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Well-posedness and Asymptotic Behavior for a Pseudo-parabolic Equation Involving p-biharmonic Operator and Logarithmic Nonlinearity

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Abstract. This paper deals with the well-posedness and asymptotic behavior for a pseudo-parabolic equation involving p-biharmonic operator and logarithmic nonlinearity under Navior boundary condition. By combining Galerkin approximation, the method of potential well, the technique of differential inequality and improved logarithmic Sobolev inequality, we establish the local and global solvability, infinite and finite time blow-up phenomena of weak solutions in different energy levels. Moreover, we obtain the growth rate of weak solutions, life span in different energy cases and also give a result of extinction phenomenon.

1. Introduction

We consider a pseudo-parabolic equation involving p-biharmonic operator and logarithmic nonlinearity

$$u_t - \Delta u_t + \Delta\big(|\Delta u|^{p-2}\Delta u\big) = |u|^{q-2}u \log |u|, \quad (x,t) \in \Omega \times (0,+\infty), \tag{1.1}$$

subject to Navior boundary and initial conditions

$$u(x,t) = \Delta u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,+\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, initial data $u_0 \in H_0^1(\Omega) \cap W_0^{2,p}(\Omega)$, parameters p and q satisfy

$$\max\left\{1, \frac{2N}{N+4}\right\} < p \leq q < p\left(1 + \frac{4}{N}\right). \tag{1.4}$$

Partial differential equations with logarithmic nonlinearities have attracted much attention in recent years, due to their wide applications in physics and other applied sciences, see [3–7, 11, 13, 15, 17, 19, 21] and references therein. Among them, many scholars have been devoted to the topic on the global existence and blow-up phenomena

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of the second-order parabolic or pseudo-parabolic equations with p-Laplacian operator $\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$ and there have been fruitful results, one can see [4, 11] (for parabolic equations) and [3, 5, 7, 13, 15, 19] (for pseudo-parabolic equations).

However, there are fewer studies on higher order equations involving p-biharmonic operator $\Delta\big(|\Delta u|^{p-2}\Delta u\big)$ which appear in many fields. For example, the parabolic biharmonic equation (the case of $p = 2$ in (1.1)) arises in the growth theory of epitaxial thin films, where $u(x,t)$ denotes the height from the surface of the film in epitaxial growth and $\Delta^2 u$ represents the capillarity-driven surface diffusion. The authors of [6, 17, 21] considered the following p-biharmonic parabolic equation with logarithmic nonlinearity

$$u_t + \Delta\big(|\Delta u|^{p-2}\Delta u\big) = |u|^{q-2}u\log|u|, \quad (x,t) \in \Omega \times (0,+\infty),$$

subject to Navior boundary condition (1.2). For the case of $p = 2$ and $2 < q < 2 + \frac{4}{N}$, Li and Liu [17] established the global existence and exponential decay estimate of the solution by virtue of the method of potential well, and obtained the finite time blow-up phenomenon with positive initial energy (the subcritical case) by using of the concavity technique. For $2 < p < q < p\big(1 + \frac{4}{N}\big)$, Wang and Liu [21] established the local and global well-posedness of solutions, and derived the sufficient conditions of finite time blow-up for the solution with positive initial energy. Moreover, the results of finite time blow-up with negative initial energy and extinction phenomenon are deduced in the case of $p < q$, $q > 2$ and $p < q < 2$, respectively. Liu and Li [17] studied the case of $p > q > \frac{p}{2} + 1$ and $p > \max\big\{\frac{N}{2}, 2\big\}$, they established the well-posedness of local weak solution and proved the long-time behavior and the propagation of perturbations, based on the methods of difference and variation.

In addition, we refer to [8, 12, 16] for the researches on global bifurcation theory, finite speed of propagation and extinction phenomenon of elliptic or parabolic equations involving p-biharmonic operators and local (or non-local) power type nonlinearities, and refer to [2, 20] for the studies on the long-time behavior, extinction phenomenon and life span estimation of solutions of Kirchhoff fractional p-Laplacian diffusion equations and polyharmonic Kirchhoff equations, so on.

In view of the works mentioned above, one can find that problem (1.1)–(1.3) for pseudo-parabolic equation involving p-biharmonic operator and logarithmic nonlinearity has not been investigated yet. The main difficulty lies in finding the influence of the interaction among the p-biharmonic operator, the third derivative term Δu_t and the logarithmic nonlinearity on the asymptotic behavior of the weak solution. Motivated by these observations, we establish the local and global well-posedness of weak solution by the methods of multiplier and potential well for the case of $\max\big\{1, \frac{2N}{N+4}\big\} < p \le q < p\big(1 + \frac{4}{N}\big)$. By combining improved logarithmic Sobolev inequality, Gronwall inequality, the techniques of differential inequalities and concavity, we obtain the phenomena of finite time blow-up with various initial energy (including arbitrary initial energy) and infinite blow-up of solutions, and further derived the estimates of blow-up rate and life-span, for the case of $1 < p \le q \le 2$; $\max\left\{1, \frac{2N}{N+4}\right\} < p \le q$, $2 < q < p\left(1+\frac{4}{N}\right)$; and $2 < p < q < p\left(1+\frac{4}{N}\right)$, respectively. Meantime, for $\max\left\{1, \frac{2N}{N+2}\right\} < p < q < 2$, we present the sufficient conditions of extinction in finite time, and obtain the extinction time and decay rate estimate (for more detailed classification of parameters and summary of main conclusions, see Figure 1.1).

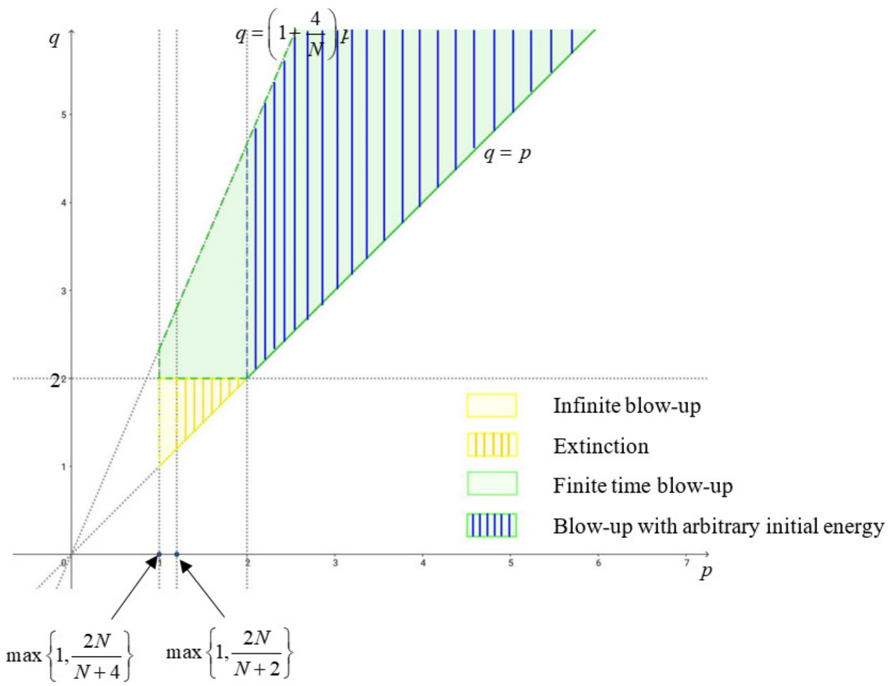

Figure 1.1: The classification of parameters.

In fact, the third derivative term Δu_t can be regarded as a damping term, which has inhibitory effect on the qualitative properties as blow-up, extinction and so on. Therefore, we research the properties of solutions in the sense of new measurements. Meantime, compared with the works in [6, 17, 21], the analysis and classification of the qualitative properties are presented more comprehensively and precisely in this paper. For example, the results on the infinite blow-up, blow-up with arbitrary initial energy, the life-span estimation of blow-up solution, the extinction rate, etc., haven't been studied in [6,17,21].

We established the well-poseness and asymptotic behavior of the solution under appropriate conditions by virtue of Galerkin approximation, improved logarithmic Sobolev inequality, the method of potential well and the technique of differential inequality, etc. Our detailed results are given below.

