nabla G_a(x,x^2)\,d\sigma\\
&=O(\eta).
\end{aligned}$$

Since

$$\nabla H_a(x,x^3)=\nabla H_a(x^3,x^3)+O(\eta),\quad \nabla G_a(x,x^1)=\nabla G_a(x^3,x^1)+O(\eta)$$

and

$$\nabla G_a(x,x^2)=\nabla G_a(x^3,x^2)+O(\eta),$$

we get

$$(2-\gamma-\xi)\nabla H_a(x^3,x^3)+\frac{1-\xi}{\gamma}\nabla G_a(x^3,x^1)+\frac{1-\gamma}{\xi}\nabla G_a(x^3,x^2)=O(\eta),$$

which means that x^3 is a critical point of the functional

$$\mathcal{E}_3\colon x\to H_a(\,\cdot\,,x^3)+\frac{1-\xi}{\gamma(2-\gamma-\xi)}G_a(\,\cdot\,,x^1)+\frac{1-\gamma}{\xi(2-\gamma-\xi)}G_a(\,\cdot\,,x^2). \tag{2.4}$$

• In $B_1=B(x^1,\eta)$, since we have $G_a(x,x^1)=-8\ln|x-x^1|+H_a(x,x^1)$ where H_a is a smooth function in Ω, then

$$u_1^*=\frac{1}{\gamma}G_a(x,x^1)+G_a(x,x^3)=-\frac{8}{\gamma}\ln|x-x^1|+K(x)\ \text{ with } K(x)=\frac{1}{\gamma}H_a(x,x^1)+G_a(x,x^3)$$

and $u_2^*=\frac{1}{\xi}G_a(x,x^2)+G_a(x,x^3)=S(x)$.

We already have

$$\begin{aligned}
&I_{\text{lhs}}\\
&=\gamma(1-\xi)\Big[\int_{\partial\omega}\Big(-\frac{1}{2}(\Delta u_1^*)^2\nu-(\nabla(\Delta u_1^*)\cdot\nabla u_1^*)\nu+\nabla u_1^*\cdot\nu\nabla(\Delta u_1^*)\,d\sigma+\nabla(\Delta u_1^*)\cdot\nu\nabla u_1^*\Big)\,d\sigma\\
&\qquad+\int_\omega\frac{1}{a}\Delta a\Delta u_1^*\cdot\nabla u_1^*+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_1^*)\cdot\nabla u_1^*-\int_\omega\frac{V(x)}{a}\nabla a\Delta u_1^*-\int_\omega V(x)\Delta u_1^*\cdot\nabla u_1^*\Big]\\
&\quad+\xi(1-\gamma)\Big[\int_{\partial\omega}\Big(-\frac{1}{2}(\Delta u_2^*)^2\nu-(\nabla(\Delta u_2^*)\cdot\nabla u_2^*)\nu+\nabla u_2^*\cdot\nu\nabla(\Delta u_2^*)\,d\sigma+\nabla(\Delta u_2^*)\cdot\nu\nabla u_2^*\Big)\,d\sigma\\
&\qquad+\int_\omega\frac{1}{a}\Delta a\Delta u_2^*\cdot\nabla u_2^*+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_2^*)\cdot\nabla u_2^*-\int_\omega\frac{V(x)}{a}\nabla a\Delta u_2^*-\int_\omega V(x)\Delta u_2^*\cdot\nabla u_2^*\Big]\\
&\quad+(1-\gamma)(1-\xi)\Big[\int_{\partial\omega}\Big(-(\Delta u_1^*\cdot\Delta u_2^*)\nu+\nabla u_2^*\cdot\nu\nabla(\Delta u_1^*)+\nabla u_1^*\cdot\nu\nabla(\Delta u_2^*)\Big)\,d\sigma\\
&\qquad+\int_\omega\frac{1}{a}\Delta a\Delta u_1^*\cdot\nabla u_2^*+\int_\omega\frac{1}{a}\Delta a\Delta u_2^*\cdot\nabla u_1^*+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_1^*)\cdot\nabla u_2^*+\int_\omega\frac{2}{a}\nabla a\nabla(\Delta u_2^*)\cdot\nabla u_1^*
\end{aligned}$$

$$- 2\int_\omega \frac{V(x)}{a}\nabla a\nabla u_1^*\nabla u_2^* - \int_\omega V(x)\Delta u_1^*\nabla u_2^* - \int_\omega V(x)\Delta u_2^*\nabla u_1^*\Big].$$

By computation, we prove that

$$\begin{aligned} I_{\rm lhs} &= -\frac{8}{\eta}\left[(1-\xi)\int_{\partial B_1}\nabla\Delta K(x)\nu\, d\sigma + \frac{(1-\xi)(1-\gamma)}{\gamma}\int_{\partial B_1}\nabla\Delta S(x)\nu\, d\sigma\right] \\ &\quad + \frac{16}{\eta^2}\left[(1-\xi)\int_{\partial B_1}\Delta K(x)\nu\, d\sigma + \frac{(1-\xi)(1-\gamma)}{\gamma}\int_{\partial B_1}\Delta S(x)\nu\, d\sigma\right] \\ &\quad + \frac{32}{\eta^3}\left[(1-\xi)\int_{\partial B_1}\nabla K(x)\nu\, d\sigma + \frac{(1-\xi)(1-\gamma)}{\gamma}\int_{\partial B_1}\nabla S(x)\nu\, d\sigma\right] + O(\eta) \end{aligned}$$

with

$$K(x) = \frac{1}{\gamma}H_a(x,x^1) + G_a(x,x^3) \quad \text{and} \quad S(x) = \frac{1}{\xi}G_a(x,x^2) + G_a(x,x^3).$$

Then

$$\begin{aligned} &(1-\xi)\int_{\partial B_1}\nabla K(x)\nu\, d\sigma + \frac{(1-\xi)(1-\gamma)}{\gamma}\int_{\partial B_1}\nabla S(x)\nu\, d\sigma \\ &= \frac{1-\xi}{\gamma}\int_{\partial B_1}\nabla H_a(x,x^1)\nu\, d\sigma + \frac{1-\xi}{\gamma}\int_{\partial B_1}\nabla G_a(x,x^3)\nu\, d\sigma + \frac{(1-\xi)(1-\gamma)}{\gamma\xi}\int_{\partial B_1}\nabla G_a(x,x^2)\nu\, d\sigma \\ &= O(\eta). \end{aligned}$$

By virtue of the relations

$$\nabla H_a(x,x^1) = \nabla H_a(x^1,x^1) + O(\eta), \quad \nabla G_a(x,x^3) = \nabla G_a(x^1,x^3) + O(\eta)$$

and

$$\nabla G_a(x,x^2) = \nabla G_a(x^1,x^2) + O(\eta),$$

then

$$\frac{1-\xi}{\gamma}\nabla H_a(x^1,x^1) + \frac{1-\xi}{\gamma}\nabla G_a(x^1,x^3) + \frac{(1-\xi)(1-\gamma)}{\gamma\xi}\nabla G_a(x^1,x^2) = O(\eta),$$

we conclude that x^1 is the critical point of the functional

$$\mathcal{E}_1\colon x \to H_a(\,\cdot\,,x^1) + G_a(\,\cdot\,,x^3) + \frac{1-\gamma}{\xi}G_a(\,\cdot\,,x^2). \tag{2.5}$$

• In $B_2 = B(x^2,\eta)$, we can prove similarly as in $B_1 = B(x^1,\eta)$ that x^2 is a critical point of the functional

$$\mathcal{E}_2\colon x \to H_a(\,\cdot\,,x^2) + G_a(\,\cdot\,,x^3) + \frac{1-\xi}{\gamma}G_a(\,\cdot\,,x^1). \tag{2.6}$$

Finally by combination of (2.4), (2.5) and (2.6), we conclude that the point (x^1,x^2,x^3) is a critical point of the functional $\mathcal{E}$ defined by

$$\begin{aligned} \mathcal{E}(x^1,x^2,x^3) &= \frac{1-\xi}{2\gamma}H_a(x^1,x^1) + \frac{1-\gamma}{2\xi}H_a(x^2,x^2) + \frac{2-\gamma-\xi}{2}H_a(x^3,x^3) \\ &\quad + \frac{1-\xi}{\gamma}G_a(x^1,x^3) + \frac{1-\gamma}{\xi}G_a(x^2,x^3) + \frac{(1-\xi)(1-\gamma)}{\gamma\xi}G_a(x^1,x^2). \end{aligned}$$

3. Proof of Theorem 1.5

3.1. Construction of the approximate solution

We denote by ε the smallest positive parameter satisfying

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

Let

$$u_\varepsilon(x) := 4\ln\frac{(1+\varepsilon^2)}{\varepsilon^2+|x|^2},$$

which is a solution of

$$\Delta^2 u = \rho^4 e^u \quad \text{in } \mathbb{R}^4. \tag{3.1}$$

Hence for all $\tau > 0$, the function

$$u_{\varepsilon,\tau}(x) := 4\ln\frac{\tau(1+\varepsilon^2)}{\varepsilon^2+|\tau x|^2} \tag{3.2}$$

is also a solution of (3.1).

3.1.1. A linearized operator

First we introduce some definitions and notations.

Definition 3.1. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$, $\mu \in \mathbb{R}$ and $|x| = r$, we introduce the weighted Hölder spaces $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)$ as the space of the functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^4)$ for which the norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)} = \|u\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0))} + \sup_{r\geq 1}\left((1+r^2)^{-\mu/2}\|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0)\setminus B_{1/2}(0))}\right)$$

is finite. Similarly, for given $\overline{r} \geq 1$, let $\mathcal{C}^{k,\alpha}_\mu(B_{\overline{r}}(0))$ be the space of functions in $\mathcal{C}^{k,\alpha}(B_{\overline{r}}(0))$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(B_{\overline{r}}(0))} = \|u\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0))} + \sup_{1\leq r\leq \overline{r}}\left(r^{-\mu}\|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0)\setminus B_{1/2}(0))}\right)$$

is finite. Finally, set $B_r^*(x^i) = B_r(x^i)\setminus\{x^i\}$, let $\mathcal{C}^{k,\alpha}_\mu(\overline{B}_1^*(0))$ be the space of the functions in $\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\overline{B}_1^*(0))$ for which the norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(\overline{B}_1^*(0))} = \sup_{r\leq 1/2}\left(r^{-\mu}\|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2(0)\setminus B_1(0))}\right)$$

is finite.

We define the linear elliptic operator $\mathbb{L}$ by

$$\mathbb{L} := \Delta^2 - \frac{384}{(1+r^2)^4},$$

which corresponds to the linearization of (3.1) about the radial symmetric solution $u_{\varepsilon=1,\tau=1}$ defined by (3.2). When $k > 2$, we let $[\mathcal{C}_\mu^{k,\alpha}(\overline{\Omega})]_0$ to be the subspace of functions $w \in \mathcal{C}_\mu^{k,\alpha}(\overline{\Omega})$ satisfying $\Delta w = w = 0$ on $\partial\Omega$.

Proposition 3.2. [4] *All bounded solution of $\mathbb{L}w = 0$ on $\mathbb{R}^4$ are linear combinations of*

$$\phi_0(x) = 4\frac{1-|x|^2}{1+|x|^2} \quad \textit{and} \quad \phi_i(x) = \frac{8x_i}{1+|x|^2} \quad \textit{for } i = 1, \ldots, 4.$$

Moreover, for $\mu > 1$, $\mu \notin \mathbb{Z}$, the operator $\mathbb{L}\colon \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \to \mathcal{C}_{\mu-4}^{0,\alpha}(\mathbb{R}^4)$ is surjective.

In the following, we denote a right inverse of $\mathbb{L}$ by $\mathcal{G}_\mu$. Similarly, using the fact that any bounded bi-harmonic solution on $\mathbb{R}^4$ is constant, we claim

Proposition 3.3. [4] *Let $\delta > 0$, $\delta \notin \mathbb{Z}$ then Δ^2 is surjective from $\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$ to $\mathcal{C}_{\delta-4}^{0,\alpha}(\mathbb{R}^4)$.*

We denote a right inverse of Δ^2 by $\mathcal{K}_\delta\colon \mathcal{C}_{\delta-4}^{0,\alpha}(\mathbb{R}^4) \to \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$ for $\delta > 0$, $\delta \notin \mathbb{Z}$.

Finally, we consider punctured domains. Given $\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3$ three distinct points in Ω, we define $\widetilde{\mathbf{x}} := (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3)$ and $\overline{\Omega}^*(\widetilde{\mathbf{x}}) := \overline{\Omega} \setminus \{\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\}$. Let $r_0 > 0$ be small such that $\overline{B}_{r_0}(\widetilde{x}^i)$ are disjoint and included in Ω. For all $r \in (0, r_0)$, we define

$$\overline{\Omega}_r(\widetilde{\mathbf{x}}) := \overline{\Omega} \setminus \bigcup_{i=1}^{3} B_r(\widetilde{x}^i).$$

Definition 3.4. Let $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the weighted Hölder space $\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(\widetilde{x}))$ as the space of the functions $w \in \mathcal{C}_{\text{loc}}^{k,\alpha}(\overline{\Omega}^*(\widetilde{x}))$ such that the norm

$$\|w\|_{\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(\widetilde{x}))} := \|w\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega}_{r_0/2}(\widetilde{x}))} + \sum_{i=1}^{3} \sup_{0<r\le r_0/2} \left(r^{-\nu}\|w(\widetilde{x}_i + r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2(0)\setminus B_1(0))}\right)$$

is finite. Furthermore, for $k \ge 2$, we denote by $\left[\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))\right]_0$ the space of all functions $w \in \mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))$ satisfying $\Delta w = w = 0$ on $\partial\Omega$.

We recall the following result.

Proposition 3.5. [4] *Let $\nu < 0$, $\nu \notin \mathbb{Z}$. Then Δ^2 is surjective from $\left[\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))\right]_0$ to $\mathcal{C}_{\nu-4}^{0,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))$.*

We denote by $\widetilde{\mathcal{K}}_\nu\colon \mathcal{C}_{\nu-4}^{0,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}})) \to \left[\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))\right]_0$ a right inverse of Δ^2 for $\nu < 0$, $\nu \notin \mathbb{Z}$.

3.1.2. Ansatz and first estimates

For all $\sigma \geq 1$, we denote by $\xi_{\mu,\sigma} \colon \mathcal{C}^{0,\alpha}_{\mu}(\overline{B}_{\sigma}(0)) \to \mathcal{C}^{0,\alpha}_{\mu}(\mathbb{R}^4)$ the extension operator defined by

$$\xi_{\mu,\sigma}(f)(x) = \begin{cases} f(x) & \text{for } |x| \leq \sigma, \\ \chi\big(\frac{|x|}{\sigma}\big) f\big(\sigma \frac{x}{|x|}\big) & \text{for } |x| \geq \sigma. \end{cases} \tag{3.3}$$

Here χ is a cut-off function over $\mathbb{R}_+$, which is equal to 1 for $t \leq 1$ and equal to 0 for $t \geq 2$.

It is easy to check that there exists a constant $\overline{c} := \overline{c}(\mu) > 0$ independent of σ such that

$$\|\xi_{\mu,\sigma}(w)\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathbb{R}^4)} \leq \overline{c} \|w\|_{\mathcal{C}^{0,\alpha}_{\mu}(\overline{B}_{\sigma}(0))}. \tag{3.4}$$

For all $\varepsilon, \beta, \lambda, \tau_i > 0$, $i = 1, 2, 3$ and $\gamma, \xi \in (0, 1)$, we define

$$r_{\varepsilon} := r_{\varepsilon,\beta,\lambda} := \max\big(\varepsilon^{1/2}, \varepsilon^{\frac{\gamma+\xi-1}{\gamma}}, \varepsilon^{\frac{\gamma+\xi-1}{\xi}}, \beta^{1/2}, \lambda^{1/2}\big) \quad \text{and} \quad R^i_{\varepsilon} := R^i_{\varepsilon,\beta,\lambda} := \tau_i \frac{r_{\varepsilon}}{\varepsilon}.$$

Here, we are interested to study in $B_{r_{\varepsilon}}(x^1)$, the system

$$\begin{aligned} \Delta_a^2 u_1 - \Delta_a u_1 &= \rho^4 e^{\gamma u_1 + (1-\gamma) u_2}, \\ \Delta_a^2 u_2 - \Delta_a u_2 &= \rho^4 e^{\xi u_2 + (1-\xi) u_1}. \end{aligned} \tag{3.5}$$

Using the transformation

$$v_1(x) = u_1\left(\frac{\varepsilon}{\tau_1} x\right) + \frac{8}{\gamma} \ln \varepsilon - \frac{4}{\gamma} \ln\left(\frac{\tau_1(1+\varepsilon^2)}{2}\right) \quad \text{and} \quad v_2(x) = u_2\left(\frac{\varepsilon}{\tau_1} x\right),$$

the previous system can be written as

$$\begin{aligned} &\Delta^2 v_1 + \Sigma^1_{\widetilde{a}_1} v_1 + \Sigma^2_{\widetilde{a}_1} v_1 = 24 e^{\gamma v_1 + (1-\gamma) v_2} && \text{in } B_{R^1_{\varepsilon}}(x^1), \\ &\Delta^2 v_2 + \Sigma^1_{\widetilde{a}_1} v_2 + \Sigma^2_{\widetilde{a}_1} v_2 = 24 \frac{2^{4\left(\frac{\gamma+\xi-1}{\gamma}\right)} \varepsilon^{8\left(\frac{\gamma+\xi-1}{\gamma}\right)}}{\big(\tau_1(1+\varepsilon^2)\big)^{4\left(\frac{\gamma+\xi-1}{\gamma}\right)}} e^{\xi v_2 + (1-\xi) v_1} && \text{in } B_{R^1_{\varepsilon}}(x^1) \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} \Sigma^1_{\widetilde{a}_1} v_i &= 2 \frac{\nabla \widetilde{a}_1}{\widetilde{a}_1} \cdot \nabla(\Delta v_i) - \widetilde{V}_1(x) \left(\frac{\varepsilon}{\tau_1}\right)^2 \nabla \log \widetilde{a}_1 \cdot \nabla v_i, \\ \Sigma^2_{\widetilde{a}_1} v_i &= \left(\frac{\Delta \widetilde{a}_1}{\widetilde{a}_1} - \left(\frac{\varepsilon}{\tau_1}\right)^2 \widetilde{V}_1(x)\right) \Delta v_i \quad \text{for } i = 1, 2, \end{aligned}$$

where $\widetilde{a}_1(x) = a\big(\frac{\varepsilon}{\tau_1} x\big)$, $\widetilde{V}_1(x) = V\big(\frac{\varepsilon}{\tau_1} x\big)$. Here $\tau_1 > 0$ is a constant which will be fixed later. We denote by $\overline{u} = u_{\varepsilon=\tau=1}$, we look for a solution of (3.6) of the form

$$\begin{aligned} v_1(x) &= \frac{1}{\gamma} \overline{u}(x - x^1) - \frac{1-\gamma}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{1-\gamma}{\gamma \xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{\ln \gamma}{\gamma} + h^1_1(x), \\ v_2(x) &= \frac{1}{\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + h^1_2(x). \end{aligned}$$

Using the fact that $e^{\overline{u}(x-x^1)} = \frac{16}{(1+|x-x^1|^2)^4}$, this amounts to solve the system

$$\begin{aligned}
\mathbb{L}h_1^1 &= \frac{384}{\gamma(1+r^2)^4}\left[e^{\gamma h_1^1+(1-\gamma)h_2^1} - \gamma h_1^1 - 1\right] - \Sigma^1_{\widetilde{a}_1} v_1 - \Sigma^2_{\widetilde{a}_1} v_1, \\
\Delta^2 h_2^1 &= \frac{24\cdot 16^{\frac{1-\xi}{\gamma}} C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}}{\gamma^{\frac{1-\xi}{\gamma}}(1+r^2)^{4\frac{1-\xi}{\gamma}}} e^{\xi h_2^1+(1-\xi)h_1^1+\frac{\gamma+\xi-1}{\gamma}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)} \\
&\quad - \Sigma^1_{\widetilde{a}_1} v_2 - \Sigma^2_{\widetilde{a}_1} v_2
\end{aligned} \tag{3.7}$$

in $B_{R_\varepsilon^1}(x^1)$, where $r = |x-x^1|$ and $C_{1,\varepsilon} = \frac{2}{\tau_1(1+\varepsilon^2)}$. We fix $\mu \in (1,2)$ and $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$. To find a solution of (3.7), it is enough to find a fixed point (h_1^1, h_2^1) in a small ball of $\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ solutions of

$$\begin{aligned}
h_1^1 &= \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon^1} \circ \mathcal{T}_1(h_1^1,h_2^1) = \mathcal{N}_1(h_1^1,h_2^1), \\
h_2^1 &= \mathcal{K}_\delta \circ \xi_{\delta,R_\varepsilon^1} \circ \mathcal{R}_1(h_1^1,h_2^1) = \mathcal{M}_1(h_1^1,h_2^1).
\end{aligned} \tag{3.8}$$

Here $\xi_{\sigma,R_\varepsilon^1}$, is defined in (3.3), $\mathcal{G}_\mu$ and $\mathcal{K}_\delta$ are defined after Propositions 3.2 and 3.3 respectively and

$$\begin{aligned}
\mathcal{T}_1(h_1^1,h_2^1) &= \frac{384}{\gamma(1+r^2)^4}\left[e^{\gamma h_1^1+(1-\gamma)h_2^1} - \gamma h_1^1 - 1\right] - \Sigma^1_{\widetilde{a}_1} v_1 - \Sigma^2_{\widetilde{a}_1} v_1, \\
\mathcal{R}_1(h_1^1,h_2^1) &= \frac{24\cdot 16^{\frac{1-\xi}{\gamma}} C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}}{\gamma^{\frac{1-\xi}{\gamma}}(1+r^2)^{4\frac{1-\xi}{\gamma}}} e^{\xi h_2^1+(1-\xi)h_1^1+\frac{\gamma+\xi-1}{\gamma}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)} \\
&\quad - \Sigma^1_{\widetilde{a}_1} v_2 - \Sigma^2_{\widetilde{a}_1} v_2.
\end{aligned}$$

Lemma 3.6. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ and $\overline{c}_\kappa > 0$ (only depending on κ) such that for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$, $\beta \in (0,\beta_\kappa)$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$ with $r_\varepsilon := r_{\varepsilon,\beta,\lambda}$. We have*

$$\begin{gathered}
\|\mathcal{N}_1(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \quad \|\mathcal{M}_1(0,0)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \\
\|\mathcal{N}_1(h_1^1,h_2^1) - \mathcal{N}_1(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le \overline{c}_\kappa r_\varepsilon^2 \|(h_1^1,h_2^1) - (k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}
\end{gathered}$$

and

$$\|\mathcal{M}_1(h_1^1,h_2^1) - \mathcal{M}_1(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|(h_1^1,h_2^1) - (k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}$$

for all $(h_1^1,h_2^1), (k_1^1,k_2^1) \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ satisfying

$$\|(h_1^1,h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2. \tag{3.9}$$

Proof. We denote $r=|x-\widetilde{x}^1|$, taking in to account that $(\Sigma^i_{\widetilde{a}_1})$, $i=1,2$ are linear operators. Given $\kappa>0$, there exists a constant $c_\kappa>0$ (which can depend only on κ) such that for $\mu\in(1,2)$, we have

$$
\begin{aligned}
&\sup_{r\le R^1_\varepsilon} r^{4-\mu}|\mathcal{T}_1(0,0)|\\
&\le \sup_{r\le R^1_\varepsilon} r^{4-\mu}\left|\Sigma^1_{\widetilde{a}_1}\left(\frac{1}{\gamma}\overline{u}-\frac{1-\gamma}{\gamma}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)-\frac{\ln\gamma}{\gamma}\right)\right|\\
&\quad+\sup_{r\le R^1_\varepsilon} r^{4-\mu}\left|\Sigma^2_{\widetilde{a}_1}\left(\frac{1}{\gamma}\overline{u}-\frac{1-\gamma}{\gamma}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)-\frac{\ln\gamma}{\gamma}\right)\right|\\
&\le c_\kappa \sup_{r\le R^1_\varepsilon} r^{4-\mu}\left(\left|\Sigma^1_{\widetilde{a}_1}\left(\frac{1}{\gamma}\overline{u}\right)\right|+\left|\Sigma^2_{\widetilde{a}_1}\left(\frac{1}{\gamma}\overline{u}\right)\right|\right)\\
&\quad+c_\kappa\frac{1-\gamma}{\gamma}\sup_{r\le R^1_\varepsilon} r^{4-\mu}\left(\left|\Sigma^1_{\widetilde{a}_1}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right|\right.\\
&\qquad\left.+\left|\Sigma^2_{\widetilde{a}_1}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right|\right)\\
&\le c_\kappa \sup_{r\le R^1_\varepsilon} r^{4-\mu}\left(2\left|\frac{\nabla\widetilde{a}_1}{\widetilde{a}_1}\right|\left|\frac{1}{\gamma}\nabla(\Delta\overline{u})\right|+\left(\frac{\varepsilon}{\tau_1}\right)^2|\widetilde{V}_1(x)||\nabla\log\widetilde{a}_1|\left|\frac{1}{\gamma}\nabla\overline{u}\right|\right)\\
&\quad+c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^2\sup_{r\le R^1_\varepsilon} r^{4-\mu}\left|\frac{1}{\widetilde{a}_1}\left(\left(\frac{\varepsilon}{\tau_1}\right)^{-2}\Delta\widetilde{a}_1-\widetilde{V}_1(x)\widetilde{a}_1\right)\right|\left|\frac{1}{\gamma}\Delta\overline{u}\right|\\
&\quad+c_\kappa\frac{1-\gamma}{\gamma}\sup_{r\le R^1_\varepsilon} r^{4-\mu}\left[2\left|\frac{\nabla\widetilde{a}_1}{\widetilde{a}_1}\right|\left|\nabla\left(\Delta\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right)\right|\right.\\
&\qquad\left.+\left(\frac{\varepsilon}{\tau_1}\right)^2|\widetilde{V}_1(x)||\nabla\log\widetilde{a}_1|\left|\nabla\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right|\right]\\
&\quad+c_\kappa\frac{1-\gamma}{\gamma}\left(\frac{\varepsilon}{\tau_1}\right)^2\sup_{r\le R^1_\varepsilon} r^{4-\mu}\left|\frac{1}{\widetilde{a}_1}\left(\left(\frac{\varepsilon}{\tau_1}\right)^{-2}\Delta\widetilde{a}_1-\widetilde{V}_1(x)\widetilde{a}_1\right)\right|\left|\Delta\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right|.
\end{aligned}
$$

Using the fact that $a(x)$ is a solution of (1.3) satisfying (H), $V(x)$ is a smooth bounded potential and that $G_a(x,y)$ verifies $|\nabla^i G_a(x,y)|\le c|x-y|^{-i}$ for $i\ge 1$, we obtain

$$
\begin{aligned}
\sup_{r\le R^1_\varepsilon} r^{4-\mu}|\mathcal{T}_1(0,0)| &\le c_\kappa\|\nabla a\|_\infty\left(\frac{\varepsilon}{\tau_1}\right)\frac{1}{\gamma}\sup_{r\le R^1_\varepsilon} r^{4-\mu}\frac{r(3+r^2)}{(1+r^2)^3}+c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^3\|\nabla a\|_\infty\frac{1}{\gamma}\sup_{r\le R^1_\varepsilon} r^{4-\mu}\frac{r}{1+r^2}\\
&\quad+c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^2\lambda\|f\|_\infty\frac{1}{\gamma}\sup_{r\le R^1_\varepsilon} r^{4-\mu}\frac{2+r^2}{(1+r^2)^2}\\
&\quad+c_\kappa\frac{1-\gamma}{\gamma}\sup_{r\le R^1_\varepsilon} r^{4-\mu}\left[\left(\frac{\varepsilon}{\tau_1}\right)\|\nabla a\|_\infty\left(\frac{1}{\xi}|x-x^2|^{-3}+|x-x^3|^{-3}\right)\right.\\
&\qquad+\left(\frac{\varepsilon}{\tau_1}\right)^3\|\nabla a\|_\infty\left(\frac{1}{\xi}|x-x^2|^{-1}+|x-x^3|^{-1}\right)\\
&\qquad\left.+\left(\frac{\varepsilon}{\tau_1}\right)^2\lambda\|f\|_\infty\left(\frac{1}{\xi}|x-x^2|^{-2}+|x-x^3|^{-2}\right)\right].
\end{aligned}
$$

Taking into account that for r very large, we have $(1+r^2)^{-\alpha}\sim r^{-2\alpha}$ and the fact that

$\|\nabla a\|_\infty < \beta$, then we get

$$\sup_{r\le R_\varepsilon^1} r^{4-\mu}|\mathcal{T}_1(0,0)| \le c_\kappa\beta\varepsilon\frac{1}{\gamma} + c_\kappa\beta\varepsilon^\mu\frac{1}{\gamma}r_\varepsilon^{3-\mu} + c_\kappa\lambda\varepsilon^\mu\frac{1}{\gamma}r_\varepsilon^{2-\mu} + c_\kappa\varepsilon^{\mu-3}\beta\frac{1-\gamma}{\gamma}r_\varepsilon^{4-\mu}$$
$$+ c_\kappa\varepsilon^{\mu-1}\beta\frac{1-\gamma}{\gamma}r_\varepsilon^{4-\mu} + c_\kappa\varepsilon^{\mu-2}\lambda\frac{1-\gamma}{\gamma}r_\varepsilon^{4-\mu} \le c_\kappa r_\varepsilon^2.$$

Making use of Proposition 3.2 together with (3.4), for $\mu\in(1,2)$, there exists a constant c_κ such that

$$\|\mathcal{N}_1(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2.$$

For the second estimate, we have

$$\mathcal{R}_1(0,0) = \frac{24\cdot 16^{\frac{1-\xi}{\gamma}} C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}}{\gamma^{\frac{1-\xi}{\gamma}}(1+r^2)^{4\frac{1-\xi}{\gamma}}} e^{\frac{\gamma+\xi-1}{\gamma}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)}$$
$$- \Sigma_{\widetilde{a}_1}^1\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right) - \Sigma_{\widetilde{a}_1}^2\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right).$$

Then

$$\sup_{r\le R_\varepsilon^1} r^{4-\delta}|\mathcal{R}_1(0,0)|$$
$$\le \sup_{r\le R_\varepsilon^1} r^{4-\delta}24C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}}\varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}\left(\frac{16}{\gamma(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}} e^{\frac{\gamma+\xi-1}{\gamma}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)}$$
$$+ \sup_{r\le R_\varepsilon^1} r^{4-\delta}\left[\left|\Sigma_{\widetilde{a}_1}^1\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right| + \left|\Sigma_{\widetilde{a}_1}^2\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right|\right]$$
$$\le c_\kappa C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}}\varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}\sup_{r\le R_\varepsilon^1} S(r)$$
$$+ c_\kappa\sup_{r\le R_\varepsilon^1} r^{4-\delta}\left(2\left|\frac{\nabla\widetilde{a}_1}{\widetilde{a}_1}\right|\left|\nabla\left(\Delta\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right)\right|\right.$$
$$\left.+\left(\frac{\varepsilon}{\tau_1}\right)^2|\widetilde{V}_1(x)||\nabla\log\widetilde{a}_1|\left|\nabla\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right|\right)$$
$$+ c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^2\sup_{r\le R_\varepsilon^1} r^{4-\delta}\left|\frac{1}{\widetilde{a}_1}\left(\left(\frac{\varepsilon}{\tau_1}\right)^{-2}\Delta\widetilde{a}_1 - \widetilde{V}_1(x)\widetilde{a}_1\right)\right|\left|\Delta\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)\right|,$$

where $S(r) = \frac{r^{4-\delta}}{(1+r^2)^{4\frac{1-\xi}{\gamma}}}$. Then we use the fact that $\|\nabla a\|_\infty < \beta$ and $|\nabla^i G_a(x,y)| \le c|x-y|^{-i}$ for $i\ge 1$, we get

$$\sup_{r\le R_\varepsilon^1} r^{4-\delta}|\mathcal{R}_1(0,0)|$$
$$\le c_\kappa C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}}\varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}\sup_{r\le R_\varepsilon^1} S(r) + c_\kappa\|\nabla a\|_\infty\left(\frac{\varepsilon}{\tau_1}\right)\sup_{r\le R_\varepsilon^1} r^{4-\delta}\left(\frac{1}{\xi}|x-x^2|^{-3}+|x-x^3|^{-3}\right)$$
$$+ c_\kappa\|\nabla a\|_\infty\left(\frac{\varepsilon}{\tau_1}\right)^3\sup_{r\le R_\varepsilon^1} r^{4-\delta}\left(\frac{1}{\xi}|x-x^2|^{-1}+|x-x^3|^{-1}\right)$$

$$+ c_\kappa \left(\frac{\varepsilon}{\tau_1}\right)^2 \lambda \|f\|_\infty \sup_{r \le R_\varepsilon^1} r^{4-\delta} \left(\frac{1}{\xi}|x - x^2|^{-2} + |x - x^3|^{-2}\right).$$

If $4 - \delta + 8(\xi - 1)\gamma^{-1} \le 0$, then S is bounded on $\mathbb{R}_+$. If $4 - \delta + 8(\xi - 1)\gamma^{-1} > 0$, then $\sup_{r \le R_\varepsilon^1} S(r) = S\left(\frac{r_\varepsilon}{\varepsilon}\right)$. Then we obtain

$$\begin{aligned}\sup_{r \le R_\varepsilon^1} r^{4-\delta} |\mathcal{R}_1(0,0)| &\le c_\kappa \max\left\{\varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}, \varepsilon^{4+\delta} r_\varepsilon^{4-\delta+8(\xi-1)\gamma^{-1}}\right\} \\ &\quad + c_\kappa \beta \varepsilon^{\delta-3} r_\varepsilon^{4-\delta} + c_\kappa \beta \varepsilon^{\delta-1} r_\varepsilon^{4-\delta} + c_\kappa \lambda \varepsilon^{\delta-2} r_\varepsilon^{4-\delta} \\ &\le c_\kappa r_\varepsilon^2.\end{aligned}$$

Using the same argument as above, we get

$$\|\mathcal{M}_1(0,0)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2.$$

Let (h_1^1, h_2^1), (k_1^1, k_2^1) in $B(0, 2c_\kappa r_\varepsilon^2)$ of $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$, then we have

$$\begin{aligned}&\sup_{r \le R_\varepsilon^1} r^{4-\mu} \left|\mathcal{T}_1(h_1^1, h_2^1) - \mathcal{T}_1(k_1^1, k_2^1)\right| \\ &\le \sup_{r \le R_\varepsilon^1} \frac{r^{4-\mu}}{\gamma(1+r^2)^4} \left|e^{\gamma h_1^1 + (1-\gamma) h_2^1} - e^{\gamma k_1^1 + (1-\gamma) k_2^1} - \gamma(h_1^1 - k_1^1)\right| \\ &\quad + \sup_{r \le R_\varepsilon^1} r^{4-\mu} \left|\Sigma_{\widetilde{a}_1}^1 (h_1^1 - k_1^1)\right| + \sup_{r \le R_\varepsilon^1} r^{4-\mu} \left|\Sigma_{\widetilde{a}_1}^2 (h_1^1 - k_1^1)\right| \\ &\le c_\kappa \sup_{r \le R_\varepsilon^1} \frac{r^{4-\mu}}{\gamma(1+r^2)^4} \left(\gamma^2 |(h_1^1)^2 - (k_1^1)^2| + (1-\gamma)|h_2^1 - k_2^1|\right) \\ &\quad + c_\kappa \sup_{r \le R_\varepsilon^1} r^{4-\mu} \left(2\left|\frac{\nabla \widetilde{a}_1}{\widetilde{a}_1}\right| \left|\nabla(\Delta(h_1^1 - k_1^1))\right| + \left(\frac{\varepsilon}{\tau_1}\right)^2 \left|\widetilde{V}_1(x)\right| \left|\nabla \log \widetilde{a}_1\right| \left|\nabla(h_1^1 - k_1^1)\right|\right) \\ &\quad + c_\kappa \left(\frac{\varepsilon}{\tau_1}\right)^2 \sup_{r \le R_\varepsilon^1} r^{4-\mu} \left|\frac{1}{\widetilde{a}_1}\left(\left(\frac{\varepsilon}{\tau_1}\right)^{-2} \Delta \widetilde{a}_1 - \widetilde{V}_1(x) \widetilde{a}_1\right)\right| \left|\Delta(h_1^1 - k_1^1)\right|.\end{aligned}$$