For the convenience of description, we denote the Sobolev spaces and norms as follows:

$$X := H_0^1(\Omega) \cap W_0^{2,p}(\Omega),$$
$$\|u\|_p := \|u\|_{L^p(\Omega)}, \quad \|u\|_{2,p} := \|u\|_{W_0^{2,p}(\Omega)} = \|\Delta u\|_p, \quad 1 < p < +\infty,$$

and

$$\|u\|_{H_0^1(\Omega)} := \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}}, \quad \forall\, u \in H_0^1(\Omega).$$

Meantime, we denote $W^{-2,p'}(\Omega)$ as the dual space of $W^{2,p}(\Omega)$ and $\langle\,\cdot\,,\cdot\,\rangle$ as the dual pairing between $W^{-2,p'}(\Omega)$ and $W^{2,p}(\Omega)$, where $p' = \frac{p}{p-1}$ is the conjugate exponent of $p > 1$.

For $u \in X$, we define the energy functional and Nehari functional as

$$J(u) := \frac{1}{p}\|\Delta u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \log|u|\,\mathrm{d}x + \frac{1}{q^2}\|u\|_q^q, \tag{1.5}$$

and

$$I(u) := \langle J'(u), u\rangle = \|\Delta u\|_p^p - \int_\Omega |u|^q \log|u|\,\mathrm{d}x. \tag{1.6}$$

Then it follows from (1.5) and (1.6) that

$$J(u) = \frac{1}{q}I(u) + \frac{q-p}{pq}\|\Delta u\|_p^p + \frac{1}{q^2}\|u\|_q^q. \tag{1.7}$$

We also need to define the depth of the potential well

$$d := \inf_{u \in \mathcal{N}} J(u), \tag{1.8}$$

where $\mathcal{N} := \{u \in X \setminus \{0\} \mid I(u) = 0\}$ is the Nehari manifold. Furthermore, we introduce a constant

$$M := \begin{cases} \frac{q-p}{pq} r_*^p & \text{if } p \neq q, \\ \frac{1}{p^2} R^p & \text{if } p = q, \end{cases} \tag{1.9}$$

where r_* and R are given in Lemmas 2.7 and 2.9 below, respectively.

Next, the potential well W and its corresponding set V are defined by

$$W := \{u \in X \mid I(u) > 0, J(u) < d\} \cup \{0\},$$
$$V := \{u \in X \mid I(u) < 0, J(u) < d\}.$$

Now, we state our main results.

- **Local and global solvability** (see Theorems 3.1 and 4.1). Let $u_0 \in X$ and p, q satisfy (1.4). Then problem (1.1)–(1.3) admits a unique local weak solution. Furthermore, if $J(u_0) \leq d$ and $I(u_0) \geq 0$, then problem (1.1)–(1.3) admits a unique global weak solution

$$u \in L^\infty(0,+\infty; X) \quad \text{and} \quad u_t \in L^2(0,+\infty; H_0^1(\Omega)).$$

- **Blow-up phenomena.**
  - **Infinite blow-up** (see Theorem 5.1). Let $u_0 \in X \setminus \{0\}$ and p, q satisfy $1 < p \le q \le 2$. If $J(u_0) \le d$ and $I(u_0) < 0$, then the weak solution $u(t)$ of problem (1.1)–(1.3) blows up in infinite time. Furthermore, if $I(u_0) < 0$ and
  $$J(u_0) \begin{cases} \le M & \text{if } M < d, \\ < M & \text{if } M = d, \end{cases}$$
  then for all $\rho \in (0,1)$, there exists a $t_\rho > 0$ such that the lower bound of blow-up rate is given by (5.1).
  - **Finite time blow-up** (see Theorem 5.3, Corollaries 5.4 and 5.6). Let $u_0 \in X \setminus \{0\}$ and p, q satisfy $\max\left\{1, \frac{2N}{N+4}\right\} < p \le q$, $2 < q < p\left(1 + \frac{4}{N}\right)$.
    (1) If $J(u_0) \le M$ and $I(u_0) < 0$, then the weak solution $u(t)$ of problem (1.1)–(1.3) blows up in finite time and the upper bound of blow-up time is given by (5.21). Moreover, if q satisfies
    $$\begin{cases} 1 < q < +\infty & \text{if } N = 1, 2, \\ 1 < q < \frac{2N}{N-2} & \text{if } N \ge 3 \end{cases}$$
    further, then the lower bound of blow-up time and blow-up rate are given by (5.23) and (5.24), respectively.
    (2) If $J(u_0) < 0$, then the weak solution $u(t)$ of problem (1.1)–(1.3) blows up in finite time. Moreover, the upper bound of blow-up time and blow-up rate are given by (5.38) and (5.39), respectively.

    Furthermore, if $J(u(t_0)) < 0$, $\forall\, t_0 \in [0, T_{\max})$, then the weak solution $u(t)$ of problem (1.1)–(1.3) blows up in finite time.
  - **Blow-up with arbitrary initial energy** (see Theorem 5.7). Let $u_0 \in X \setminus \{0\}$ and p, q satisfy $2 < p < q < p\left(1 + \frac{4}{N}\right)$. If
  $$J(u_0) < \frac{q-p}{2q\kappa_p^p\left(1 + \overline{B}^2\right)} \|u_0\|_{H_0^1(\Omega)}^2 - \frac{(p-2)(q-p)}{2pq} |\Omega|,$$
  where κ_p and $\overline{B}$ are the optimal embedding constants of $W_0^{2,p}(\Omega) \subset\subset W_0^{1,p}(\Omega)$ and $H_0^1(\Omega) \subset\subset L^2(\Omega)$, respectively, then the weak solution $u(t)$ of problem (1.1)–(1.3) blows up in finite time.
- **Extinction phenomenon** (see Theorem 6.2). Let p, q satisfy $\max\left\{1, \frac{2N}{N+4}\right\} < p < q < 2$ and $0 < \|u_0\|_{H_0^1(\Omega)} < B_p^{-p} |\Omega|^{\frac{q+\alpha-2}{2}}$, where B_p is given in (6.4), then the weak solution $u(t)$ of problem (1.1)–(1.3) becomes extinct in finite time. Moreover, decay rate and extinction time are given by (6.1) and (6.2), respectively.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we establish the local solvability by using Galerkin approximation and some energy estimates. In Sections 4 and 5, we present the detailed proofs of global existence and blow-up properties. Finally, the extinction and decay estimate are derived in Section 6.

2. Preliminaries

In this section, we introduce some definitions, lemmas and corollaries needed in the proofs of main results.

To begin with, we present the definitions of weak solution, finite time blow-up and infinite blow-up of problem (1.1)–(1.3).

Definition 2.1 (Weak solution)**.** Let $u_0 \in X$ and $T > 0$. $u = u(t) \in L^\infty(0,T;X)$ with $u_t \in L^2(0,T;H_0^1(\Omega))$ is called a weak solution of problem (1.1)–(1.3), if $u(0) = u_0$ a.e. in Ω and the following equality

$$(u_t, v) + (\nabla u_t, \nabla v) + \big(|\Delta u|^{p-2}\Delta u, \Delta v\big) = \big(|u|^{q-2}u\log|u|, v\big), \quad \text{a.e. } t \in (0,T)$$

holds for all $v \in X$, where $(\,\cdot\,,\cdot\,)$ means the inner product of $L^2(\Omega)$.

Definition 2.2 (Finite time blow-up)**.** Let $u = u(t)$ be a weak solution of problem (1.1)–(1.3). We call u finite time blow-up if the maximal existence time $T_{\max} < +\infty$ and

$$\lim_{t\to T_{\max}^-} \int_0^t \|u(s)\|_{H_0^1(\Omega)}^2 \,\mathrm{d}s = +\infty.$$

Definition 2.3 (Infinite blow-up)**.** Let $u = u(t)$ be a weak solution of problem (1.1)–(1.3). We call u infinite blow-up if the maximal existence time $T_{\max} = +\infty$ and

$$\lim_{t\to+\infty} \|u(t)\|_{H_0^1(\Omega)}^2 = +\infty.$$

Next, we prove some necessary lemmas and corollaries. By applying the Rellich–Kondrachov Theorem, we improve the classical logarithmic Sobolev inequality.

Lemma 2.4 (Improved logarithmic Sobolev inequality)**.** *For all $u \in W_0^{2,\gamma}(\Omega)$ with $\gamma \in (1,+\infty)$ and $\forall\,\mu > 0$, we have*

$$\gamma \int_\Omega |u|^\gamma \log\left(\frac{|u|}{\|u\|_\gamma}\right) \mathrm{d}x + \frac{N}{\gamma}\log\left(\frac{\gamma\mu e}{N\mathcal{L}_\gamma}\right)\|u\|_\gamma^\gamma \le \mu\kappa_\gamma^\gamma \|\Delta u\|_\gamma^\gamma, \tag{2.1}$$

where

$$\mathcal{L}_\gamma = \frac{\gamma}{N}\left(\frac{\gamma-1}{e}\right)^{\gamma-1} \pi^{-\frac{\gamma}{2}} \left[\frac{\Gamma\big(\frac{N}{2}+1\big)}{\Gamma\big(\frac{N(\gamma-1)}{\gamma}+1\big)}\right]^{\frac{\gamma}{N}},$$

κ_γ is the optimal embedding constant of $W_0^{2,\gamma}(\Omega) \subset\subset W_0^{1,\gamma}(\Omega)$ and Γ is the Gamma function.