Recall that for all functions w in $\mathcal{C}_\mu^{k,\alpha}(\mathbb{R}^4)$ bounded by a constant times $(1+r^2)^{\mu/2}$, have their ℓ-th partial derivatives that are bounded by $(1+r^2)^{(\mu-\ell)/2}$, for $\ell = 1, \ldots, k+\alpha, \ldots$ (a.e. $|\nabla^\ell w| \le c_\kappa r^{\mu-\ell} \|w\|_{\mathcal{C}_\mu^{k,\alpha}(\mathbb{R}^4)}$, $(1+r^2)^{(\mu-\ell)/2} \sim r^{\mu-\ell}$ for r very large), then there exists a constant $c_\kappa > 0$, using the fact that $a(x)$ is a solution of (1.3) satisfying (H) and $V(x)$ is a smooth bounded potential, we deduce that

$$\begin{aligned}&\sup_{r \le R_\varepsilon^1} r^{4-\mu} \left|\mathcal{T}_1(h_1^1, h_2^1) - \mathcal{T}_1(k_1^1, k_2^1)\right| \\ &\le c_\kappa \sup_{r \le R_\varepsilon^1} \frac{r^{4-\mu}}{\gamma(1+r^2)^4} \left[\gamma^2 r^{2\mu} \left(\|h_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + \|k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\right) \|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \right. \\ &\quad \left. + (1-\gamma) r^\delta \|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}\right]\end{aligned}$$

$$
\begin{aligned}
&+ c_\kappa\beta\varepsilon R^1_\varepsilon\|h^1_1-k^1_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa\beta\varepsilon^3(R^1_\varepsilon)^3\|h^1_1-k^1_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\\
&+ c_\kappa\lambda\varepsilon^2(R^1_\varepsilon)^2\|h^1_1-k^1_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\\
\le{}& c_\kappa r_\varepsilon^2\|h^1_1-k^1_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|h^1_2-k^1_2\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} + c_\kappa\beta r_\varepsilon\|h^1_1-k^1_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\\
&+ c_\kappa\beta r_\varepsilon^3\|h^1_1-k^1_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa\lambda r_\varepsilon^2\|h^1_1-k^1_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}
$$

Provided $h^1_i, k^1_i \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$, $i=1,2$, satisfies (3.9) and making use of Proposition 3.2 together with (3.4), we conclude that

$$
\begin{aligned}
&\|\mathcal{N}_1(h^1_1,h^1_2)-\mathcal{N}_1(k^1_1,k^1_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\\
\le{}& c_\kappa r_\varepsilon^2\|h^1_1-k^1_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|h^1_2-k^1_2\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}\\
\le{}& \max\left\{c_\kappa r_\varepsilon^2, c_\kappa(1-\gamma)\right\}\|(h^1_1,h^1_2)-(k^1_1,k^1_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}.
\end{aligned}
$$

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \le 1/2$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. There exists a constant $\gamma_0\in(0,1)$ such that $c_\kappa(1-\gamma)\le 1/2$ for all $\gamma\in(\gamma_0,1)$, we can find a constant $\overline{c}_\kappa>0$ such that

$$
\|\mathcal{N}_1(h^1_1,h^1_2)-\mathcal{N}_1(k^1_1,k^1_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le \overline{c}_\kappa r_\varepsilon^2\|(h^1_1,h^1_2)-(k^1_1,k^1_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}. \tag{3.10}
$$

Similarly we get the estimate for $\mathcal{M}_1$, then

$$
\begin{aligned}
&\sup_{r\le R^1_\varepsilon} r^{4-\delta}\left|\mathcal{R}_1(h^1_1,h^1_2)-\mathcal{R}_1(k^1_1,k^1_2)\right|\\
\le{}& \sup_{r\le R^1_\varepsilon}\frac{24\cdot 16^{\frac{1-\xi}{\gamma}}C_{1,\varepsilon}^{4^{\frac{\gamma+\xi-1}{\gamma}}}\varepsilon^{8^{\frac{\gamma+\xi-1}{\gamma}}}}{\gamma^{\frac{1-\xi}{\gamma}}(1+r^2)^{4^{\frac{1-\xi}{\gamma}}}} r^{4-\delta}e^{\frac{\gamma+\xi-1}{\gamma}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)}\\
&\quad\times\left|e^{\xi h^1_2+(1-\xi)h^1_1}-e^{\xi k^1_2+(1-\xi)k^1_1}\right|\\
&+\sup_{r\le R^1_\varepsilon} r^{4-\delta}\left|\Sigma^1_{\widetilde{a}_1}(h^1_2-k^1_2)\right| + \sup_{r\le R^1_\varepsilon} r^{4-\delta}\left|\Sigma^2_{\widetilde{a}_1}(h^1_2-k^1_2)\right|\\
\le{}& c_\kappa 16^{\frac{1-\xi}{\gamma}}\gamma^{-\frac{1-\xi}{\gamma}}C_{1,\varepsilon}^{4^{\frac{\gamma+\xi-1}{\gamma}}}\varepsilon^{8^{\frac{\gamma+\xi-1}{\gamma}}}\sup_{r\le R^1_\varepsilon}\frac{r^{4-\delta}}{(1+r^2)^{4^{\frac{1-\xi}{\gamma}}}}\left(\xi|h^1_2-k^1_2|+(1-\xi)|h^1_1-k^1_1|\right)\\
&+c_\kappa\sup_{r\le R^1_\varepsilon} r^{4-\delta}\left(2\left|\frac{\nabla\widetilde{a}_1}{\widetilde{a}_1}\right|\left|\nabla(\Delta(h^1_2-k^1_2))\right|+\left(\frac{\varepsilon}{\tau_1}\right)^2\left|\widetilde{V}_1(x)\right|\left|\nabla\log\widetilde{a}_1\right|\left|\nabla(h^1_2-k^1_2)\right|\right)\\
&+c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^2\sup_{r\le R^1_\varepsilon} r^{4-\delta}\left|\frac{1}{\widetilde{a}_1}\left(\left(\frac{\varepsilon}{\tau_1}\right)^{-2}\Delta\widetilde{a}_1-\widetilde{V}_1(x)\widetilde{a}_1\right)\right|\left|\Delta(h^1_2-k^1_2)\right|\\
\le{}& c_\kappa 16^{\frac{1-\xi}{\gamma}}\gamma^{-\frac{1-\xi}{\gamma}}C_{1,\varepsilon}^{4^{\frac{\gamma+\xi-1}{\gamma}}}\varepsilon^{8^{\frac{\gamma+\xi-1}{\gamma}}}\\
&\times\sup_{r\le R^1_\varepsilon}\frac{r^{4-\delta}}{(1+r^2)^{4^{\frac{1-\xi}{\gamma}}}}\left(\xi r^\delta\|h^1_2-k^1_2\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}+(1-\xi)r^\mu\|h^1_1-k^1_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\right)\\
&+c_\kappa\beta r_\varepsilon\|h^1_2-k^1_2\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}+c_\kappa\beta r_\varepsilon^3\|h^1_2-k^1_2\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}+c_\kappa\lambda r_\varepsilon^2\|h^1_2-k^1_2\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}.
\end{aligned}
$$

We conclude that

$$\|\mathcal{M}_1(h_1^1,h_2^1)-\mathcal{M}_1(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|(h_1^1,h_2^1)-(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}. \qquad \square \tag{3.11}$$

Therefore, (3.10) and (3.11) are enough to show that

$$(h_1^1,h_2^1)\mapsto \big(\mathcal{N}_1(h_1^1,h_2^1),\mathcal{M}_1(h_1^1,h_2^1)\big)$$

is a contraction from the ball

$$\big\{(h_1^1,h_2^1)\in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4) : \|(h_1^1,h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2\big\}$$

into itself and hence a unique fixed point (h_1^1,h_2^1) exists in this set. This fixed point is a solution of (3.8). Then we have

Proposition 3.7. *Let $\mu\in(1,2)$ and $\delta\in\big(0,\min\big\{1,\frac{\gamma+\xi-1}{\gamma},\frac{\gamma+\xi-1}{\xi}\big\}\big)$. Given $\kappa>0$, there exist $\varepsilon_\kappa>0$, $\lambda_\kappa>0$, $\beta_\kappa>0$, $c_\kappa>0$ (which can depend only on κ) and $\gamma_0\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\lambda\in(0,\lambda_\kappa)$, $\beta\in(0,\beta_\kappa)$ and $\gamma\in(\gamma_0,1)$, there exists a unique $(h_1^1,h_2^1)\in\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ solution of (3.8) such that*

$$\begin{aligned} v_1(x) &:= \frac{1}{\gamma}\overline{u}(x-x^1)-\frac{1-\gamma}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)-\frac{1-\gamma}{\gamma\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)-\frac{\ln\gamma}{\gamma}+h_1^1(x),\\ v_2(x) &:= \frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)+h_2^1(x)\end{aligned}$$

solves (3.6) in $B_{R_\varepsilon^1}(x^1)$. In addition,

$$\|(h_1^1,h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

- In $B_{R_\varepsilon^2}(x^2)$, using the following transformation

$$v_1(x)=u_1\left(\frac{\varepsilon}{\tau_2}x\right) \quad\text{and}\quad v_2(x)=u_2\left(\frac{\varepsilon}{\tau_2}x\right)+\frac{8}{\xi}\ln\varepsilon-\frac{4}{\xi}\ln\left(\frac{\tau_2(1+\varepsilon^2)}{2}\right),$$

the system (3.5) can be written as

$$\begin{aligned} &\Delta^2 v_1+\Sigma^1_{\widetilde{a}_2}v_1+\Sigma^2_{\widetilde{a}_2}v_1 = 24\frac{2^{4\left(\frac{\gamma+\xi-1}{\xi}\right)}\varepsilon^{8\left(\frac{\gamma+\xi-1}{\xi}\right)}}{\big(\tau_2(1+\varepsilon^2)\big)^{4\left(\frac{\gamma+\xi-1}{\xi}\right)}}e^{\gamma v_1+(1-\gamma)v_2} && \text{in } B_{R_\varepsilon^2}(x^2),\\ &\Delta^2 v_2+\Sigma^1_{\widetilde{a}_2}v_2+\Sigma^2_{\widetilde{a}_2}v_2 = 24e^{\xi v_2+(1-\xi)v_1} && \text{in } B_{R_\varepsilon^2}(x^2)\end{aligned} \tag{3.12}$$

with

$$\begin{aligned}\Sigma^1_{\widetilde{a}_2}v_i &= 2\frac{\nabla\widetilde{a}_2}{\widetilde{a}_2}\cdot\nabla(\Delta v_i)-\widetilde{V}_2(x)\left(\frac{\varepsilon}{\tau_2}\right)^2\nabla\log\widetilde{a}_2\cdot\nabla v_i,\\ \Sigma^2_{\widetilde{a}_2}v_i &= \left(\frac{\Delta\widetilde{a}_2}{\widetilde{a}_2}-\left(\frac{\varepsilon}{\tau_2}\right)^2\widetilde{V}_2(x)\right)\Delta v_i \quad \text{for } i=1,2,\end{aligned}$$

where $\widetilde{a}_2(x) = a\left(\frac{\varepsilon}{\tau_2}x\right)$, $\widetilde{V}_2(x) = V\left(\frac{\varepsilon}{\tau_2}x\right)$. Here $\tau_2 > 0$ is a constant which will be fixed later. We look for a solution of (3.12) of the form

$$\begin{aligned} v_1(x) &= \frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_1^2(x), \\ v_2(x) &= \frac{1}{\xi}\overline{u}(x - x^2) - \frac{1-\xi}{\xi}G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) - \frac{1-\xi}{\gamma\xi}G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{\ln\xi}{\xi} + h_2^2(x), \end{aligned}$$

this amounts to solve the system

$$\begin{aligned} \Delta^2 h_1^2 &= \frac{24\cdot 16^{\frac{1-\gamma}{\xi}} C_{2,\varepsilon}^{4\frac{\gamma+\xi-1}{\xi}} \varepsilon^{8\frac{\gamma+\xi-1}{\xi}}}{\xi^{\frac{1-\gamma}{\xi}}(1+r^2)^{4\frac{1-\gamma}{\xi}}} e^{\gamma h_1^2 + (1-\gamma)h_2^2 + \frac{\gamma+\xi-1}{\xi}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_1}, x^1\right) + G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right)\right)} \\ &\quad - \Sigma^1_{\widetilde{a}_1} v_1 - \Sigma^2_{\widetilde{a}_1} v_1, \\ \mathbb{L}h_2^2 &= \frac{384}{\xi(1+r^2)^4}\left[e^{\xi h_2^2 + (1-\xi)h_1^2} - \xi h_2^2 - 1\right] - \Sigma^1_{\widetilde{a}_1} v_2 - \Sigma^2_{\widetilde{a}_1} v_2 \end{aligned} \tag{3.13}$$

in $B_{R_\varepsilon^2}(x^2)$, where $C_{2,\varepsilon} = \frac{2}{\tau_2(1+\varepsilon^2)}$. We denote by

$$\Delta^2 h_1^2 = \mathcal{T}_2(h_1^2, h_2^2) \quad \text{and} \quad \mathbb{L}h_2^2 = \mathcal{R}_2(h_1^2, h_2^2).$$

To find a solution of (3.13), it is enough to find a fixed point (h_1^2, h_2^2) in a small ball of $\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ solutions of

$$\begin{aligned} h_1^2 &= \mathcal{K}_\delta \circ \xi_{\delta, R_\varepsilon^2} \circ \mathcal{T}_2(h_1^2, h_2^2) = \mathcal{N}_2(h_1^2, h_2^2), \\ h_2^2 &= \mathcal{G}_\mu \circ \xi_{\mu, R_\varepsilon^2} \circ \mathcal{R}_2(h_1^2, h_2^2) = \mathcal{M}_2(h_1^2, h_2^2). \end{aligned} \tag{3.14}$$

Then, we have the following result.

Lemma 3.8. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ and $\overline{c}_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$ with $r_\varepsilon := r_{\varepsilon,\beta,\lambda}$. We have*

$$\begin{gathered} \|\mathcal{N}_2(0,0)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \quad \|\mathcal{M}_2(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \\ \|\mathcal{N}_2(h_1^2, h_2^2) - \mathcal{N}_2(k_1^2, k_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|(h_1^2, h_2^2) - (k_1^2, k_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \end{gathered}$$

and

$$\|\mathcal{M}_2(h_1^2, h_2^2) - \mathcal{M}_2(k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le \overline{c}_\kappa r_\varepsilon^2 \|(h_1^2, h_2^2) - (k_1^2, k_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)},$$

provided $(h_1^2, h_2^2), (k_1^2, k_2^2) \in \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ satisfying

$$\|(h_1^2, h_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(k_1^2, k_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Proof. The proof is similar to Lemma 3.6, in fact we have only to respect the symmetry of systems obtained and their appropriate spaces in $B_{R^1_\varepsilon}(x^1)$ and $B_{R^2_\varepsilon}(x^2)$. We deduce that

$$\|\mathcal{N}_2(h^2_1, h^2_2) - \mathcal{N}_2(k^2_1, k^2_2)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leq c_\kappa r^2_\varepsilon \|(h^2_1, h^2_2) - (k^2_1, k^2_2)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}. \tag{3.15}$$

Following the same arguments as the first case, we prove the second estimate

$$\|\mathcal{M}_2(h^2_1, h^2_2) - \mathcal{M}_2(k^2_1, k^2_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq \overline{c}_\kappa r^2_\varepsilon \|(h^2_1, h^2_2) - (k^2_1, k^2_2)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}. \tag{3.16}$$ □

Therefore, (3.15) and (3.16) are enough to show that

$$(h^2_1, h^2_2) \mapsto \big(\mathcal{N}_2(h^2_1, h^2_2), \mathcal{M}_2(h^2_1, h^2_2)\big)$$

is a contraction from the ball

$$\big\{(h^2_1, h^2_2) \in \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) : \|(h^2_1, h^2_2)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq 2c_\kappa r^2_\varepsilon\big\}$$

into itself and hence a unique fixed point (h^2_1, h^2_2) exists in this set, which is a solution of (3.14). We summarize this in the following proposition.

Proposition 3.9. *Given $\kappa > 0$, $\mu \in (1,2)$ and $\delta \in \big(0, \min\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\}\big)$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ (only depending on κ) and $\xi_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$ and $\xi \in (\xi_0, 1)$, there exists a unique $(h^2_1, h^2_2) \in \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ solution of* (3.14) *such that*

$$\begin{aligned} v_1(x) &= \frac{1}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h^2_1(x), \\ v_2(x) &= \frac{1}{\xi}\overline{u}(x - x^2) - \frac{1-\xi}{\xi} G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) - \frac{1-\xi}{\gamma\xi} G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{\ln\xi}{\xi} + h^2_2(x) \end{aligned}$$

solves (3.12) *in $B_{R^2_\varepsilon}(x^2)$. In addition,*

$$\|(h^2_1, h^2_2)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq 2c_\kappa r^2_\varepsilon.$$

- In $B_{R^3_\varepsilon}(x^3)$, using the following transformation

$$\begin{aligned} v_1(x) &= u_1\left(\frac{\varepsilon}{\tau_3}x\right) + 8\ln\varepsilon - 4\ln\left(\frac{\tau_3(1+\varepsilon^2)}{2}\right), \\ v_2(x) &= u_2\left(\frac{\varepsilon}{\tau_3}x\right) + 8\ln\varepsilon - 4\ln\left(\frac{\tau_3(1+\varepsilon^2)}{2}\right), \end{aligned}$$

the system (3.5) can be written as

$$\begin{aligned} \Delta^2 v_1 + \Sigma^1_{\tilde{a}_3} v_1 + \Sigma^2_{\tilde{a}_3} v_1 &= 24 e^{\gamma v_1 + (1-\gamma)v_2} && \text{in } B_{R^3_\varepsilon}(x^3), \\ \Delta^2 v_2 + \Sigma^1_{\tilde{a}_3} v_2 + \Sigma^2_{\tilde{a}_3} v_2 &= 24 e^{\xi v_2 + (1-\xi)v_1} && \text{in } B_{R^3_\varepsilon}(x^3) \end{aligned} \tag{3.17}$$

with

$$\Sigma^1_{\widetilde{a}_3} v_i = 2\frac{\nabla \widetilde{a}_3}{\widetilde{a}_3}\cdot\nabla(\Delta v_i) - \widetilde{V}_3(x)\left(\frac{\varepsilon}{\tau_3}\right)^2 \nabla\log\widetilde{a}_3\cdot\nabla v_i,$$
$$\Sigma^2_{\widetilde{a}_3} v_i = \left(\frac{\Delta\widetilde{a}_3}{\widetilde{a}_3} - \left(\frac{\varepsilon}{\tau_3}\right)^2 \widetilde{V}_3(x)\right)\Delta v_i \quad \text{for } i = 1, 2,$$

where $\widetilde{a}_3(x) = a\big(\frac{\varepsilon}{\tau_3}x\big)$, $\widetilde{V}_3(x) = V\big(\frac{\varepsilon}{\tau_3}x\big)$. Here $\tau_3 > 0$ is a constant which will be fixed later.

We look for a solution of (3.17) of the form

$$v_1(x) = \overline{u}(x - x^3) + h^3_1(x),$$
$$v_2(x) = \overline{u}(x - x^3) + h^3_2(x),$$

this amounts to solve the system

$$\begin{aligned} \mathbb{L}h^3_1 &= \frac{384}{(1+r^2)^4}\big[e^{\gamma h^3_1+(1-\gamma)h^3_2} - h^3_1 - 1\big] - \Sigma^1_{\widetilde{a}_3}(\overline{u}+h^3_1) - \Sigma^2_{\widetilde{a}_3}(\overline{u}+h^3_1), \\ \mathbb{L}h^3_2 &= \frac{384}{(1+r^2)^4}\big[e^{\xi h^3_2+(1-\xi)h^3_1} - h^3_2 - 1\big] - \Sigma^1_{\widetilde{a}_3}(\overline{u}+h^3_2) - \Sigma^2_{\widetilde{a}_3}(\overline{u}+h^3_2) \end{aligned} \tag{3.18}$$

in $B_{R^3_\varepsilon}(x^3)$. We denote by

$$\mathbb{L}h^3_1 = \mathcal{T}_3(h^3_1, h^3_2) \quad \text{and} \quad \mathbb{L}h^3_2 = \mathcal{R}_3(h^3_1, h^3_2).$$

To find a solution of (3.18), it is enough to find a fixed point (h^3_1, h^3_2) in a small ball of $\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ solutions of

$$\begin{aligned} h^3_1 &= \mathcal{G}_\mu\circ\xi_{\mu,R^3_\varepsilon}\circ\mathcal{T}_3(h^3_1,h^3_2) = \mathcal{N}_3(h^3_1,h^3_2), \\ h^3_2 &= \mathcal{G}_\mu\circ\xi_{\mu,R^3_\varepsilon}\circ\mathcal{R}_3(h^3_1,h^3_2) = \mathcal{M}_3(h^3_1,h^3_2). \end{aligned} \tag{3.19}$$

Then, we have the following result.

Lemma 3.10. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$, $\overline{c}_\kappa > 0$ and $\overline{\overline{c}}_\kappa > 0$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\lambda\in(0,\lambda_\kappa)$, $\beta\in(0,\beta_\kappa)$ and $\mu\in(1,2)$ with $r_\varepsilon := r_{\varepsilon,\beta,\lambda}$. We have*

$$\|\mathcal{N}_3(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r^2_\varepsilon, \quad \|\mathcal{M}_3(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r^2_\varepsilon,$$
$$\|\mathcal{N}_3(h^3_1,h^3_2) - \mathcal{N}_3(k^3_1,k^3_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le \overline{c}_\kappa r^2_\varepsilon\|(h^3_1,h^3_2)-(k^3_1,k^3_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}$$

and

$$\|\mathcal{M}_3(h^3_1,h^3_2) - \mathcal{M}_3(k^3_1,k^3_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le \overline{\overline{c}}_\kappa r^2_\varepsilon\|(h^3_1,h^3_2)-(k^3_1,k^3_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)},$$

provided $(h^3_1,h^3_2), (k^3_1,k^3_2)\in\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ satisfying

$$\|(h^3_1,h^3_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r^2_\varepsilon, \quad \|(k^3_1,k^3_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r^2_\varepsilon. \tag{3.20}$$

Proof. We have

$$
\begin{aligned}
&\sup_{r\le R_\varepsilon^3} r^{4-\mu}|\mathcal{T}_3(0,0)|\\
&\le \sup_{r\le R_\varepsilon^3} r^{4-\mu}\big(\big|\Sigma^1_{\widetilde{a}_3}(\overline{u})\big|+\big|\Sigma^2_{\widetilde{a}_3}(\overline{u})\big|\big)\\
&\le c_\kappa \sup_{r\le R_\varepsilon^3} r^{4-\mu}\left(2\left|\frac{\nabla\widetilde{a}_3}{\widetilde{a}_3}\right|\big|\nabla(\Delta\overline{u})\big|+\left(\frac{\varepsilon}{\tau_3}\right)^2\big|\widetilde{V}_3(x)\big|\big|\nabla\log\widetilde{a}_3\big|\big|\nabla\overline{u}\big|\right)\\
&\quad + c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^2\sup_{r\le R_\varepsilon^3} r^{4-\mu}\left|\frac{1}{\widetilde{a}_3}\left(\left(\frac{\varepsilon}{\tau_3}\right)^{-2}\Delta\widetilde{a}_3-\widetilde{V}_3(x)\widetilde{a}_3\right)\right|\big|\Delta\overline{u}\big|\\
&\le c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)\|\nabla a\|_\infty \sup_{r\le R_\varepsilon^3} r^{4-\mu}\frac{r(3+r^2)}{(1+r^2)^3}+c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^3\|\nabla a\|_\infty\sup_{r\le R_\varepsilon^3} r^{4-\mu}\frac{r}{1+r^2}\\
&\quad + c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^2\lambda\|f\|_\infty\sup_{r\le R_\varepsilon^3} r^{4-\mu}\frac{2+r^2}{(1+r^2)^2}\\
&\le c_\kappa\beta\varepsilon+c_\kappa\beta\varepsilon^\mu r_\varepsilon^{3-\mu}+c_\kappa\lambda\varepsilon^\mu r_\varepsilon^{2-\mu}\le c_\kappa r_\varepsilon^2.
\end{aligned}
$$

Making use of Proposition 3.2 together with (3.4), for $\mu\in(1,2)$, there exists a constant c_κ such that

$$\|\mathcal{N}_3(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2.$$

For the second estimate, we use the same argument as above and we get

$$\|\mathcal{M}_3(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2.$$

To derive the third estimate, for (h_1^3,h_2^3), (k_1^3,k_2^3) verifying (3.20), we have

$$
\begin{aligned}
&\sup_{r\le R_\varepsilon^3} r^{4-\mu}\big|\mathcal{T}_3(h_1^3,h_2^3)-\mathcal{T}_3(k_1^3,k_2^3)\big|\\
&\le \sup_{r\le R_\varepsilon^3}\frac{384r^{4-\mu}}{(1+r^2)^4}\big|\big(e^{\gamma h_1^3+(1-\gamma)h_2^3}-h_1^3-1\big)-\big(e^{\gamma k_1^3+(1-\gamma)k_2^3}-k_1^3-1\big)\big|\\
&\quad+\sup_{r\le R_\varepsilon^3} r^{4-\mu}\big|\Sigma^1_{\widetilde{a}_3}(h_1^3-k_1^3)\big|+\sup_{r\le R_\varepsilon^3} r^{4-\mu}\big|\Sigma^2_{\widetilde{a}_3}(h_1^3-k_1^3)\big|\\
&\le c_\kappa\sup_{r\le R_\varepsilon^3}\frac{384r^{4-\mu}}{(1+r^2)^4}\big|(\gamma-1)(h_1^3-k_1^3)+(1-\gamma)(h_2^3-k_2^3)\big|\\
&\quad+c_\kappa\sup_{r\le R_\varepsilon^3} r^{4-\mu}\left(2\left|\frac{\nabla\widetilde{a}_3}{\widetilde{a}_3}\right|\big|\nabla(\Delta(h_1^3-k_1^3))\big|+\left(\frac{\varepsilon}{\tau_3}\right)^2\big|\widetilde{V}_3(x)\big|\big|\nabla\log\widetilde{a}_3\big|\big|\nabla(h_1^3-k_1^3)\big|\right)\\
&\quad+c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^2\sup_{r\le R_\varepsilon^3} r^{4-\mu}\left|\frac{1}{\widetilde{a}_3}\left(\left(\frac{\varepsilon}{\tau_3}\right)^{-2}\Delta\widetilde{a}_3-\widetilde{V}_3(x)\widetilde{a}_3\right)\right|\big|\Delta(h_1^3-k_1^3)\big|\\
&\le c_\kappa\sup_{r\le R_\varepsilon^3}\frac{384r^{4-\mu}}{(1+r^2)^4}(1-\gamma)\big[r^\mu\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}+r^\mu\|h_2^3-k_2^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\big]\\
&\quad+c_\kappa\beta r_\varepsilon\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}+c_\kappa\beta r_\varepsilon^3\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}+c_\kappa\lambda r_\varepsilon^2\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}
$$

Making use of Proposition 3.2 together with (3.4), we conclude that

$$\begin{aligned}&\|\mathcal{N}_3(h_1^3,h_2^3)-\mathcal{N}_3(k_1^3,k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\\ &\leq \big(c_\kappa r_\varepsilon^2+c_\kappa(1-\gamma)\big)\|h_1^3-k_1^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}+c_\kappa(1-\gamma)\|h_2^3-k_2^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\\ &\leq \big(c_\kappa r_\varepsilon^2+c_\kappa(1-\gamma)\big)\|(h_1^3,h_2^3)-(k_1^3,k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}.\end{aligned}$$

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2\leq 1/2$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. There exists a constant $\gamma_0\in(0,1)$ such that $c_\kappa(1-\gamma)\leq 1/2$ for all $\gamma\in(\gamma_0,1)$, we can find a constant $\overline{c}_\kappa>0$ such that

$$\|\mathcal{N}_3(h_1^3,h_2^3)-\mathcal{N}_3(k_1^3,k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\leq \overline{c}_\kappa r_\varepsilon^2\|(h_1^3,h_2^3)-(k_1^3,k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{3.21}$$

Similarly, we get

$$\begin{aligned}&\|\mathcal{M}_3(h_1^3,h_2^3)-\mathcal{M}_3(k_1^3,k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\\ &\leq \big(c_\kappa r_\varepsilon^2+c_\kappa(1-\xi)\big)\|h_2^3-k_2^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}+c_\kappa(1-\xi)\|h_1^3-k_1^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\\ &\leq \big(c_\kappa r_\varepsilon^2+c_\kappa(1-\xi)\big)\|(h_1^3,h_2^3)-(k_1^3,k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}.\end{aligned}$$

Then, reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2\leq 1/2$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. There exists a constant $\xi_0\in(0,1)$ such that $c_\kappa(1-\xi)\leq 1/2$ for all $\xi\in(\xi_0,1)$, we can find a constant $\overline{\overline{c}}_\kappa>0$ such that

$$\|\mathcal{M}_3(h_1^3,h_2^3)-\mathcal{M}_3(k_1^3,k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\leq \overline{\overline{c}}_\kappa r_\varepsilon^2\|(h_1^3,h_2^3)-(k_1^3,k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{3.22}$$

□

Therefore, (3.21) and (3.22) are enough to show that

$$(h_1^3,h_2^3)\mapsto\big(\mathcal{N}_3(h_1^3,h_2^3),\mathcal{M}_3(h_1^3,h_2^3)\big)$$

is a contraction from the ball

$$\big\{(h_1^3,h_2^3)\in\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4):\|(h_1^3,h_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\leq 2c_\kappa r_\varepsilon^2\big\}$$

into itself and hence a unique fixed point (h_1^3,h_2^3) exists in this set, which is a solution of (3.19). Then we have

Proposition 3.11. *Given $\kappa>0$, $\mu\in(1,2)$, there exist $\varepsilon_\kappa>0$, $\lambda_\kappa>0$, $\beta_\kappa>0$, $c_\kappa>0$ (which depend only on κ), $\gamma_0\in(0,1)$ and $\xi_0\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\lambda\in(0,\lambda_\kappa)$, $\beta\in(0,\beta_\kappa)$, $\gamma\in(\gamma_0,1)$ and $\xi\in(\xi_0,1)$ there exists a unique $(h_1^3,h_2^3)\in\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ solution of (3.19) such that*

$$\begin{aligned}v_1(x)&:=\overline{u}(x-x^3)+h_1^3(x),\\ v_2(x)&:=\overline{u}(x-x^3)+h_2^3(x)\end{aligned}$$

solves (3.17) in $B_{R_\varepsilon^3}(x^3)$. In addition,

$$\|(h_1^3,h_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\leq 2c_\kappa r_\varepsilon^2.$$

3.1.3. Bi-harmonic extensions

Next, we will study the properties of interior and exterior bi-harmonic extensions. Given $(\varphi, \psi), (\widetilde{\varphi}, \widetilde{\psi}) \in \mathcal{C}^{4,\alpha}(S^3) \times \mathcal{C}^{2,\alpha}(S^3)$, where S^3 is the three-dimensional unit sphere, we define respectively $H^{\text{int}} = H^{\text{int}}(\varphi, \psi \cdot) = H^{\text{int}}_{\varphi,\psi}$ to be the solution of

$$\begin{cases} \Delta^2 H^{\text{int}} = 0 & \text{in } B_1(0), \\ H^{\text{int}} = \varphi & \text{on } \partial B_1(0), \\ \Delta H^{\text{int}} = \psi & \text{on } \partial B_1(0) \end{cases}$$

and $H^{\text{ext}} = H^{\text{ext}}(\widetilde{\varphi}, \widetilde{\psi} \cdot) = H^{\text{ext}}_{\widetilde{\varphi},\widetilde{\psi}}$ to be the solution of

$$\begin{cases} \Delta^2 H^{\text{ext}} = 0 & \text{in } \mathbb{R}^4 \setminus B_1(0), \\ H^{\text{ext}} = \widetilde{\varphi} & \text{on } \partial B_1(0), \\ \Delta H^{\text{ext}} = \widetilde{\psi} & \text{on } \partial B_1(0), \end{cases}$$

which decays at infinity. We will use also

Definition 3.12. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the space $\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1(0))$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\text{loc}}(\mathbb{R}^4 - B_1(0))$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1(0))} = \sup_{r \geq 1} \left(r^{-\nu} \|w(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}_\nu(\overline{B}_2(0) - B_1(0))} \right)$$

is finite.

We denote by $e_1, \ldots, e_4$ the coordinate functions on S^3.

Lemma 3.13. [2] *Assume that*

$$\int_{S^3} (8\varphi - \psi)\, dv_{S^3} = 0 \quad \textit{and} \quad \int_{S^3} (12\varphi - \psi) e_\ell \, dv_{S^3} = 0 \quad \textit{for } \ell = 1, \ldots, 4. \tag{3.23}$$

Then there exists a constant $c > 0$ such that

$$\|H^{\text{int}}_{\varphi,\psi}\|_{\mathcal{C}^{4,\alpha}_2(\overline{B}^*_1(0))} \leq c\big(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}\big).$$

Moreover, there exists a constant $c > 0$ such that if

$$\int_{S^3} \widetilde{\psi}\, dv_{S^3} = 0, \tag{3.24}$$

then

$$\|H^{\text{ext}}_{\widetilde{\varphi},\widetilde{\psi}}\|_{\mathcal{C}^{4,\alpha}_{-1}(\mathbb{R}^4 - B_1(0))} \leq c\big(\|\widetilde{\varphi}\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\widetilde{\psi}\|_{\mathcal{C}^{2,\alpha}(S^3)}\big).$$

If $F \subset L^2(S^3)$ is a subspace of $L^2(S^3)$, we denote by $F^\perp$ the subspace of all elements which are orthogonal to $1, e_1, \dots, e_4$. We will need the following result.

Lemma 3.14. [2] *The mapping*

$$\begin{array}{rccc} \mathcal{P}\colon & \mathcal{C}^{4,\alpha}(S^3)^\perp \times \mathcal{C}^{2,\alpha}(S^3)^\perp & \to & \mathcal{C}^{3,\alpha}(S^3)^\perp \times \mathcal{C}^{1,\alpha}(S^3)^\perp \\ & (\varphi,\psi) & \mapsto & \big(\partial_r(H^{\mathrm{int}}_{\varphi,\psi} - H^{\mathrm{ext}}_{\varphi,\psi}), \partial_r(\Delta H^{\mathrm{int}}_{\varphi,\psi} - \Delta H^{\mathrm{ext}}_{\varphi,\psi})\big) \end{array}$$

is an isomorphism.

3.2. The nonlinear interior problem

Here, we look for a solution of the following system as in Section 3.1.2, we will just add the interior harmonic extension and the perturbation term v_i for $i = 1, 2$:

$$\begin{aligned} &\Delta^2 v_1 + \Sigma^1_{\tilde{a}_1} v_1 + \Sigma^2_{\tilde{a}_1} v_1 = 24 e^{\gamma v_1 + (1-\gamma) v_2} && \text{in } B_{R^1_\varepsilon}(x^1), \\ &\Delta^2 v_2 + \Sigma^1_{\tilde{a}_1} v_2 + \Sigma^2_{\tilde{a}_1} v_2 = 24 \frac{2^{4(\frac{\gamma+\xi-1}{\gamma})} \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{\big(\tau_1(1+\varepsilon^2)\big)^{4(\frac{\gamma+\xi-1}{\gamma})}} e^{\xi v_2 + (1-\xi) v_1} && \text{in } B_{R^1_\varepsilon}(x^1). \end{aligned}$$

Given $\varphi^1 := (\varphi^1_1, \varphi^1_2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^1 := (\psi^1_1, \psi^1_2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ^1_1, ψ^1_1) and (φ^1_2, ψ^1_2) satisfy (3.23). We write for $x \in B_{R^1_\varepsilon}(x^1)$ the following system

$$\begin{aligned} v_1(x) &= \frac{1}{\gamma} \overline{u}(x - x^1) - \frac{1-\gamma}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{1-\gamma}{\gamma\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{\ln\gamma}{\gamma} + h^1_1(x) \\ &\quad + H^{\mathrm{int},1}_1\left(\varphi^1_1, \psi^1_1; \frac{x - x^1}{R^1_\varepsilon}\right) + v^1_1(x), \\ v_2(x) &= \frac{1}{\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + h^1_2(x) + H^{\mathrm{int},1}_2\left(\varphi^1_2, \psi^1_2; \frac{x - x^1}{R^1_\varepsilon}\right) + v^1_2(x). \end{aligned}$$

Using the fact that H^{int} is bi-harmonic and taking into account that $\big(\Sigma^i_{\tilde{a}_1}\big)_i$, $i = 1, 2$ are linear operators, this amounts to solve the system

$$\begin{aligned} \mathbb{L} v^1_1 &= \frac{384}{\gamma(1+r^2)^4} \big[e^{\gamma(h^1_1 + H^{\mathrm{int},1}_1 + v^1_1) + (1-\gamma)(h^1_2 + H^{\mathrm{int},1}_2 + v^1_2)} - e^{\gamma h^1_1 + (1-\gamma) h^1_2} - \gamma v^1_1\big] \\ &\quad - \Sigma^1_{\tilde{a}_1}\big(H^{\mathrm{int},1}_1 + v^1_1\big) - \Sigma^2_{\tilde{a}_1}\big(H^{\mathrm{int},1}_1 + v^1_1\big), \\ \Delta^2 v^1_2 &= \frac{24 \cdot 16^{\frac{1-\xi}{\gamma}} C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}}{\gamma^{\frac{1-\xi}{\gamma}} (1+r^2)^{4\frac{1-\xi}{\gamma}}} e^{\frac{\gamma+\xi-1}{\gamma}\left(\frac{1}{\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right)\right)} \\ &\quad \times \big[e^{\xi(h^1_2 + H^{\mathrm{int},1}_2 + v^1_2) + (1-\xi)(h^1_1 + H^{\mathrm{int},1}_1 + v^1_1)} - e^{\xi h^1_2 + (1-\xi) h^1_1}\big] \\ &\quad - \Sigma^1_{\tilde{a}_1}\big(H^{\mathrm{int},1}_2 + v^1_2\big) - \Sigma^2_{\tilde{a}_1}\big(H^{\mathrm{int},1}_2 + v^1_2\big). \end{aligned} \tag{3.25}$$

We denote by

$$\mathbb{L} v^1_1 = \mathcal{S}_1(v^1_1, v^1_2) \quad \text{and} \quad \Delta^2 v^1_2 = \mathcal{P}_1(v^1_1, v^1_2).$$

To find a solution of (3.25), it is enough to find a fixed point (v_1^1, v_2^1) in a small ball of $\mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)$ solutions of

$$
\begin{aligned}
v_1^1 &= \mathcal{G}_{\mu} \circ \xi_{\mu,R_\varepsilon^1} \circ \mathcal{S}_1(v_1^1, v_2^1) = \aleph_1(v_1^1, v_2^1), \\
v_2^1 &= \mathcal{K}_{\delta} \circ \xi_{\delta,R_\varepsilon^1} \circ \mathcal{P}_1(v_1^1, v_2^1) = \Upsilon_1(v_1^1, v_2^1).
\end{aligned} \tag{3.26}
$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that the functions $(\varphi_j^1, \psi_j^1) \in (\mathcal{C}^{4,\alpha} \times \mathcal{C}^{2,\alpha})$ for $j \in \{1,2\}$ and the constant $\tau_1 > 0$ satisfy

$$
\frac{1}{\ln(1/r_\varepsilon^2)} |\ln(\tau_1/\tau_1^*)| \leq \kappa r_\varepsilon^2, \quad \|\varphi_j^1\|_{\mathcal{C}^{4,\alpha}(S^3)} \leq \kappa r_\varepsilon^2 \quad \text{and} \quad \|\psi_j^1\|_{\mathcal{C}^{2,\alpha}(S^3)} \leq \kappa r_\varepsilon^2. \tag{3.27}
$$

Then, we have the following result.

Lemma 3.15. *Let $\varphi^1 := (\varphi_1^1, \varphi_2^1) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^1 := (\psi_1^1, \psi_2^1) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ_1^1, ψ_1^1) and (φ_2^1, ψ_2^1) satisfy (3.23) and (3.27). Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$ with $r_\varepsilon := r_{\varepsilon,\beta,\lambda}$. We have*

$$
\begin{gathered}
\|\aleph_1(0,0)\|_{\mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2, \quad \|\Upsilon_1(0,0)\|_{\mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2, \\
\|\aleph_1(v_1^1, v_2^1) - \aleph_1(t_1^1, t_2^1)\|_{\mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4)} \leq \bar{c}_\kappa r_\varepsilon^2 \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{\mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)}
\end{gathered}
$$

and

$$
\|\Upsilon_1(v_1^1, v_2^1) - \Upsilon_1(t_1^1, t_2^1)\|_{\mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2 \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{\mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)},
$$

provided $(v_1^1, v_2^1), (t_1^1, t_2^1) \in \mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)$ satisfying

$$
\|(v_1^1, v_2^1)\|_{\mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2, \quad \|(t_1^1, t_2^1)\|_{\mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2. \tag{3.28}
$$

Proof. The first and second estimates follows from the result of Lemma 3.13 together with the assumption on the norms of φ_j^1 and ψ_j^1 given by (3.27). Indeed, we have

$$
\left\| H^{\mathrm{int}}_{\varphi_j^1, \psi_j^1}\left(\frac{r}{R_\varepsilon^1} \,\cdot\,\right) \right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0) - B_1(0))} \leq C r^2 (R_\varepsilon^1)^{-2} \left(\|\varphi_j^1\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi_j^1\|_{\mathcal{C}^{2,\alpha}(S^3)} \right)
$$

for all $r \leq R_\varepsilon^1/2$. Then by (3.27), we get

$$
\left\| H^{\mathrm{int}}_{\varphi_j^1, \psi_j^1}\left(\frac{r}{R_\varepsilon^1} \,\cdot\,\right) \right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0) - B_1(0))} \leq c_\kappa \varepsilon^2 r^2. \tag{3.29}
$$

On the other hand,

$$\begin{aligned}
&\sup_{r\le R_\varepsilon^1} r^{4-\mu}|\mathcal{S}_1(0,0)|\\
&\le \sup_{r\le R_\varepsilon^1}\frac{384r^{4-\mu}}{\gamma(1+r^2)^4}\left|e^{\gamma(h_1^1+H_1^{\mathrm{int},1})+(1-\gamma)(h_2^1+H_2^{\mathrm{int},1})}-e^{\gamma h_1^1+(1-\gamma)h_2^1}\right|\\
&\quad+\sup_{r\le R_\varepsilon^1} r^{4-\mu}\left(\left|\Sigma_{\widetilde{a}_1}^1\left(H_1^{\mathrm{int},1}\right)\right|+\left|\Sigma_{\widetilde{a}_1}^2\left(H_1^{\mathrm{int},1}\right)\right|\right)\\
&\le c_\kappa\sup_{r\le R_\varepsilon^1}\frac{384r^{4-\mu}}{\gamma(1+r^2)^4}\left|\gamma H_1^{\mathrm{int},1}+(1-\gamma)H_2^{\mathrm{int},1}\right|\\
&\quad+c_\kappa\sup_{r\le R_\varepsilon^1} r^{4-\mu}\left(2\left|\frac{\nabla\widetilde{a}_1}{\widetilde{a}_1}\right|\left|\nabla(\Delta H_1^{\mathrm{int},1})\right|+\left(\frac{\varepsilon}{\tau_1}\right)^2\left|\widetilde{V}_1(x)\right|\left|\nabla\log\widetilde{a}_1\right|\left|\nabla(H_1^{\mathrm{int},1})\right|\right)\\
&\quad+c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^2\sup_{r\le R_\varepsilon^1} r^{4-\mu}\left|\frac{1}{\widetilde{a}_1}\left(\left(\frac{\varepsilon}{\tau_1}\right)^{-2}\Delta\widetilde{a}_1-\widetilde{V}_1(x)\widetilde{a}_1\right)\right|\left|\Delta(H_1^{\mathrm{int},1})\right|.
\end{aligned}$$

From the asymptotic behavior of H^{int} given by the estimate (3.29) and since $a(x)$ is solution of (1.3) satisfying (H) and $V(x)$ is smooth bounded potential, we get

$$\begin{aligned}
\sup_{r\le R_\varepsilon^1} r^{4-\mu}|\mathcal{S}_1(0,0)| &\le c_\kappa\sup_{r\le R_\varepsilon^1}\frac{384r^{4-\mu}}{\gamma(1+r^2)^4}\left(\gamma r^2\|H_1^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}+(1-\gamma)r^2\|H_2^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}\right)\\
&\quad+c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^3\|\nabla a\|_\infty\sup_{r\le R_\varepsilon^1} r^{5-\mu}\|H_1^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}\\
&\quad+c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^2\lambda\|f\|_\infty\sup_{r\le R_\varepsilon^1} r^{4-\mu}\|H_1^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}.
\end{aligned}$$

Making use of Proposition 3.2 together with (3.4), for $\mu\in(1,2)$, there exists a constant $c_\kappa>0$ such that

$$\|\aleph_1(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2.$$

For the second estimate, we have

$$\begin{aligned}
&\sup_{r\le R_\varepsilon^1} r^{4-\delta}|\mathcal{P}_1(0,0)|\\
&\le\sup_{r\le R_\varepsilon^1} 24C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}}\varepsilon^{8\left(\frac{\gamma+\xi-1}{\gamma}\right)}r^{4-\delta}\left(\frac{16}{\gamma(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}}\\
&\qquad\times e^{\frac{\gamma+\xi-1}{\gamma}\left(\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_1},x^2\right)+G_a\left(\frac{\varepsilon x}{\tau_1},x^3\right)\right)}\left|e^{\xi(h_2^1+H_2^{\mathrm{int},1})+(1-\xi)(h_1^1+H_1^{\mathrm{int},1})}-e^{\xi h_2^1+(1-\xi)h_1^1}\right|\\
&\quad+\sup_{r\le R_\varepsilon^1} r^{4-\delta}\left|\Sigma_{\widetilde{a}_1}^1\left(H_2^{\mathrm{int},1}\right)\right|+\sup_{r\le R_\varepsilon^1} r^{4-\delta}\left|\Sigma_{\widetilde{a}_1}^2\left(H_2^{\mathrm{int},1}\right)\right|\\
&\le c_\kappa\sup_{r\le R_\varepsilon^1} 24C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}}\varepsilon^{8\left(\frac{\gamma+\xi-1}{\gamma}\right)}r^{4-\delta}\left(\frac{16}{\gamma(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}}\left(\xi r^2\|H_2^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}}+(1-\xi)r^2\|H_1^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}}\right)\\
&\quad+c_\kappa\sup_{r\le R_\varepsilon^1} r^{4-\delta}\left(2\left|\frac{\nabla\widetilde{a}_1}{\widetilde{a}_1}\right|\left|\nabla(\Delta H_2^{\mathrm{int},1})\right|+\left(\frac{\varepsilon}{\tau_1}\right)^2\left|\widetilde{V}_1(x)\right|\left|\nabla\log\widetilde{a}_1\right|\left|\nabla(H_2^{\mathrm{int},1})\right|\right)
\end{aligned}$$

$$
\begin{aligned}
&+ c_\kappa \left(\frac{\varepsilon}{\tau_1}\right)^2 \sup_{r\le R_\varepsilon^1} r^{4-\delta}\left|\frac{1}{\widetilde{a}_1}\left(\left(\frac{\varepsilon}{\tau_1}\right)^{-2}\Delta\widetilde{a}_1 - \widetilde{V}_1(x)\widetilde{a}_1\right)\right|\left|\Delta(H_2^{\mathrm{int},1})\right| \\
&\le c_\kappa \sup_{r\le R_\varepsilon^1} 24 C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})} r^{4-\delta}\left(\frac{16}{\gamma(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}} \left(\xi r^2\|H_2^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}} + (1-\xi)r^2\|H_1^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}}\right) \\
&+ c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^3\|\nabla a\|_\infty \sup_{r\le R_\varepsilon^1} r^{5-\delta}\|H_2^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}} + c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^2\lambda\|f\|_\infty \sup_{r\le R_\varepsilon^1} r^{4-\delta}\|H_2^{\mathrm{int},1}\|_{\mathcal{C}_2^{4,\alpha}}.
\end{aligned}
$$

Using the same argument as above, we get

$$
\|\Upsilon_1(0,0)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2.
$$

To derive the third estimate, for (v_1^1, v_2^1), (t_1^1, t_2^1) verifying (3.28), we have

$$
\begin{aligned}
&\sup_{r\le R_\varepsilon^1} r^{4-\mu}\left|\mathcal{S}_1(v_1^1,v_2^1) - \mathcal{S}_1(t_1^1,t_2^1)\right| \\
&\le \sup_{r\le R_\varepsilon^1} \frac{384 r^{4-\mu}}{\gamma(1+r^2)^4}\Big|\left(e^{\gamma(h_1^1+H_1^{\mathrm{int},1}+v_1^1)+(1-\gamma)(h_2^1+H_2^{\mathrm{int},1}+v_2^1)} - e^{\gamma h_1^1+(1-\gamma)h_2^1} - \gamma v_1^1\right) \\
&\quad - \left(e^{\gamma(h_1^1+H_1^{\mathrm{int},1}+t_1^1)+(1-\gamma)(h_2^1+H_2^{\mathrm{int},1}+t_2^1)} - e^{\gamma h_1^1+(1-\gamma)h_2^1} - \gamma t_1^1\right)\Big| \\
&\quad + \sup_{r\le R_\varepsilon^1} r^{4-\mu}\left|\Sigma_{\widetilde{a}_1}^1(v_1^1 - t_1^1)\right| + \sup_{r\le R_\varepsilon^1} r^{4-\mu}\left|\Sigma_{\widetilde{a}_1}^2(v_1^1 - t_1^1)\right| \\
&\le c_\kappa \sup_{r\le R_\varepsilon^1}\frac{384 r^{4-\mu}}{\gamma(1+r^2)^4}\left(\gamma^2\left|(v_1^1)^2 - (t_1^1)^2\right| + (1-\gamma)\left|v_2^1 - t_2^1\right|\right) \\
&\quad + c_\kappa \sup_{r\le R_\varepsilon^1} r^{4-\mu}\left(2\left|\frac{\nabla\widetilde{a}_1}{\widetilde{a}_1}\right|\left|\nabla(\Delta(v_1^1-t_1^1))\right| + \left(\frac{\varepsilon}{\tau_1}\right)^2\left|\widetilde{V}_1(x)\right|\left|\nabla\log\widetilde{a}_1\right|\left|\nabla(v_1^1-t_1^1)\right|\right) \\
&\quad + c_\kappa\left(\frac{\varepsilon}{\tau_1}\right)^2 \sup_{r\le R_\varepsilon^1} r^{4-\mu}\left|\frac{1}{\widetilde{a}_1}\left(\left(\frac{\varepsilon}{\tau_1}\right)^{-2}\Delta\widetilde{a}_1 - \widetilde{V}_1(x)\widetilde{a}_1\right)\right|\left|\Delta(v_1^1 - t_1^1)\right|.
\end{aligned}
$$

Using the fact that $a(x)$ is a solution of (1.3) satisfying (H), $V(x)$ is a smooth bounded potential and for $\ell = 1,\dots,4+\alpha$, we have $|\nabla^\ell w| \le c_\kappa r^{\mu-\ell}\|w\|_{\mathcal{C}_\mu^{k,\alpha}(\mathbb{R}^4)}$. Then we get

$$
\begin{aligned}
&\sup_{r\le R_\varepsilon^1} r^{4-\mu}\left|\mathcal{S}_1(v_1^1,v_2^1) - \mathcal{S}_1(t_1^1,t_2^1)\right| \\
&\le c_\kappa \sup_{r\le R_\varepsilon^1}\frac{384 r^{4-\mu}}{\gamma(1+r^2)^4} \\
&\quad \times\left[\gamma^2 r^{2\mu}\left(\|v_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + \|t_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\right)\|v_1^1 - t_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + (1-\gamma)r^\delta\|v_2^1 - t_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}\right] \\
&\quad + c_\kappa\beta r_\varepsilon\|v_1^1 - t_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + c_\kappa\beta r_\varepsilon^3\|v_1^1 - t_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + c_\kappa\lambda r_\varepsilon^2\|v_1^1 - t_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&\|\aleph_1(v_1^1,v_2^1) - \aleph_1(t_1^1,t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \\
&\le c_\kappa r_\varepsilon^2\|v_1^1 - t_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|v_2^1 - t_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \\
&\le \max\left\{c_\kappa r_\varepsilon^2, c_\kappa(1-\gamma)\right\}\|(v_1^1,v_2^1) - (t_1^1,t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}.
\end{aligned}
$$

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \le 1/2$ for all $\varepsilon \in (0, \varepsilon_\kappa)$. There exists a constant $\gamma_0 \in (0,1)$ such that $c_\kappa(1-\gamma) \le 1/2$ for all $\gamma \in (\gamma_0, 1)$, we can find a constant $\overline{c}_\kappa > 0$ such that

$$\|\aleph_1(v_1^1, v_2^1) - \aleph_1(t_1^1, t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le \overline{c}_\kappa r_\varepsilon^2 \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}. \tag{3.30}$$

On the other hand, we have

$$\begin{aligned}
&\sup_{r \le R_\varepsilon^1} r^{4-\delta} \big|\mathcal{P}_1(v_1^1, t_2^1) - \mathcal{P}_1(v_1^1, t_2^1)\big| \\
&\le \sup_{r \le R_\varepsilon^1} 24 C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})} r^{4-\delta} \left(\frac{16}{\gamma(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}} e^{\frac{\gamma+\xi-1}{\gamma}\left(\frac{1}{\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right)\right)} \\
&\quad \times \left| e^{\xi(h_2^1 + H_2^{\mathrm{int},1} + v_2^1) + (1-\xi)(h_1^1 + H_1^{\mathrm{int},1} + v_1^1)} - e^{\xi(h_2^1 + H_2^{\mathrm{int},1} + t_2^1) + (1-\xi)(h_1^1 + H_1^{\mathrm{int},1} + t_1^1)} \right| \\
&\quad + \sup_{r \le R_\varepsilon^1} r^{4-\delta} \big|\Sigma_{\widetilde{a}_1}^1 (v_2^1 - t_2^1)\big| + \sup_{r \le R_\varepsilon^1} r^{4-\delta} \big|\Sigma_{\widetilde{a}_1}^2 (v_2^1 - t_2^1)\big| \\
&\le c_\kappa \sup_{r \le R_\varepsilon^1} 24 C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})} r^{4-\delta} \left(\frac{16}{\gamma(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}} \left(\xi|v_2^1 - t_2^1| + (1-\xi)|v_1^1 - t_1^1|\right) \\
&\quad + c_\kappa \sup_{r \le R_\varepsilon^1} r^{4-\delta} \left(2\left|\frac{\nabla \widetilde{a}_1}{\widetilde{a}_1}\right| \big|\nabla(\Delta(v_2^1 - t_2^1))\big| + \left(\frac{\varepsilon}{\tau_1}\right)^2 \big|\widetilde{V}_1(x)\big| \big|\nabla \log \widetilde{a}_1\big| \big|\nabla(v_2^1 - t_2^1)\big| \right) \\
&\quad + c_\kappa \left(\frac{\varepsilon}{\tau_1}\right)^2 \sup_{r \le R_\varepsilon^1} r^{4-\delta} \left| \frac{1}{\widetilde{a}_1} \left(\left(\frac{\varepsilon}{\tau_1}\right)^{-2} \Delta \widetilde{a}_1 - \widetilde{V}_1(x) \widetilde{a}_1 \right) \right| \big|\Delta(v_2^1 - t_2^1)\big| \\
&\le c_\kappa \sup_{r \le R_\varepsilon^1} 24 C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})} r^{4-\delta} \left(\frac{16}{\gamma(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}} \\
&\quad \times \left[\xi r^\delta \|v_2^1 - t_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} + (1-\xi) r^\mu \|v_1^1 - t_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \right] \\
&\quad + c_\kappa \beta r_\varepsilon \|v_2^1 - t_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} + c_\kappa \beta r_\varepsilon^3 \|v_2^1 - t_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} + c_\kappa \lambda r_\varepsilon^2 \|v_2^1 - t_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}.
\end{aligned}$$

We conclude that

$$\|\Upsilon_1(v_1^1, v_2^1) - \Upsilon_1(t_1^1, t_2^1)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}. \tag{3.31}$$ □

Hence, (3.30) and (3.31) are enough to show that

$$(v_1^1, v_2^1) \mapsto \big(\aleph_1(v_1^1, v_2^1), \Upsilon_1(v_1^1, v_2^1)\big)$$

is a contraction from the ball

$$\big\{(v_1^1, v_2^1) \in \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) : \|(v_1^1, v_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2 c_\kappa r_\varepsilon^2\big\}$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

Proposition 3.16. *Given $\kappa > 0$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ (only depending on κ) and $\gamma_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$ and $\gamma \in (\gamma_0, 1)$, for all τ_1 in some fixed compact subset of $[\tau_1^-, \tau_1^+] \subset (0, \infty)$ and for φ_j^1 and ψ_j^1 satisfying (3.23) and (3.27), there exists a unique $(v_1^1, v_2^1) := (v^1_{1,\varepsilon,\lambda,\beta,\tau_1,\varphi_1^1,\psi_1^1}, v^1_{2,\varepsilon,\lambda,\beta,\tau_1,\varphi_2^1,\psi_2^1})$ solution of (3.26) such that*

$$\|(v_1^1, v_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2.$$

Hence

$$\begin{aligned}
v_1(x) &= \frac{1}{\gamma}\overline{u}(x - x^1) - \frac{1-\gamma}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{1-\gamma}{\gamma\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{\ln\gamma}{\gamma} + h_1^1(x) \\
&\quad + H_1^{\mathrm{int},1}\left(\varphi_1^1, \psi_1^1; \frac{x - x^1}{R_\varepsilon^1}\right) + v_1^1(x), \\
v_2(x) &= \frac{1}{\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + h_2^1(x) + H_2^{\mathrm{int},1}\left(\varphi_2^1, \psi_2^1; \frac{x - x^1}{R_\varepsilon^1}\right) + v_2^1(x)
\end{aligned}$$

solves (3.6) *in* $B_{R_\varepsilon^1}(x^1)$.

• In $B_{R_\varepsilon^2}(x^2)$, following the same arguments as the first case by reversing the roles of the functions u_1 and u_2 and by respecting the changes of the coefficients, we can prove that there exists $(v_1^2, v_2^2) \in \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ such that

$$\|(v_1^2, v_2^2)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2.$$

Furthermore, (v_1^2, v_2^2) solves the following system

$$\begin{aligned}
\Delta^2 v_1^2 &= \frac{24 \cdot 16^{\frac{1-\gamma}{\xi}} C_{2,\varepsilon}^{4\frac{\gamma+\xi-1}{\xi}} \varepsilon^{8\frac{\gamma+\xi-1}{\xi}}}{\xi^{\frac{1-\gamma}{\xi}} (1+r^2)^{4\frac{1-\gamma}{\xi}}} e^{\frac{\gamma+\xi-1}{\xi}\left(\frac{1}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right)\right)} \\
&\quad \times \left[e^{\gamma(h_1^2 + H_1^{\mathrm{int},2} + v_1^2) + (1-\gamma)(h_2^2 + H_2^{\mathrm{int},2} + v_2^2)} - e^{\gamma h_1^2 + (1-\gamma) h_2^2}\right] \\
&\quad - \Sigma^1_{\widetilde{a}_2}\left(H_1^{\mathrm{int},2} + v_1^2\right) - \Sigma^2_{\widetilde{a}_2}\left(H_1^{\mathrm{int},2} + v_1^2\right), \\
\mathbb{L} v_2^2 &= \frac{384}{\xi(1+r^2)^4}\left[e^{\xi(h_2^2 + H_2^{\mathrm{int},2} + v_2^2) + (1-\xi)(h_1^2 + H_1^{\mathrm{int},2} + v_1^2)} - e^{\xi h_2^2 + (1-\xi) h_1^2} - \xi v_2^2\right] \\
&\quad - \Sigma^1_{\widetilde{a}_2}\left(H_2^{\mathrm{int},2} + v_2^2\right) - \Sigma^2_{\widetilde{a}_2}\left(H_2^{\mathrm{int},2} + v_2^2\right).
\end{aligned} \tag{3.32}$$

We denote by

$$\Delta^2 v_1^2 = \mathcal{S}_2(v_1^2, v_2^2) \quad \text{and} \quad \mathbb{L} v_2^2 = \mathcal{P}_2(v_1^2, v_2^2).$$

To find a solution of (3.32), it is enough to find a fixed point (v_1^2, v_2^2) in a small ball of $\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ solutions of

$$\begin{aligned}
v_1^2 &= \mathcal{K}_\delta \circ \xi_{\delta, R_\varepsilon^2} \circ \mathcal{S}_2(v_1^2, v_2^2) = \aleph_2(v_1^2, v_2^2), \\
v_2^2 &= \mathcal{G}_\mu \circ \xi_{\mu, R_\varepsilon^2} \circ \mathcal{P}_2(v_1^2, v_2^2) = \Upsilon_2(v_1^2, v_2^2).
\end{aligned} \tag{3.33}$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that the functions $(\varphi_j^2, \psi_j^2) \in (\mathcal{C}^{4,\alpha} \times \mathcal{C}^{2,\alpha})$ for $j \in \{1,2\}$ and the constant $\tau_2 > 0$ satisfy

$$\frac{1}{\ln(1/r_\varepsilon^2)}|\ln(\tau_2/\tau_2^*)| \le \kappa r_\varepsilon^2, \quad \|\varphi_j^2\|_{\mathcal{C}^{4,\alpha}(S^3)} \le \kappa r_\varepsilon^2 \quad \text{and} \quad \|\psi_j^2\|_{\mathcal{C}^{2,\alpha}(S^3)} \le \kappa r_\varepsilon^2. \tag{3.34}$$

Then, we have the following result.

Proposition 3.17. *Given $\kappa > 0$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ (only depending on κ) and $\xi_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$ and $\xi \in (\xi_0, 1)$, for all τ_2 in some fixed compact subset of $[\tau_2^-, \tau_2^+] \subset (0, \infty)$ and for φ_j^2 and ψ_j^2 satisfying* (3.23) *and* (3.34)*, there exists a unique $(v_1^2, v_2^2) := (v_{1,\varepsilon,\lambda,\beta,\tau_2,\varphi_1^2,\psi_1^2}, v_{2,\varepsilon,\lambda,\beta,\tau_2,\varphi_2^2,\psi_2^2})$ solution of* (3.33) *such that*

$$\|(v_1^2, v_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Hence

$$\begin{aligned} v_1(x) &= \frac{1}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_1^2(x) + H_1^{\text{int},2}\left(\varphi_1^2, \psi_1^2; \frac{x - x^2}{R_\varepsilon^2}\right) + v_1^2(x), \\ v_2(x) &= \frac{1}{\xi}\overline{u}(x - x^2) - \frac{1-\xi}{\xi} G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) - \frac{1-\xi}{\gamma\xi} G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{\ln \xi}{\xi} + h_2^2(x) \\ &\quad + H_2^{\text{int},2}\left(\varphi_2^2, \psi_2^2; \frac{x - x^2}{R_\varepsilon^2}\right) + v_2^2(x) \end{aligned}$$

solves (3.12) *in $B_{R_\varepsilon^2}(x^2)$.*

- In $B_{R_\varepsilon^3}(x^3)$, we are interested to study the system (3.17).

Given $\varphi^3 := (\varphi_1^3, \varphi_2^3) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^3 := (\psi_1^3, \psi_2^3) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ_1^3, ψ_1^3) and (φ_2^3, ψ_2^3) satisfy (3.23). For $x \in B_{R_\varepsilon^3}(x^3)$, we look for a solution of (3.17) of the form

$$\begin{aligned} v_1(x) &= \overline{u}(x - x^3) + h_1^3(x) + H_1^{\text{int},3}\left(\varphi_1^3, \psi_1^3; \frac{x - x^3}{R_\varepsilon^3}\right) + v_1^3(x), \\ v_2(x) &= \overline{u}(x - x^3) + h_2^3(x) + H_2^{\text{int},3}\left(\varphi_2^3, \psi_2^3; \frac{x - x^3}{R_\varepsilon^3}\right) + v_2^3(x). \end{aligned}$$

This amounts to solve the system

$$\begin{aligned} \mathbb{L}v_1^3 &= \frac{384}{(1+r^2)^4}\left[e^{\gamma(h_1^3 + H_1^{\text{int},3} + v_1^3) + (1-\gamma)(h_2^3 + H_2^{\text{int},3} + v_2^3)} - e^{\gamma h_1^3 + (1-\gamma)h_2^3} - v_1^3\right] \\ &\quad - \Sigma_{\tilde{a}_3}^1\left(H_1^{\text{int},3} + v_1^3\right) - \Sigma_{\tilde{a}_3}^2\left(H_1^{\text{int},3} + v_1^3\right), \\ \mathbb{L}v_2^3 &= \frac{384}{(1+r^2)^4}\left[e^{\xi(h_2^3 + H_2^{\text{int},3} + v_2^3) + (1-\xi)(h_1^3 + H_1^{\text{int},3} + v_1^3)} - e^{\xi h_2^3 + (1-\xi)h_1^3} - v_2^3\right] \\ &\quad - \Sigma_{\tilde{a}_3}^1\left(H_2^{\text{int},3} + v_2^3\right) - \Sigma_{\tilde{a}_3}^2\left(H_2^{\text{int},3} + v_2^3\right). \end{aligned} \tag{3.35}$$

We denote by

$$\mathbb{L}v_1^3 = \mathcal{S}_3(v_1^3, v_2^3) \quad \text{and} \quad \mathbb{L}v_2^3 = \mathcal{P}_3(v_1^3, v_2^3).$$

To find a solution of (3.35), it is enough to find a fixed point (v_1^3, v_2^3) in a small ball of $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ solutions of

$$\begin{aligned} v_1^3 &= \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon^3} \circ \mathcal{S}_3(v_1^3, v_2^3) = \aleph_3(v_1^3, v_2^3), \\ v_2^3 &= \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon^3} \circ \mathcal{P}_3(v_1^3, v_2^3) = \Upsilon_3(v_1^3, v_2^3). \end{aligned} \tag{3.36}$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that the functions $(\varphi_j^3, \psi_j^3) \in (\mathcal{C}^{4,\alpha} \times \mathcal{C}^{2,\alpha})$ for $j \in \{1, 2\}$ and the constant $\tau_3 > 0$ satisfy

$$\frac{1}{\ln(1/r_\varepsilon^2)} |\ln(\tau_3/\tau_3^*)| \leq \kappa r_\varepsilon^2, \quad \|\varphi_j^3\|_{\mathcal{C}^{4,\alpha}(S^3)} \leq \kappa r_\varepsilon^2 \quad \text{and} \quad \|\psi_j^3\|_{\mathcal{C}^{2,\alpha}(S^3)} \leq \kappa r_\varepsilon^2. \tag{3.37}$$

Then, we have the following result.