Proof. For all $u \in W_0^{1,\gamma}(\Omega)$, it follows from the classical logarithmic Sobolev inequality in [10] that

$$\gamma \int_\Omega |u|^\gamma \log\left(\frac{|u|}{\|u\|_\gamma}\right) \mathrm{d}x + \frac{N}{\gamma} \log\left(\frac{\gamma\mu e}{N\mathcal{L}_\gamma}\right) \|u\|_\gamma^\gamma \leq \mu \|\nabla u\|_\gamma^\gamma. \tag{2.2}$$

On the other hand, from Rellich–Kondrachov Theorem (see [1, p. 168]), we can see

$$W_0^{2,\gamma}(\Omega) \subset\subset W_0^{1,\gamma}(\Omega), \quad \forall \gamma > 1,$$

i.e., there exists a positive constant κ_γ such that

$$\|\nabla u\|_\gamma \leq \kappa_\gamma \|\Delta u\|_\gamma, \quad \forall u \in W_0^{2,\gamma}(\Omega). \tag{2.3}$$

Therefore, we can derive (2.1) by combining (2.2) with (2.3). Then Lemma 2.4 is proved completely. □

Furthermore, we present some auxiliary results as follows.

Lemma 2.5. *Let $u \in X \setminus \{0\}$ and p, q satisfy*

$$\max\left\{1, \frac{2N}{N+4}\right\} < p < q < p\left(1 + \frac{4}{N}\right). \tag{2.4}$$

Then for all α with

$$0 < \alpha \leq p\left(1 + \frac{4}{N}\right) - q, \tag{2.5}$$

we have

(1) *if $0 < \|\Delta u\|_p \leq r(\alpha)$, then $I(u) > 0$,*

(2) *if $I(u) \leq 0$, then $\|\Delta u\|_p > r(\alpha)$,*

where

$$r(\alpha) = \left(\frac{\alpha}{B_\alpha^{q+\alpha}}\right)^{\frac{1}{q+\alpha-p}}, \tag{2.6}$$

and B_α is the optimal embedding constant of $W_0^{2,p}(\Omega) \subset\subset L^{q+\alpha}(\Omega)$, i.e.,

$$\frac{1}{B_\alpha} = \inf_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_p}{\|u\|_{q+\alpha}} \quad \textit{and} \quad \|u\|_{q+\alpha} \leq B_\alpha \|\Delta u\|_p, \quad \forall u \in W_0^{2,p}(\Omega). \tag{2.7}$$

Proof. By simple calculations, we obtain

$$\log|u(x)| < \frac{|u(x)|^\alpha}{\alpha} \quad \text{a.e. } x \in \Omega,\ \forall\, \alpha > 0. \tag{2.8}$$

Then using the definition of $I(u)$ and the inequality above, we have

$$I(u) = \|\Delta u\|_p^p - \int_\Omega |u|^q \log|u|\,\mathrm{d}x > \|\Delta u\|_p^p - \frac{\|u\|_{q+\alpha}^{q+\alpha}}{\alpha}.$$

Since α satisfies (2.5), it follows from the embedding inequality (2.7) that

$$I(u) > \|\Delta u\|_p^p - \frac{B_\alpha^{q+\alpha}}{\alpha}\|\Delta u\|_p^{q+\alpha} = \|\Delta u\|_p^p \left(1 - \frac{B_\alpha^{q+\alpha}}{\alpha}\|\Delta u\|_p^{q+\alpha-p}\right),$$

from which we can derive (1) and (2). Then Lemma 2.5 is proved completely. □

Remark 2.6. From $p > \frac{2N}{N+4}$ we can deduce

$$p\left(1+\frac{4}{N}\right) < \begin{cases} \frac{Np}{N-2p} & \text{if } N > 2p, \\ +\infty & \text{if } N \le 2p. \end{cases}$$

Then by Rellich–Kondrachov Theorem (see [1, p. 168]), we have $W_0^{2,p}(\Omega) \subset\subset L^{q+\alpha}(\Omega)$ for all $p > 1$ and all $\alpha \ge 0$. Therefore, the constant B_α in Lemma 2.5 is well defined.

Lemma 2.7. *Let*

$$r_* := \sup_{\alpha \in \left(0, p\left(1+\frac{4}{N}\right)-q\right]} r(\alpha) \quad \textit{and} \quad r^* := \sup_{\alpha \in \left(0, p\left(1+\frac{4}{N}\right)-q\right]} \sigma(\alpha), \tag{2.9}$$

where

$$\sigma(\alpha) = \left(\frac{\alpha}{B_{pq}^{q+\alpha}}\right)^{\frac{1}{q+\alpha-p}} |\Omega|^{\frac{\alpha}{q(q+\alpha-p)}},$$

and B_{pq} is the optimal embedding constant of $W_0^{2,p}(\Omega) \subset\subset L^q(\Omega)$, i.e.,

$$\frac{1}{B_{pq}} = \inf_{W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_p}{\|u\|_q} \quad \textit{and} \quad \|u\|_q \le B_{pq}\|\Delta u\|_p, \quad \forall\, u \in W_0^{2,p}(\Omega). \tag{2.10}$$

Then r_* exists and satisfies

$$0 < r_* \le r^* < +\infty.$$

Proof. From (2.5), (2.6) and (2.9) we can see that if r_* exists, then $r_* > 0$. Thus, in order to prove Lemma 2.7, we only need to prove $r(\alpha) \le \sigma(\alpha)$, the existence of r^* and $r^* < +\infty$.

First of all, we prove $r(\alpha) \le \sigma(\alpha)$. For all $u \in X \setminus \{0\}$, since (2.4) and (2.5) hold, we have $u \in L^{q+\alpha}(\Omega) \cap L^q(\Omega)$. By Hölder's inequality, we obtain

$$\int_\Omega |u|^q \, \mathrm{d}x \le |\Omega|^{\frac{\alpha}{q+\alpha}} \left(\int_\Omega |u|^{q+\alpha} \, \mathrm{d}x \right)^{\frac{q}{q+\alpha}}. \tag{2.11}$$

Combining (2.7), (2.10) and (2.11) to derive

$$\frac{1}{B_\alpha} = \inf_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_p}{\|u\|_{q+\alpha}} \le |\Omega|^{\frac{\alpha}{q(q+\alpha)}} \inf_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_p}{\|u\|_q} = \frac{1}{B_{pq}} |\Omega|^{\frac{\alpha}{q(q+\alpha)}}.$$

Therefore,

$$r(\alpha) = \left(\frac{\alpha}{B_\alpha^{q+\alpha}} \right)^{\frac{1}{q+\alpha-p}} \le \sigma(\alpha).$$

On the other hand, it follows from the continuity of $\sigma(\alpha)$ on $\left[0, p\left(1 + \frac{4}{N}\right) - q\right]$ that r^* exists and satisfies

$$r^* = \sup_{\alpha \in \left(0, p\left(1+\frac{4}{N}\right) - q\right]} \sigma(\alpha) \le \max_{\alpha \in \left[0, p\left(1+\frac{4}{N}\right) - q\right]} \sigma(\alpha) < +\infty.$$

Then Lemma 2.7 is proved completely. $\square$

Corollary 2.8. *Let $u \in X \setminus \{0\}$ and p, q satisfy* (2.4).

(1) *If $0 < \|\Delta u\|_p \le r_*$, then $I(u) > 0$;*

(2) *If $I(u) \le 0$, then $\|\Delta u\|_p > r_*$,*

where r_* is defined in (2.9).

Proof. We only need to prove (1) since (2) is the direct result of (1). For $u \in X \setminus \{0\}$, if $0 < \|\Delta u\|_p \le r_*$, then we can derive from the definition of r_* that there exists a α_0 satisfying (2.5) such that $0 < \|\Delta u\|_p \le r(\alpha_0)$. Therefore, (1) can be deduced easily by Lemma 2.5. Then Corollary 2.8 is proved completely. $\square$

Lemma 2.9. *Let $u \in X \setminus \{0\}$ and p, q satisfy*

$$\max\left\{1, \frac{2N}{N+4}\right\} < p = q < p\left(1 + \frac{4}{N}\right).$$

(1) *If $0 < \|u\|_p < R$, then $I(u) > 0$;*

(2) *If $I(u) < 0$, then $\|u\|_p > R$;*

(3) *If $I(u) = 0$, then $\|u\|_p \ge R$,*

where

$$R = \left(\frac{p^2 e}{N \kappa_p^p \mathcal{L}_p} \right)^{\frac{N}{p^2}}.$$

Proof. By using (1.6) and (2.1) (taking $\gamma = p = q$), we obtain

$$\begin{aligned} I(u) &= \|\Delta u\|_p^p - \int_\Omega |u|^q \log|u| \, \mathrm{d}x \\ &\geq \left(1 - \frac{\mu \kappa_p^p}{p}\right) \|\Delta u\|_p^p + \left[\frac{N}{p^2} \log\left(\frac{p\mu e}{N \mathcal{L}_p} \right) - \log \|u\|_p \right] \|u\|_p^p. \end{aligned}$$

Now, we choose $\mu = \frac{p}{\kappa_p^p}$ to get

$$I(u) \geq \left[\frac{N}{p^2} \log\left(\frac{p^2 e}{N \kappa_p^p \mathcal{L}_p} \right) - \log \|u\|_p \right] \|u\|_p^p,$$

from which we can verify the results (1)–(3) directly. Lemma 2.9 is proved completely. □

Lemma 2.10. *Let $u \in X \setminus \{0\}$ and p, q satisfy* (1.4)*. Then we have*

$$d \geq M,$$

where d and M are defined by (1.8) *and* (1.9)*, respectively.*

Proof. For all $u \in \mathcal{N}$, we have $u \in X \setminus \{0\}$ and $I(u) = 0$. Then it follows from (1.7) that

$$\begin{aligned} J(u) &= \frac{1}{q} I(u) + \frac{q-p}{pq} \|\Delta u\|_p^p + \frac{1}{q^2} \|u\|_q^q \\ &= \frac{q-p}{pq} \|\Delta u\|_p^p + \frac{1}{q^2} \|u\|_q^q \\ &\begin{cases} > \frac{q-p}{pq} \|\Delta u\|_p^p & \text{if } p \neq q, \\ = \frac{1}{q^2} \|u\|_q^q & \text{if } p = q. \end{cases} \end{aligned}$$

Therefore, we can deduce the result by the definition of d (1.8), Corollary 2.8(2) and Lemma 2.9(3). Then Lemma 2.10 is proved completely. □

Lemma 2.11. *Let $u \in X \setminus \{0\}$ satisfy $I(u) < 0$ and p, q satisfy* (1.4)*. Then there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$.*

Proof. For all $\lambda > 0$, we have

$$I(u) = \lambda^p \|\Delta u\|_p^p - \lambda^q \int_\Omega |u|^q \log|u| \, \mathrm{d}x - \lambda^q \log\lambda \|u\|_q^q = \lambda^p \big(\|\Delta u\|_p^p - \phi(\lambda) \big), \tag{2.12}$$

where

$$\phi(\lambda) = \lambda^{q-p} \int_\Omega |u|^q \log|u| \, \mathrm{d}x - \lambda^{q-p} \log\lambda \|u\|_q^q.$$

By combining $I(u) < 0$, (2.12), Corollary 2.8 and Lemma 2.9, we can see

$$\phi(1) = \int_\Omega |u|^q \log |u| \,\mathrm{d}x > \|\Delta u\|_p^p \geq r_0^p := \begin{cases} r_*^p & \text{if } p < q, \\ B_{pp}^p R^p & \text{if } p = q \end{cases} > 0, \tag{2.13}$$

where we have used the case of $p = q$ in (2.10) to derive the last inequality, and B_{pp} is the optimal embedding constant of $W_0^{2,p}(\Omega) \subset\subset L^p(\Omega)$.

On the other hand, it follows the condition $p \leq q$ that

$$\begin{aligned} \phi(\lambda) &= \lambda^{q-p} \int_\Omega |u|^q \log |u| \,\mathrm{d}x - \lambda^{q-p} \log \lambda \|u\|_q^q \\ &\to \begin{cases} -\infty & \text{if } p = q, \\ 0 & \text{if } p < q, \end{cases} \quad \text{as } \lambda \to 0^+. \end{aligned}$$

Combining (2.12) with (2.13), we obtain there exists a $\lambda^* \in (0,1)$ such that $\phi(\lambda^*) = \|\Delta u\|_p^p$ and $I(\lambda^* u) = 0$. Then Lemma 2.11 is proved completely. □

Lemma 2.12. *Let $u \in X \setminus \{0\}$ satisfy $I(u) < 0$ and p, q satisfy* (1.4)*. Then we have*

$$I(u) < q(J(u) - d).$$

Proof. From Lemma 2.11, we can see that there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$. Now, we define

$$f(\lambda) := qJ(\lambda u) - I(\lambda u), \quad \lambda > 0.$$

Calculating directly, we obtain

$$f(\lambda) = \frac{q-p}{p} \lambda^p \|\Delta u\|_p^p + \frac{\lambda^q}{q} \|u\|_q^q,$$

and

$$f'(\lambda) = (q-p)\lambda^{p-1} \|\Delta u\|_p^p + \lambda^{q-1} \|u\|_q^q.$$

Then it follows from Corollary 2.8 and Lemma 2.9 that $f'(\lambda) > 0$, f is non-deceasing with respect to $\lambda > 0$ and $f(1) > f(\lambda^*)$. Therefore,

$$qJ(u) - I(u) > qJ(\lambda^* u) - I(\lambda^* u) = qJ(\lambda^* u) \geq qd > 0,$$

where we have used the fact that $\lambda^* u \in \mathcal{N}$ and the definition of d (1.8) to derive the last inequality. Then Lemma 2.12 is proved completely. □

3. Local solvability

In this section, we present the local solvability of problem (1.1)–(1.3) by virtue of Galerkin approximation.

Theorem 3.1 (Local solvability)**.** *Let $u_0 \in X$ and p, q satisfy* (1.4)*. Then there exists a $T_0 > 0$ such that problem* (1.1)*–*(1.3) *admits a unique weak solution on* $[0, T_0)$ *and*

$$u \in L^\infty(0, T_0; X), \quad u_t \in L^2(0, T_0; H_0^1(\Omega)).$$

Moreover, u satisfies the following energy inequality:

$$\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, \mathrm{d}s + J(u(t)) \le J(u_0), \quad t \in [0, T_0). \tag{3.1}$$

Proof. We divide the proof into 5 steps.

Step 1: Approximate problem. Let $\{\omega_j\}_{j=1}^{+\infty}$ be a completed orthogonal basis of X. We define the finite dimensional space $V_m := \operatorname{span}\{\omega_1, \omega_2, \ldots, \omega_m\}$, $m \in \mathbb{N}_+$, and construct the approximate solution

$$u_m(x,t) := \sum_{j=1}^m g_{jm}(t) \omega_j(x),$$

where $u_m(x,t)$ satisfies the following Cauchy problem:

$$(u_{mt}, \omega_j) + (\nabla u_{mt}, \nabla \omega_j) + \big(|\Delta u_m|^{p-2} \Delta u_m, \Delta \omega_j\big) = \big(|u_m|^{q-2} u_m \log |u_m|, \omega_j\big), \tag{3.2}$$

$$u_{0m} = \sum_{j=1}^m g_{jm}(0) \omega_j \to u_0 \quad \text{in } X. \tag{3.3}$$

The standard theory of ODEs yields that Cauchy problem (3.2)–(3.3) possesses local solutions.

Step 2: Priori estimates. We discuss the following two cases:

Case 1: $\max\big\{1, \frac{2N}{N+4}\big\} < p \le q$, $2 \le q < p\big(1 + \frac{4}{N}\big)$.