Lemma 3.18. *Let $\mu \in (1, 2)$. Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$, $\bar{c}_\kappa > 0$ and $\bar{\bar{c}}_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\beta \in (0, \beta_\kappa)$ with $r_\varepsilon := r_{\varepsilon,\beta,\lambda}$. We have*

$$\|\aleph_3(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2, \quad \|\Upsilon_3(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2,$$
$$\|\aleph_3(v_1^3, v_2^3) - \aleph_3(t_1^3, t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq \bar{c}_\kappa r_\varepsilon^2 \|(v_1^3, v_2^3) - (t_1^3, t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}$$

and

$$\|\Upsilon_3(v_1^3, v_2^3) - \Upsilon_3(t_1^3, t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq \bar{\bar{c}}_\kappa r_\varepsilon^2 \|(v_1^3, v_2^3) - (t_1^3, t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)},$$

provided $(v_1^3, v_2^3), (t_1^3, t_2^3) \in \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ satisfying

$$\|(v_1^3, v_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2, \quad \|(t_1^3, t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2. \tag{3.38}$$

Proof. The first and second estimates follows from the estimate of H^{int} given by Lemma 3.13 together with the assumption on the norms of φ_j^3 and ψ_j^3 given by (3.37), we have

$$\left\| H^{\text{int}}_{\varphi_j^3, \psi_j^3}\left(\frac{r}{R_\varepsilon^3}\ \cdot\ \right) \right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0) - B_1(0))} \leq C r^2 (R_\varepsilon^3)^{-2} \big(\|\varphi_j^3\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi_j^3\|_{\mathcal{C}^{2,\alpha}(S^3)} \big)$$

for all $r \leq R_\varepsilon^3/2$. Then by (3.37), we get

$$\left\| H^{\text{int}}_{\varphi_j^3, \psi_j^3}\left(\frac{r}{R_\varepsilon^3}\ \cdot\ \right) \right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0) - B_1(0))} \leq c_\kappa \varepsilon^2 r^2. \tag{3.39}$$

On the other hand,

$$
\begin{aligned}
&\sup_{r\le R_\varepsilon^3} r^{4-\mu}|\mathcal{S}_3(0,0)|\\
&\le \sup_{r\le R_\varepsilon^3}\frac{384r^{4-\mu}}{(1+r^2)^4}\big|e^{\gamma(h_1^3+H_1^{\mathrm{int},3})+(1-\gamma)(h_2^3+H_2^{\mathrm{int},3})}-e^{\gamma h_1^3+(1-\gamma)h_2^3}\big|\\
&\quad+\sup_{r\le R_\varepsilon^3} r^{4-\mu}\big(\big|\Sigma^1_{\widetilde{a}_3}\big(H_1^{\mathrm{int},3}\big)\big|+\big|\Sigma^2_{\widetilde{a}_3}\big(H_1^{\mathrm{int},3}\big)\big|\big)\\
&\le c_\kappa\sup_{r\le R_\varepsilon^3}\frac{384r^{4-\mu}}{(1+r^2)^4}\big|\gamma H_1^{\mathrm{int},3}+(1-\gamma)H_2^{\mathrm{int},3}\big|\\
&\quad+c_\kappa\sup_{r\le R_\varepsilon^3} r^{4-\mu}\left(2\left|\frac{\nabla\widetilde{a}_3}{\widetilde{a}_3}\right|\big|\nabla(\Delta H_1^{\mathrm{int},3})\big|+\left(\frac{\varepsilon}{\tau_3}\right)^2\big|\widetilde{V}_3(x)\big|\big|\nabla\log\widetilde{a}_3\big|\big|\nabla(H_1^{\mathrm{int},3})\big|\right)\\
&\quad+c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^2\sup_{r\le R_\varepsilon^3} r^{4-\mu}\left|\frac{1}{\widetilde{a}_3}\left(\left(\frac{\varepsilon}{\tau_3}\right)^{-2}\Delta\widetilde{a}_3-\widetilde{V}_3(x)\widetilde{a}_3\right)\right|\big|\Delta(H_1^{\mathrm{int},3})\big|.
\end{aligned}
$$

From the asymptotic behavior of H^{int} given by the estimate (3.39) and since $a(x)$ is solution of (1.3) satisfying (H) and $V(x)$ is smooth bounded potential, we get

$$
\begin{aligned}
\sup_{r\le R_\varepsilon^3} r^{4-\mu}|\mathcal{S}_3(0,0)| &\le c_\kappa\sup_{r\le R_\varepsilon^3}\frac{384r^{4-\mu}}{(1+r^2)^4}\big(\gamma r^2\|H_1^{\mathrm{int},3}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}+(1-\gamma)r^2\|H_2^{\mathrm{int},3}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}\big)\\
&\quad+c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^3\|\nabla a\|_\infty\sup_{r\le R_\varepsilon^3} r^{5-\mu}\|H_1^{\mathrm{int},3}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}\\
&\quad+c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^2\lambda\|f\|_\infty\sup_{r\le R_\varepsilon^3} r^{4-\mu}\|H_1^{\mathrm{int},3}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}.
\end{aligned}
$$

Making use of Proposition 3.2 together with (3.4), for $\mu\in(1,2)$, there exists a constant $c_\kappa>0$ such that

$$\|\aleph_3(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2.$$

For the second estimate, we use the same argument as above and we get

$$\|\Upsilon_3(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2.$$

To derive the third estimate, for (v_1^3,v_2^3), (t_1^3,t_2^3) verifying (3.38), we have

$$
\begin{aligned}
&\sup_{r\le R_\varepsilon^3} r^{4-\mu}\big|\mathcal{S}_3(v_1^3,v_2^3)-\mathcal{S}_3(t_1^3,t_2^3)\big|\\
&\le \sup_{r\le R_\varepsilon^3}\frac{384r^{4-\mu}}{(1+r^2)^4}\big|\big(e^{\gamma(h_1^3+H_1^{\mathrm{int},3}+v_1^3)+(1-\gamma)(h_2^3+H_2^{\mathrm{int},3}+v_2^3)}-e^{\gamma h_1^3+(1-\gamma)h_2^3}-v_1^3\big)\\
&\qquad-\big(e^{\gamma(h_1^3+H_1^{\mathrm{int},3}+t_1^3)+(1-\gamma)(h_2^3+H_2^{\mathrm{int},3}+t_2^3)}-e^{\gamma h_1^3+(1-\gamma)h_2^3}-t_1^3\big)\big|\\
&\quad+\sup_{r\le R_\varepsilon^3} r^{4-\mu}\big|\Sigma^1_{\widetilde{a}_3}(v_1^3-t_1^3)\big|+\sup_{r\le R_\varepsilon^3} r^{4-\mu}\big|\Sigma^2_{\widetilde{a}_3}(v_1^3-t_1^3)\big|
\end{aligned}
$$

$$\leq c_\kappa \sup_{r\leq R_\varepsilon^3} \frac{384 r^{4-\mu}}{(1+r^2)^4}\big|(\gamma-1)(v_1^3-t_1^3)+(1-\gamma)(v_2^3-t_2^3)\big|$$
$$+ c_\kappa \sup_{r\leq R_\varepsilon^3} r^{4-\mu}\left(2\left|\frac{\nabla\widetilde{a}_3}{\widetilde{a}_3}\right|\big|\nabla(\Delta(v_1^3-t_1^3))\big| + \left(\frac{\varepsilon}{\tau_3}\right)^2\big|\widetilde{V}_3(x)\big|\big|\nabla\log\widetilde{a}_3\big|\big|\nabla(v_1^3-t_1^3)\big|\right)$$
$$+ c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^2 \sup_{r\leq R_\varepsilon^3} r^{4-\mu}\left|\frac{1}{\widetilde{a}_3}\left(\left(\frac{\varepsilon}{\tau_3}\right)^{-2}\Delta\widetilde{a}_3 - \widetilde{V}_3(x)\widetilde{a}_3\right)\right|\big|\Delta(v_1^3-t_1^3)\big|.$$

Using the fact that $a(x)$ is a solution of (1.3) satisfying (H), $V(x)$ is a smooth bounded potential and for $\ell = 1,\dots,4+\alpha$, we have $|\nabla^\ell w| \leq c_\kappa r^{\mu-\ell}\|w\|_{\mathcal{C}_\mu^{k,\alpha}(\mathbb{R}^4)}$. Then we get

$$\sup_{r\leq R_\varepsilon^3} r^{4-\mu}\big|\mathcal{S}_3(v_1^3,v_2^3)-\mathcal{S}_3(t_1^3,t_2^3)\big|$$
$$\leq c_\kappa \sup_{r\leq R_\varepsilon^3}\frac{384r^{4-\mu}}{(1+r^2)^4}(1-\gamma)\big[r^\mu\|v_1^3-t_1^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + r^\mu\|v_2^3-t_2^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\big]$$
$$+ c_\kappa\beta r_\varepsilon\|v_1^3-t_1^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + c_\kappa\beta r_\varepsilon^3\|v_1^3-t_1^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + c_\kappa\lambda r_\varepsilon^2\|v_1^3-t_1^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}.$$

We conclude that

$$\|\aleph_3(v_1^3,v_2^3)-\aleph_3(t_1^3,t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}$$
$$\leq \big(c_\kappa r_\varepsilon^2 + c_\kappa(1-\gamma)\big)\|v_1^3-t_1^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|v_2^3-t_2^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}$$
$$\leq \big(c_\kappa r_\varepsilon^2 + c_\kappa(1-\gamma)\big)\|(v_1^3,v_2^3)-(t_1^3,t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}.$$

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \leq 1/2$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. There exists a constant $\gamma_0\in(0,1)$ such that $c_\kappa(1-\gamma)\leq 1/2$ for all $\gamma\in(\gamma_0,1)$, we can find a constant $\overline{c}_\kappa > 0$ such that

$$\|\aleph_3(v_1^3,v_2^3)-\aleph_3(t_1^3,t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq \overline{c}_\kappa r_\varepsilon^2\|(v_1^3,v_2^3)-(t_1^3,t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{3.40}$$

Similarly, we get

$$\|\Upsilon_3(v_1^3,v_2^3)-\Upsilon_3(t_1^3,t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}$$
$$\leq \big(c_\kappa r_\varepsilon^2 + c_\kappa(1-\xi)\big)\|v_2^3-t_2^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + c_\kappa(1-\xi)\|v_1^3-t_1^3\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}$$
$$\leq \big(c_\kappa r_\varepsilon^2 + c_\kappa(1-\xi)\big)\|(v_1^3,v_2^3)-(t_1^3,t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}.$$

Then, reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \leq 1/2$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. There exists a constant $\xi_0\in(0,1)$ such that $c_\kappa(1-\xi)\leq 1/2$ for all $\xi\in(\xi_0,1)$, we can find a constant $\overline{\overline{c}}_\kappa > 0$ such that

$$\|\Upsilon_3(v_1^3,v_2^3)-\Upsilon_3(t_1^3,t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq \overline{\overline{c}}_\kappa r_\varepsilon^2\|(v_1^3,v_2^3)-(t_1^3,t_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{3.41}$$

□

Therefore, (3.40) and (3.41) are enough to show that

$$(v_1^3, v_2^3) \mapsto \big(\aleph_3(v_1^3, v_2^3), \Upsilon_3(v_1^3, v_2^3)\big)$$

is a contraction from the ball

$$\big\{(v_1^3, v_2^3) \in \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) : \|(v_1^3, v_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2\big\}$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

Proposition 3.19. *Given $\kappa > 0$, $\mu \in (1,2)$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ (only depending on κ), $\gamma_0 \in (0,1)$ and $\xi_0 \in (0,1)$ such that for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$, $\beta \in (0,\beta_\kappa)$, $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, for all τ_3 in some fixed compact subset of $[\tau_3^-, \tau_3^+] \subset (0,\infty)$ and for φ_j^3 and ψ_j^3 satisfying* (3.23) *and* (3.37)*, there exists a unique $(v_1^3, v_2^3) := (v_{1,\varepsilon,\lambda,\beta,\tau_3,\varphi_1^3,\psi_1^3}, v_{2,\varepsilon,\lambda,\beta,\tau_3,\varphi_2^3,\psi_2^3})$ solution of* (3.36) *such that*

$$\|(v_1^3, v_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2.$$

Hence

$$\begin{aligned} v_1(x) &= \overline{u}(x - x^3) + h_1^3(x) + H_1^{\mathrm{int},3}\left(\varphi_1^3, \psi_1^3; \frac{x - x^3}{R_\varepsilon^3}\right) + v_1^3(x), \\ v_2(x) &= \overline{u}(x - x^3) + h_2^3(x) + H_2^{\mathrm{int},3}\left(\varphi_2^3, \psi_2^3; \frac{x - x^3}{R_\varepsilon^3}\right) + v_2^3(x) \end{aligned}$$

solves (3.17) *in $B_{R_\varepsilon^3}(x^3)$.*

Remark also that the functions $(v_1^i, v_2^i) := (v_{1,\varepsilon,\lambda,\beta,\tau_i,\varphi_1^i,\psi_1^i}, v_{2,\varepsilon,\lambda,\beta,\tau_i,\varphi_2^i,\psi_2^i})$ $(i \in \{1,2,3\})$ depend continuously on the parameter τ_i.

3.3. The nonlinear exterior problem

Given $\widetilde{\mathbf{x}} := (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) \in \Omega^3$ close to $\mathbf{x} := (x^1, x^2, x^3)$, $\boldsymbol{\eta} := (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ close to 0, $\widetilde{\boldsymbol{\varphi}}_1 := (\widetilde{\varphi}_1^1, \widetilde{\varphi}_1^2, \widetilde{\varphi}_1^3) \in (\mathcal{C}^{4,\alpha}(S^3))^3$, $\widetilde{\boldsymbol{\varphi}}_2 := (\widetilde{\varphi}_2^1, \widetilde{\varphi}_2^2, \widetilde{\varphi}_2^3) \in (\mathcal{C}^{4,\alpha}(S^3))^3$, $\widetilde{\boldsymbol{\psi}}_1 := (\widetilde{\psi}_1^1, \widetilde{\psi}_1^2, \widetilde{\psi}_1^3) \in (\mathcal{C}^{2,\alpha}(S^3))^3$ and $\widetilde{\boldsymbol{\psi}}_2 := (\widetilde{\psi}_2^1, \widetilde{\psi}_2^2, \widetilde{\psi}_2^3) \in (\mathcal{C}^{2,\alpha}(S^3))^3$ satisfying (3.24). Let $\widetilde{\mathbf{w}}_1$ and $\widetilde{\mathbf{w}}_2$ be defined by

$$\begin{aligned} \widetilde{\mathbf{w}}_1(x) &= \frac{1+\eta_1}{\gamma} G_a(x, \widetilde{x}^1) + (1+\eta_3) G_a(x, \widetilde{x}^3) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x - \widetilde{x}^i}{r_\varepsilon}\right), \\ \widetilde{\mathbf{w}}_2(x) &= \frac{1+\eta_2}{\xi} G_a(x, \widetilde{x}^2) + (1+\eta_3) G_a(x, \widetilde{x}^3) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x - \widetilde{x}^i}{r_\varepsilon}\right). \end{aligned}$$

Here χ_{r_0} is a cut-off function identically equal to 1 in $B_{r_0/2}(0)$ and identically equal to 0 outside $B_{r_0}(0)$.

We would like to find a solution of the system

$$\begin{aligned} \Delta^2 u_1 + \Sigma_a^1 u_1 + \Sigma_a^2 u_1 &= \rho^4 e^{\gamma u_1 + (1-\gamma) u_2}, \\ \Delta^2 u_2 + \Sigma_a^1 u_2 + \Sigma_a^2 u_2 &= \rho^4 e^{\xi u_2 + (1-\xi) u_1} \end{aligned} \tag{3.42}$$

in the domain $\overline{\Omega}_{r_\varepsilon}(\widetilde{x})$ with $u_k = \widetilde{\mathbf{w}}_k + \widetilde{v}_k$ a perturbation of $\widetilde{\mathbf{w}}_k$, $k = 1, 2$.

We recall that for $\rho > 0$, we have

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4},$$

$$\Sigma_a^1 u_i = 2\frac{\nabla a}{a} \cdot \nabla(\Delta u_i) - V(x) \nabla \log a \cdot \nabla u_i \quad \text{and} \quad \Sigma_a^2 u_i = \left(\frac{\Delta a}{a} - V(x) \right) \Delta u_i, \quad i = 1, 2.$$

This amounts to solve in $\overline{\Omega}_{r_\varepsilon}(\widetilde{x})$,

$$\begin{aligned} \Delta^2 \widetilde{v}_1 &= \rho^4 e^{\gamma(\widetilde{\mathbf{w}}_1 + \widetilde{v}_1) + (1-\gamma)(\widetilde{\mathbf{w}}_2 + \widetilde{v}_2)} - \Delta^2 \widetilde{\mathbf{w}}_1 - \Sigma_a^1(\widetilde{\mathbf{w}}_1 + \widetilde{v}_1) - \Sigma_a^2(\widetilde{\mathbf{w}}_1 + \widetilde{v}_1), \\ \Delta^2 \widetilde{v}_2 &= \rho^4 e^{\xi(\widetilde{\mathbf{w}}_2 + \widetilde{v}_2) + (1-\xi)(\widetilde{\mathbf{w}}_1 + \widetilde{v}_1)} - \Delta^2 \widetilde{\mathbf{w}}_2 - \Sigma_a^1(\widetilde{\mathbf{w}}_2 + \widetilde{v}_2) - \Sigma_a^2(\widetilde{\mathbf{w}}_2 + \widetilde{v}_2). \end{aligned} \tag{3.43}$$

For all $\sigma \in (0, r_0/2)$ and all $\widetilde{\mathbf{x}} = (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) \in \Omega^3$ such that $\|\mathbf{x} - \widetilde{\mathbf{x}}\| \le r_0/2$, where $\mathbf{x} = (x^1, x^2, x^3)$, we denote by $\widetilde{\xi}_{\sigma,\widetilde{\mathbf{x}}} \colon \mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}_\sigma(\widetilde{\mathbf{x}})) \to \mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))$ the extension operator defined by

$$\begin{cases} \widetilde{\xi}_{\sigma,\widetilde{\mathbf{x}}}(f) \equiv f & \text{in } \overline{\Omega}_\sigma(\widetilde{\mathbf{x}}), \\ \widetilde{\xi}_{\sigma,\widetilde{\mathbf{x}}}(f)(\widetilde{x}_j + x) = \widetilde{\chi}\big(\frac{|x|}{\sigma}\big) f\big(\widetilde{x}^j + \sigma \frac{x}{|x|}\big) & \text{in } B_\sigma(\widetilde{x}^j) - B_{\sigma/2}(\widetilde{x}^j), \, \forall\, 1 \le j \le 3, \\ \widetilde{\xi}_{\sigma,\widetilde{\mathbf{x}}}(f) \equiv 0 & \text{in } B_{\sigma/2}(\widetilde{x}^1) \cup B_{\sigma/2}(\widetilde{x}^2) \cup B_{\sigma/2}(\widetilde{x}^3). \end{cases}$$

Here $\widetilde{\chi}$ is a cut-off function over $\mathbb{R}_+$ which is equal to 1 for $t \ge 1$ and equal to 0 for $t \le 1/2$. Obviously, there exists a constant $\bar{c} = \bar{c}(\nu) > 0$ only depending on ν such that

$$\big\| \widetilde{\xi}_{\sigma,\widetilde{\mathbf{x}}}(w) \big\|_{\mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))} \le \bar{c} \|w\|_{\mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}_\sigma(\widetilde{\mathbf{x}}))}. \tag{3.44}$$

We fix $\nu \in (-1, 0)$. In order to solve (3.43), it is enough to find $(\widetilde{v}_1, \widetilde{v}_2) \in (\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}})))^2$ solution of

$$\widetilde{v}_1 = \widetilde{\mathcal{K}}_\nu \circ \widetilde{\xi}_{r_\varepsilon,\widetilde{\mathbf{x}}} \circ \widetilde{Q}_1(\widetilde{v}_1, \widetilde{v}_2) \quad \text{and} \quad \widetilde{v}_2 = \widetilde{\mathcal{K}}_\nu \circ \widetilde{\xi}_{r_\varepsilon,\widetilde{\mathbf{x}}} \circ \widetilde{Q}_2(\widetilde{v}_1, \widetilde{v}_2), \tag{3.45}$$

where

$$\begin{aligned} \widetilde{Q}_1(\widetilde{v}_1, \widetilde{v}_2) &= \rho^4 e^{\gamma(\widetilde{\mathbf{w}}_1 + \widetilde{v}_1) + (1-\gamma)(\widetilde{\mathbf{w}}_2 + \widetilde{v}_2)} - \Delta^2 \widetilde{\mathbf{w}}_1 - \Sigma_a^1(\widetilde{\mathbf{w}}_1 + \widetilde{v}_1) - \Sigma_a^2(\widetilde{\mathbf{w}}_1 + \widetilde{v}_1), \\ \widetilde{Q}_2(\widetilde{v}_1, \widetilde{v}_2) &= \rho^4 e^{\xi(\widetilde{\mathbf{w}}_2 + \widetilde{v}_2) + (1-\xi)(\widetilde{\mathbf{w}}_1 + \widetilde{v}_1)} - \Delta^2 \widetilde{\mathbf{w}}_2 - \Sigma_a^1(\widetilde{\mathbf{w}}_2 + \widetilde{v}_2) - \Sigma_a^2(\widetilde{\mathbf{w}}_2 + \widetilde{v}_2). \end{aligned}$$

We denote by

$$\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2)=\widetilde{\mathcal{K}}_\nu\circ\widetilde{\xi}_{r_\varepsilon,\widetilde{\mathbf{x}}}\circ\widetilde{Q}_1(\widetilde{v}_1,\widetilde{v}_2)\quad\text{and}\quad\widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2)=\widetilde{\mathcal{K}}_\nu\circ\widetilde{\xi}_{r_\varepsilon,\widetilde{\mathbf{x}}}\circ\widetilde{Q}_2(\widetilde{v}_1,\widetilde{v}_2).$$

Given $\kappa>0$ (whose value will be fixed later on), we further assume that for $i\in\{1,2,3\}$ and $j\in\{1,2\}$ the functions $\widetilde{\varphi}^i_j$, $\widetilde{\psi}^i_j$, the parameters η_i and the point $\widetilde{\mathbf{x}}=(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3)$ satisfy

$$\|\widetilde{\varphi}^i_j\|_{\mathcal{C}^{4,\alpha}(S^3)}\le\kappa r_\varepsilon^2,\quad\|\widetilde{\psi}^i_j\|_{\mathcal{C}^{2,\alpha}(S^3)}\le\kappa r_\varepsilon^2,\tag{3.46}$$

$$|\eta_i|\le\kappa r_\varepsilon^2,\quad|\widetilde{x}^i-x^i|\le\kappa r_\varepsilon.\tag{3.47}$$

Then, the following result holds.

Lemma 3.20. *Under the above assumptions, there exists a constant $c_\kappa>0$ such that*

$$\|\widetilde{\mathcal{N}}(0,0)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\le c_\kappa r_\varepsilon^2,\quad\|\widetilde{\mathcal{M}}(0,0)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\le c_\kappa r_\varepsilon^2,$$
$$\|\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{\mathcal{N}}(\widetilde{v}'_1,\widetilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\le c_\kappa r_\varepsilon^2\|(\widetilde{v}_1,\widetilde{v}_2)-(\widetilde{v}'_1,\widetilde{v}'_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}})))^2}$$

and

$$\|\widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{\mathcal{M}}(\widetilde{v}'_1,\widetilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\le c_\kappa r_\varepsilon^2\|(\widetilde{v}_1,\widetilde{v}_2)-(\widetilde{v}'_1,\widetilde{v}'_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}})))^2},$$

provided $(\widetilde{v}_1,\widetilde{v}_2,\widetilde{v}'_1,\widetilde{v}'_2)\in\left(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))\right)^4$ *satisfy*

$$\|(\widetilde{v}_1,\widetilde{v}_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}})))^2}\le 2c_\kappa r_\varepsilon^2\quad\textit{and}\quad\|(\widetilde{v}'_1,\widetilde{v}'_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}})))^2}\le 2c_\kappa r_\varepsilon^2.\tag{3.48}$$

Proof. As for the interior problem, the proof of the two first estimates follows from the asymptotic behavior of H^{ext} together with the assumption on the norm of boundary data $\widetilde{\varphi}^i_j$ and $\widetilde{\psi}^i_j$ given by (3.46). Indeed, let c_κ be a constant depending only on κ, by Lemma 3.13,

$$\left|H^{\text{ext}}\left(\widetilde{\varphi}^i_j,\widetilde{\psi}^i_j;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right|\le c_\kappa r_\varepsilon^3 r^{-1}.\tag{3.49}$$

On the other hand,

$$\widetilde{Q}_1(0,0)=\rho^4e^{\gamma\widetilde{\mathbf{w}}_1+(1-\gamma)\widetilde{\mathbf{w}}_2}-\Delta^2\widetilde{\mathbf{w}}_1-\Sigma^1_a\widetilde{\mathbf{w}}_1-\Sigma^2_a\widetilde{\mathbf{w}}_1$$

and

$$\widetilde{Q}_2(0,0)=\rho^4e^{\xi\widetilde{\mathbf{w}}_2+(1-\xi)\widetilde{\mathbf{w}}_2}-\Delta^2\widetilde{\mathbf{w}}_2-\Sigma^1_a\widetilde{\mathbf{w}}_2-\Sigma^2_a\widetilde{\mathbf{w}}_2.$$

We will estimate $\widetilde{Q}_1(0,0)$ in different sub-regions of $\overline{\Omega}^*(\widetilde{\mathbf{x}})$.

• In $B_{r_0/2}(\widetilde{x}^1) \setminus B_{r_\varepsilon}(\widetilde{x}^1)$, we have $\chi_{r_0}(x-\widetilde{x}^1)=1$, $\chi_{r_0}(x-\widetilde{x}^2)=0$, $\chi_{r_0}(x-\widetilde{x}^3)=0$ and $\Delta^2\widetilde{\mathbf{w}}_1=0$, we denote

$$\begin{aligned}
[\nabla,\chi_{r_0}]w &= \nabla\chi_{r_0}\cdot w+\chi_{r_0}\cdot\nabla w,\\
[\Delta,\chi_{r_0}]w &= w\Delta\chi_{r_0}+\chi_{r_0}\Delta w+2\nabla\chi_{r_0}\cdot\nabla w,\\
[\Delta^2,\chi_{r_0}]w &= w\Delta^2\chi_{r_0}+2\Delta w\Delta\chi_{r_0}+4\nabla(\Delta w)\cdot\nabla\chi_{r_0}\\
&\quad +4\nabla w\cdot\nabla(\Delta\chi_{r_0})+4\sum_{i,j=1}^{4}\frac{\partial^2\chi_{r_0}}{\partial z_i\partial z_j}\frac{\partial^2 w}{\partial z_i\partial z_j}.
\end{aligned}$$

Then

$$\begin{aligned}
&|\widetilde{Q}_1(0,0)|\\
&\le c_\kappa\varepsilon^4|x-\widetilde{x}^1|^{-8(1+\eta_1)}+\left(\frac{1+\eta_1}{\gamma}\right)\big|\Sigma_a^1 G_a(x,\widetilde{x}^1)\big|+(1+\eta_3)\big|\Sigma_a^1 G_a(x,\widetilde{x}^3)\big|\\
&\quad+\left(\frac{1+\eta_1}{\gamma}\right)\big|\Sigma_a^2 G_a(x,\widetilde{x}^1)\big|+(1+\eta_3)\big|\Sigma_a^2 G_a(x,\widetilde{x}^3)\big|\\
&\quad+\left|\Sigma_a^1 H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^1,\widetilde{\psi}_1^1;\frac{x-\widetilde{x}^1}{r_\varepsilon}\right)\right|+\left|\Sigma_a^2 H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^1,\widetilde{\psi}_1^1;\frac{x-\widetilde{x}^1}{r_\varepsilon}\right)\right|\\
&\le c_\kappa\varepsilon^4|x-\widetilde{x}^1|^{-8(1+\eta_1)}\\
&\quad+c_\kappa\left(\frac{1+\eta_1}{\gamma}\right)\left[2\left|\frac{\nabla a}{a}\right|\big|\nabla(\Delta G_a(x,\widetilde{x}^1))\big|\big|\widetilde{V}(x)\big|\big|\nabla\log a\big|\big|\nabla(G_a(x,\widetilde{x}^1))\big|\right]\\
&\quad+c_\kappa(1+\eta_3)\left[2\left|\frac{\nabla a}{a}\right|\big|\nabla(\Delta G_a(x,\widetilde{x}^3))\big||V(x)||\nabla\log a|\big|\nabla(G_a(x,\widetilde{x}^3))\big|\right]\\
&\quad+c_\kappa\left(\frac{1+\eta_1}{\gamma}\right)\left|\frac{\Delta a}{a}-V(x)\right|\big|\Delta G_a(x,\widetilde{x}^1)\big|+c_\kappa(1+\eta_3)\left|\frac{\Delta a}{a}-V(x)\right|\big|\Delta G_a(x,\widetilde{x}^3)\big|\\
&\quad+c_\kappa\left(2\left|\frac{\nabla a}{a}\right|\left|\nabla\left(\Delta H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^1,\widetilde{\psi}_1^1;\frac{x-\widetilde{x}^1}{r_\varepsilon}\right)\right)\right|+|V(x)||\nabla\log a|\left|\nabla H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^1,\widetilde{\psi}_1^1;\frac{x-\widetilde{x}^1}{r_\varepsilon}\right)\right|\right)\\
&\quad+c_\kappa\left|\frac{\Delta a}{a}-V(x)\right|\left|\Delta H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^1,\widetilde{\psi}_1^1;\frac{x-\widetilde{x}^1}{r_\varepsilon}\right)\right|.
\end{aligned}$$

Since $a(x)$ is a solution of (1.3) satisfying $\|\nabla a\|_\infty\le\beta$ and $V(x)$ is a smooth bounded potential and using the fact $|\nabla^i G_a(x,y)|\le|x-y|^{-i}$ for $i\ge1$, we obtain

$$\begin{aligned}
|\widetilde{Q}_1(0,0)| &\le c_\kappa\varepsilon^4|x-\widetilde{x}^1|^{-8(1+\eta_1)}\\
&\quad+c_\kappa\left(\frac{1+\eta_1}{\gamma}\right)\big[\beta|x-\widetilde{x}^1|^{-3}+\beta|x-\widetilde{x}^1|^{-1}\big]\\
&\quad+c_\kappa(1+\eta_3)\big[\beta|x-\widetilde{x}^3|^{-3}+\beta|x-\widetilde{x}^3|^{-1}\big]\\
&\quad+c_\kappa\left[\left(\frac{1+\eta_1}{\gamma}\right)\lambda|x-\widetilde{x}^1|^{-2}+(1+\eta_3)\lambda|x-\widetilde{x}^3|^{-2}\right]\\
&\quad+c_\kappa\big[\beta r_\varepsilon^3|x-\widetilde{x}^1|^{-4}+\beta r_\varepsilon^3|x-\widetilde{x}^1|^{-2}+\lambda r_\varepsilon^3|x-\widetilde{x}^1|^{-3}\big].
\end{aligned}$$

Hence, for $r = |x - \widetilde{x}^1|$, $\nu \in (-1, 0)$ and η_1 small enough, we get

$$\begin{aligned}\|\widetilde{Q}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0/2}(\widetilde{x}^1))} &\leq \sup_{r_\varepsilon \leq r \leq r_0/2} r^{4-\nu}|\widetilde{Q}_1(0,0)| \\ &\leq c_\kappa \varepsilon^4 r_\varepsilon^{-4} + 4c_\kappa\beta + 2c_\kappa\lambda + 2c_\kappa\beta r_\varepsilon^3 + c_\kappa\lambda r_\varepsilon^3 \leq c_\kappa r_\varepsilon^2.\end{aligned}$$

• In $B_{r_0}(\widetilde{x}^1) \setminus B_{r_0/2}(\widetilde{x}^1)$, taking into account that $\Delta^2 G_a(x, \widetilde{x}^1) = \Delta^2 G_a(x, \widetilde{x}^3) = 0$, using the fact that $a(x)$ is a solution of (1.3) satisfying (H), $V(x)$ is a smooth bounded potential, then we have

$$\begin{aligned}|\widetilde{Q}_1(0,0)| &\leq c_\kappa\varepsilon^4|x-\widetilde{x}^1|^{-8(1+\eta_1)} + |\Delta^2\widetilde{\mathbf{w}}_1| + |\Sigma_a^1\widetilde{\mathbf{w}}_1| + |\Sigma_a^2\widetilde{\mathbf{w}}_1| \\ &\leq c_\kappa\varepsilon^4|x-\widetilde{x}^1|^{-8(1+\eta_1)} + \sum_{i=1}^3\left|[\Delta^2, \chi_{r_0}(x-\widetilde{x}^i)]H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right| \\ &\quad + \frac{1+\eta_1}{\gamma}\left[2\left|\frac{\nabla a}{a}\right|\left|\nabla(\Delta G_a(x,\widetilde{x}^1))\right| + |V(x)||\nabla\log a|\left|\nabla(G_a(x,\widetilde{x}^1))\right|\right] \\ &\quad + (1+\eta_3)\left[2\left|\frac{\nabla a}{a}\right|\left|\nabla(\Delta(G_a(x,\widetilde{x}^3)))\right| + |V(x)||\nabla\log a|\left|\nabla(G_a(x,\widetilde{x}^3))\right|\right] \\ &\quad + \left|\sum_{i=1}^3 2\frac{\nabla a}{a}\cdot\nabla\left(\Delta\left(\chi_{r_0}(x-\widetilde{x}^i)H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right)\right)\right| \\ &\quad + \left|\sum_{i=1}^3 V(x)\nabla\log a\cdot\nabla\left(\chi_{r_0}(x-\widetilde{x}^i)H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right)\right| \\ &\quad + \left(\frac{1+\eta_1}{\gamma}\right)\left|\frac{\Delta a}{a} - V(x)\right|\left|\Delta G_a(x,\widetilde{x}^1)\right| + c_\kappa(1+\eta_3)\left|\frac{\Delta a}{a} - V(x)\right|\left|\Delta G_a(x,\widetilde{x}^3)\right| \\ &\quad + \left|\sum_{i=1}^3\left(\frac{\Delta a}{a} - V(x)\right)\Delta\left(\chi_{r_0}(x-\widetilde{x}^i)H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right)\right| \\ &\leq c_\kappa\varepsilon^4|x-\widetilde{x}^1|^{-8(1+\eta_1)} + c_\kappa r_\varepsilon^3|x-\widetilde{x}^1|^{-5} + c_\kappa\frac{1+\eta_1}{\gamma}\left[\beta|x-\widetilde{x}^1|^{-3} + \beta|x-\widetilde{x}^1|^{-1}\right] \\ &\quad + c_\kappa(1+\eta_3)\left[\beta|x-\widetilde{x}^3|^{-3} + \beta|x-\widetilde{x}^3|^{-1}\right] + c_\kappa\beta r_\varepsilon^3|x-\widetilde{x}^1|^{-4} + c_\kappa\beta r_\varepsilon^3|x-\widetilde{x}^1|^{-2} \\ &\quad + c_\kappa\frac{1+\eta_1}{\gamma}\lambda|x-\widetilde{x}^1|^{-2} + c_\kappa(1+\eta_3)\lambda|x-\widetilde{x}^3|^{-2} + c_\kappa\lambda r_\varepsilon^3|x-\widetilde{x}^1|^{-3}.\end{aligned}$$

Hence for $\nu \in (-1, 0)$ and η_1 small enough, we get

$$\begin{aligned}\|\widetilde{Q}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^1) - B_{r_0/2}(\widetilde{x}^1))} &\leq \sup_{r_0/2 \leq r \leq r_0} r^{4-\nu}|\widetilde{Q}_1(0,0)| \\ &\leq c_\kappa\varepsilon^4 + c_\kappa r_\varepsilon^3 + 5c_\kappa\beta + 2c_\kappa\beta r_\varepsilon^3 + 2c_\kappa\lambda + c_\kappa\lambda r_\varepsilon^3 \leq c_\kappa r_\varepsilon^2.\end{aligned}$$

• In $B_{r_0/2}(\widetilde{x}^2) \setminus B_{r_\varepsilon}(\widetilde{x}^2)$, we have $\chi_{r_0}(x - \widetilde{x}^1) = 0$, $\chi_{r_0}(x - \widetilde{x}^2) = 1$, $\chi_{r_0}(x - \widetilde{x}^3) = 0$

and $\Delta^2\widetilde{\mathbf{w}}_1 = 0$, so $\widetilde{Q}_1(0,0) = \rho^4 e^{\gamma\widetilde{\mathbf{w}}_1+(1-\gamma_1)\widetilde{\mathbf{w}}_2} - \Sigma_a^1\widetilde{\mathbf{w}}_1 - \Sigma_a^2\widetilde{\mathbf{w}}_1$. Then

$$\begin{aligned}|\widetilde{Q}_1(0,0)| &\le \varepsilon^4|x-\widetilde{x}^2|^{-8\frac{(1-\gamma)(1+\eta_2)}{\xi}} + \frac{1+\eta_1}{\gamma}\big|\Sigma_a^1 G_a(x,\widetilde{x}^1)\big| + (1+\eta_3)\big|\Sigma_a^1 G_a(x,\widetilde{x}^3)\big| \\ &\quad + \frac{1+\eta_1}{\gamma}\big|\Sigma_a^2 G_a(x,\widetilde{x}^1)\big| + (1+\eta_3)\big|\Sigma_a^2 G_a(x,\widetilde{x}^3)\big| + \left|\Sigma_a^1 H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^2,\widetilde{\psi}_1^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\right| \\ &\quad + \left|\Sigma_a^2 H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^2,\widetilde{\psi}_1^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\right|.\end{aligned}$$

Hence, $\nu \in (-1,0)$ and η_2 small enough, we get

$$\begin{aligned}\|\widetilde{Q}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0/2}(\widetilde{x}^2))} &\le \sup_{r_\varepsilon \le r \le r_0/2} r^{4-\nu}|\widetilde{Q}_1(0,0)| \\ &\le c_\kappa r_\varepsilon^2 + 4c_\kappa\beta + 2c_\kappa\lambda + 2c_\kappa\beta r_\varepsilon^2 + c_\kappa\lambda r_\varepsilon^3 \\ &\le c_\kappa r_\varepsilon^2.\end{aligned}$$

- In $B_{r_0}(\widetilde{x}^2)\setminus B_{r_0/2}(\widetilde{x}^2)$, using the estimate (3.49), there holds

$$\begin{aligned}|\widetilde{Q}_1(0,0)| &\le \varepsilon^4|x-\widetilde{x}^2|^{-8\frac{(1-\gamma)(1+\eta_2)}{\xi}} + \sum_{i=1}^{3}\left|[\Delta^2,\chi_{r_0}(x-\widetilde{x}^i)]H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i,\widetilde{\psi}_1^i;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right| \\ &\quad + |\Sigma_a^1\widetilde{\mathbf{w}}_1| + |\Sigma_a^2\widetilde{\mathbf{w}}_1|.\end{aligned}$$

Hence for $\nu \in (-1,0)$ and η_2 small, we find

$$\begin{aligned}\|\widetilde{Q}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^1)-B_{r_0/2}(\widetilde{x}^1))} &\le \sup_{r_0/2\le r\le r_0} r^{4-\nu}|\widetilde{Q}_1(0,0)| \\ &\le c_\kappa\varepsilon^4 + c_\kappa r_\varepsilon^3 + 5c_\kappa\beta + 2c_\kappa\beta r_\varepsilon^3 + 2c_\kappa\lambda + c_\kappa\lambda r_\varepsilon^3 \le c_\kappa r_\varepsilon^2.\end{aligned}$$

Similarly, for $\nu \in (-1,0)$ and η_3 small enough, we can prove the same result for $\widetilde{x}^3$.

- In $\Omega \setminus \big(B_{r_0}(\widetilde{x}^1)\cup B_{r_0}(\widetilde{x}^2)\cup B_{r_0}(\widetilde{x}^3)\big)$, we have $\chi_{r_0}(x-\widetilde{x}^1)=0$, $\chi_{r_0}(x-\widetilde{x}^2)=0$, $\chi_{r_0}(x-\widetilde{x}^3)=0$ and $\Delta^2\widetilde{\mathbf{w}}_1=0$. Thus

$$\begin{aligned}|\widetilde{Q}_1(0,0)| &\le \varepsilon^4 e^{\gamma\widetilde{\mathbf{w}}_1+(1-\gamma)\widetilde{\mathbf{w}}_2} + |\Sigma_a^1\widetilde{\mathbf{w}}_1| + |\Sigma_a^2\widetilde{\mathbf{w}}_1| \\ &\le c_\kappa\varepsilon^4|x-\widetilde{x}^1|^{-8(1+\eta_1)}|x-\widetilde{x}^2|^{-8\frac{(1-\gamma)(1-\eta_2)}{\xi}}|x-\widetilde{x}^3|^{-8(1+\eta_3)} \\ &\quad + c_\kappa\frac{1+\eta_1}{\gamma}\big|\Sigma_a^1 G_a(x,\widetilde{x}^1)\big| + c_\kappa(1+\eta_3)\big|\Sigma_a^1 G_a(x,\widetilde{x}^3)\big| \\ &\quad + c_\kappa\frac{1+\eta_1}{\gamma}\big|\Sigma_a^2 G_a(x,\widetilde{x}^1)\big| + c_\kappa(1+\eta_3)\big|\Sigma_a^2 G_a(x,\widetilde{x}^3)\big| \\ &\le c_\kappa\varepsilon^4|x-\widetilde{x}^1|^{-8(1+\eta_1)} \\ &\quad + c_\kappa\frac{1+\eta_1}{\gamma}\left[2\left|\frac{\nabla a}{a}\right|\big|\nabla(\Delta G_a(x,\widetilde{x}^1))\big| + |V(x)||\nabla\log a|\big|\nabla(G_a(x,\widetilde{x}^1))\big|\right] \\ &\quad + c_\kappa(1+\eta_3)\left[2\left|\frac{\nabla a}{a}\right|\big|\nabla(\Delta G_a(x,\widetilde{x}^3))\big| + |V(x)||\nabla\log a|\big|\nabla(G_a(x,\widetilde{x}^3))\big|\right]\end{aligned}$$

$$
\begin{aligned}
&+ c_\kappa \frac{1+\eta_1}{\gamma}\left|\frac{\Delta a}{a} - V(x)\right|\left|\Delta G_a(x,\widetilde{x}^1)\right| + c_\kappa(1+\eta_3)\left|\frac{\Delta a}{a} - V(x)\right|\left|\Delta G_a(x,\widetilde{x}^3)\right| \\
&\le c_\kappa \varepsilon^4 |x-\widetilde{x}^1|^{-8(1+\eta_1)} + c_\kappa \beta \frac{1+\eta_1}{\gamma}|x-\widetilde{x}^1|^{-3} + c_\kappa \beta |x-\widetilde{x}^1|^{-1} \\
&\quad + c_\kappa \beta(1+\eta_3)|x-\widetilde{x}^3|^{-3} + c_\kappa\beta(1+\eta_3)|x-\widetilde{x}^3|^{-1} + c_\kappa \frac{1+\eta_1}{\gamma}\lambda|x-\widetilde{x}^1|^{-2} \\
&\quad + c_\kappa \lambda(1+\eta_3)|x-\widetilde{x}^3|^{-2}.
\end{aligned}
$$

So, for $\nu \in (-1,0)$, $r = |x-\widetilde{x}^1|$ and η_1 small enough, we get

$$
\|\widetilde{Q}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}\left(\overline{\Omega} - \bigcup_{i=1}^3 B_{r_0}(\widetilde{x}^i)\right)} \le \sup_{r\ge r_0} r^{4-\nu}|\widetilde{Q}_1(0,0)| \le c_\kappa \varepsilon^4 + 4c_\kappa\beta + 2c_\kappa\lambda \le c_\kappa r_\varepsilon^2.
$$

Now, we are interested in the second equation of the previous system.

• In $B_{r_0/2}(\widetilde{x}^1)\setminus B_{r_\varepsilon}(\widetilde{x}^1)$, we have $\chi_{r_0}(x-\widetilde{x}^1) = 1$, $\chi_{r_0}(x-\widetilde{x}^2)=0$, $\chi_{r_0}(x-\widetilde{x}^3)=0$ and $\Delta^2\widetilde{\mathbf{w}}_2 = 0$ so that $\widetilde{Q}_2(0,0) = \rho^4 e^{\xi\widetilde{\mathbf{w}}_2 + (1-\xi)\widetilde{\mathbf{w}}_1} - \Sigma_a^1\widetilde{\mathbf{w}}_2 - \Sigma_a^2\widetilde{\mathbf{w}}_2$. Then

$$
\begin{aligned}
|\widetilde{Q}_2(0,0)| &\le \varepsilon^4|x-\widetilde{x}^1|^{-8\frac{(1-\xi)(1+\eta_1)}{\gamma}} + \frac{1+\eta_2}{\xi}\left|\Sigma_a^1 G_a(x,\widetilde{x}^2)\right| + (1+\eta_3)\left|\Sigma_a^1 G_a(x,\widetilde{x}^3)\right| \\
&\quad + \frac{1+\eta_2}{\xi}\left|\Sigma_a^2 G_a(x,\widetilde{x}^2)\right| + (1+\eta_3)\left|\Sigma_a^2 G_a(x,\widetilde{x}^3)\right| \\
&\quad + \left|\Sigma_a^1 H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^1, \widetilde{\psi}_2^1; \frac{x-\widetilde{x}^1}{r_\varepsilon}\right)\right| + \left|\Sigma_a^2 H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^1, \widetilde{\psi}_2^1; \frac{x-\widetilde{x}^1}{r_\varepsilon}\right)\right|.
\end{aligned}
$$

Hence for $\nu \in (-1,0)$, $r = |x-\widetilde{x}^1|$ and η_1 small enough, we get

$$
\begin{aligned}
\|\widetilde{Q}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0/2}(\widetilde{x}^1))} &\le \sup_{r_\varepsilon \le r \le r_0/2} r^{4-\nu}|\widetilde{Q}_2(0,0)| \\
&\le c_\kappa r_\varepsilon^2 + 4c_\kappa\beta + 2c_\kappa\lambda + 2c_\kappa\beta r_\varepsilon^3 + c_\kappa\lambda r_\varepsilon^3 \le c_\kappa r_\varepsilon^2.
\end{aligned}
$$

• In $B_{r_0}(\widetilde{x}^1)\setminus B_{r_0/2}(\widetilde{x}^1)$, using the estimate (3.49), then we have

$$
\begin{aligned}
|\widetilde{Q}_2(0,0)| &\le c_\kappa\varepsilon^4|x-\widetilde{x}^1|^{-8\frac{(1-\xi)(1+\eta_1)}{\gamma}} + \left|\sum_{i=1}^3[\Delta^2, \chi_{r_0}(x-\widetilde{x}^i)] H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right| \\
&\quad + c_\kappa\frac{1+\eta_2}{\xi}\left|\Sigma_a^1 G_a(x,\widetilde{x}^2)\right| + c_\kappa(1+\eta_3)\left|\Sigma_a^1 G_a(x,\widetilde{x}^3)\right| + c_\kappa\frac{1+\eta_2}{\xi}\left|\Sigma_a^2 G_a(x,\widetilde{x}^2)\right| \\
&\quad + c_\kappa(1+\eta_3)\left|\Sigma_a^2 G_a(x,\widetilde{x}^3)\right| + \left|\sum_{i=1}^3 \Sigma_a^1\left(\chi_{r_0}(x-\widetilde{x}^i) H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right)\right| \\
&\quad + \left|\sum_{i=1}^3 \Sigma_a^2\left(\chi_{r_0}(x-\widetilde{x}^i) H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right)\right|.
\end{aligned}
$$

So, for $\nu \in (-1,0)$ and η_1 small enough, we get

$$
\begin{aligned}
\|\widetilde{Q}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^1) - B_{r_0/2}(\widetilde{x}^1))} &\le \sup_{r_0/2 \le r \le r_0} r^{4-\nu}|\widetilde{Q}_2(0,0)| \\
&\le c_\kappa\varepsilon^4 + c_\kappa r_\varepsilon^2 + 5c_\kappa\beta + 2c_\kappa\beta r_\varepsilon^3 + 2c_\kappa\lambda + c_\kappa\lambda r_\varepsilon^3 \le c_\kappa r_\varepsilon^2.
\end{aligned}
$$

• In $B_{r_0/2}(\widetilde{x}^2) \setminus B_{r_\varepsilon}(\widetilde{x}^2)$, we have $\chi_{r_0}(x-\widetilde{x}^1)=0$, $\chi_{r_0}(x-\widetilde{x}^2)=1$, $\chi_{r_0}(x-\widetilde{x}^3)=0$ and $\Delta^2\widetilde{\mathbf{w}}_2=0$ so that $\widetilde{Q}_2(0,0)=\rho^4 e^{\xi\widetilde{\mathbf{w}}_2+(1-\xi)\widetilde{\mathbf{w}}_1}-\Sigma_a^1\widetilde{\mathbf{w}}_2-\Sigma_a^2\widetilde{\mathbf{w}}_2$. Then

$$\begin{aligned}|\widetilde{Q}_2(0,0)| &\le c_\kappa\varepsilon^4|x-\widetilde{x}^2|^{-8(1+\eta_2)}+c_\kappa\frac{1+\eta_2}{\xi}\left|\Sigma_a^1 G_a(x,\widetilde{x}^2)\right|+c_\kappa(1+\eta_3)\left|\Sigma_a^1 G_a(x,\widetilde{x}^3)\right|\\ &\quad+c_\kappa\frac{1+\eta_2}{\xi}\left|\Sigma_a^2 G_a(x,\widetilde{x}^2)\right|+c_\kappa(1+\eta_3)\left|\Sigma_a^2 G_a(x,\widetilde{x}^3)\right|\\ &\quad+\left|\Sigma_a^1 H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^2,\widetilde{\psi}_2^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\right|+\left|\Sigma_a^2 H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^2,\widetilde{\psi}_2^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\right|.\end{aligned}$$

We get

$$\begin{aligned}\|\widetilde{Q}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0/2}(\widetilde{x}^1))} &\le \sup_{r_\varepsilon\le r\le r_0/2} r^{4-\nu}|\widetilde{Q}_2(0,0)|\\ &\le c_\kappa\varepsilon^4 r_\varepsilon^{-4}+4c_\kappa\beta+2c_\kappa\lambda+2c_\kappa\beta r_\varepsilon^2+c_\kappa\lambda r_\varepsilon^3\le c_\kappa r_\varepsilon^2.\end{aligned}$$

• In $B_{r_0}(\widetilde{x}^2)\setminus B_{r_0/2}(\widetilde{x}^2)$, using the estimate (3.49), there holds

$$\begin{aligned}|\widetilde{Q}_2(0,0)| &\le c_\kappa\varepsilon^4|x-\widetilde{x}^2|^{-8(1+\eta_2)}+\left|\sum_{i=1}^3[\Delta^2,\chi_{r_0}(x-\widetilde{x}^i)]H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i,\widetilde{\psi}_2^i;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right|\\ &\quad+c_\kappa\frac{1+\eta_2}{\xi}\left|\Sigma_a^1 G_a(x,\widetilde{x}^2)\right|+c_\kappa(1+\eta_3)\left|\Sigma_a^1 G_a(x,\widetilde{x}^3)\right|+c_\kappa\frac{1+\eta_2}{\xi}\left|\Sigma_a^2 G_a(x,\widetilde{x}^2)\right|\\ &\quad+c_\kappa(1+\eta_3)\left|\Sigma_a^2 G_a(x,\widetilde{x}^3)\right|+\left|\sum_{i=1}^3\Sigma_a^1\left(\chi_{r_0}(x-\widetilde{x}^i)H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i,\widetilde{\psi}_2^i;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right)\right|\\ &\quad+\left|\sum_{i=1}^3\Sigma_a^2\left(\chi_{r_0}(x-\widetilde{x}^i)H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i,\widetilde{\psi}_2^i;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right)\right|.\end{aligned}$$

Hence, for $\nu\in(-1,0)$ and η_2 small enough, we get

$$\|\widetilde{Q}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^2)-B_{r_0/2}(\widetilde{x}^2))}\le\sup_{r_0/2\le r\le r_0} r^{4-\nu}|\widetilde{Q}_2(0,0)|\le c_\kappa r_\varepsilon^2.$$

Similarly, for $\nu\in(-1,0)$ and η_3 small enough, we can prove the same result for $\widetilde{x}^3$.

• In $\Omega\setminus\left(B_{r_0}(\widetilde{x}^1)\cup B_{r_0}(\widetilde{x}^2)\cup B_{r_0}(\widetilde{x}^3)\right)$, we have $\chi_{r_0}(x-\widetilde{x}^1)=0$, $\chi_{r_0}(x-\widetilde{x}^2)=0$, $\chi_{r_0}(x-\widetilde{x}^3)=0$, and $\Delta^2\widetilde{\mathbf{w}}_2=0$. So, for $\nu\in(-1,0)$, we have

$$\|\widetilde{Q}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}\left(\overline{\Omega}-\bigcup_{i=1}^3 B_{r_0}(\widetilde{x}^i)\right)}\le\sup_{r\ge r_0} r^{4-\nu}|\widetilde{Q}_2(0,0)|\le c_\kappa r_\varepsilon^2.$$

Making use of Proposition 3.5 together with (3.44), we conclude that

$$\|\widetilde{\mathcal{N}}(0,0)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\le c_\kappa r_\varepsilon^2\quad\text{and}\quad\|\widetilde{\mathcal{M}}(0,0)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\le c_\kappa r_\varepsilon^2.$$

For the proof of the third estimate, let $\widetilde{v}_1$, $\widetilde{v}_2$, $\widetilde{v}_1'$ and $\widetilde{v}_2' \in \mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*)$ satisfy (3.48), we have

$$
\begin{aligned}
&\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big|\widetilde{S}_1(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{S}_1(\widetilde{v}_1',\widetilde{v}_2')\big| \\
&\le c_\kappa\varepsilon^4\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big|e^{\gamma\widetilde{\mathbf{w}}_1+(1-\gamma)\widetilde{\mathbf{w}}_2}\big|\big|e^{\gamma\widetilde{v}_1+(1-\gamma)\widetilde{v}_2}-e^{\gamma\widetilde{v}_1'+(1-\gamma)\widetilde{v}_2'}\big| + c_\kappa\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big|\Sigma_a^1\widetilde{v}_1-\Sigma_a^1\widetilde{v}_1'\big| \\
&\quad + c_\kappa\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big|\Sigma_a^2\widetilde{v}_1-\Sigma_a^2\widetilde{v}_1'\big| \\
&\le c_\kappa\varepsilon^4\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}|x-\widetilde{x}^1|^{-8(1+\eta_1)}|x-\widetilde{x}^2|^{-8\frac{(1-\gamma)(1+\eta_2)}{\xi}}|x-\widetilde{x}^3|^{-8(1+\eta_3)} \\
&\qquad\times e^{(1+\eta_1)H_a(x,\widetilde{x}^1)+(1+\eta_3)H_a(x,\widetilde{x}^3)+(1-\gamma)\frac{1+\eta_2}{\xi}H_a(x,\widetilde{x}^2)} \\
&\qquad\times\prod_{i=1}^3\Big(e^{\gamma\chi_{r_0}(x-\widetilde{x}^i)H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i,\widetilde{\psi}_1^i;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)}e^{(1-\gamma)\chi_{r_0}(x-\widetilde{x}^i)H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i,\widetilde{\psi}_2^i;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)}\Big) \\
&\qquad\times\big|e^{\gamma\widetilde{v}_1+(1-\gamma)\widetilde{v}_2}-e^{\gamma\widetilde{v}_1'+(1-\gamma)\widetilde{v}_2'}\big| \\
&\quad + c_\kappa\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big|\Sigma_a^1(\widetilde{v}_1-\widetilde{v}_1')\big| + c_\kappa\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big|\Sigma_a^2(\widetilde{v}_1-\widetilde{v}_1')\big| \\
&\le c_\kappa\varepsilon^4\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}|x-\widetilde{x}^1|^{-8(1+\eta_1)}\big(\gamma|\widetilde{v}_1-\widetilde{v}_1'|+(1-\gamma)|\widetilde{v}_2-\widetilde{v}_2'|\big) \\
&\quad + c_\kappa\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\left|2\frac{\nabla a}{a}\cdot\nabla(\Delta(\widetilde{v}_1-\widetilde{v}_1'))-V(x)\nabla\log a\cdot\nabla(\widetilde{v}_1-\widetilde{v}_1')\right| \\
&\quad + c_\kappa\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\left|\left(\frac{\Delta a}{a}-V(x)\right)\cdot\Delta(\widetilde{v}_1-\widetilde{v}_1')\right|.
\end{aligned}
$$

Using the fact for all $w\in\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}_{r_\varepsilon}(\widetilde{x}))$, there exists a constant $c>0$ such that $|\nabla^\ell w|\le c_\kappa r^{\nu-\ell}\|w\|_{\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(\widetilde{x}))}$ and the function $a(x)$ satisfying (H), $V(x)$ is a smooth bounded potential and $\|\nabla a\|_\infty\le\beta$, then

$$
\begin{aligned}
&\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big|\widetilde{S}_1(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{S}_1(\widetilde{v}_1',\widetilde{v}_2')\big| \\
&\le c_\kappa\varepsilon^4\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}|x-\widetilde{x}^1|^{-8(1+\eta_1)}\big(\gamma r^\nu\|\widetilde{v}_1-\widetilde{v}_1'\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{x}))}+(1-\gamma)r^\nu\|\widetilde{v}_2-\widetilde{v}_2'\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{x}))}\big) \\
&\quad + c_\kappa\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big(\beta r^{\nu-3}\|\widetilde{v}_1-\widetilde{v}_1'\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{x}))}+\beta r^{\nu-1}\|\widetilde{v}_1-\widetilde{v}_1'\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{x}))}\big) \\
&\quad + c_\kappa\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\lambda\|f\|_\infty r^{\nu-2}\|\widetilde{v}_1-\widetilde{v}_1'\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{x}))}.
\end{aligned}
$$

For $r=|x-\widetilde{x}^1|$ and η_1 small enough, we have

$$
\begin{aligned}
&\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big|\widetilde{S}_1(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{S}_1(\widetilde{v}_1',\widetilde{v}_2')\big| \\
&\le c_\kappa\varepsilon^4 r_\varepsilon^{-4}\big(\gamma\|\widetilde{v}_1-\widetilde{v}_1'\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{x}))}+(1-\gamma)\|\widetilde{v}_2-\widetilde{v}_2'\|_{\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{x}))}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2c_\kappa\beta\|\widetilde{v}_1-\widetilde{v}_1'\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{x}))} + c_\kappa\lambda\|\widetilde{v}_1-\widetilde{v}_1'\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{x}))}\\
&\le c_\kappa r_\varepsilon^2\big(\|\widetilde{v}_1-\widetilde{v}_1'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))} + \|\widetilde{v}_2-\widetilde{v}_2'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\big).
\end{aligned}
$$

Using (3.44) and Proposition 3.5, we conclude that

$$
\|\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{\mathcal{N}}(\widetilde{v}_1',\widetilde{v}_2')\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))} \le c_\kappa r_\varepsilon^2\big(\|\widetilde{v}_1-\widetilde{v}_1'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}+\|\widetilde{v}_2-\widetilde{v}_2'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\big). \tag{3.50}
$$

Similarly we can use the same argument to prove

$$
\begin{aligned}
&\|\widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{\mathcal{M}}(\widetilde{v}_1',\widetilde{v}_2')\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\\
&\le c_\kappa r_\varepsilon^2\big(\|\widetilde{v}_1-\widetilde{v}_1'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))} + \|\widetilde{v}_2-\widetilde{v}_2'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}}))}\big).
\end{aligned} \tag{3.51}
$$

This proves the lemma. □

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \le 1/2$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. Then, (3.50) and (3.51) are enough to show that

$$
(\widetilde{v}_1,\widetilde{v}_2)\mapsto\big(\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2),\widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2)\big)
$$

is a contraction from the ball

$$
\big\{(\widetilde{v}_1,\widetilde{v}_2)\in\big(\mathcal{C}^{4,\alpha}_\nu(\mathbb{R}^4)\big)^2 : \|(\widetilde{v}_1,\widetilde{v}_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\mathbb{R}^4))^2}\le 2c_\kappa r_\varepsilon^2\big\}
$$

into itself. Hence there exist a unique fixed point $(\widetilde{v}_1,\widetilde{v}_2)$ in this set, which is a solution of (3.45). Applying a fixed point Theorem for contraction mappings, we conclude the following proposition.

Proposition 3.21. *Given $\kappa>0$, there exist $\varepsilon_\kappa>0$, $\lambda_\kappa>0$, $\beta_\kappa>0$ (depending on κ) such that for any $\varepsilon\in(0,\varepsilon_\kappa)$, $\lambda\in(0,\lambda_\kappa)$, $\beta\in(0,\beta_\kappa)$, η_i and $\widetilde{x}^i$ satisfying* (3.47) *and functions $\widetilde{\varphi}^i_j$ and $\widetilde{\psi}^i_j$ satisfying* (3.24) *and* (3.46)*, there exists a unique $(\widetilde{v}_1,\widetilde{v}_2)$ $(:=(\widetilde{v}_{1,\varepsilon,\lambda,\beta,\eta_1,\eta_3,\widetilde{\mathbf{x}},\widetilde{\varphi}^i_1,\widetilde{\psi}^i_1},\widetilde{v}_{2,\varepsilon,\lambda,\beta,\eta_2,\eta_3,\widetilde{\mathbf{x}},\widetilde{\varphi}^i_2,\widetilde{\psi}^i_2}))$ solution of* (3.45) *so that for (v_1,v_2) defined by*

$$
\begin{aligned}
v_1(x) &:= \frac{1+\eta_1}{\gamma}G_a(x,\widetilde{x}^1)+(1+\eta_3)G_a(x,\widetilde{x}^3)+\sum_{i=1}^3\chi_{r_0}(x-\widetilde{x}^i)H_1^{\text{ext}}\left(\widetilde{\varphi}^i_1,\widetilde{\psi}^i_1;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\\
&\quad+\widetilde{v}_1(x),\\
v_2(x) &:= \frac{1+\eta_2}{\xi}G_a(x,\widetilde{x}^2)+(1+\eta_3)G_a(x,\widetilde{x}^3)+\sum_{i=1}^3\chi_{r_0}(x-\widetilde{x}^i)H_2^{\text{ext}}\left(\widetilde{\varphi}^i_2,\widetilde{\psi}^i_2;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\\
&\quad+\widetilde{v}_2(x)
\end{aligned}
$$

solve (3.42) *in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\mathbf{x}})$. In addition, we have*

$$
\|(\widetilde{v}_1,\widetilde{v}_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{x}})))^2}\le 2c_\kappa r_\varepsilon^2.
$$

3.4. The nonlinear Cauchy-data matching

We will gather the results of the previous sections. Using the previous notations, assume that $\widetilde{\mathbf{x}} := (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) \in \Omega^3$ are given close to $\mathbf{x} := (x^1, x^2, x^3)$. Assume also that

$$\boldsymbol{\tau} := (\tau_1, \tau_2, \tau_3) \in [\tau_1^-, \tau_1^+] \times [\tau_2^-, \tau_2^+] \times [\tau_3^-, \tau_3^+] \subset (0, \infty)^3$$

are given (the values of τ_l^- and τ_l^+ for $l = 1, 2, 3$ will be fixed later). First, we consider some set of boundary data $\boldsymbol{\varphi}^i := (\varphi_1^i, \varphi_2^i) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\boldsymbol{\psi}^i := (\psi_1^i, \psi_2^i) \in (\mathcal{C}^{2,\alpha}(S^3))^2$. According to the results of Propositions 3.16, 3.17 and 3.19 and provided $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\beta \in (0, \beta_\kappa)$, we can find $u_{\mathrm{int}} := (u_{\mathrm{int},1}, u_{\mathrm{int},2})$ a solution of (3.5) in $B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3)$, which can be decomposed as

$$u_{\mathrm{int},1}(x) := \begin{cases} \frac{1}{\gamma} u_{\varepsilon,\tau_1}(x - \widetilde{x}^1) - \frac{1-\gamma}{\gamma\xi} G_a(x, \widetilde{x}^2) - \frac{1-\gamma}{\gamma} G_a(x, \widetilde{x}^3) - \frac{\ln\gamma}{\gamma} \\ \quad + h_1^1\big(\frac{R_\varepsilon^1(x-\widetilde{x}^1)}{r_\varepsilon}\big) + H_1^{\mathrm{int},1}\big(\varphi_1^1, \psi_1^1; \frac{x-\widetilde{x}^1}{r_\varepsilon}\big) + v_1^1\big(\frac{R_\varepsilon^1(x-\widetilde{x}^1)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^1), \\ \frac{1}{\gamma} G_a(x, \widetilde{x}^1) + G_a(x, \widetilde{x}^3) + h_1^2\big(\frac{R_\varepsilon^2(x-\widetilde{x}^2)}{r_\varepsilon}\big) \\ \quad + H_1^{\mathrm{int},2}\big(\varphi_1^2, \psi_1^2; \frac{x-\widetilde{x}^2}{r_\varepsilon}\big) + v_1^2\big(\frac{R_\varepsilon^2(x-\widetilde{x}^2)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^2), \\ u_{\varepsilon,\tau_3}(x - \widetilde{x}^3) + h_1^3\big(\frac{R_\varepsilon^3(x-\widetilde{x}^3)}{r_\varepsilon}\big) \\ \quad + H_1^{\mathrm{int},3}\big(\varphi_1^3, \psi_1^3; \frac{x-\widetilde{x}^3}{r_\varepsilon}\big) + v_1^3\big(\frac{R_\varepsilon^3(x-\widetilde{x}^3)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^3) \end{cases}$$

and

$$u_{\mathrm{int},2}(x) := \begin{cases} \frac{1}{\xi} G_a(x, \widetilde{x}^2) + G_a(x, \widetilde{x}^3) + h_2^1\big(\frac{R_\varepsilon^1(x-\widetilde{x}^1)}{r_\varepsilon}\big) \\ \quad + H_2^{\mathrm{int},1}\big(\varphi_2^1, \psi_2^1; \frac{x-\widetilde{x}^1}{r_\varepsilon}\big) + v_2^1\big(\frac{R_\varepsilon^1(x-\widetilde{x}^1)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^1), \\ \frac{1}{\xi} u_{\varepsilon,\tau_2}(x - \widetilde{x}^2) - \frac{1-\xi}{\xi} G_a(x, \widetilde{x}^3) - \frac{1-\xi}{\gamma\xi} G_a(x, \widetilde{x}^1) - \frac{\ln\xi}{\xi} \\ \quad + h_2^2\big(\frac{R_\varepsilon^2(x-\widetilde{x}^2)}{r_\varepsilon}\big) + H_2^{\mathrm{int},2}\big(\varphi_2^2, \psi_2^2; \frac{x-\widetilde{x}^2}{r_\varepsilon}\big) + v_2^2\big(\frac{R_\varepsilon^2(x-\widetilde{x}^2)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^2), \\ u_{\varepsilon,\tau_3}(x - \widetilde{x}^3) + h_2^3\big(\frac{R_\varepsilon^3(x-\widetilde{x}^3)}{r_\varepsilon}\big) \\ \quad + H_2^{\mathrm{int},3}\big(\varphi_1^3, \psi_1^3; \frac{x-\widetilde{x}^3}{r_\varepsilon}\big) + v_2^3\big(\frac{R_\varepsilon^3(x-\widetilde{x}^3)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^3), \end{cases}$$

where for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$, $R_\varepsilon^i = \tau_i \frac{r_\varepsilon}{\varepsilon}$ and the functions h_j^i and v_j^i satisfy

$$\|(h_1^1, h_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2, \quad \|(h_1^2, h_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2,$$
$$\|(h_1^3, h_2^3)\|_{(\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4))^2} \leq 2c_\kappa r_\varepsilon^2$$

and

$$\|(v_1^1, v_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2, \quad \|(v_1^2, v_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2,$$
$$\|(v_1^3, v_2^3)\|_{(\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4))^2} \leq 2c_\kappa r_\varepsilon^2.$$

Similarly, given some boundary data $\widetilde{\varphi}_j^i \in C^{4,\alpha}(S^3)$, $\widetilde{\psi}_j^i \in C^{2,\alpha}(S^3)$ satisfying (3.24), $(\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3$ satisfying (3.47), provided $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$ and $\beta \in (0,\beta_\kappa)$, by Proposition 3.21, we find a solution $u_{\text{ext}} := (u_{\text{ext},1}, u_{\text{ext},2})$ of (3.42) in $\overline{\Omega} \setminus (B_{r_\varepsilon}\widetilde{x}^1 \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3))$ which can be decomposed as

$$\begin{aligned} u_{\text{ext},1}(x) :=& \frac{1+\eta_1}{\gamma} G_a(x,\widetilde{x}^1) + (1+\eta_3) G_a(x,\widetilde{x}^3) + \sum_{i=1}^{3} \chi_{r_0}(x-\widetilde{x}^i) H_1^{\text{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right) \\ &+ \widetilde{v}_1(x), \\ u_{\text{ext},2}(x) :=& \frac{1+\eta_2}{\xi} G_a(x,\widetilde{x}^2) + (1+\eta_3) G_a(x,\widetilde{x}^3) + \sum_{i=1}^{3} \chi_{r_0}(x-\widetilde{x}^i) H_2^{\text{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right) \\ &+ \widetilde{v}_2(x) \end{aligned}$$

with $\widetilde{v}_1, \widetilde{v}_2 \in \mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))$ satisfying

$$\|(\widetilde{v}_1, \widetilde{v}_2)\|_{(\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}})))^2} \le 2c_\kappa r_\varepsilon^2.$$

It remains to determine the parameters and the boundary data in such a way that the function which is equal to u_{int} in $B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3)$ and to u_{ext} in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\mathbf{x}})$ is a smooth function. This amounts to find the boundary data and the parameters so that, for each $j = 1, 2$,

$$u_{\text{int},j} = u_{\text{ext},j}, \quad \partial_r u_{\text{int},j} = \partial_r u_{\text{ext},j}, \quad \Delta u_{\text{int},j} = \Delta u_{\text{ext},j}, \quad \partial_r \Delta u_{\text{int},j} = \partial_r \Delta u_{\text{ext},j} \tag{3.52}$$

on $\partial B_{r_\varepsilon}(\widetilde{x}^1)$, $\partial B_{r_\varepsilon}(\widetilde{x}^2)$ and $\partial B_{r_\varepsilon}(\widetilde{x}^3)$.

Suppose that (3.52) is verified, this provides that for each ε small enough $u_\varepsilon \in \mathcal{C}^{4,\alpha}$ (which is obtained by matching together the functions u_{int} and the function u_{ext}), a weak solution of our system and elliptic regularity theory implies that this solution is in fact smooth. That will complete the proof since, as ε tends to 0, the sequence of solutions we have obtained satisfies the required singular limit behavior.