Priori estimate I: Multiplying (3.2) by $g_{jm}(t)$, summing on $j = 1, 2, \ldots, m$ and then integrating on $[0, t]$, we know that

$$S_m(t) = S_m(0) + \int_0^t \int_\Omega |u_m(x,s)|^q \log |u_m(x,s)| \, \mathrm{d}x\mathrm{d}s, \tag{3.4}$$

where

$$S_m(t) = \frac{1}{2}\|u_m\|_2^2 + \frac{1}{2}\|\nabla u_m\|_2^2 + \int_0^t \|\Delta u_m(s)\|_p^p \, \mathrm{d}s. \tag{3.5}$$

On the other hand, we can see from (2.8) that for all $\alpha > 0$,

$$\int_\Omega |u_m|^q \log |u_m| \, \mathrm{d}x \le \frac{1}{\alpha} \|u_m\|_{q+\alpha}^{q+\alpha}, \tag{3.6}$$

where α is chosen to satisfy $\alpha < p\left(1 + \frac{4}{N}\right) - q$. Then by the Nirenberg interpolation inequality and Young's inequality, we obtain

$$\begin{aligned}\int_\Omega |u_m|^q \log |u_m| \, \mathrm{d}x &\le C \|\Delta u_m\|_p^{\theta(q+\alpha)} \|u_m\|_2^{(1-\theta)(q+\alpha)} \\ &\le \varepsilon \|\Delta u_m\|_p^p + C(\varepsilon) \|u_m\|_2^{\frac{p(1-\theta)(q+\alpha)}{p-\theta(q+\alpha)}},\end{aligned} \tag{3.7}$$

where $\varepsilon \in (0,1)$ and

$$\theta = \left(\frac{1}{2} - \frac{1}{q+\alpha}\right)\left(\frac{2}{N} - \frac{1}{p} + \frac{1}{2}\right)^{-1}.$$

Now, we set

$$\beta := \frac{p(1-\theta)(q+\alpha)}{2[p-\theta(q+\alpha)]} = \frac{p(2q+2\alpha+N) - N(q+\alpha)}{p(4+N) - N(q+\alpha)},$$

then $\beta > 1$ because $\max\left\{1, \frac{2N}{N+4}\right\} < p \le q$ and $2 \le q < p\left(1 + \frac{4}{N}\right)$. Combining (3.4)–(3.7), we obtain

$$S_m(t) \le C_1 + C_2 \int_0^t S_m^\beta(s) \, \mathrm{d}s,$$

where C_1 and C_2 are positive constants independent of m. Therefore, by means of Gronwall inequality (see [9]), there exists a positive constant $T_0 > 0$ such that

$$S_m(t) \le C_{T_0}. \tag{3.8}$$

Priori estimate II: Multiplying (3.2) by $g'_{jm}(t)$, summing on $j = 1, 2, \ldots, m$ and then integrating on $[0,t]$, we know that

$$J(u_m(t)) + \int_0^t \left(\|u_{ms}(s)\|_2^2 + \|\nabla u_{ms}(s)\|_2^2\right) \mathrm{d}s = J(u_m(0)) = J(u_{0m}). \tag{3.9}$$

By the continuity of J and (3.3), we can see there exists a constant $C > 0$ such that

$$J(u_{0m}) \le C, \quad \forall\, m \in \mathbb{N}_+. \tag{3.10}$$

Combining (1.5) with (3.7)–(3.10), we can derive

$$\begin{aligned} C \ge J(u_m) &= \frac{1}{p}\|\Delta u_m\|_p^p - \frac{1}{q}\int_\Omega |u_m|^q \log |u_m| \, \mathrm{d}x + \frac{1}{q^2}\|u_m\|_q^q \\ &\ge \left(\frac{1}{p} - \frac{\varepsilon}{q}\right)\|\Delta u_m\|_p^p - \frac{C(\varepsilon)}{q}\|u_m\|_2^{2\beta} + \frac{1}{q^2}\|u_m\|_q^q \\ &\ge \left(\frac{1}{p} - \frac{\varepsilon}{q}\right)\|\Delta u_m\|_p^p - \frac{C(\varepsilon)}{q} 2^\beta S_m^\beta(t) + \frac{1}{q^2}\|u_m\|_q^q, \end{aligned}$$

i.e.,

$$\|\Delta u_m\|_p^p + \|u_m\|_q^q \le C_{T_0}. \tag{3.11}$$

Case 2*:* $1 < p \le q < 2$.

Priori estimate I: Combining (3.4) and (3.6), taking $\alpha = 2 - q$, we obtain

$$\begin{aligned} S_m(t) &\le S_m(0) + \frac{1}{2-q}\int_0^t \|u_m(s)\|_2^2\,\mathrm{d}s \\ &\le S_m(0) + \frac{2}{2-q}\int_0^t S_m(s)\,\mathrm{d}s. \end{aligned}$$

Then by means of Gronwall inequality, there exists a positive constant $T_0 > 0$ such that

$$S_m(t) \le C_{T_0}. \tag{3.12}$$

Priori estimate II: From (1.5), (3.9), (3.10) and (3.12), we have

$$\begin{aligned} &\frac{1}{p}\|\Delta u_m\|_p^p + \frac{1}{q^2}\|u_m\|_q^q + \int_0^t \left(\|u_{ms}(s)\|_2^2 + \|\nabla u_{ms}(s)\|_2^2\right)\mathrm{d}s \\ &\le C + \frac{1}{q}\int_\Omega |u_m|^q \log|u_m|\,\mathrm{d}x \le C + \frac{1}{q(2-q)}\|u_m\|_2^2 \le C_{T_0}. \end{aligned} \tag{3.13}$$

Therefore, by combining (3.5), (3.8) and (3.11)–(3.13), we can derive

$$\|u_m\|_{L^\infty(0,T_0;X)} \le C, \quad \forall\, m \in \mathbb{N}_+, \tag{3.14}$$

$$\|u_{mt}\|_{L^2(0,T_0;H_0^1(\Omega))} \le C, \quad \forall\, m \in \mathbb{N}_+, \tag{3.15}$$

$$\left\||\Delta u_m|^{p-2}\Delta u_m\right\|_{L^\infty(0,T_0;W_0^{-2,p'}(\Omega))} \le C, \quad \forall\, m \in \mathbb{N}_+. \tag{3.16}$$

Step 3*: Pass to the limit.* It follows from (3.14)–(3.16) that there exist functions u and χ and a subsequence of $\{u_m\}_{m=1}^{+\infty}$, which we still denote by $\{u_m\}_{m=1}^{+\infty}$ for convenience, such that

$$u_m \xrightarrow{W^*} u \quad \text{in } L^\infty(0,T_0;X), \tag{3.17}$$

$$u_{mt} \xrightarrow{W} u_t \quad \text{in } L^2(0,T_0;H_0^1(\Omega)), \tag{3.18}$$

$$|\Delta u_m|^{p-2}\Delta u_m \xrightarrow{W^*} \chi \quad \text{in } L^\infty(0,T_0;W_0^{-2,p'}(\Omega)).$$

Using (3.17), (3.18) and Aubin–Lions Theorem (see [18]), we can obtain

$$u_m \to u \quad \text{strongly in } C([0,T_0];H_0^1(\Omega)). \tag{3.19}$$

Therefore, $u_m \to u$ a.e. $(x,t) \in \Omega \times (0,T_0)$, which implies

$$|u_m|^{q-2}u_m \log|u_m| \to |u|^{q-2}u\log|u| \quad \text{a.e. } (x,t) \in \Omega \times (0,T_0).$$

On the other hand, by a direct calculation, we have

$$\begin{aligned} \int_\Omega \left||u_m|^{q-2}u_m\log|u_m|\right|^{\frac{2q}{2q-1}}\mathrm{d}x &= \int_{\{x\in\Omega \mid |u_m|\le 1\}} \left||u_m|^{q-2}u_m\log|u_m|\right|^{\frac{2q}{2q-1}}\mathrm{d}x \\ &\quad + \int_{\{x\in\Omega \mid |u_m|> 1\}} \left||u_m|^{q-2}u_m\log|u_m|\right|^{\frac{2q}{2q-1}}\mathrm{d}x \\ &\le \left[\frac{1}{(q-1)e}\right]^{\frac{2q}{2q-1}}|\Omega| + 2^{\frac{2q}{2q-1}}\|u_m\|_q^q \le C, \end{aligned} \tag{3.20}$$

where we have used the fact that $|x^{q-1}\log x| \le \frac{1}{(q-1)e}$ for $0<x<1$ and $\log x \le 2x^{\frac{1}{2}}$ for $x \in (0,+\infty)$. Thus, from (3.19) and (3.20) we have

$$|u_m|^{q-2}u_m \log|u_m| \xrightarrow{W^*} |u|^{q-2}u\log|u| \quad \text{in } L^\infty(0,T_0;L^{\frac{2q}{2q-1}}(\Omega)),$$

and we can pass the limit in (3.2) to derive

$$(u_t,\omega) + (\nabla u_t,\nabla\omega) + (\chi,\Delta\omega) = \big(|u|^{q-2}u\log|u|,\omega\big), \quad \forall\,\omega \in X.$$

Finally, by the well known arguments of the theory of monotone operators, we know that

$$\chi = |\Delta u|^{p-2}\Delta u,$$

and

$$(u_t,\omega) + (\nabla u_t,\nabla\omega) + \big(|\Delta u|^{p-2}\Delta u,\Delta\omega\big) = \big(|u|^{q-2}u\log|u|,\omega\big), \quad \forall\,\omega \in X.$$

Step 4*: Uniqueness.* Assume that there are two solutions u_1 and u_2 to problem (1.1)–(1.3) with the same initial condition $u_1(x,0) = u_2(x,0) = u_0(x) \in X$. Let $v = u_1 - u_2$, then v satisfies $v(0) = 0$ and

$$(v_t,\omega) + (\nabla v_t,\nabla\omega) + \big(|\Delta v|^{p-2}\Delta v,\Delta\omega\big) = \big(|v|^{q-2}v\log|v|,\omega\big) \tag{3.21}$$

for all $\omega \in X$. Now, we choose the test function in (3.21) as

$$\omega(s) := \begin{cases} u_1(s) - u_2(s) & \text{if } s \in [0,t], \\ 0 & \text{if } s \in (t,T_0), \end{cases}$$

then it follows from the monotonicity of p-biharmonic operator that

$$\frac{1}{2}\|v(t)\|_{H_0^1(\Omega)}^2 \le \int_0^t \int_\Omega [h(u_1(s)) - h(u_2(s))][u_1(s) - u_2(s)]\,\mathrm{d}x\mathrm{d}s,$$

where $h(u) = |u|^{q-2}u\log|u|$. Therefore, the uniqueness of problem (1.1)–(1.3) can be deduced by the Lipschitz continuity of $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ and Gronwall inequality.

Step 5*: Energy inequality.* Let $\zeta \in C[0,T_0]$ be a nonnegative function. From (3.9) we have

$$\int_0^{T_0} \zeta(t)J(u_m(t))\,\mathrm{d}t + \int_0^{T_0} \zeta(t)\int_0^t \|u_{ms}(s)\|_{H_0^1(\Omega)}^2\,\mathrm{d}s\mathrm{d}t = \int_0^{T_0}\zeta(t)J(u_m(0))\,\mathrm{d}t.$$

It is clear that the right side of the inequality above converges to $\int_0^{T_0}\zeta(t)J(u_0)\,\mathrm{d}t$ as $m \to +\infty$, and the first term on the left side is lower semi-continuous with respect to the weak topology of $L^2(0,T_0;X)$, i.e.,

$$\int_0^{T_0}\zeta(t)J(u(t))\,\mathrm{d}t \le \liminf_{m\to+\infty}\int_0^{T_0}\zeta(t)J(u_m(t))\,\mathrm{d}t,$$

which implies

$$\int_0^{T_0} \zeta(t)J(u(t))\,\mathrm{d}t + \int_0^{T_0} \zeta(t)\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,\mathrm{d}s\mathrm{d}t \le \int_0^{T_0} \zeta(t)J(u_0)\,\mathrm{d}t.$$

Then we can obtain the energy inequality (3.1) by the arbitrariness of $\zeta(t)$, and Theorem 3.1 is proved completely. □

4. Global solvability

In this section, we present the detailed proof of the global existence of solution to problem (1.1)–(1.3).

Theorem 4.1 (Global solvability)**.** *Let $u_0 \in X \setminus \{0\}$ and p, q satisfy* (1.4)*. If $J(u_0) \le d$ and $I(u_0) \ge 0$, then problem* (1.1)–(1.3) *admits a global weak solution*

$$u \in L^\infty(0,+\infty;X) \quad \textit{and} \quad u_t \in L^2(0,+\infty;H_0^1(\Omega)).$$

Proof. We divide the proof into 2 steps.

Step 1*: $J(u_0) < d$.* Let $\{u_m\}_{m=1}^{+\infty}$, $\{u_{0m}\}_{m=1}^{+\infty}$ and $\{\omega_j\}_{j=1}^{m}$ be the same as in Theorem 3.1 and T_m is the maximal existence time of u_m. Then by the continuity of $J(u)$ and $I(u)$, we have $J(u_{0m}) \le d$ and $I(u_{0m}) \ge 0$.

Next, we only need to prove the case of $0 < J(u_{0m}) < d$ and $I(u_{0m}) > 0$. In fact,

(i) the case of $J(u_{0m}) < 0$ and $I(u_{0m}) \ge 0$ contradicts with (1.7),

(ii) the case of $0 < J(u_{0m}) < d$ and $I(u_{0m}) = 0$ contradicts with the definition of d,

(iii) the case of $J(u_{0m}) = 0$ and $I(u_{0m}) \ge 0$ is trivial.

Multiplying (3.2) by $g'_{jm}(t)$, summing on $j = 1, 2, \ldots, m$ and then integrating on $[0,t]$, we know that

$$J(u_m(t)) + \int_0^t \|u_{ms}(s)\|^2_{H_0^1(\Omega)}\,\mathrm{d}s \le J(u_{0m}) < d, \quad t > 0. \tag{4.1}$$

Now, we claim that

$$u_m(x,t) \in W, \quad \forall\, t > 0. \tag{4.2}$$

In fact, if it is false, then there exists a $t_0 > 0$ such that $u_m(t_0) \in \partial W$, i.e., $u_m(t_0) \in X\setminus\{0\}$, and $J(u_m(t_0)) = d$ or $I(u_m(t_0)) = 0$. From (4.1), $J(u_m(t_0)) = d$ is not true. Thus $u_m(t_0) \in \mathcal{N}$, and then we have $J(u_m(t_0)) \ge d$ by the definition of d in (1.8), which contradicts with (4.1).

Case 1*:* $p \neq q$. Combining (1.7), (4.1) and (4.2) to derive

$$\int_0^t \|u_{ms}(s)\|_{H_0^1(\Omega)}^2 \,\mathrm{d}s + \frac{q-p}{pq}\|\Delta u_m(t)\|_p^p + \frac{1}{q^2}\|u_m(t)\|_q^q < d, \quad t > 0,$$

which implies

$$\int_0^t \|u_{ms}(s)\|_{H_0^1(\Omega)}^2 \,\mathrm{d}s < d, \tag{4.3}$$

$$\|\Delta u_m(t)\|_p^p < \frac{pqd}{q-p},$$

and

$$\|u_m(t)\|_q^q < q^2 d. \tag{4.4}$$

Case 2*:* $p = q$. Similar to Case 1, we can derive (4.3) and (4.4). Moreover, taking $\gamma = p$ and $\mu = \frac{p}{2\kappa_p^p}$ in (2.1), we have

$$\begin{aligned}
\|\Delta u_m\|_p^p &= I(u_m) + \int_\Omega |u_m|^p \log|u_m| \,\mathrm{d}x \\
&= 2I(u_m) + 2\int_\Omega |u_m|^p \log|u_m| \,\mathrm{d}x - \|\Delta u_m\|_p^p \\
&\leq 2I(u_m) + 2\|u_m\|_p^p \log\|u_m\|_p - \frac{2N}{p^2}\log\left(\frac{p^2 e}{2N\kappa_p^p \mathcal{L}_p}\right)\|u_m\|_p^p \\
&= 2pJ(u_m) + \left[2\log\|u_m\|_p - \frac{2}{p} - \frac{2N}{p^2}\log\left(\frac{p^2 e}{2N\kappa_p^p \mathcal{L}_p}\right)\right]\|u_m\|_p^p \\
&\leq Cd.
\end{aligned}$$

Combining the two cases above, we obtain

$$\int_0^t \|u_{ms}(s)\|_{H_0^1(\Omega)}^2 \,\mathrm{d}s + \|\Delta u_m(t)\|_p^p + \|u_m(t)\|_q^q \leq Cd, \quad t > 0. \tag{4.5}$$

On the other hand, multiplying (3.2) by $g_{jm}(t)$, summing on $j = 1, 2, \ldots, m$ and then integrating with respect to time variable on $[0, t]$, we know that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_{H_0^1(\Omega)}^2 = -I(u_m(t)) < 0,$$

which implies

$$\|u_m(t)\|_{H_0^1(\Omega)}^2 \leq \|u_{0m}\|_{H_0^1(\Omega)}^2 \leq C. \tag{4.6}$$

Clearly, the constants on the right side of (4.5) and (4.6) are independent of T_m, then for $T > 0$, we can choose $T_m = T$ and it follows from the arbitrariness of T that $u(t)$ is the global weak solution of problem (1.1)–(1.3).