Before we proceed, the following remarks are due. First it will be convenient to observe that the function u_{ε,τ_i} can be expanded as

$$u_{\varepsilon,\tau_i}(x) = -4\ln\tau_i - 8\ln|x| + \mathcal{O}\left(\frac{\varepsilon^2\tau_i^{-2}}{|x|^2}\right) \quad \text{on } \partial B_{r_\varepsilon}(x^i).$$

- For x on $\partial B_{r_\varepsilon}(\widetilde{x}^1)$, we have

$$\begin{aligned} &(u_{\text{int},1} - u_{\text{ext},1})(x) \\ =& -\frac{4}{\gamma}\ln\tau_1 + \frac{8\eta_1}{\gamma}\ln|x-\widetilde{x}^1| - \frac{1-\gamma}{\gamma\xi} G_a(x,\widetilde{x}^2) - \frac{\ln\gamma}{\gamma} \\ &+ H_1^{\text{int},1}\left(\varphi_1^1, \psi_1^1, \frac{x-\widetilde{x}^1}{r_\varepsilon}\right) - H_1^{\text{ext}}\left(\widetilde{\varphi}_1^1, \widetilde{\psi}_1^1, \frac{x-\widetilde{x}^1}{r_\varepsilon}\right) - \frac{1+\eta_1}{\gamma} H_a(x,\widetilde{x}^1) \\ &- \left(1+\eta_3+\frac{1-\gamma}{\gamma}\right) G_a(x,\widetilde{x}^3) + \mathcal{O}\left(\frac{\varepsilon^2\tau_1^{-2}}{|x-\widetilde{x}^1|^2}\right) + \mathcal{O}(r_\varepsilon^2). \end{aligned} \tag{3.53}$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{x}^1)$ in (3.52), it will be more convenient to solve on S^3 the following set of equations

$$(3.54)\qquad \begin{aligned} (u_{\mathrm{int},1}-u_{\mathrm{ext},1})(\widetilde{x}^1+r_\varepsilon\,\cdot\,)&=0, & \partial_r(u_{\mathrm{int},1}-u_{\mathrm{ext},1})(\widetilde{x}^1+r_\varepsilon\,\cdot\,)&=0,\\ \Delta(u_{\mathrm{int},1}-u_{\mathrm{ext},1})(\widetilde{x}^1+r_\varepsilon\,\cdot\,)&=0, & \partial_r\Delta(u_{\mathrm{int},1}-u_{\mathrm{ext},1})(\widetilde{x}^1+r_\varepsilon\,\cdot\,)&=0.\end{aligned}$$

Since the boundary data are chosen to satisfy (3.23) or (3.24), we decompose

$$\begin{aligned} \varphi_1^1&=\varphi_{1,0}^1+\varphi_{1,1}^1+\varphi_1^{1,\perp}, & \psi_1^1&=8\varphi_{1,0}^1+12\varphi_{1,1}^1+\psi_1^{1,\perp},\\ \widetilde{\varphi}_1^1&=\widetilde{\varphi}_{1,0}^1+\widetilde{\varphi}_{1,1}^1+\widetilde{\varphi}_1^{1,\perp}, & \widetilde{\psi}_1^1&=\widetilde{\psi}_{1,1}^1+\widetilde{\psi}_1^{1,\perp},\end{aligned}$$

where $\varphi_{1,0}^1,\widetilde{\varphi}_{1,0}^1\in\mathbb{E}_0=\mathbb{R}$ are constant on S^3, $\varphi_{1,1}^1$, $\widetilde{\varphi}_{1,1}^1$, $\widetilde{\psi}_{1,1}^1$ belong to $\mathbb{E}_1=\mathrm{Span}\{e_1,e_2,e_3,e_4\}$ and $\varphi_1^{1,\perp}$, $\widetilde{\varphi}_1^{1,\perp}$, $\psi_1^{1,\perp}$, $\widetilde{\psi}_1^{1,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

Using (3.53), we have for $x\in S^3$,

$$\begin{aligned} &(u_{\mathrm{int},1}-u_{\mathrm{ext},1})(\widetilde{x}^1+r_\varepsilon x)\\ &=-\frac{4}{\gamma}\ln\tau_1+\frac{8\eta_1}{\gamma}\ln(r_\varepsilon|x|)-\frac{\ln\gamma}{\gamma}+H_1^{\mathrm{int},1}(\varphi_1^1,\psi_1^1,x)-H_1^{\mathrm{ext}}(\widetilde{\varphi}_1^1,\widetilde{\psi}_1^1,x)\\ &\quad-\frac{1}{\gamma}\left(H_a(\widetilde{x}^1,\widetilde{x}^1)+\frac{1-\gamma}{\xi}G_a(\widetilde{x}^1,\widetilde{x}^2)+G_a(\widetilde{x}^1,\widetilde{x}^3)\right)-\frac{\eta_1}{\gamma}H_a(\widetilde{x}^1,\widetilde{x}^1)\\ &\quad-\eta_3G_a(\widetilde{x}^1,\widetilde{x}^3)+\mathcal{O}(r_\varepsilon^2).\end{aligned}$$

Then, the projection of the set equations (3.54) over $\mathbb{E}_0$ will yield

$$(3.55)\qquad \begin{aligned} -4\ln\tau_1+8\eta_1\ln r_\varepsilon-\ln\gamma+\gamma\varphi_{1,0}^1-\gamma\widetilde{\varphi}_{1,0}^1-\mathcal{E}_1(\widetilde{x}^1,\widetilde{\mathbf{x}})+\mathcal{O}(r_\varepsilon^2)&=0,\\ 8\eta_1+2\gamma\varphi_{1,0}^1+2\gamma\widetilde{\varphi}_{1,0}^1+\mathcal{O}(r_\varepsilon^2)&=0,\\ 16\eta_1+8\gamma\varphi_{1,0}^1+\mathcal{O}(r_\varepsilon^2)&=0,\\ -32\eta_1+\mathcal{O}(r_\varepsilon^2)&=0,\end{aligned}$$

where

$$\mathcal{E}_1(\,\cdot\,,\widetilde{\mathbf{x}}):=H_a(\,\cdot\,,\widetilde{x}^1)+\frac{1-\gamma}{\xi}G_a(\,\cdot\,,\widetilde{x}^2)+G_a(\,\cdot\,,\widetilde{x}^3).$$

The system (3.55) can be simply written as

$$\eta_1=\mathcal{O}(r_\varepsilon^2),\ \ \varphi_{1,0}^1=\mathcal{O}(r_\varepsilon^2),\ \ \widetilde{\varphi}_{1,0}^1=\mathcal{O}(r_\varepsilon^2)\ \text{ and }\ \frac{1}{\ln r_\varepsilon}[4\ln\tau_1+\ln\gamma+\mathcal{E}_1(\widetilde{x}^1,\widetilde{\mathbf{x}})]=\mathcal{O}(r_\varepsilon^2).$$

We are now in a position to define τ_1^- and τ_1^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\widetilde{x}^i$ will converge to x^i for $i\in\{1,2,3\}$ and τ_1 will converge to τ_1^* satisfying

$$4\ln\tau_1^*=-\ln\gamma-\mathcal{E}_1(x^1,\mathbf{x}).$$

Hence it is enough to choose τ_1^- and τ_1^+ in such a way that

$$4\ln(\tau_1^-)<-\ln\gamma-\mathcal{E}_1(x^1,\mathbf{x})<4\ln(\tau_1^+).$$

Consider now the projection of (3.54) over $\mathbb{E}_1$. Give a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \dots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$:

$$\overline{\nabla} f = \sum_{i=1}^{4} \partial_{x_i} f \, e_i.$$

With these notations in mind, we obtain the system of equations

$$\begin{aligned}
\varphi_{1,1}^{1} - \widetilde{\varphi}_{1,1}^{1} - \frac{1}{\gamma}\overline{\nabla}\mathcal{E}_1(\widetilde{x}^1, \widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
3\varphi_{1,1}^{1} + 3\widetilde{\varphi}_{1,1}^{1} + \frac{1}{2}\widetilde{\psi}_{1,1}^{1} - \frac{1}{\gamma}\overline{\nabla}\mathcal{E}_1(\widetilde{x}^1, \widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi_{1,1}^{1} - 3\widetilde{\varphi}_{1,1}^{1} - \widetilde{\psi}_{1,1}^{1} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi_{1,1}^{1} + 15\widetilde{\varphi}_{1,1}^{1} + \frac{18}{4}\widetilde{\psi}_{1,1}^{1} + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned}$$

which can be simplified as follows:

$$\varphi_{1,1}^{1} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_{1,1}^{1} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\psi}_{1,1}^{1} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \overline{\nabla}\mathcal{E}_1(\widetilde{x}^1, \widetilde{\mathbf{x}}) = \mathcal{O}(r_\varepsilon^2).$$

Finally, we consider the projection onto $L^2(S^3)^{\perp}$. Indeed, this is the step in the proof which makes use of the nondegeneracy condition assumed on the critical point of the functional $\mathcal{F}$, see also Remark 3.22 at the end of the section. This assumption is needed in order to obtain the order r_ε^2 in the following estimates. For more details on this condition, we refer the reader to [4]. This yields the system

$$\begin{aligned}
\varphi_1^{1,\perp} - \widetilde{\varphi}_1^{1,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\big(H^{\mathrm{int}}_{\varphi_1^{1,\perp}, \psi_1^{1,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_1^{1,\perp}, \widetilde{\psi}_1^{1,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\psi_1^{1,\perp} - \widetilde{\psi}_1^{1,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\Delta\big(H^{\mathrm{int}}_{\varphi_1^{1,\perp}, \psi_1^{1,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_1^{1,\perp}, \widetilde{\psi}_1^{1,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0.
\end{aligned}$$

Applying Lemma 3.14, this last system can be rewritten as

$$\varphi_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2).$$

If we define the parameter $t_1 \in \mathbb{R}$ by

$$t_1 := \frac{1}{\ln r_\varepsilon}\big[4 \ln \tau_1 + \ln \gamma + \mathcal{E}_1(\widetilde{x}^1, \widetilde{\mathbf{x}})\big],$$

then the systems found by projecting (3.54) gather in this equality

$$T_\varepsilon^1 = \big(t_1, \eta_1, \varphi_{1,0}^1, \widetilde{\varphi}_{1,0}^1, \varphi_{1,1}^1, \widetilde{\varphi}_{1,1}^1, \widetilde{\psi}_{1,1}^1, \overline{\nabla}\mathcal{E}_1(\widetilde{x}^1, \widetilde{\mathbf{x}}), \varphi_1^{1,\perp}, \widetilde{\varphi}_1^{1,\perp}, \psi_1^{1,\perp}, \widetilde{\psi}_1^{1,\perp}\big) = \mathcal{O}(r_\varepsilon^2). \tag{3.56}$$

As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left-hand side, but are bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

• On $\partial B_{r_\varepsilon}(\widetilde{x}^1)$, we have

$$\begin{aligned}(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(x) = &-\frac{\eta_2}{\xi} G_a(x, \widetilde{x}^2) - \eta_3 G_a(x, \widetilde{x}^3) + H_2^{\mathrm{int},1}\left(\varphi_2^1, \psi_2^1, \frac{x - \widetilde{x}^1}{r_\varepsilon}\right)\\ &- H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^1, \widetilde{\psi}_2^1, \frac{x - \widetilde{x}^1}{r_\varepsilon}\right) + \mathcal{O}(r_\varepsilon^2).\end{aligned}$$

In the same manner as above, we will solve on S^3 the following system

$$\begin{aligned}(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon \,\cdot\,) = 0, &\qquad \partial_r (u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon \,\cdot\,) = 0,\\ \Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon \,\cdot\,) = 0, &\qquad \partial_r \Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon \,\cdot\,) = 0.\end{aligned} \tag{3.57}$$

We decompose

$$\begin{aligned}\varphi_2^1 &= \varphi_{2,0}^1 + \varphi_{2,1}^1 + \varphi_2^{1,\perp}, &\qquad \psi_2^1 &= 8\varphi_{2,0}^1 + 12\varphi_{2,1}^1 + \psi_2^{1,\perp},\\ \widetilde{\varphi}_2^1 &= \widetilde{\varphi}_{2,0}^1 + \widetilde{\varphi}_{2,1}^1 + \widetilde{\varphi}_2^{1,\perp}, &\qquad \widetilde{\psi}_2^1 &= \widetilde{\psi}_{2,1}^1 + \widetilde{\psi}_2^{1,\perp},\end{aligned}$$

where $\varphi_{2,0}^1, \widetilde{\varphi}_{2,0}^1 \in \mathbb{E}_0$, $\varphi_{2,1}^1, \widetilde{\varphi}_{2,1}^1, \widetilde{\psi}_{2,1}^1 \in \mathbb{E}_1$ and $\varphi_2^{1,\perp}$, $\widetilde{\varphi}_2^{1,\perp}$, $\psi_2^{1,\perp}$, $\widetilde{\psi}_2^{1,\perp}$ belong to $(L^2(S^3))^\perp$. Projecting the set of equations (3.57) over $\mathbb{E}_0$, we get

$$\begin{aligned}\varphi_{2,0}^1 - \widetilde{\varphi}_{2,0}^1 + \mathcal{O}(r_\varepsilon^2) &= 0,\\ 2\varphi_{2,0}^1 + 2\widetilde{\varphi}_{2,0}^1 + \mathcal{O}(r_\varepsilon^2) &= 0,\\ 8\varphi_{2,0}^1 + \mathcal{O}(r_\varepsilon^2) &= 0.\end{aligned}$$

From the L^2-projection of (3.57) over $\mathbb{E}_1$, we obtain the system of equations

$$\begin{aligned}\varphi_{2,1}^1 - \widetilde{\varphi}_{2,1}^1 + \mathcal{O}(r_\varepsilon^2) &= 0,\\ 3\varphi_{2,1}^1 + 3\widetilde{\varphi}_{2,1}^1 + \frac{1}{2}\widetilde{\psi}_{2,1}^1 + \mathcal{O}(r_\varepsilon^2) &= 0,\\ 15\varphi_{2,1}^1 - 3\widetilde{\varphi}_{2,1}^1 - \widetilde{\psi}_{2,1}^1 + \mathcal{O}(r_\varepsilon^2) &= 0,\\ 15\varphi_{2,1}^1 + 15\widetilde{\varphi}_{2,1}^1 + \frac{18}{4}\widetilde{\psi}_{2,1}^1 + \mathcal{O}(r_\varepsilon^2) &= 0.\end{aligned}$$

Finally, we consider the L^2-projection onto $(L^2(S^3))^\perp$. This yields the system

$$\begin{aligned}\varphi_2^{1,\perp} - \widetilde{\varphi}_2^{1,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\ \partial_r\big(H^{\mathrm{int}}_{\varphi_2^{1,\perp}, \psi_2^{1,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_2^{1,\perp}, \widetilde{\psi}_2^{1,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0,\\ \psi_2^{1,\perp} - \widetilde{\psi}_2^{1,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\ \partial_r\Delta\big(H^{\mathrm{int}}_{\varphi_2^{1,\perp}, \psi_2^{1,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_2^{1,\perp}, \widetilde{\psi}_2^{1,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0.\end{aligned}$$

Using Lemma 3.14 again, the above system can be rewritten as

$$\varphi_2^{1,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_2^{1,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_2^{1,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_2^{1,\perp} = \mathcal{O}(r_\varepsilon^2).$$

Then, the systems found by projecting (3.57) gather in this equality

$$\big(\varphi_{2,0}^1, \widetilde{\varphi}_{2,0}^1, \varphi_{2,1}^1, \widetilde{\varphi}_{2,1}^1, \widetilde{\psi}_{2,1}^1, \varphi_2^{1,\perp}, \widetilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \widetilde{\psi}_2^{1,\perp}\big) = \mathcal{O}(r_\varepsilon^2).$$

- On $\partial B_{r_\varepsilon}(\widetilde{x}^2)$, we have

$$\begin{aligned}(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(x) = -\frac{\eta_1}{\gamma} G_a(x, \widetilde{x}^1) - \eta_3 G_a(x, \widetilde{x}^3) + H_1^{\mathrm{int},2}\left(\varphi_1^2, \psi_1^2, \frac{x - \widetilde{x}^2}{r_\varepsilon}\right) \\ - H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^2, \widetilde{\psi}_1^2, \frac{x - \widetilde{x}^2}{r_\varepsilon}\right) + \mathcal{O}(r_\varepsilon^2).\end{aligned}$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{x}^2)$ in (3.52), it will be more convenient to solve on S^3 the following set of equations

$$\begin{aligned}(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) = 0, \qquad \partial_r(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) = 0, \\ \Delta(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) = 0, \qquad \partial_r\Delta(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) = 0.\end{aligned} \tag{3.58}$$

We decompose

$$\begin{aligned}\varphi_1^2 = \varphi_{1,0}^2 + \varphi_{1,1}^2 + \varphi_1^{2,\perp}, \qquad \psi_1^2 = 8\varphi_{1,0}^2 + 12\varphi_{1,1}^2 + \psi_1^{2,\perp}, \\ \widetilde{\varphi}_1^2 = \widetilde{\varphi}_{1,0}^2 + \widetilde{\varphi}_{1,1}^2 + \widetilde{\varphi}_1^{2,\perp}, \qquad \widetilde{\psi}_1^2 = \widetilde{\psi}_{1,1}^2 + \widetilde{\psi}_1^{2,\perp},\end{aligned}$$

where $\varphi_{1,0}^2, \widetilde{\varphi}_{1,0}^2 \in \mathbb{E}_0$, $\varphi_{1,1}^2, \widetilde{\varphi}_{1,1}^2, \widetilde{\psi}_{1,1}^2 \in \mathbb{E}_1 = \ker(\Delta_{S^3} + 1) = \operatorname{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_1^{2,\perp}$, $\widetilde{\varphi}_1^{2,\perp}$, $\psi_1^{2,\perp}$, $\widetilde{\psi}_1^{2,\perp}$ belong to $(L^2(S^3))^\perp$. Projecting the set of equations (3.58) over $\mathbb{E}_0$, we get

$$\begin{aligned}\varphi_{1,0}^2 - \widetilde{\varphi}_{1,0}^2 + \mathcal{O}(r_\varepsilon^2) = 0, \\ 2\varphi_{1,0}^2 + 2\widetilde{\varphi}_{1,0}^2 + \mathcal{O}(r_\varepsilon^2) = 0, \\ 8\varphi_{1,0}^2 + \mathcal{O}(r_\varepsilon^2) = 0.\end{aligned}$$

From the L^2-projection of (3.58) over $\mathbb{E}_1$, we obtain the system of equations

$$\begin{aligned}\varphi_{1,1}^2 - \widetilde{\varphi}_{1,1}^2 + \mathcal{O}(r_\varepsilon^2) = 0, \\ 3\varphi_{1,1}^2 + 3\widetilde{\varphi}_{1,1}^2 + \frac{1}{2}\widetilde{\psi}_{1,1}^2 + \mathcal{O}(r_\varepsilon^2) = 0, \\ 15\varphi_{1,1}^2 - 3\widetilde{\varphi}_{1,1}^2 - \widetilde{\psi}_{1,1}^2 + \mathcal{O}(r_\varepsilon^2) = 0, \\ 15\varphi_{1,1}^2 + 15\widetilde{\varphi}_{1,1}^2 + \frac{18}{4}\widetilde{\psi}_{1,1}^2 + \mathcal{O}(r_\varepsilon^2) = 0.\end{aligned}$$

Finally, we consider the L^2-projection onto $(L^2(S^3))^{\perp}$. This yields the system

$$\begin{aligned}
\varphi_1^{2,\perp} - \widetilde{\varphi}_1^{2,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\big(H^{\rm int}_{\varphi_1^{2,\perp},\psi_1^{2,\perp}} - H^{\rm ext}_{\widetilde{\varphi}_1^{2,\perp},\widetilde{\psi}_1^{2,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\psi_1^{2,\perp} - \widetilde{\psi}_1^{2,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\Delta\big(H^{\rm int}_{\varphi_1^{2,\perp},\psi_1^{2,\perp}} - H^{\rm ext}_{\widetilde{\varphi}_1^{2,\perp},\widetilde{\psi}_1^{2,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0.
\end{aligned}$$

Applying Lemma 3.14, this last system can be written as

$$\varphi_1^{2,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_1^{2,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_1^{2,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_1^{2,\perp} = \mathcal{O}(r_\varepsilon^2).$$

Then, the systems found by projecting (3.58) gather in this equality

$$\big(\varphi_{1,0}^2, \widetilde{\varphi}_{1,0}^2, \varphi_{1,1}^2, \widetilde{\varphi}_{1,1}^2, \widetilde{\psi}_{1,1}^2, \varphi_1^{2,\perp}, \widetilde{\varphi}_1^{2,\perp}, \psi_1^{2,\perp}, \widetilde{\psi}_1^{2,\perp}\big) = \mathcal{O}(r_\varepsilon^2).$$

- On $\partial B_{r_\varepsilon}(\widetilde{x}^2)$, we have

$$\begin{aligned}
&(u_{{\rm int},2} - u_{{\rm ext},2})(x)\\
&= -\frac{4}{\xi}\ln\tau_2 + \frac{8\eta_2}{\xi}\ln|x-\widetilde{x}^2| - \frac{1-\xi}{\gamma\xi}G_a(x,\widetilde{x}^1) - \frac{\ln\xi}{\xi} + H_2^{{\rm int},2}\left(\varphi_2^2, \psi_2^2, \frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\\
&\quad - H_2^{\rm ext}\left(\widetilde{\varphi}_2^2, \widetilde{\psi}_2^2, \frac{x-\widetilde{x}^2}{r_\varepsilon}\right) - \frac{1+\eta_2}{\xi}H_a(x,\widetilde{x}^2) - \left(1+\eta_3+\frac{1-\xi}{\xi}\right)G_a(x,\widetilde{x}^3)\\
&\quad + \mathcal{O}\left(\frac{\varepsilon^2\tau_2^{-2}}{|x-\widetilde{x}^2|^2}\right) + \mathcal{O}(r_\varepsilon^2).
\end{aligned} \tag{3.59}$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{x}^2)$ in (3.52), it will be more convenient to solve on S^3 the following set of equations

$$\begin{aligned}
(u_{{\rm int},2} - u_{{\rm ext},2})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0, & \partial_r(u_{{\rm int},2} - u_{{\rm ext},2})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0,\\
\Delta(u_{{\rm int},2} - u_{{\rm ext},2})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0, & \partial_r\Delta(u_{{\rm int},2} - u_{{\rm ext},2})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0.
\end{aligned} \tag{3.60}$$

Since the boundary data are chosen to satisfy (3.23) or (3.24), we decompose

$$\begin{aligned}
\varphi_2^2 &= \varphi_{2,0}^2 + \varphi_{2,1}^2 + \varphi_2^{2,\perp}, & \psi_2^2 &= 8\varphi_{2,0}^2 + 12\varphi_{2,1}^2 + \psi_2^{2,\perp},\\
\widetilde{\varphi}_2^2 &= \widetilde{\varphi}_{2,0}^2 + \widetilde{\varphi}_{2,1}^2 + \widetilde{\varphi}_2^{2,\perp}, & \widetilde{\psi}_2^2 &= \widetilde{\psi}_{2,1}^2 + \widetilde{\psi}_2^{2,\perp},
\end{aligned}$$

where $\varphi_{2,0}^2, \widetilde{\varphi}_{2,0}^2 \in \mathbb{E}_0 = \mathbb{R}$ are constants on S^3, $\varphi_{2,1}^2$, $\widetilde{\varphi}_{2,1}^2$, $\widetilde{\psi}_{2,1}^2$ belong to $\mathbb{E}_1 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_2^{2,\perp}$, $\widetilde{\varphi}_2^{2,\perp}$, $\psi_2^{2,\perp}$, $\widetilde{\psi}_2^{2,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

Using (3.59), we have for $x \in S^3$,

$$\begin{aligned}
&(u_{{\rm int},2} - u_{{\rm ext},2})(\widetilde{x}^2 + r_\varepsilon x)\\
&= -\frac{4}{\xi}\ln\tau_2 + \frac{8\eta_2}{\xi}\ln(r_\varepsilon|x|) - \frac{\ln\xi}{\xi} + H_2^{{\rm int},2}(\varphi_2^2, \psi_2^2, x) - H_1^{\rm ext}(\widetilde{\varphi}_2^2, \widetilde{\psi}_2^2, x)\\
&\quad - \frac{1}{\xi}\left(H_a(\widetilde{x}^2,\widetilde{x}^2) + \frac{1-\xi}{\gamma}G_a(\widetilde{x}^2,\widetilde{x}^1) + G_a(\widetilde{x}^2,\widetilde{x}^3)\right) - \frac{\eta_2}{\xi}H_a(\widetilde{x}^2,\widetilde{x}^2)\\
&\quad - \eta_3 G_a(\widetilde{x}^2,\widetilde{x}^3) + \mathcal{O}(r_\varepsilon^2).
\end{aligned}$$

Then, the projection of the set equations (3.60) over $\mathbb{E}_0$ will yield

$$
\begin{aligned}
-4\ln\tau_2 + 8\eta_2 \ln r_\varepsilon - \ln\xi + \xi\varphi^2_{2,0} - \xi\widetilde{\varphi}^2_{2,0} - \mathcal{E}_2(\widetilde{x}^2, \widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
8\eta_2 + 2\xi\varphi^2_{2,0} + 2\xi\widetilde{\varphi}^2_{2,0} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
16\eta_2 + 8\xi\varphi^2_{2,0} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
-32\eta_2 + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned}
\tag{3.61}
$$

where

$$
\mathcal{E}_2(\,\cdot\,, \widetilde{\mathbf{x}}) := H_a(\,\cdot\,, \widetilde{x}^2) + \frac{1-\xi}{\gamma} G_a(\,\cdot\,, \widetilde{x}^1) + G_a(\,\cdot\,, \widetilde{x}^3).
$$

The system (3.61) can be simply written as

$$
\eta_2 = \mathcal{O}(r_\varepsilon^2), \;\; \varphi^2_{2,0} = \mathcal{O}(r_\varepsilon^2), \;\; \widetilde{\varphi}^2_{2,0} = \mathcal{O}(r_\varepsilon^2) \;\text{ and }\; \frac{1}{\ln r_\varepsilon}[4\ln\tau_2 + \ln\xi + \mathcal{E}_2(\widetilde{x}^2, \widetilde{\mathbf{x}})] = \mathcal{O}(r_\varepsilon^2).
$$

We are now in a position to define τ_2^- and τ_2^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\widetilde{x}^i$ will converge to x^i for $i \in \{1,2,3\}$ and τ_2 will converge to τ_2^* satisfying

$$
4\ln\tau_2^* = -\ln\xi - \mathcal{E}_2(x^2, \mathbf{x}).
$$

Hence it is enough to choose τ_2^- and τ_2^+ in such a way that

$$
4\ln(\tau_2^-) < -\ln\xi - \mathcal{E}_2(x^2, \mathbf{x}) < 4\ln(\tau_2^+).
$$

Consider now the projection of (3.60) over $\mathbb{E}_1$. Give a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$:

$$
\overline{\nabla} f = \sum_{i=1}^{4} \partial_{x_i} f\, e_i.
$$

With these notations in mind, we obtain the system of equations

$$
\begin{aligned}
\varphi^2_{2,1} - \widetilde{\varphi}^2_{2,1} - \frac{1}{\xi}\overline{\nabla}\mathcal{E}_2(\widetilde{x}^2, \widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
3\varphi^2_{2,1} + 3\widetilde{\varphi}^2_{2,1} + \frac{1}{2}\widetilde{\psi}^2_{2,1} - \frac{1}{\xi}\overline{\nabla}\mathcal{E}_2(\widetilde{x}^2, \widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi^2_{2,1} - 3\widetilde{\varphi}^2_{2,1} - \widetilde{\psi}^2_{2,1} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi^2_{2,1} + 15\widetilde{\varphi}^2_{2,1} + \frac{18}{4}\widetilde{\psi}^2_{2,1} + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned}
$$

which can be simplified as follows:

$$
\varphi^2_{2,1} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}^2_{2,1} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\psi}^2_{2,1} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \overline{\nabla}\mathcal{E}_2(\widetilde{x}^2, \widetilde{\mathbf{x}}) = \mathcal{O}(r_\varepsilon^2).
$$

Finally, we consider the projection onto $L^2(S^3)^\perp$. This yields the system

$$\begin{aligned} \varphi_2^{2,\perp} - \widetilde{\varphi}_2^{2,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0, \\ \partial_r \big(H^{\mathrm{int}}_{\varphi_2^{2,\perp},\psi_2^{2,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_2^{2,\perp},\widetilde{\psi}_2^{2,\perp}} \big) + \mathcal{O}(r_\varepsilon^2) &= 0, \\ \psi_2^{2,\perp} - \widetilde{\psi}_2^{2,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0, \\ \partial_r \Delta \big(H^{\mathrm{int}}_{\varphi_2^{2,\perp},\psi_2^{2,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_2^{2,\perp},\widetilde{\psi}_2^{2,\perp}} \big) + \mathcal{O}(r_\varepsilon^2) &= 0. \end{aligned}$$

Applying Lemma 3.14, this last system can be rewritten as

$$\varphi_2^{2,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_2^{2,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_2^{2,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_2^{2,\perp} = \mathcal{O}(r_\varepsilon^2).$$

If we define the parameter $t_2 \in \mathbb{R}$ by

$$t_2 := \frac{1}{\ln r_\varepsilon} [4 \ln \tau_2 + \ln \xi + \mathcal{E}_2(\widetilde{x}^2, \widetilde{\mathbf{x}})],$$

then the systems found by projecting (3.54) gather in this equality

$$T_\varepsilon^2 = \big(t_2, \eta_2, \varphi_{2,0}^2, \widetilde{\varphi}_{2,0}^2, \varphi_{2,1}^2, \widetilde{\varphi}_{2,1}^2, \widetilde{\psi}_{2,1}^2, \overline{\nabla} \mathcal{E}_2(\widetilde{z}_2, \widetilde{\mathbf{z}}), \varphi_2^{2,\perp}, \widetilde{\varphi}_2^{2,\perp}, \psi_2^{2,\perp}, \widetilde{\psi}_2^{2,\perp}\big) = \mathcal{O}(r_\varepsilon^2). \tag{3.62}$$

As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left-hand side, but are bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

- On $\partial B_{r_\varepsilon}(\widetilde{x}^3)$, we have

$$\begin{aligned} &(1-\xi)(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(x) + (1-\gamma)(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(x) \\ &= -4(2-\gamma-\xi)\ln\tau_3 + 8(2-\gamma-\xi)\eta_3 \ln|x - \widetilde{x}^3| + (1-\xi)H^{\mathrm{int}}\left(\varphi_1^3, \psi_1^3, \frac{x-\widetilde{x}^3}{r_\varepsilon}\right) \\ &\quad + (1-\gamma)H^{\mathrm{int}}\left(\varphi_2^3, \psi_2^3, \frac{x-\widetilde{x}^3}{r_\varepsilon}\right) - (1-\xi)H^{\mathrm{ext}}\left(\widetilde{\varphi}_1^3, \widetilde{\psi}_1^3, \frac{x-\widetilde{x}^3}{r_\varepsilon}\right) - (1-\gamma)H^{\mathrm{ext}}\left(\widetilde{\varphi}_2^3, \widetilde{\psi}_2^3, \frac{x-\widetilde{x}^3}{r_\varepsilon}\right) \\ &\quad - \left((2-\gamma-\xi)H_a(x, \widetilde{x}^3) + \frac{1-\xi}{\gamma}G_a(x, \widetilde{x}^1) + \frac{1-\gamma}{\xi}G_a(x, \widetilde{x}^2)\right) + \mathcal{O}\left(\frac{\varepsilon^2 \tau_3^{-2}}{|x-\widetilde{x}^3|^2}\right) + \mathcal{O}(r_\varepsilon^2). \end{aligned}$$

We denote

$$\begin{aligned} \varphi^3 &= (1-\xi)\varphi_1^3 + (1-\gamma)\varphi_2^3, & \psi^3 &= (1-\xi)\psi_1^3 + (1-\gamma)\psi_2^3, \\ \widetilde{\varphi}^3 &= (1-\xi)\widetilde{\varphi}_1^3 + (1-\gamma)\widetilde{\varphi}_2^3, & \widetilde{\psi}^3 &= (1-\xi)\widetilde{\psi}_1^3 + (1-\gamma)\widetilde{\psi}_2^3. \end{aligned}$$

Then, we have

$$\begin{aligned} &(1-\xi)(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(x) + (1-\gamma)(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(x) \\ &= -4(2-\gamma-\xi)\ln\tau_3 + 8(2-\gamma-\xi)\eta_3 \ln|x-\widetilde{x}^3| \\ &\quad + H^{\mathrm{int}}\left(\varphi^3, \psi^3, \frac{x-\widetilde{x}^3}{r_\varepsilon}\right) - H^{\mathrm{ext}}\left(\widetilde{\varphi}^3, \widetilde{\psi}^3, \frac{x-\widetilde{x}^3}{r_\varepsilon}\right) \\ &\quad - \left((2-\gamma-\xi)H_a(x, \widetilde{x}^3) + \frac{1-\xi}{\gamma}G_a(x, \widetilde{x}^1) + \frac{1-\gamma}{\xi}G_a(x, \widetilde{x}^2)\right) \\ &\quad + \mathcal{O}\left(\frac{\varepsilon^2 \tau_3^{-2}}{|x-\widetilde{x}^3|^2}\right) + \mathcal{O}(r_\varepsilon^2). \end{aligned} \tag{3.63}$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{x}^3)$ in (3.52), it will be more convenient to solve on S^3, the following set of equations

$$
\begin{aligned}
\big((1-\xi)(u_{\mathrm{int},1}-u_{\mathrm{ext},1})+(1-\gamma)(u_{\mathrm{int},2}-u_{\mathrm{ext},2})\big)(\widetilde{x}^3+r_\varepsilon\,\cdot\,)&=0,\\
\partial_r\big((1-\xi)(u_{\mathrm{int},1}-u_{\mathrm{ext},1})+(1-\gamma)(u_{\mathrm{int},2}-u_{\mathrm{ext},2})\big)(\widetilde{x}^3+r_\varepsilon\,\cdot\,)&=0,\\
\Delta\big((1-\xi)(u_{\mathrm{int},1}-u_{\mathrm{ext},1})+(1-\gamma)(u_{\mathrm{int},2}-u_{\mathrm{ext},2})\big)(\widetilde{x}^3+r_\varepsilon\,\cdot\,)&=0,\\
\partial_r\Delta\big((1-\xi)(u_{\mathrm{int},1}-u_{\mathrm{ext},1})+(1-\gamma)(u_{\mathrm{int},2}-u_{\mathrm{ext},2})\big)(\widetilde{x}^3+r_\varepsilon\,\cdot\,)&=0.
\end{aligned}
\tag{3.64}
$$

Since the boundary data are chosen to satisfy (3.23) or (3.24), we decompose

$$
\begin{aligned}
\varphi^3&=\varphi^3_0+\varphi^3_1+\varphi^{3,\perp}, &\qquad \psi^3&=8\varphi^3_0+12\varphi^3_1+\psi^{3,\perp},\\
\widetilde{\varphi}^3&=\widetilde{\varphi}^3_0+\widetilde{\varphi}^3_1+\widetilde{\varphi}^{3,\perp}, &\qquad \widetilde{\psi}^3&=\widetilde{\psi}^3_1+\widetilde{\psi}^{3,\perp},
\end{aligned}
$$

where $\varphi^3_0,\widetilde{\varphi}^3_0\in\mathbb{E}_0=\mathbb{R}$ are constant on S^3, φ^3_1, $\widetilde{\varphi}^3_1$, $\widetilde{\psi}^3_1$ belong to $\mathbb{E}_1=\mathrm{Span}\{e_1,e_2,e_3,e_4\}$ and $\varphi^{3,\perp}$, $\widetilde{\varphi}^{3,\perp}$, $\psi^{3,\perp}$, $\widetilde{\psi}^{3,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

We insist that for $x\in S^3$, both equations (3.63) involve the same relation of the parameter τ_3 and the appropriate energy $\mathcal{E}_3$. Then we have

$$
\begin{aligned}
&\big((1-\xi)(u_{\mathrm{int},1}-u_{\mathrm{ext},1})+(1-\gamma)(u_{\mathrm{int},2}-u_{\mathrm{ext},2})\big)(\widetilde{x}^3+r_\varepsilon x)\\
&=-4(2-\gamma-\xi)\ln\tau_3+8(2-\gamma-\xi)\eta_3\ln r_\varepsilon|x|+H^{\mathrm{int}}(\varphi^3,\psi^3,x)-H^{\mathrm{ext}}(\widetilde{\varphi}^3,\widetilde{\psi}^3,x)\\
&\quad-\left((2-\gamma-\xi)H_a(\widetilde{x}^3,\widetilde{x}^3)+\frac{1-\xi}{\gamma}G_a(\widetilde{x}^3,\widetilde{x}^1)+\frac{1-\gamma}{\xi}G_a(\widetilde{x}^3,\widetilde{x}^2)\right)\\
&\quad+\mathcal{O}\left(\frac{\varepsilon^2\tau_3^{-2}}{|x-\widetilde{x}^3|^2}\right)+\mathcal{O}(r_\varepsilon^2).
\end{aligned}
$$

Projecting the set of equations (3.64) over $\mathbb{E}_0$, we get

$$
\begin{aligned}
-4(2-\gamma-\xi)\ln\tau_3+8(2-\gamma-\xi)\eta_3\ln r_\varepsilon+\varphi^3_0-\widetilde{\varphi}^3_0-\mathcal{E}_3(\widetilde{x}^3,\widetilde{\mathbf{x}})+\mathcal{O}(r_\varepsilon^2)&=0,\\
8(2-\gamma-\xi)\eta_3+2\varphi^3_0+2\widetilde{\varphi}^3_0+\mathcal{O}(r_\varepsilon^2)&=0,\\
16(2-\gamma-\xi)\eta_3+8\varphi^3_0+\mathcal{O}(r_\varepsilon^2)&=0,\\
-32(2-\gamma-\xi)\eta_3+\mathcal{O}(r_\varepsilon^2)&=0,
\end{aligned}
\tag{3.65}
$$

where

$$
\mathcal{E}_3(\,\cdot\,,\widetilde{\mathbf{x}}):=(2-\gamma-\xi)H_a(\,\cdot\,,\widetilde{x}^3)+\frac{1-\xi}{\gamma}G_a(\,\cdot\,,\widetilde{x}^1)+\frac{1-\gamma}{\xi}G_a(\,\cdot\,,\widetilde{x}^2).
$$

The system (3.65) can be simply written as

$$
\eta_3=\mathcal{O}(r_\varepsilon^2),\ \ \varphi^3_0=\mathcal{O}(r_\varepsilon^2),\ \ \widetilde{\varphi}^3_0=\mathcal{O}(r_\varepsilon^2)\ \text{ and }\ \frac{1}{\ln r_\varepsilon}[(2-\gamma-\xi)4\ln\tau_3+\mathcal{E}_3(\widetilde{x}^3,\widetilde{\mathbf{x}})]=\mathcal{O}(r_\varepsilon^2).
$$

We are now in a position to define τ_3^- and τ_3^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\widetilde{x}^i$ will converge to x^i for $i\in\{1,2,3\}$ and τ_3 will converge to τ_3^* satisfying

$$
4(2-\gamma-\xi)\ln\tau_3^*=-\mathcal{E}_3(x^3,\mathbf{x}).
$$

Hence it is enough to choose τ_3^- and τ_3^+ in such a way that

$$4(2-\gamma-\xi)\ln(\tau_3^-) < -\mathcal{E}_3(x^3,\mathbf{x}) < 4(2-\gamma-\xi)\ln(\tau_3^+).$$

Consider now the projection of (3.64) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$:

$$\overline{\nabla} f = \sum_{i=1}^{4} \partial_{x_i} f\, e_i.$$

With these notations in mind, we obtain the system of equations

$$\begin{aligned}
\varphi_1^3 - \widetilde{\varphi}_1^3 - \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3,\widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
3\varphi_1^3 + 3\widetilde{\varphi}_1^3 + \frac{1}{2}\widetilde{\psi}_1^3 - \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3,\widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi_1^3 - 3\widetilde{\varphi}_1^3 - \widetilde{\psi}_1^3 + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi_1^3 + 15\widetilde{\varphi}_1^3 + \frac{18}{4}\widetilde{\psi}_1^3 + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned}$$

which can be simplified as follows:

$$\varphi_1^3 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_1^3 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\psi}_1^3 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3,\widetilde{\mathbf{x}}) = \mathcal{O}(r_\varepsilon^2).$$

Finally, we consider the projection onto $L^2(S^3)^\perp$. This yields the system

$$\begin{aligned}
\varphi^{3,\perp} - \widetilde{\varphi}^{3,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\big(H^{\text{int}}_{\varphi^{3,\perp},\psi^{3,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}^{3,\perp},\widetilde{\psi}^{3,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\psi^{3,\perp} - \widetilde{\psi}^{3,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\Delta\big(H^{\text{int}}_{\varphi^{3,\perp},\psi^{3,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}^{3,\perp},\widetilde{\psi}^{3,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0.
\end{aligned}$$

Thanks to the result of Lemma 3.14, this last system can be rewritten as

$$\varphi^{3,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}^{3,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi^{3,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}^{3,\perp} = \mathcal{O}(r_\varepsilon^2).$$

If we define the parameter $t_3 \in \mathbb{R}$ by

$$t_3 := \frac{1}{\ln r_\varepsilon}[(2-\gamma-\xi)4\ln\tau_3 + \mathcal{E}_3(\widetilde{x}^3,\widetilde{\mathbf{x}})],$$

then the systems found by projecting (3.64) gather in this equality

$$T_\varepsilon^3 = \big(t_3, \eta_3, \varphi_0^3, \widetilde{\varphi}_0^3, \varphi_1^3, \widetilde{\varphi}_1^3, \widetilde{\psi}_1^3, \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3,\widetilde{\mathbf{x}}), \varphi^{3,\perp}, \widetilde{\varphi}^{3,\perp}, \psi^{3,\perp}, \widetilde{\psi}^{3,\perp}\big) = \mathcal{O}(r_\varepsilon^2). \tag{3.66}$$

As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left-hand side, but are bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

We recall that $\mathbf{d} = r_\varepsilon(\widetilde{\mathbf{x}} - \mathbf{x})$, in addition the previous systems can be written as for $i = 1, 2, 3$:

$$\big(\mathbf{d}, t_i, \eta_i, \varphi^i, \widetilde{\varphi}^i, \psi^i, \widetilde{\psi}^i, \overline{\nabla}\mathcal{E}_i\big) = \mathcal{O}(r_\varepsilon^2).$$

Combining (3.56), (3.62) and (3.66), we have

$$T_\varepsilon = (T_\varepsilon^1, T_\varepsilon^2, T_\varepsilon^3) = \big(\mathcal{O}(r_\varepsilon^2), \mathcal{O}(r_\varepsilon^2), \mathcal{O}(r_\varepsilon^2)\big). \tag{3.67}$$

Then the nonlinear mapping which appears on the right-hand side of (3.67) is continuous, compact. In addition, reducing ε_κ if necessary, this nonlinear mapping sends the ball of radius κr_ε^2 (for the natural product norm) into itself, provided κ is fixed large enough. Applying Schauder's fixed point theorem in the ball of radius κr_ε^2 in the product space where the entries live, we obtain the existence of a solution of equation (3.67).