Step 2: $J(u_0)=d$. Let $\delta_m := 1-\frac{1}{m}$ and $u_{m0} := \delta_m u_0$, $m\in\mathbb{N}_+$ and $m\geq 2$. We consider the following problem:

$$\begin{cases} u_t-\Delta u_t+\Delta\big(|\Delta u|^{p-2}\Delta u\big)=|u|^{q-2}u\log|u|, & (x,t)\in\Omega\times(0,+\infty),\\ u(x,t)=\Delta u(x,t)=0, & (x,t)\in\partial\Omega\times(0,+\infty),\\ u(x,0)=u_{m0}(x), & x\in\Omega. \end{cases}$$

First of all, we claim that $J(u_{0m})<d$ and $I(u_{0m})>0$. In fact, from $u_0\in X$, $\delta_m\in(0,1)$ and $I(u_0)\geq 0$, we can see

$$\begin{aligned} I(u_{0m}) &= \delta_m^p\|\Delta u_0\|_p^p-\delta_m^q\log|\delta_m|\|u_0\|_q^q-\delta_m^q\int_\Omega |u_0|^q\log|u_0|\,\mathrm{d}x\\ &> \delta_m^p\left(\|\Delta u_0\|_p^p-\delta_m^{q-p}\int_\Omega |u_0|^q\log|u_0|\,\mathrm{d}x\right)\\ &\geq \begin{cases} \delta_m^p\|\Delta u_0\|_p^p\geq 0 & \text{if } \int_\Omega |u_0|^q\log|u_0|\,\mathrm{d}x\leq 0,\\ \delta_m^p\big(1-\delta_m^{q-p}\big)\int_\Omega |u_0|^q\log|u_0|\,\mathrm{d}x\geq 0 & \text{if } \int_\Omega |u_0|^q\log|u_0|\,\mathrm{d}x> 0. \end{cases} \end{aligned}$$

On the other hand, by direct calculations, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\delta_m}J(\delta_m u) &= \frac{1}{\delta_m}\left(\delta_m^p\|\Delta u_0\|_p^p-\delta_m^q\log|\delta_m|\|u_0\|_q^q-\delta_m^q\int_\Omega |u_0|^q\log|u_0|\,\mathrm{d}x\right)\\ &= \frac{1}{\delta_m}I(\delta_m u). \end{aligned}$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}\delta_m}J(\delta_m u_0)=\frac{1}{\delta_m}I(\delta_m u_0)=\frac{1}{\delta_m}I(u_{0m})>0,$$

which implies that $J(\delta_m u_0)$ is strictly increasing with respect to δ_m and

$$J(u_{0m})=J(\delta_m u_0)<J(u_0)=d.$$

Since $u_{0m}\to u_0$ as $m\to+\infty$, our result can be derived by the same processes as in the proof of Theorem 3.1 and Step 1. Then Theorem 4.1 is proved completely. $\square$

5. Blow-up phenomena

In this section, we present infinite and finite time blow-up phenomena of the solution to problem (1.1)–(1.3) in different energy levels.

5.1. Infinite blow-up

This subsection is devoted to infinite blow-up and the lower bound of blow-up rate for problem (1.1)–(1.3).

Theorem 5.1 (Infinite blow-up)**.** *Let $u_0 \in X \setminus \{0\}$ and p, q satisfy $1 < p \le q \le 2$.*

(1) *If $J(u_0) \le d$ and $I(u_0) < 0$, then the weak solution $u(t)$ of problem* (1.1)–(1.3) *blows up in infinite time.*

(2) *Furthermore, if $I(u_0) < 0$ and*

$$J(u_0) \begin{cases} \le M & \text{if } M < d, \\ < M & \text{if } M = d, \end{cases}$$

then for all $\rho \in (0,1)$, there exists a $t_\rho > 0$ such that the weak solution $u(t)$ of problem (1.1)–(1.3) *satisfies*

$$\|u(t)\|^2_{H^1_0(\Omega)} \ge C_\rho \frac{(t-t_\rho)^{\frac{2}{2-q\rho}}}{t}, \quad \forall\, t \ge t_\rho, \tag{5.1}$$

where

$$C_\rho = \left[\left(1 - \frac{q\rho}{2}\right) G^{-\frac{q\rho}{2}}(t_\rho) G'(t_\rho)\right]^{\frac{2}{2-q\rho}} \quad \text{and} \quad G(t) = \int_0^t \|u(s)\|^2_{H^1_0(\Omega)}\, \mathrm{d}s.$$

Proof. (1) We divide the proof into 2 steps.

Step 1*: $J(u_0) < d$.* We begin with claiming that $u(t) \in V$, $\forall\, t \in [0, T_{\max})$. In fact, if it is false, then there exists a $t_0 \in [0, T_{\max})$ such that $u(t_0) \in \partial V$, i.e., $J(u(t_0)) = d$ or $I(u(t_0)) = 0$. From $J(u(t_0)) \le J(u_0) < d$ we know that $J(u(t_0)) = d$ is not true, then there exists a $t_0 \in [0, T_{\max})$ such that

$$I(u(t_0)) = 0 \quad \text{and} \quad I(u(t)) < 0, \quad t \in [0, t_0).$$

Thus, it follows from Corollary 2.8 and Lemma 2.9 that $\|\Delta u(t)\|_p \ge r_* > 0$, $t \in [0, t_0)$ if $p < q$, while $\|\Delta u(t)\|_p \ge B_{pp}\|u(t)\|_p > B_{pp}R > 0$, $t \in [0, t_0)$ if $p = q$. Meantime, it follows from the continuity of $\|\Delta u(t)\|_p$ with respect to t that $\|\Delta u(t_0)\|_p > 0$. Therefore, $u(t_0) \in \mathcal{N}$, and $J(u(t_0)) \ge d$ by the definition of d in (1.8), which is a contradiction.

Let

$$G(t) := \int_0^t \|u(s)\|^2_{H^1_0(\Omega)}\, \mathrm{d}s, \quad t \in [0, T_{\max}).$$

By direct calculations, we obtain

$$G'(t) = \|u(t)\|^2_{H^1_0(\Omega)} = \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2, \tag{5.2}$$

and

$$G''(t) = 2\int_\Omega u u_t \,\mathrm{d}x + 2\int_\Omega \nabla u\cdot\nabla u_t \,\mathrm{d}x = -2I(u(t)). \tag{5.3}$$

By means of Lemma 2.12, $I(u) < 0$ and $J(u(t)) \le J(u_0) < d$, $\forall\, t \in [0, T_{\max})$, we can see

$$\begin{aligned} G''(t) &= -2I(u(t)) > 2q(d - J(u(t))) \\ &\ge 2q(d - J(u_0)) := C_0, \quad t \in [0, T_{\max}). \end{aligned} \tag{5.4}$$

Combining (5.2), (5.4) and

$$G'(t) = G'(0) + \int_0^t G''(s)\,\mathrm{d}s,$$

we can derive

$$\|u(t)\|^2_{H_0^1(\Omega)} \ge \|u_0\|^2_{H_0^1(\Omega)} + C_0 t > 0, \quad t \in [0, T_{\max}). \tag{5.5}$$

Now, we prove that u cannot blow up in finite time. Arguing by contradiction, we assume that u blows up in finite time, i.e., $T_{\max} < +\infty$ and

$$\lim_{t\to T_{\max}^-} G(t) = \lim_{t\to T_{\max}^-} \int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\,\mathrm{d}s = +\infty,$$

which implies that

$$\lim_{t\to T_{\max}^-} \|u(t)\|^2_{H_0^1(\Omega)} = +\infty. \tag{5.6}$$

Meantime, by (5.2) and (5.4), we have

$$G'(t)\log G'(t) - G''(t) = \|u(t)\|^2_{H_0^1(\Omega)} \log \|u(t)\|^2_{H_0^1(\Omega)} + 2I(u(t)). \tag{5.7}$$

Next, we discuss the following two cases:

Case 1*:* $1 < q < 2$. Taking $\alpha = 2 - q$ in (2.8), we have

$$I(u(t)) \ge \|\Delta u(t)\|_p^p - \frac{1}{2-q}\|u(t)\|_2^2. \tag{5.8}$$

On the other hand, from (5.6) we obtain that there exists a $t_1 \in (0, T_{\max})$ such that

$$\|u(t_1)\|^2_{H_0^1(\Omega)} = \|u(t_1)\|_2^2 + \|\nabla u(t_1)\|_2^2 > e^{\frac{2}{2-q}}, \quad t \in (t_1, T_{\max}). \tag{5.9}$$