This completes the proof of Theorem 1.5. □

Remark 3.22. In order to inverse problem (3.56), (3.62) and (3.66), we remark that the fact that x^i is a nondegenerate critical point of $\mathcal{E}_i(\,\cdot\,, \mathbf{x})$, $i \in \{1, 3\}$ is equivalent to say that (x^1, x^2, x^3) is a nondegenerate critical point of the function $\mathcal{F}$ defined by

$$\begin{aligned}\mathcal{F}(x^1, x^2, x^3) = {} & \frac{1-\xi}{2\gamma} H_a(x^1, x^1) + \frac{1-\gamma}{2\xi} H_a(x^2, x^2) + \frac{2-\gamma-\xi}{2} H_a(x^3, x^3) \\ & + \frac{1-\xi}{\gamma} G_a(x^1, x^3) + \frac{1-\gamma}{\xi} G_a(x^2, x^3) + \frac{(1-\xi)(1-\gamma)}{\gamma\xi} G_a(x^1, x^2).\end{aligned}$$

Indeed, we have

$$\nabla\mathcal{F}(x^1, x^2, x^3) = \left(\frac{\partial\mathcal{F}}{\partial x^1}(x^1, x^2, x^3), \frac{\partial\mathcal{F}}{\partial x^2}(x^1, x^2, x^3), \frac{\partial\mathcal{F}}{\partial x^3}(x^1, x^2, x^3)\right).$$

On the other hand,

$$\begin{aligned}\mathcal{E}_1(x, \mathbf{x}) &= H_a(x, \widetilde{x}^1) + G_a(x, \widetilde{x}^3) + \frac{1-\gamma}{\xi} G_a(x, \widetilde{x}^2), \\ \mathcal{E}_2(x, \mathbf{x}) &= H_a(x, \widetilde{x}^2) + G_a(x, \widetilde{x}^3) + \frac{1-\xi}{\gamma} G_a(x, \widetilde{x}^1)\end{aligned}$$

and

$$\mathcal{E}_3(x, \mathbf{x}) = (2-\gamma-\xi) H_a(x, \widetilde{x}^3) + \frac{1-\xi}{\gamma} G_a(x, \widetilde{x}^1) + \frac{1-\gamma}{\xi} G_a(x, \widetilde{x}^2),$$

then

$$\begin{aligned}\frac{\partial\mathcal{E}_1}{\partial x}(x^1, \mathbf{x}) &= \frac{\partial H_a}{\partial x}(x^1, x^1) + \frac{\partial G_a}{\partial x}(x^1, x^3) + \frac{1-\gamma}{\xi}\frac{\partial G_a}{\partial x}(x^1, x^2) = \frac{\partial\mathcal{F}}{\partial x^1}(x^1, x^2, x^3), \\ \frac{\partial\mathcal{E}_2}{\partial x}(x^2, \mathbf{x}) &= \frac{\partial H_a}{\partial x}(x^2, x^2) + \frac{\partial G_a}{\partial x}(x^2, x^3) + \frac{1-\xi}{\gamma}\frac{\partial G_a}{\partial x}(x^1, x^2) = \frac{\partial\mathcal{F}}{\partial x^2}(x^1, x^2, x^3)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\mathcal{E}_3}{\partial x}(x^3, \mathbf{x}) &= (2-\gamma-\xi)\frac{\partial H_a}{\partial x}(x^3, x^3) + \frac{1-\xi}{\gamma}\frac{\partial G_a}{\partial x}(x^1, x^3) + \frac{1-\gamma}{\xi}\frac{\partial G_a}{\partial x}(x^2, x^3) \\ &= \frac{\partial\mathcal{F}}{\partial x^3}(x^1, x^2, x^3).\end{aligned}$$

4. Proof of Theorem 1.6

4.1. Construction of the approximate solution

4.1.1. Ansatz and first estimates

Here, we are interested to study the system

$$\begin{aligned}\Delta_a^2 u_1 - \Delta_a u_1 &= \rho^4 e^{\gamma u_1 + (1-\gamma)u_2},\\ \Delta_a^2 u_2 - \Delta_a u_2 &= \rho^4 e^{\xi u_2 + (1-\xi)u_1}.\end{aligned}$$

Using the following transformations

$$v_1(x) = \begin{cases} u_1\left(\frac{\varepsilon}{\tau_1}x\right) + \frac{8}{\gamma}\ln\varepsilon - \frac{4}{\gamma}\ln\left(\frac{\tau_1(1+\varepsilon^2)}{2}\right) & \text{in } B_{R_\varepsilon^1}(x^1),\\ u_1\left(\frac{\varepsilon}{\tau_2}x\right) & \text{in } B_{R_\varepsilon^2}(x^2),\\ u_1\left(\frac{\varepsilon}{\tau_3}x\right) + 8\ln\varepsilon - 4\ln\left(\frac{\tau_3(1+\varepsilon^2)}{2}\right) & \text{in } B_{R_\varepsilon^3}(x^3)\end{cases}$$

and

$$v_2(x) = \begin{cases} u_2\left(\frac{\varepsilon}{\tau_1}x\right) & \text{in } B_{R_\varepsilon^1}(x^1),\\ u_2\left(\frac{\varepsilon}{\tau_2}x\right) + \frac{8}{\xi}\ln\varepsilon - \frac{4}{\xi}\ln\left(\frac{\tau_2(1+\varepsilon^2)}{2}\right) & \text{in } B_{R_\varepsilon^2}(x^2),\\ u_2\left(\frac{\varepsilon}{\tau_3}x\right) + 8\ln\varepsilon - 4\ln\left(\frac{\tau_3(1+\varepsilon^2)}{2}\right) & \text{in } B_{R_\varepsilon^3}(x^3).\end{cases}$$

So, the previous systems can be written as

$$(4.1)\qquad \begin{aligned}\Delta^2 v_1 + \Sigma^1_{\tilde a_1} v_1 + \Sigma^2_{\tilde a_1} v_1 &= 24 e^{\gamma v_1 + (1-\gamma)v_2} && \text{in } B_{R_\varepsilon^1}(x^1),\\ \Delta^2 v_2 + \Sigma^1_{\tilde a_1} v_2 + \Sigma^2_{\tilde a_1} v_2 &= 24 C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} e^{\xi v_2 + (1-\xi)v_1} && \text{in } B_{R_\varepsilon^1}(x^1),\end{aligned}$$

$$(4.2)\qquad \begin{aligned}\Delta^2 v_1 + \Sigma^1_{\tilde a_2} v_1 + \Sigma^2_{\tilde a_2} v_1 &= 24 C_{2,\varepsilon}^{4\frac{\gamma+\xi-1}{\xi}} \varepsilon^{8\frac{\gamma+\xi-1}{\xi}} e^{\gamma v_1 + (1-\gamma)v_2} && \text{in } B_{R_\varepsilon^2}(x^2),\\ \Delta^2 v_2 + \Sigma^1_{\tilde a_2} v_2 + \Sigma^2_{\tilde a_2} v_2 &= 24 e^{\xi v_2 + (1-\xi)v_1} && \text{in } B_{R_\varepsilon^2}(x^2)\end{aligned}$$

and

$$(4.3)\qquad \begin{aligned}\Delta^2 v_1 + \Sigma^1_{\tilde a_3} v_1 + \Sigma^2_{\tilde a_3} v_1 &= 24 e^{\gamma v_1 + (1-\gamma)v_2} && \text{in } B_{R_\varepsilon^3}(x^3),\\ \Delta^2 v_2 + \Sigma^1_{\tilde a_3} v_2 + \Sigma^2_{\tilde a_3} v_2 &= 24 e^{\xi v_2 + (1-\xi)v_1} && \text{in } B_{R_\varepsilon^3}(x^3),\end{aligned}$$

where $C_{i,\varepsilon} = \frac{2}{\tau_i(1+\varepsilon^2)}$ for $i = 1, 2$. Here $\tau_i > 0$ is constant which be fixed later. We denote by $\overline{u} = u_{\varepsilon=1,\tau_i=1}$. In $B_{R_\varepsilon^1}(x^1)$ and $B_{R_\varepsilon^2}(x^2)$, we reproduce exactly the same as in the proof of Theorem 1.5, so we have the following propositions.

Proposition 4.1. *Let $\mu \in (1,2)$ and $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$. Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ (depending on κ) and $\gamma_0 \in (0,1)$ such* *that for all* $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$ *and* $\gamma \in (\gamma_0, 1)$, *there exists a unique* $(h_1^1, h_2^1) \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ *solution of* (3.8) *such that*

$$\begin{aligned} v_1(x) &:= \frac{1}{\gamma}\overline{u}(x - x^1) - \frac{1-\gamma}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{1-\gamma}{\gamma\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{\ln\gamma}{\gamma} + h_1^1(x), \\ v_2(x) &:= \frac{1}{\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + h_2^1(x) \end{aligned}$$

solves (4.1) *in* $B_{R_\varepsilon^1}(x^1)$. *In addition,*

$$\|(h_1^1, h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Proposition 4.2. *Given* $\kappa > 0$, $\mu \in (1,2)$ *and* $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$, *there exist* $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ *(which can depend only on* κ*) and* $\xi_0 \in (0,1)$ *such that for all* $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$ *and* $\xi \in (\xi_0, 1)$, *there exists a unique* $(h_1^2, h_2^2) \in \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ *solution of* (3.14) *such that*

$$\begin{aligned} v_1(x) &= \frac{1}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_1^2(x), \\ v_2(x) &= \frac{1}{\xi}\overline{u}(x - x^2) - \frac{1-\xi}{\xi} G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) - \frac{1-\xi}{\gamma\xi} G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{\ln\xi}{\xi} + h_2^2(x) \end{aligned}$$

solves (4.2) *in* $B_{R_\varepsilon^2}(x^2)$. *In addition,*

$$\|(h_1^2, h_2^2)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

- In $B_{R_\varepsilon^3}(x^3)$, we look for a solution of (4.3) of the form

$$\begin{aligned} v_1(x) &= \overline{u}(x - x^3) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_1^3(x), \\ v_2(x) &= \overline{u}(x - x^3) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_2^3(x). \end{aligned}$$

This amounts to solve the equations

$$\begin{aligned} \mathbb{L}h_1^3 &= \frac{384}{(1+r^2)^4}\left[e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right)\right) + \gamma h_1^3 + (1-\gamma)h_2^3} - h_1^3 - 1\right] \\ &\quad - \Sigma^1_{\tilde{a}_3} v_1 - \Sigma^2_{\tilde{a}_3} v_1, \\ \mathbb{L}h_2^3 &= \frac{384}{(1+r^2)^4}\left[e^{\frac{(1-\xi)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(-\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + \frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right)\right) + \xi h_2^3 + (1-\xi)h_1^3} - h_2^3 - 1\right] \\ &\quad - \Sigma^1_{\tilde{a}_3} v_2 - \Sigma^2_{\tilde{a}_3} v_2. \end{aligned} \tag{4.4}$$

We denote by

$$\mathbb{L}h_1^3 = \mathcal{T}_3(h_1^3, h_2^3) \quad \text{and} \quad \mathbb{L}h_2^3 = \mathcal{R}_3(h_1^3, h_2^3).$$

To find a solution of (4.4), it is enough to find a fixed point (h_1^3, h_2^3) in a small ball of $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$, solutions of

$$\begin{aligned} h_1^3 &= \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon^3} \circ \mathcal{T}_3(h_1^3, h_2^3) = \mathcal{N}_3(h_1^3, h_2^3),\\ h_2^3 &= \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon^3} \circ \mathcal{R}_3(h_1^3, h_2^3) = \mathcal{M}_3(h_1^3, h_2^3). \end{aligned} \tag{4.5}$$

Then, we have the following result.

Lemma 4.3. *Let $\mu \in (1,2)$. Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$, $\overline{c}_\kappa > 0$ and $\overline{\overline{c}}_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\beta \in (0, \beta_\kappa)$ with $r_\varepsilon := r_{\varepsilon,\beta,\lambda}$. We have*

$$\|\mathcal{N}_3(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \quad \|\mathcal{M}_3(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2,$$
$$\|\mathcal{N}_3(h_1^3, h_2^3) - \mathcal{N}_3(k_1^3, k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le \overline{c}_\kappa r_\varepsilon^2 \|(h_1^3, h_2^3) - (k_1^3, k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}$$

and

$$\|\mathcal{M}_3(h_1^3, h_2^3) - \mathcal{M}_3(k_1^3, k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le \overline{\overline{c}}_\kappa r_\varepsilon^2 \|(h_1^3, h_2^3) - (k_1^3, k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)},$$

provided $(h_1^3, h_2^3), (k_1^3, k_2^3) \in \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ satisfying

$$\|(h_1^3, h_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(k_1^3, k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2. \tag{4.6}$$

Proof. We have

$$\begin{aligned} &\sup_{r \le R_\varepsilon^3} r^{4-\mu} |\mathcal{T}_3(0,0)| \\ &\le \sup_{r \le R_\varepsilon^3} \frac{384}{(1+r^2)^4} r^{4-\mu} \left| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)} \left(\frac{1}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{1}{\xi} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right)\right)} - 1 \right| \\ &\quad + \sup_{r \le R_\varepsilon^3} r^{4-\mu} \left| \Sigma_{\widetilde{a}_3}^1 \left(\overline{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right)\right)\right| \\ &\quad + \sup_{r \le R_\varepsilon^3} r^{4-\mu} \left| \Sigma_{\widetilde{a}_3}^1 \left(\overline{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right)\right)\right|. \end{aligned}$$

Using (1.12), we obtain

$$\begin{aligned} &\sup_{r \le R_\varepsilon^3} r^{4-\mu} |\mathcal{T}_3(0,0)| \\ &\le c_\kappa \sup_{r \le R_\varepsilon^3} r^{4-\mu} \left(\left|\Sigma_{\widetilde{a}_3}^1(\overline{u})\right| + \left|\Sigma_{\widetilde{a}_3}^2(\overline{u})\right| \right) \\ &\le c_\kappa \left(\frac{\varepsilon}{\tau_3}\right) \|\nabla a\|_\infty \sup_{r \le R_\varepsilon^3} r^{4-\mu} \frac{r(3+r^2)}{(1+r^2)^3} + c_\kappa \left(\frac{\varepsilon}{\tau_3}\right)^3 \|\nabla a\|_\infty \sup_{r \le R_\varepsilon^3} r^{4-\mu} \frac{r}{1+r^2} \\ &\quad + c_\kappa \left(\frac{\varepsilon}{\tau_3}\right)^2 \lambda \|f\|_\infty \sup_{r \le R_\varepsilon^3} r^{4-\mu} \frac{2+r^2}{(1+r^2)^2} \\ &\le c_\kappa \beta \varepsilon + c_\kappa \beta \varepsilon^\mu r_\varepsilon^{3-\mu} + c_\kappa \lambda \varepsilon^\mu r_\varepsilon^{2-\mu} \le c_\kappa r_\varepsilon^2. \end{aligned}$$

Making use of Proposition 3.2 together with (3.4), for $\mu \in (1,2)$, there exists a constant c_κ such that

$$\|\mathcal{N}_3(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2.$$

For the second estimate, we use the same techniques to prove

$$\|\mathcal{M}_3(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2.$$

To derive the third estimate, for (h_1^3,h_2^3), (k_1^3,k_2^3) verifying (4.6), we have

$$\begin{aligned}
&\sup_{r\leq R_\varepsilon^3} r^{4-\mu}\big|\mathcal{T}_3(h_1^3,h_2^3)-\mathcal{T}_3(k_1^3,k_2^3)\big| \\
&\leq \sup_{r\leq R_\varepsilon^3} \frac{384}{(1+r^2)^4} r^{4-\mu}\bigg| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)-\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\gamma h_1^3+(1-\gamma)h_2^3} - h_1^3 \\
&\qquad - e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)-\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\gamma k_1^3+(1-\gamma)k_2^3} + k_1^3\bigg| \\
&\quad + \sup_{r\leq R_\varepsilon^3} r^{4-\mu}\big|\Sigma^1_{\widetilde{a}_3}(h_1^3-k_1^3)\big| + \sup_{r\leq R_\varepsilon^3} r^{4-\mu}\big|\Sigma^2_{\widetilde{a}_3}(h_1^3-k_1^3)\big| \\
&\leq c_\kappa \sup_{r\leq R_\varepsilon^3} \frac{384 r^{4-\mu}}{(1+r^2)^4}\big|(\gamma-1)(h_1^3-k_1^3)+(1-\gamma)(h_2^3-k_2^3)\big| \\
&\quad + c_\kappa \sup_{r\leq R_\varepsilon^3} r^{4-\mu}\big|\Sigma^1_{\widetilde{a}_3}(h_1^3-k_1^3)\big| + c_\kappa \sup_{r\leq R_\varepsilon^3} r^{4-\mu}\big|\Sigma^2_{\widetilde{a}_3}(h_1^3-k_1^3)\big| \\
&\leq c_\kappa \sup_{r\leq R_\varepsilon^3} \frac{384 r^{4-\mu}}{(1+r^2)^4}(1-\gamma)\big[r^\mu\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + r^\mu\|h_2^3-k_2^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\big] \\
&\quad + c_\kappa\beta r_\varepsilon\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa\beta r_\varepsilon^3\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa\lambda r_\varepsilon^2\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}$$

Making use of Proposition 3.2 together with (3.4), we conclude that

$$\begin{aligned}
&\|\mathcal{N}_3(h_1^3,h_2^3)-\mathcal{N}_3(k_1^3,k_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\leq \big(c_\kappa r_\varepsilon^2 + c_\kappa(1-\gamma)\big)\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|h_2^3-k_2^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\leq \big(c_\kappa r_\varepsilon^2 + c_\kappa(1-\gamma)\big)\|(h_1^3,h_2^3)-(k_1^3,k_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}$$

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \leq 1/2$ for all $\varepsilon \in (0,\varepsilon_\kappa)$. There exists a constant $\gamma_0 \in (0,1)$ such that $c_\kappa(1-\gamma) \leq 1/2$ for all $\gamma \in (\gamma_0,1)$, we can find a constant $\overline{c}_\kappa > 0$ such that

$$\|\mathcal{N}_3(h_1^3,h_2^3)-\mathcal{N}_3(k_1^3,k_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq \overline{c}_\kappa r_\varepsilon^2\|(h_1^3,h_2^3)-(k_1^3,k_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}. \tag{4.7}$$

Similarly, we get

$$\begin{aligned}
&\|\mathcal{M}_3(h_1^3,h_2^3)-\mathcal{M}_3(k_1^3,k_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\leq \big(c_\kappa r_\varepsilon^2 + c_\kappa(1-\xi)\big)\|h_2^3-k_2^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa(1-\xi)\|h_1^3-k_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\leq \big(c_\kappa r_\varepsilon^2 + c_\kappa(1-\xi)\big)\|(h_1^3,h_2^3)-(k_1^3,k_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}$$

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \leq 1/2$ for all $\varepsilon \in (0, \varepsilon_\kappa)$. There exists a constant $\xi_0 \in (0,1)$ such that $c_\kappa(1-\xi) \leq 1/2$ for all $\xi \in (\xi_0, 1)$, we can find a constant $\overline{\overline{c}}_\kappa > 0$ such that

$$\|\mathcal{M}_3(h_1^3, h_2^3) - \mathcal{M}_3(k_1^3, k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq \overline{\overline{c}}_\kappa r_\varepsilon^2 \|(h_1^3, h_2^3) - (k_1^3, k_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{4.8}$$ □

Consequently, (4.7) and (4.8) are enough to show that

$$(h_1^3, h_2^3) \mapsto \big(\mathcal{N}_3(h_1^3, h_2^3), \mathcal{M}_3(h_1^3, h_2^3)\big)$$

is a contraction from the ball

$$\big\{(h_1^3, h_2^3) \in \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) : \|(h_1^3, h_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2\big\}$$

into itself and hence a unique fixed point (h_1^3, h_2^3) exists in this set, which is a solution of (4.5). Then we have

Proposition 4.4. *Given $\kappa > 0$, $\mu \in (1,2)$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ (depending on κ), $\gamma_0 \in (0,1)$ and $\xi_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$, $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$ there exists a unique $(h_1^3, h_2^3) \in \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ solution of* (4.5) *such that*

$$\begin{aligned} v_1(x) &= \overline{u}(x - x^3) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_1^3(x), \\ v_2(x) &= \overline{u}(x - x^3) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_2^3(x) \end{aligned}$$

solves (4.3) *in $B_{R_\varepsilon^3}(x^3)$. In addition,*

$$\|(h_1^3, h_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2.$$

4.2. The nonlinear interior problem

Here, we are interested to study the following system in $B_{R_\varepsilon^3}(x^3)$

$$\begin{aligned} \Delta^2 v_1 + \Sigma_{\tilde{a}_3}^1 v_1 + \Sigma_{\tilde{a}_3}^2 v_1 &= 24 e^{\gamma v_1 + (1-\gamma) v_2}, \\ \Delta^2 v_2 + \Sigma_{\tilde{a}_3}^1 v_2 + \Sigma_{\tilde{a}_3}^2 v_2 &= 24 e^{\xi v_2 + (1-\xi) v_1}. \end{aligned}$$

Remark 4.5. Given $\varphi^i := (\varphi_1^i, \varphi_2^i) \in \big(\mathcal{C}^{4,\alpha}(S^3)\big)^2$ and $\psi^i := (\psi_1^i, \psi_2^i) \in \big(\mathcal{C}^{2,\alpha}(S^3)\big)^2$ for $i = 1, 2$ such that (φ_1^i, ψ_1^i) and (φ_2^i, ψ_2^i) satisfy (3.23). In $B_{R_\varepsilon^1}(x^1)$ and $B_{R_\varepsilon^2}(x^2)$, we proceed in the same way as the proof of Theorem 1.5, so we have the following propositions.

Proposition 4.6. *Given $\kappa > 0$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ (depending on κ) and $\gamma_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \varepsilon_\kappa)$, $\beta \in (0, \beta_\kappa)$, $\gamma \in (\gamma_0, 1)$, for all τ_1 in some fixed compact subset of $[\tau_1^-, \tau_1^+] \subset (0, \infty)$ and for φ_j^1 and ψ_j^1 satisfying (3.23) and (3.27), there exists a unique $(v_1^1, v_2^1) := (v_{1,\varepsilon,\lambda,\beta,\tau_1,\varphi_1^1,\psi_1^1}, v_{2,\varepsilon,\lambda,\beta,\tau_1,\varphi_2^1,\psi_2^1})$ solution of (3.26) such that*

$$\|(v_1^1, v_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2.$$

Hence

$$\begin{aligned} v_1(x) &= \frac{1}{\gamma}\overline{u}(x-x^1) - \frac{1-\gamma}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{1-\gamma}{\gamma\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{\ln\gamma}{\gamma} + h_1^1(x) \\ &\quad + H_1^{\mathrm{int},1}\left(\varphi_1^1, \psi_1^1; \frac{x-x^1}{R_\varepsilon^1}\right) + v_1^1(x), \\ v_2(x) &= \frac{1}{\xi} G_a\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + G_a\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + h_2^1(x) + H_2^{\mathrm{int},1}\left(\varphi_2^1, \psi_2^1; \frac{x-x^1}{R_\varepsilon^1}\right) + v_2^1(x) \end{aligned}$$

solves (4.1) in $B_{R_\varepsilon^1}(x^1)$.

Proposition 4.7. *Given $\kappa > 0$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\left\{1, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ (depending on κ) and $\xi_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$, $\xi \in (\xi_0, 1)$, for all τ_2 in some fixed compact subset of $[\tau_2^-, \tau_2^+] \subset (0, \infty)$ and for φ_j^2 and ψ_j^2 satisfying (3.23) and (3.34), there exists a unique $(v_1^2, v_2^2) := (v_{1,\varepsilon,\lambda,\beta,\tau_2,\varphi_1^2,\psi_1^2}, v_{2,\varepsilon,\lambda,\beta,\tau_2,\varphi_2^2,\psi_2^2})$ solution of (3.33) such that*

$$\|(v_1^2, v_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2.$$

Hence

$$\begin{aligned} v_1(x) &= \frac{1}{\gamma} G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_1^2(x) + H_1^{\mathrm{int},2}\left(\varphi_1^2, \psi_1^2; \frac{x-x^2}{R_\varepsilon^2}\right) + v_1^2(x), \\ v_2(x) &= \frac{1}{\xi}\overline{u}(x-x^2) - \frac{1-\xi}{\xi} G_a\left(\frac{\varepsilon x}{\tau_2}, x^3\right) - \frac{1-\xi}{\gamma\xi} G_a\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{\ln\xi}{\xi} + h_2^2(x) \\ &\quad + H_2^{\mathrm{int},2}\left(\varphi_2^2, \psi_2^2; \frac{x-x^2}{R_\varepsilon^2}\right) + v_2^2(x) \end{aligned}$$

solves (4.2) in $B_{R_\varepsilon^2}(x^2)$.

• In $B_{R_\varepsilon^3}(x^3)$, we look for a solution of (4.3) as in Section 4.1.1 by adding the interior harmonic extension and the disturbance term v_i^3, $i = 1,2$ as follows:

$$\begin{aligned} v_1(x) &= \overline{u}(x-x^3) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_1^3(x) \\ &\quad + H_1^{\mathrm{int},3}\left(\varphi_1^3, \psi_1^3; \frac{x-x^3}{R_\varepsilon^3}\right) + v_1^3(x), \end{aligned}$$

$$v_2(x) = \overline{u}(x-x^3) - \frac{1-\xi}{\gamma(2-\gamma-\xi)}G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)}G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_2^3(x)$$
$$+ H_2^{\mathrm{int},3}\left(\varphi_2^3, \psi_2^3; \frac{x-x^3}{R_\varepsilon^3}\right) + v_2^3(x).$$

This amounts to solve the equations

(4.9)

$$\mathbb{L}v_1^3 = \frac{384}{(1+r^2)^4}\Big[e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)-\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\gamma(h_1^3+H_1^{\mathrm{int},3}+v_1^3)+(1-\gamma)(h_2^3+H_2^{\mathrm{int},3}+v_2^3)}$$
$$- e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)-\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\gamma h_1^3+(1-\gamma)h_2^3} - v_1^3\Big]$$
$$- \Sigma^1_{\widetilde{a}_3}(H_1^{\mathrm{int},3}+v_1^3) - \Sigma^2_{\widetilde{a}_3}(H_1^{\mathrm{int},3}+v_1^3),$$
$$\mathbb{L}v_2^3 = \frac{384}{(1+r^2)^4}\Big[e^{\frac{(1-\xi)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(-\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)+\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\xi(h_2^3+H_2^{\mathrm{int},3}+v_2^3)+(1-\xi)(h_1^3+H_1^{\mathrm{int},3}+v_1^3)}$$
$$- e^{\frac{(1-\xi)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(-\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)+\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\xi h_2^3+(1-\xi)h_1^3} - v_2^3\Big]$$
$$- \Sigma^1_{\widetilde{a}_3}(H_2^{\mathrm{int},3}+v_2^3) - \Sigma^2_{\widetilde{a}_3}(H_2^{\mathrm{int},3}+v_2^3).$$

We denote by

$$\mathbb{L}v_1^3 = \mathcal{S}_3(v_1^3, v_2^3) \quad \text{and} \quad \mathbb{L}v_2^3 = \mathcal{P}_3(v_1^3, v_2^3).$$

To find a solution of (4.9), it is enough to find a fixed point (v_1^3, v_2^3) in a small ball of $\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ solutions of

$$\begin{aligned} v_1^3 &= \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon^3} \circ \mathcal{S}_3(v_1^3, v_2^3) = \aleph_3(v_1^3, v_2^3), \\ v_2^3 &= \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon^3} \circ \mathcal{P}_3(v_1^3, v_2^3) = \Upsilon_3(v_1^3, v_2^3). \end{aligned} \tag{4.10}$$

Then, we have the following result.

Lemma 4.8. *Let $\mu \in (1,2)$. Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$, $\overline{c}_\kappa > 0$ and $\overline{\overline{c}}_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$, $\beta \in (0, \beta_\kappa)$ with $r_\varepsilon := r_{\varepsilon,\beta,\lambda}$. We have*

$$\|\aleph_3(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \quad \|\Upsilon_3(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2,$$
$$\|\aleph_3(v_1^3, v_2^3) - \aleph_3(t_1^3, t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le \overline{c}_\kappa r_\varepsilon^2 \|(v_1^3, v_2^3) - (t_1^3, t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}$$

and

$$\|\Upsilon_3(v_1^3, v_2^3) - \Upsilon_3(t_1^3, t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le \overline{\overline{c}}_\kappa r_\varepsilon^2 \|(v_1^3, v_2^3) - (t_1^3, t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)},$$

provided $(v_1^3, v_2^3), (t_1^3, t_2^3) \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ satisfying

$$\|(v_1^3, v_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(t_1^3, t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2. \tag{4.11}$$

Proof. The first and second estimates follows from the estimate of H^{int} given by Lemma 3.13 together with the assumption on the norms of φ_j^3 and ψ_j^3 given by (3.37), we have

$$\left\| H^{\text{int}}_{\varphi_j^3,\psi_j^3}\left(\frac{r}{R_\varepsilon^3}\ \cdot\ \right)\right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))} \leq Cr^2(R_\varepsilon^3)^{-2}\big(\|\varphi_j^3\|_{\mathcal{C}^{4,\alpha}(S^3)}+\|\psi_j^3\|_{\mathcal{C}^{2,\alpha}(S^3)}\big)$$

for all $r \leq R_\varepsilon^3/2$. Then by (3.37), we get

$$\left\| H^{\text{int}}_{\varphi_j^3,\psi_j^3}\left(\frac{r}{R_\varepsilon^3}\ \cdot\ \right)\right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))} \leq c_\kappa \varepsilon^2 r^2.$$

On the other hand,

$$\begin{aligned}
&\sup_{r\leq R_\varepsilon^3} r^{4-\mu}|\mathcal{S}_3(0,0)| \\
&\leq \sup_{r\leq R_\varepsilon^3} \frac{384}{(1+r^2)^4} r^{4-\mu}\bigg| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)-\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\gamma(h_1^3+H_1^{\text{int},3})+(1-\gamma)(h_2^3+H_2^{\text{int},3})} \\
&\qquad - e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)-\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\gamma h_1^3+(1-\gamma)h_2^3}\bigg| \\
&\quad + \sup_{r\leq R_\varepsilon^3} r^{4-\mu}\big(\big|\Sigma^1_{\widetilde{a}_3}(H_1^{\text{int},3})\big|+\big|\Sigma^2_{\widetilde{a}_3}(H_1^{\text{int},3})\big|\big) \\
&\leq c_\kappa \sup_{r\leq R_\varepsilon^3} \frac{384}{(1+r^2)^4} r^{4-\mu}\big(\gamma r^2\|H_1^{\text{int},3}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}+(1-\gamma)r^2\|H_2^{\text{int},3}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}\big) \\
&\quad + c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^3\|\nabla a\|_\infty \sup_{r\leq R_\varepsilon^3} r^{5-\mu}\|H_1^{\text{int},3}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)} \\
&\quad + c_\kappa\left(\frac{\varepsilon}{\tau_3}\right)^2\lambda\|f\|_\infty \sup_{r\leq R_\varepsilon^3} r^{4-\mu}\|H_1^{\text{int},3}\|_{\mathcal{C}_2^{4,\alpha}(\overline{B}_1^*)}.
\end{aligned}$$

Making use of Proposition 3.2 together with (3.4), for $\mu \in (1,2)$, there exists a constant $c_\kappa > 0$ such that

$$\|\aleph_3(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2.$$

For the second estimate, we use the same argument as above and we get

$$\|\Upsilon_3(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2.$$

To derive the third estimate, for (v_1^3,v_2^3), (t_1^3,t_2^3) verifying (4.11), we have

$$\begin{aligned}
&\sup_{r\leq R_\varepsilon^3} r^{4-\mu}\big|\mathcal{S}_3(v_1^3,v_2^3)-\mathcal{S}_3(t_1^3,t_2^3)\big| \\
&\leq \sup_{r\leq R_\varepsilon^3} \frac{384}{(1+r^2)^4} r^{4-\mu} \\
&\qquad \times \bigg| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)-\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\gamma(h_1^3+H_1^{\text{int},3}+v_1^3)+(1-\gamma)(h_2^3+H_2^{\text{int},3}+v_2^3)} - v_1^3
\end{aligned}$$

$$
\begin{aligned}
&- e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G_a\left(\frac{\varepsilon x}{\tau_3},x^1\right)-\frac{1}{\xi}G_a\left(\frac{\varepsilon x}{\tau_3},x^2\right)\right)+\gamma(h_1^3+H_1^{\mathrm{int},3}+t_1^3)+(1-\gamma)(h_2^3+H_2^{\mathrm{int},3}+t_2^3)} + t_1^3\Big| \\
&+ \sup_{r\le R_\varepsilon^3} r^{4-\mu}\left|\Sigma^1_{\widetilde{a}_3}(v_1^3-t_1^3)\right| + \sup_{r\le R_\varepsilon^3} r^{4-\mu}\left|\Sigma^2_{\widetilde{a}_3}(v_1^3-t_1^3)\right| \\
\le{}& c_\kappa \sup_{r\le R_\varepsilon^3}\frac{384 r^{4-\mu}}{(1+r^2)^4}(1-\gamma)\left[r^\mu\|v_1^3-t_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + r^\mu\|v_2^3-t_2^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\right] \\
&+ c_\kappa\beta r_\varepsilon\|v_1^3-t_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa\beta r_\varepsilon^3\|v_1^3-t_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa\lambda r_\varepsilon^2\|v_1^3-t_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&\|\aleph_3(v_1^3,v_2^3)-\aleph_3(t_1^3,t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\le \left(c_\kappa r_\varepsilon^2 + c_\kappa(1-\gamma)\right)\|v_1^3-t_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|v_2^3-t_2^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\le \left(c_\kappa r_\varepsilon^2 + c_\kappa(1-\gamma)\right)\|(v_1^3,v_2^3)-(t_1^3,t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}
$$

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \le 1/2$ for all $\varepsilon \in (0,\varepsilon_\kappa)$. There exists a constant $\gamma_0 \in (0,1)$ such that $c_\kappa(1-\gamma) \le 1/2$ for all $\gamma \in (\gamma_0,1)$, we can find a constant $\overline{c}_\kappa > 0$ such that

$$
\|\aleph_3(v_1^3,v_2^3)-\aleph_3(t_1^3,t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le \overline{c}_\kappa r_\varepsilon^2\|(v_1^3,v_2^3)-(t_1^3,t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}. \tag{4.12}
$$

Similarly, we get

$$
\begin{aligned}
&\|\Upsilon_3(v_1^3,v_2^3)-\Upsilon_3(t_1^3,t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\le \left(c_\kappa r_\varepsilon^2 + c_\kappa(1-\xi)\right)\|v_2^3-t_2^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa(1-\xi)\|v_1^3-t_1^3\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\le \left(c_\kappa r_\varepsilon^2 + c_\kappa(1-\xi)\right)\|(v_1^3,v_2^3)-(t_1^3,t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}
$$

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \le 1/2$ for all $\varepsilon \in (0,\varepsilon_\kappa)$. There exists a constant $\xi_0 \in (0,1)$ such that $c_\kappa(1-\xi) \le 1/2$ for all $\xi \in (\xi_0,1)$, we can find a constant $\overline{\overline{c}}_\kappa > 0$ such that

$$
\|\Upsilon_3(v_1^3,v_2^3)-\Upsilon_3(t_1^3,t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le \overline{\overline{c}}_\kappa r_\varepsilon^2\|(v_1^3,v_2^3)-(t_1^3,t_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}. \tag{4.13}
$$

□

Therefore, (4.12) and (4.13) are enough to show that

$$
(v_1^3,v_2^3) \mapsto \left(\aleph_3(v_1^3,v_2^3),\Upsilon_3(v_1^3,v_2^3)\right)
$$

is a contraction from the ball

$$
\left\{(v_1^3,v_2^3)\in\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) : \|(v_1^3,v_2^3)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2\right\}
$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

Proposition 4.9. *Given $\kappa > 0$, $\mu \in (1,2)$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$, $c_\kappa > 0$ (depending on κ), $\gamma_0 \in (0,1)$ and $\xi_0 \in (0,1)$ such that for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$, $\beta \in (0,\beta_\kappa)$, $\gamma \in (\gamma_0,1)$ and $\xi \in (\xi_0,1)$, for all τ_3 in some fixed compact subset of $[\tau_3^-,\tau_3^+] \subset (0,\infty)$ and for φ_j^3 and ψ_j^3 satisfying* (3.23) *and* (3.37)*, there exists a unique $(v_1^3, v_2^3) := (v_{1,\varepsilon,\lambda,\beta,\tau_3,\varphi_1^3,\psi_1^3}, v_{2,\varepsilon,\lambda,\beta,\tau_3,\varphi_2^3,\psi_2^3})$ solution of* (4.10) *such that*

$$\|(v_1^3, v_2^3)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Hence

$$\begin{aligned} v_1(x) &= \overline{u}(x-x^3) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_1^3(x) \\ &\quad + H_1^{\mathrm{int},3}\left(\varphi_1^3, \psi_1^3; \frac{x-x^3}{R_\varepsilon^3}\right) + v_1^3(x), \\ v_2(x) &= \overline{u}(x-x^3) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G_a\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_2^3(x) \\ &\quad + H_2^{\mathrm{int},3}\left(\varphi_2^3, \psi_2^3; \frac{x-x^3}{R_\varepsilon^3}\right) + v_2^3(x) \end{aligned}$$

solves (4.3) *in $B_{R_\varepsilon^3}(x^3)$.*

Remark also that the functions $(v_1^i, v_2^i) := (v_{1,\varepsilon,\lambda,\beta,\tau_i,\varphi_1^i,\psi_1^i}, v_{2,\varepsilon,\lambda,\beta,\tau_i,\varphi_2^i,\psi_2^i})$ ($i \in \{1,2,3\}$) depend continuously on the parameter τ_i.

4.3. The nonlinear exterior problem

We reproduce exactly the same nonlinear exterior problem as in the proof of Theorem 1.5, so we obtain the following proposition.

Proposition 4.10. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $\beta_\kappa > 0$ (depending on κ) such that for any $\varepsilon \in (0,\varepsilon_\kappa)$, $\lambda \in (0,\lambda_\kappa)$, $\beta \in (0,\beta_\kappa)$, η_i and $\widetilde{x}^i$ satisfying* (3.47) *and functions $\widetilde{\varphi}_j^i$ and $\widetilde{\psi}_j^i$ satisfying* (3.24) *and* (3.46)*, there exists a unique $(\widetilde{v}_1, \widetilde{v}_2)$ (:= $(\widetilde{v}_{1,\varepsilon,\lambda,\beta,\eta_1,\eta_3,\widetilde{\mathbf{x}},\widetilde{\varphi}_1^i,\widetilde{\psi}_1^i}, \widetilde{v}_{2,\varepsilon,\lambda,\beta,\eta_2,\eta_3,\widetilde{\mathbf{x}},\widetilde{\varphi}_2^i,\widetilde{\psi}_2^i})$) solution of* (3.45) *so that for (v_1, v_2) defined by*

$$\begin{aligned} v_1(x) &:= \frac{1+\eta_1}{\gamma} G_a(x, \widetilde{x}^1) + (1+\eta_3) G_a(x, \widetilde{x}^3) + \sum_{i=1}^{3} \chi_{r_0}(x-\widetilde{x}^i) H_1^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right) \\ &\quad + \widetilde{v}_1(x), \\ v_2(x) &:= \frac{1+\eta_2}{\xi} G_a(x, \widetilde{x}^2) + (1+\eta_3) G_a(x, \widetilde{x}^3) + \sum_{i=1}^{3} \chi_{r_0}(x-\widetilde{x}^i) H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right) \\ &\quad + \widetilde{v}_2(x) \end{aligned}$$

solve (3.42) *in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\mathbf{x}})$. In addition, we have*

$$\|(\widetilde{v}_1, \widetilde{v}_2)\|_{(\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}})))^2} \le 2c_\kappa r_\varepsilon^2.$$

4.4. The nonlinear Cauchy-data matching

We will gather the results of the previous sections. Using the previous notations, assume that $\widetilde{\mathbf{x}} := (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) \in \Omega^3$ are given close to $\mathbf{x} := (x^1, x^2, x^3)$. Assume also that

$$\boldsymbol{\tau} := (\tau_1, \tau_2, \tau_3) \in [\tau_1^-, \tau_1^+] \times [\tau_2^-, \tau_2^+] \times [\tau_3^-, \tau_3^+] \subset (0, \infty)^3$$

are given (the values of τ_l^- and τ_l^+ for $l = 1, 2, 3$ will be fixed later). First, we consider some set of boundary data $\boldsymbol{\varphi}^i := (\varphi_1^i, \varphi_2^i) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\boldsymbol{\psi}^i := (\psi_1^i, \psi_2^i) \in (\mathcal{C}^{2,\alpha}(S^3))^2$. According to the results of Propositions 4.6, 4.7 and 4.9 and provided $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\beta \in (0, \beta_\kappa)$, we can find $u_{\text{int}} := (u_{\text{int},1}, u_{\text{int},2})$ a solution of (3.5) in $B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3)$, which can be decomposed as

$$u_{\text{int},1}(x) := \begin{cases} \frac{1}{\gamma} u_{\varepsilon,\tau_1}(x - \widetilde{x}^1) - \frac{1-\gamma}{\gamma\xi} G_a(x, \widetilde{x}^2) - \frac{1-\gamma}{\gamma} G_a(x, \widetilde{x}^3) - \frac{\ln\gamma}{\gamma} \\ \quad + h_1^1\big(\frac{R_\varepsilon^1(x-\widetilde{x}^1)}{r_\varepsilon}\big) + H_1^{\text{int},1}\big(\varphi_1^1, \psi_1^1; \frac{x-\widetilde{x}^1}{r_\varepsilon}\big) + v_1^1\big(\frac{R_\varepsilon^1(x-\widetilde{x}^1)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^1), \\ \frac{1}{\gamma} G_a(x, \widetilde{x}^1) + G_a(x, \widetilde{x}^3) + h_1^2\big(\frac{R_\varepsilon^2(x-\widetilde{x}^2)}{r_\varepsilon}\big) \\ \quad + H_1^{\text{int},2}\big(\varphi_1^2, \psi_1^2; \frac{x-\widetilde{x}^2}{r_\varepsilon}\big) + v_1^2\big(\frac{R_\varepsilon^2(x-\widetilde{x}^2)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^2), \\ u_{\varepsilon,\tau_3}(x - \widetilde{x}^3) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G_a(x, \widetilde{x}^1) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a(x, \widetilde{x}^2) \\ \quad + h_1^3\big(\frac{R_\varepsilon^3(x-\widetilde{x}^3)}{r_\varepsilon}\big) + H_1^{\text{int},3}\big(\varphi_1^3, \psi_1^3; \frac{x-\widetilde{x}^3}{r_\varepsilon}\big) + v_1^3\big(\frac{R_\varepsilon^3(x-\widetilde{x}^3)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^3) \end{cases}$$

and

$$u_{\text{int},2}(x) := \begin{cases} \frac{1}{\xi} G_a(x, \widetilde{x}^2) + G_a(x, \widetilde{x}^3) + h_2^1\big(\frac{R_\varepsilon^1(x-\widetilde{x}^1)}{r_\varepsilon}\big) \\ \quad + H_2^{\text{int},1}\big(\varphi_2^1, \psi_2^1; \frac{x-\widetilde{x}^1}{r_\varepsilon}\big) + v_2^1\big(\frac{R_\varepsilon^1(x-\widetilde{x}^1)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^1), \\ \frac{1}{\xi} u_{\varepsilon,\tau_2}(x - \widetilde{x}^2) - \frac{1-\xi}{\xi} G_a(x, \widetilde{x}^3) - \frac{1-\xi}{\gamma\xi} G_a(x, \widetilde{x}^1) - \frac{\ln\xi}{\xi} \\ \quad + h_2^2\big(\frac{R_\varepsilon^2(x-\widetilde{x}^2)}{r_\varepsilon}\big) + H_2^{\text{int},2}\big(\varphi_2^2, \psi_2^2; \frac{x-\widetilde{x}^2}{r_\varepsilon}\big) + v_2^2\big(\frac{R_\varepsilon^2(x-\widetilde{x}^2)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^2), \\ u_{\varepsilon,\tau_3}(x - \widetilde{x}^3) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G_a(x, \widetilde{x}^1) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G_a(x, \widetilde{x}^2) \\ \quad + h_2^3\big(\frac{R_\varepsilon^3(x-\widetilde{x}^3)}{r_\varepsilon}\big) + H_2^{\text{int},3}\big(\varphi_2^3, \psi_2^3; \frac{x-\widetilde{x}^3}{r_\varepsilon}\big) + v_2^3\big(\frac{R_\varepsilon^3(x-\widetilde{x}^3)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^3), \end{cases}$$

where for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$, $R_\varepsilon^i = \tau_i \frac{r_\varepsilon}{\varepsilon}$ and the functions h_j^i and v_j^i satisfy

$$\|(h_1^1, h_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(h_1^2, h_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2,$$
$$\|(h_1^3, h_2^3)\|_{(\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4))^2} \le 2c_\kappa r_\varepsilon^2$$

and

$$\|(v_1^1, v_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(v_1^2, v_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2,$$
$$\|(v_1^3, v_2^3)\|_{(\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4))^2} \le 2c_\kappa r_\varepsilon^2.$$

Similarly, given some boundary data $\widetilde{\varphi}_j^i \in C^{4,\alpha}(S^3)$, $\widetilde{\psi}_j^i \in C^{2,\alpha}(S^3)$ satisfying (3.24), $(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ satisfying (3.47), provided $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ and $\beta \in (0, \beta_\kappa)$ by Proposition 3.21, we find a solution $u_{\text{ext}} := (u_{\text{ext},1}, u_{\text{ext},2})$ of (3.42) in $\overline{\Omega} \setminus (B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3))$ which can be decomposed as

$$
\begin{aligned}
u_{\text{ext},1}(x) &:= \frac{1+\eta_1}{\gamma} G_a(x, \widetilde{x}^1) + (1+\eta_3) G_a(x, \widetilde{x}^3) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H_1^{\text{ext}} \left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x - \widetilde{x}^i}{r_\varepsilon} \right) \\
&\quad + \widetilde{v}_1(x), \\
u_{\text{ext},2}(x) &:= \frac{1+\eta_2}{\xi} G_a(x, \widetilde{x}^2) + (1+\eta_3) G_a(x, \widetilde{x}^3) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H_2^{\text{ext}} \left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x - \widetilde{x}^i}{r_\varepsilon} \right) \\
&\quad + \widetilde{v}_2(x)
\end{aligned}
$$

with $\widetilde{v}_1, \widetilde{v}_2 \in \mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}}))$ satisfying

$$
\|(\widetilde{v}_1, \widetilde{v}_2)\|_{(\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\mathbf{x}})))^2} \leq 2 c_\kappa r_\varepsilon^2.
$$

It remains to determine the parameters and the boundary data in such a way that the function which is equal to u_{int} in $B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3)$ and to u_{ext} in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\mathbf{x}})$ is a smooth function. This amounts to find the boundary data and the parameters so that, for each $j = 1, 2$,

$$
\begin{aligned}
u_{\text{int},j} &= u_{\text{ext},j}, & \partial_r u_{\text{int},j} &= \partial_r u_{\text{ext},j}, \\
\Delta u_{\text{int},j} &= \Delta u_{\text{ext},j}, & \partial_r \Delta u_{\text{int},j} &= \partial_r \Delta u_{\text{ext},j}
\end{aligned} \tag{4.14}
$$

on $\partial B_{r_\varepsilon}(\widetilde{x}^1)$, $\partial B_{r_\varepsilon}(\widetilde{x}^2)$ and $\partial B_{r_\varepsilon}(\widetilde{x}^3)$.

Suppose that (4.14) is verified, this provides that for each ε small enough $u_\varepsilon \in \mathcal{C}^{4,\alpha}$ (which is obtained by matching together the functions u_{int} and the function u_{ext}), a weak solution of our system and elliptic regularity theory implies that this solution is in fact smooth. That will complete the proof since, as ε tends to 0, the sequence of solutions we have obtained satisfies the required singular limit behavior.

Before we proceed, the following remarks are due. First it will be convenient to observe that the function u_{ε,τ_i} can be expanded as

$$
u_{\varepsilon,\tau_i}(x) = -4 \ln \tau_i - 8 \ln |x| + \mathcal{O}\left(\frac{\varepsilon^2 \tau_i^{-2}}{|x|^2} \right) \quad \text{on } \partial B_{r_\varepsilon}(x^i).
$$

• On $\partial B_{r_\varepsilon}(\widetilde{x}^1)$ according to the proof of Theorem 1.5 and since when ε tends to 0, it is enough to choose τ_1^- and τ_1^+ in such a way that

$$
4 \ln(\tau_1^-) < -\ln \gamma - \mathcal{E}_1(x^1, \mathbf{x}) < 4 \ln(\tau_1^+),
$$

where

$$\mathcal{E}_1(\,\cdot\,,\widetilde{\mathbf{x}}) := H_a(\,\cdot\,,\widetilde{x}^1) + \frac{1-\gamma}{\xi}G_a(\,\cdot\,,\widetilde{x}^2) + G_a(\,\cdot\,,\widetilde{x}^3).$$

Also using the fact that

$$\varphi_1^1 = \varphi_{1,0}^1 + \varphi_{1,1}^1 + \varphi_1^{1,\perp}, \qquad \psi_1^1 = 8\varphi_{1,0}^1 + 12\varphi_{1,1}^1 + \psi_1^{1,\perp},$$
$$\widetilde{\varphi}_1^1 = \widetilde{\varphi}_{1,0}^1 + \widetilde{\varphi}_{1,1}^1 + \widetilde{\varphi}_1^{1,\perp}, \qquad \widetilde{\psi}_1^1 = \widetilde{\psi}_{1,1}^1 + \widetilde{\psi}_1^{1,\perp},$$

where $\varphi_{1,0}^1, \widetilde{\varphi}_{1,0}^1 \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi_{1,1}^1$, $\widetilde{\varphi}_{1,1}^1$, $\widetilde{\psi}_{1,1}^1$ belong to $\mathbb{E}_1 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_1^{1,\perp}$, $\widetilde{\varphi}_1^{1,\perp}$, $\psi_1^{1,\perp}$, $\widetilde{\psi}_1^{1,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$. We can prove that

$$(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0, \qquad \partial_r(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0,$$
$$\Delta(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0, \qquad \partial_r\Delta(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0$$

on S^3 yield to

$$T_\varepsilon^1 = \big(t_1, \eta_1, \varphi_{1,0}^1, \widetilde{\varphi}_{1,0}^1, \varphi_{1,1}^1, \widetilde{\varphi}_{1,1}^1, \widetilde{\psi}_{1,1}^1, \overline{\nabla}\mathcal{E}_1(\widetilde{x}^1, \widetilde{\mathbf{x}}), \varphi_1^{1,\perp}, \widetilde{\varphi}_1^{1,\perp}, \psi_1^{1,\perp}, \widetilde{\psi}_1^{1,\perp}\big) = \mathcal{O}(r_\varepsilon^2), \tag{4.15}$$

where

$$t_1 := \frac{1}{\ln r_\varepsilon}[4\ln\tau_1 + \ln\gamma + \mathcal{E}_1(\widetilde{x}^1, \widetilde{\mathbf{x}})].$$

On the other hand, using the fact that

$$\varphi_2^1 = \varphi_{2,0}^1 + \varphi_{2,1}^1 + \varphi_2^{1,\perp}, \qquad \psi_2^1 = 8\varphi_{2,0}^1 + 12\varphi_{2,1}^1 + \psi_2^{1,\perp},$$
$$\widetilde{\varphi}_2^1 = \widetilde{\varphi}_{2,0}^1 + \widetilde{\varphi}_{2,1}^1 + \widetilde{\varphi}_2^{1,\perp}, \qquad \widetilde{\psi}_2^1 = \widetilde{\psi}_{2,1}^1 + \widetilde{\psi}_2^{1,\perp},$$

where $\varphi_{2,0}^1, \widetilde{\varphi}_{2,0}^1 \in \mathbb{E}_0$, $\varphi_{2,1}^1, \widetilde{\varphi}_{2,1}^1, \widetilde{\psi}_{2,1}^1 \in \mathbb{E}_1$ and $\varphi_2^{1,\perp}$, $\widetilde{\varphi}_2^{1,\perp}$, $\psi_2^{1,\perp}$, $\widetilde{\psi}_2^{1,\perp}$ belong to $(L^2(S^3))^\perp$. We can prove that

$$(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0, \qquad \partial_r(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0,$$
$$\Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0, \qquad \partial_r\Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0$$

on S^3 yield to

$$\big(\varphi_{2,0}^1, \widetilde{\varphi}_{2,0}^1, \varphi_{2,1}^1, \widetilde{\varphi}_{2,1}^1, \widetilde{\psi}_{2,1}^1, \varphi_2^{1,\perp}, \widetilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \widetilde{\psi}_2^{1,\perp}\big) = \mathcal{O}(r_\varepsilon^2).$$

As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left-hand side, but are bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

- On $\partial B_{r_\varepsilon}(\widetilde{x}^2)$, according to the proof of Theorem 1.5, using the fact that

$$\varphi_1^2 = \varphi_{1,0}^2 + \varphi_{1,1}^2 + \varphi_1^{2,\perp}, \qquad \psi_1^2 = 8\varphi_{1,0}^2 + 12\varphi_{1,1}^2 + \psi_1^{2,\perp},$$
$$\widetilde{\varphi}_1^2 = \widetilde{\varphi}_{1,0}^2 + \widetilde{\varphi}_{1,1}^2 + \widetilde{\varphi}_1^{2,\perp}, \qquad \widetilde{\psi}_1^2 = \widetilde{\psi}_{1,1}^2 + \widetilde{\psi}_1^{2,\perp},$$

where $\varphi^2_{1,0}, \widetilde{\varphi}^2_{1,0} \in \mathbb{E}_0$, $\varphi^2_{1,1}, \widetilde{\varphi}^2_{1,1}, \widetilde{\psi}^2_{1,1} \in \mathbb{E}_1 = \ker(\Delta_{S^3}+1) = \mathrm{Span}\{e_1,e_2,e_3,e_4\}$ and $\varphi^{2,\perp}_1$, $\widetilde{\varphi}^{2,\perp}_1$, $\psi^{2,\perp}_1$, $\widetilde{\psi}^{2,\perp}_1$ belong to $(L^2(S^3))^\perp$. We can prove that

$$\big(\varphi^2_{1,0}, \widetilde{\varphi}^2_{1,0}, \varphi^2_{1,1}, \widetilde{\varphi}^2_{1,1}, \widetilde{\psi}^2_{1,1}, \varphi^{2,\perp}_1, \widetilde{\varphi}^{2,\perp}_1, \psi^{2,\perp}_1, \widetilde{\psi}^{2,\perp}_1\big) = \mathcal{O}(r^2_\varepsilon).$$

Finally, according to the proof of Theorem 1.5 and since ε tends to 0, it is enough to choose τ^-_2, τ^+_2 in such a way that

$$4\ln(\tau^-_2) < -\ln\xi - \mathcal{E}_2(x^2, \mathbf{x}) < 4\ln(\tau^+_2),$$

where

$$\mathcal{E}_2(\,\cdot\,, \widetilde{\mathbf{x}}) := H_a(\,\cdot\,, \widetilde{x}^2) + \frac{1-\xi}{\gamma} G_a(\,\cdot\,, \widetilde{x}^1) + G_a(\,\cdot\,, \widetilde{x}^3).$$

Also using the fact that

$$\begin{aligned} \varphi^2_2 &= \varphi^2_{2,0} + \varphi^2_{2,1} + \varphi^{2,\perp}_2, & \psi^2_2 &= 8\varphi^2_{2,0} + 12\varphi^2_{2,1} + \psi^{2,\perp}_2, \\ \widetilde{\varphi}^2_2 &= \widetilde{\varphi}^2_{2,0} + \widetilde{\varphi}^2_{2,1} + \widetilde{\varphi}^{2,\perp}_2, & \widetilde{\psi}^2_2 &= \widetilde{\psi}^2_{2,1} + \widetilde{\psi}^{2,\perp}_2, \end{aligned}$$

where $\varphi^2_{2,0}, \widetilde{\varphi}^2_{2,0} \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi^2_{2,1}$, $\widetilde{\varphi}^2_{2,1}$, $\widetilde{\psi}^2_{2,1}$ belong to $\mathbb{E}_1 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi^{2,\perp}_2$, $\widetilde{\varphi}^{2,\perp}_2$, $\psi^{2,\perp}_2$, $\widetilde{\psi}^{2,\perp}_2$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$. We can prove that

$$\begin{aligned} (u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0, & \partial_r(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0, \\ \Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0, & \partial_r\Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0 \end{aligned}$$

on S^3 yield to

$$T^2_\varepsilon = \big(t_2, \eta_2, \varphi^2_{2,0}, \widetilde{\varphi}^2_{2,0}, \varphi^2_{2,1}, \widetilde{\varphi}^2_{2,1}, \widetilde{\psi}^2_{2,1}, \overline{\nabla}\mathcal{E}_2(\widetilde{z}_2, \widetilde{\mathbf{z}}), \varphi^{2,\perp}_2, \widetilde{\varphi}^{2,\perp}_2, \psi^{2,\perp}_2, \widetilde{\psi}^{2,\perp}_2\big) = \mathcal{O}(r^2_\varepsilon), \tag{4.16}$$

where

$$t_2 := \frac{1}{\ln r_\varepsilon}[4\ln\tau_2 + \ln\xi + \mathcal{E}_2(\widetilde{x}^2, \widetilde{\mathbf{x}})].$$

As usual, the terms $\mathcal{O}(r^2_\varepsilon)$ depend nonlinearly on all the variables on the left-hand side, but are bounded (in the appropriate norm) by a constant (independent of ε and κ) times r^2_ε, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

- On $\partial B_{r_\varepsilon}(\widetilde{x}^3)$, we have

$$\begin{aligned} &(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(x) \\ &= -4\ln\tau_3 + 8\eta_3 \ln|x - \widetilde{x}^3| + H^{\mathrm{int},3}_1\left(\varphi^3_1, \psi^3_1; \frac{x - \widetilde{x}^3}{r_\varepsilon}\right) - H^{\mathrm{ext}}_1\left(\widetilde{\varphi}^3_1, \widetilde{\psi}^3_1; \frac{x - \widetilde{x}^3}{r_\varepsilon}\right) \\ &\quad - \left[\frac{1-\xi}{\gamma(2-\gamma-\xi)} G_a(x, \widetilde{x}^1) + \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a(x, \widetilde{x}^2) + H_a(x, \widetilde{x}^3)\right] \\ &\quad + \mathcal{O}\left(\frac{\varepsilon^2\tau_3^{-2}}{|x - \widetilde{x}^3|^2}\right) + \mathcal{O}(r^2_\varepsilon) \end{aligned} \tag{4.17}$$

and

$$
\begin{aligned}
&(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(x) \\
&= -4\ln\tau_3 + 8\eta_3 \ln|x - \widetilde{x}^3| + H_2^{\mathrm{int},3}\left(\varphi_2^3, \psi_2^3; \frac{x - \widetilde{x}^3}{r_\varepsilon}\right) - H_2^{\mathrm{ext}}\left(\widetilde{\varphi}_2^3, \widetilde{\psi}_2^3; \frac{x - \widetilde{x}^3}{r_\varepsilon}\right) \\
&\quad - \left[\frac{1-\xi}{\gamma(2-\gamma-\xi)} G_a(x, \widetilde{x}^1) + \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a(x, \widetilde{x}^2) + H_a(x, \widetilde{x}^3)\right] \\
&\quad + \mathcal{O}\left(\frac{\varepsilon^2 \tau_3^{-2}}{|x - \widetilde{x}^3|^2}\right) + \mathcal{O}(r_\varepsilon^2).
\end{aligned} \tag{4.18}
$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{x}^3)$ in (4.14), it will be more convenient to solve on S^3, for $i = 1, 2$, the following set of equations

$$
\begin{aligned}
(u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) = 0, &\qquad \partial_r(u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) = 0, \\
\Delta(u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) = 0, &\qquad \partial_r\Delta(u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) = 0.
\end{aligned} \tag{4.19}
$$

Since the boundary data are chosen to satisfy (3.23) or (3.24), we decompose

$$
\begin{aligned}
\varphi_i^3 &= \varphi_{i,0}^3 + \varphi_{i,1}^3 + \varphi_i^{3,\perp}, &\qquad \psi_i^3 &= 8\varphi_{i,0}^3 + 12\varphi_{i,1}^3 + \psi_i^{3,\perp}, \\
\widetilde{\varphi}_i^3 &= \widetilde{\varphi}_{i,0}^3 + \widetilde{\varphi}_{i,1}^3 + \widetilde{\varphi}_i^{3,\perp}, &\qquad \widetilde{\psi}_i^3 &= \widetilde{\psi}_{i,1}^3 + \widetilde{\psi}_i^{3,\perp},
\end{aligned}
$$

where $\varphi_{i,0}^3, \widetilde{\varphi}_{i,0}^3 \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi_{i,1}^3$, $\widetilde{\varphi}_{i,1}^3$, $\widetilde{\psi}_{i,1}^3$ belong to $\mathbb{E}_1 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_i^{3,\perp}$, $\widetilde{\varphi}_i^{3,\perp}$, $\psi_i^{3,\perp}$, $\widetilde{\psi}_i^{3,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

We insist that for $x \in S^3$, both equations (4.17) and (4.18) involve the same relation of the parameter τ_3 and the appropriate energy $\mathcal{E}_3$. Then we have

$$
\begin{aligned}
&(u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^3 + r_\varepsilon x) \\
&= -4\ln\tau_3 + 8\eta_3 \ln r_\varepsilon|x| + H_i^{\mathrm{int},3}(\varphi_i^3, \psi_i^3, x) - H_i^{\mathrm{ext}}(\widetilde{\varphi}_i^3, \widetilde{\psi}_i^3, x) \\
&\quad - \left[\frac{1-\xi}{\gamma(2-\gamma-\xi)} G_a(\widetilde{x}^3, \widetilde{x}^1) + \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a(\widetilde{x}^3, \widetilde{x}^2) + H_a(\widetilde{x}^3, \widetilde{x}^3)\right] + \mathcal{O}(r_\varepsilon^2).
\end{aligned}
$$

Projecting the set of equations (4.19) over $\mathbb{E}_0$, we get

$$
\begin{aligned}
-4\ln\tau_3 + 8\eta_3 \ln r_\varepsilon + \varphi_{i,0}^3 - \widetilde{\varphi}_{i,0}^3 - \mathcal{E}_3(\widetilde{x}^3, \widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0, \\
8\eta_3 + 2\varphi_{i,0}^3 + 2\widetilde{\varphi}_{i,0}^3 + \mathcal{O}(r_\varepsilon^2) &= 0, \\
16\eta_3 + 8\varphi_{i,0}^3 + \mathcal{O}(r_\varepsilon^2) &= 0, \\
-32\eta_3 + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned} \tag{4.20}
$$

where

$$
\mathcal{E}_3(\,\cdot\,, \widetilde{\mathbf{x}}) := H_a(\,\cdot\,, \widetilde{x}^3) + \frac{1-\xi}{\gamma(2-\gamma-\xi)} G_a(\,\cdot\,, \widetilde{x}^1) + \frac{1-\gamma}{\xi(2-\gamma-\xi)} G_a(\,\cdot\,, \widetilde{x}^2).
$$

The system (4.20) can be simply written as

$$\eta_3 = \mathcal{O}(r_\varepsilon^2), \quad \varphi_{i,0}^3 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_{i,0}^3 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \frac{1}{\ln r_\varepsilon}[4\ln \tau_3 + \mathcal{E}_3(\widetilde{x}^3, \widetilde{\mathbf{x}})] = \mathcal{O}(r_\varepsilon^2).$$

We are now in a position to define τ_3^- and τ_3^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\widetilde{x}^i$ will converge to x^i for $i \in \{1, 2, 3\}$ and τ_3 will converge to τ_3^* satisfying

$$4\ln \tau_3^* = -\mathcal{E}_3(x^3, \mathbf{x}).$$

Hence it is enough to choose τ_3^- and τ_3^+ in such a way that

$$4\ln(\tau_3^-) < -\mathcal{E}_3(x^3, \mathbf{x}) < 4\ln(\tau_3^+).$$

Consider now the projection of (4.19) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \dots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$:

$$\overline{\nabla} f = \sum_{i=1}^{4} \partial_{x_i} f \, e_i.$$

Keeping these notations in mind, we obtain the system of equations

$$\begin{aligned}
\varphi_{i,1}^3 - \widetilde{\varphi}_{i,1}^3 - \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3, \widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
3\varphi_{i,1}^3 + 3\widetilde{\varphi}_{i,1}^3 + \frac{1}{2}\widetilde{\psi}_{i,1}^3 - \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3, \widetilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi_{i,1}^3 - 3\widetilde{\varphi}_{i,1}^3 - \widetilde{\psi}_{i,1}^3 + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi_{i,1}^3 + 15\widetilde{\varphi}_{i,1}^3 + \frac{18}{4}\widetilde{\psi}_{i,1}^3 + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned}$$

which can be simplified as follows:

$$\varphi_{i,1}^3 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_{i,1}^3 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\psi}_{i,1}^3 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3, \widetilde{\mathbf{x}}) = \mathcal{O}(r_\varepsilon^2).$$

Finally, we consider the projection onto $L^2(S^3)^\perp$. This yields the system

$$\begin{aligned}
\varphi_i^{3,\perp} - \widetilde{\varphi}_i^{3,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\big(H^{\text{int}}_{\varphi_i^{3,\perp}, \psi_i^{3,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}_i^{3,\perp}, \widetilde{\psi}_i^{3,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\psi_i^{3,\perp} - \widetilde{\psi}_i^{3,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\Delta\big(H^{\text{int}}_{\varphi_i^{3,\perp}, \psi_i^{3,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}_i^{3,\perp}, \widetilde{\psi}_i^{3,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0.
\end{aligned}$$

Thanks to the result of Lemma 3.14, this last system can be rewritten as

$$\varphi_i^{3,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_i^{3,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_i^{3,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_i^{3,\perp} = \mathcal{O}(r_\varepsilon^2).$$

If we define the parameter $t_3 \in \mathbb{R}$ by

$$t_3 := \frac{1}{\ln r_\varepsilon}[4 \ln \tau_3 + \mathcal{E}_3(\widetilde{x}^3, \widetilde{\mathbf{x}})],$$

then the systems found by projecting (4.19) gather in this equality

$$T_\varepsilon^3 = \left(t_3, \eta_3, \varphi_{i,0}^3, \widetilde{\varphi}_{i,0}^3, \varphi_{i,1}^3, \widetilde{\varphi}_{i,1}^3, \widetilde{\psi}_{i,1}^3, \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3, \widetilde{\mathbf{x}}), \varphi_i^{3,\perp}, \widetilde{\varphi}_i^{3,\perp}, \psi_i^{3,\perp}, \widetilde{\psi}_i^{3,\perp}\right) = \mathcal{O}(r_\varepsilon^2) \tag{4.21}$$

for $i = 1, 2$. As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left-hand side, but are bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$. We recall that $\mathbf{d} = r_\varepsilon(\widetilde{\mathbf{x}} - \mathbf{x})$, in addition the previous systems can be written as for $i = 1, 2, 3$:

$$\left(\mathbf{d}, t_i, \eta_i, \varphi^i, \widetilde{\varphi}^i, \psi^i, \widetilde{\psi}^i, \overline{\nabla}\mathcal{E}_i\right) = \mathcal{O}(r_\varepsilon^2).$$

Combining (4.15), (4.16) and (4.21), we have

$$T_\varepsilon = (T_\varepsilon^1, T_\varepsilon^2, T_\varepsilon^3) = \left(\mathcal{O}(r_\varepsilon^2), \mathcal{O}(r_\varepsilon^2), \mathcal{O}(r_\varepsilon^2)\right). \tag{4.22}$$

Then the nonlinear mapping which appears on the right-hand side of (4.22) is continuous, compact. In addition, reducing ε_κ if necessary, this nonlinear mapping sends the ball of radius κr_ε^2 (for the natural product norm) into itself, provided κ is fixed large enough. Applying Schauder's fixed point theorem in the ball of radius κr_ε^2 in the product space where the entries live, we obtain the existence of a solution of equation (4.22).

This completes the proof of Theorem 1.6. □

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