Then by combining (5.7)–(5.9), we can derive

$$\begin{aligned} G'(t)\log G'(t) - G''(t) &\ge \|u(t)\|^2_{H_0^1(\Omega)} \log \|u(t)\|^2_{H_0^1(\Omega)} + 2\|\Delta u(t)\|_p^p - \frac{2}{2-q}\|u(t)\|_2^2 \\ &\ge \|u(t)\|^2_{H_0^1(\Omega)} \log \|u(t)\|^2_{H_0^1(\Omega)} - \frac{2}{2-q}\|u(t)\|_2^2 \\ &\ge \frac{2}{2-q}\big(\|u(t)\|^2_{H_0^1(\Omega)} - \|u(t)\|_2^2\big) \ge 0. \end{aligned} \tag{5.10}$$

Case 2 $q = 2$. Taking $\gamma = 2$ in classical logarithmic Sobolev inequality (2.2) and choosing $\mu > 0$ such that

$$\frac{N}{2}\log\left(\frac{2\mu e}{N\mathcal{L}_2}\right) \geq 0, \tag{5.11}$$

and combining (2.2) and (5.7) to obtain

$$\begin{aligned}
&G'(t)\log G'(t) - G''(t)\\
&\geq 2\|u(t)\|^2_{H^1_0(\Omega)}\log\|u(t)\|_{H^1_0(\Omega)} + 2\|\Delta u(t)\|_p^p - 2\|u(t)\|_2^2\log\|u(t)\|_2\\
&\quad + \frac{N}{2}\log\left(\frac{2\mu e}{N\mathcal{L}_2}\right)\|u(t)\|_2^2 - \mu\|\nabla u(t)\|_2^2\\
&\geq 2\big(\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2\big)\log\|u(t)\|_{H^1_0(\Omega)} - 2\|u(t)\|_2^2\log\|u(t)\|_2\\
&\quad + \frac{N}{2}\log\left(\frac{2\mu e}{N\mathcal{L}_2}\right)\|u(t)\|_2^2 - \mu\|\nabla u(t)\|_2^2\\
&= \left[2\big(\log\|u(t)\|_{H^1_0(\Omega)} - \log\|u(t)\|_2\big) + \frac{N}{2}\log\left(\frac{2\mu e}{N\mathcal{L}_2}\right)\right]\|u(t)\|_2^2\\
&\quad + \big(2\log\|u(t)\|_{H^1_0(\Omega)} - \mu\big)\|\nabla u(t)\|_2^2.
\end{aligned} \tag{5.12}$$

On the other hand, it follows from (5.6) that there exists a $t_2 \in (0, T_{\max})$ such that

$$2\log\|u(t_2)\|_{H^1_0(\Omega)} \geq \mu \quad \text{and} \quad \|u(t)\|^2_{H^1_0(\Omega)} > 0, \quad t \in [t_2, T_{\max}).$$

Meantime, by virtue of (5.3) and $I(u(t)) < 0$, we know that $\log\|u(t)\|_{H^1_0(\Omega)}$ is strictly increasing on $[t_2, T_{\max})$ and

$$2\log\|u(t)\|_{H^1_0(\Omega)} \geq \mu, \quad t \in [t_2, T_{\max}). \tag{5.13}$$

Combining (5.11)–(5.13) and $\log\|u(t)\|_{H^1_0(\Omega)} \geq \log\|u(t)\|_2$, we obtain

$$G'(t)\log G'(t) - G''(t) \geq 0, \quad t \in [t_2, T_{\max}). \tag{5.14}$$

Let $\bar{t} := \max\{t_1, t_2\}$, by (5.10) and (5.14), we have

$$\log G'(t) \geq \frac{G''(t)}{G'(t)} = [\log G'(t)]', \quad t \in [\bar{t}, T_{\max}).$$

Then by means of Gronwall's inequality, we get

$$\log G'(t) \geq e^{t-\bar{t}}\log G'(\bar{t}), \quad t \in [\bar{t}, T_{\max}),$$

i.e.,

$$\|u(t)\|^2_{H^1_0(\Omega)} \leq \|u(\bar{t})\|^{2e^{t-\bar{t}}}_{H^1_0(\Omega)}, \quad t \in [\bar{t}, T_{\max}),$$

which contradicts with (5.6). Therefore, $T_{\max} = +\infty$ and u cannot blow up in finite time and u blows up in infinite time.

Step 2: $J(u_0) = d$. First of all, we claim that $I(u(t)) < 0$, $\forall\, t \in [0, T_{\max})$. In fact, if it is false, then there exists a $t_0 \in (0, T_{\max})$ such that

$$I(u(t_0)) = 0 \quad \text{and} \quad I(u(t)) < 0, \quad t \in [0, t_0).$$

Thus, it follows from Corollary 2.8 and Lemma 2.9 that $\|\Delta u(t)\|_p \geq r_* > 0$, $t \in [0, t_0)$ if $p < q$, while $\|\Delta u(t)\|_p \geq B_{pp}\|u(t)\|_p > B_{pp}R > 0$, $t \in [0, t_0)$ if $p = q$. Meantime, it follows from the continuity of $\|\Delta u(t)\|_p$ with respect to t that $\|\Delta u(t_0)\|_p > 0$. Therefore, $u(t_0) \in \mathcal{N}$, and by the definition of d in (1.8), we have

$$J(u(t_0)) \geq d. \tag{5.15}$$

On the other hand, from $\int_\Omega u u_t \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla u_t \,\mathrm{d}x = -I(u(t)) > 0$, $t \in [0, t_0)$, we know that $u_t \neq 0$, $\nabla u_t \neq 0$ and $\int_0^{t_0} \|u_s(s)\|^2_{H_0^1(\Omega)} \,\mathrm{d}s > 0$. Meantime, it follows from energy inequality (3.1) that

$$J(u(t_0)) \leq J(u_0) - \int_0^{t_0} \|u_s(s)\|^2_{H_0^1(\Omega)} \,\mathrm{d}s < d,$$

which contradicts with (5.15).

Therefore, we have $\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \,\mathrm{d}s > 0$, $t \in (0, T_{\max})$, and we can take $t_1 \in (0, T_{\max})$ such that

$$J(u(t_1)) \leq J(u_0) - \int_0^{t_1} \|u_s(s)\|^2_{H_0^1(\Omega)} \,\mathrm{d}s < d.$$

If we take t_1 as the initial time, then similar to Step 1, we can obtain that the weak solution $u(t)$ of problem (1.1)–(1.3) blows up in infinite time.

(2) By the condition $d \geq M$ and the processes similar to (1), we can derive $I(u(t)) < 0$, $\forall\, t \in [0, +\infty)$. Therefore, from (1.7), (3.1), (5.2), (5.3), Corollary 2.8 and Lemma 2.9, we can derive

$$\begin{aligned}
G''(t) &= -2qJ(u(t)) + \frac{2(q-p)}{p}\|\Delta u(t)\|_p^p + \frac{2}{q}\|u(t)\|_q^q \\
&\geq -2qJ(u_0) + 2q\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \,\mathrm{d}s + \frac{2(q-p)}{p}\|\Delta u(t)\|_p^p + \frac{2}{q}\|u(t)\|_q^q \\
&\geq \begin{cases} -2qJ(u_0) + 2q\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \,\mathrm{d}s + \frac{2(q-p)}{p} r_*^p & \text{if } p < q, \\ -2qJ(u_0) + 2q\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \,\mathrm{d}s + \frac{2}{q} R^q & \text{if } p = q \end{cases} \\
&\geq 2q[M - J(u_0)] + 2q\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \,\mathrm{d}s.
\end{aligned} \tag{5.16}$$

On the other hand, it follows from

$$\begin{aligned}&\left[\int_0^t\int_\Omega (u(s)u_s(s)+\nabla u(s)\cdot\nabla u_s(s))\,\mathrm{d}x\mathrm{d}s\right]^2\\&=\frac{1}{4}\left(\int_0^t\|u_s(s)\|_{H_0^1(\Omega)}^2\,\mathrm{d}s\right)^2=\frac{1}{4}[G'(t)-G'(0)]^2\\&=\frac{1}{4}\left[(G'(t))^2-2G'(t)G'(0)+(G'(0))^2\right]\end{aligned}$$

that

$$\begin{aligned}(G'(t))^2=&\,4\left[\int_0^t\int_\Omega (u(s)u_s(s)+\nabla u(s)\cdot\nabla u_s(s))\,\mathrm{d}x\mathrm{d}s\right]^2\\&+2\big(\|u_0\|_2^2+\|\nabla u_0\|_2^2\big)G'(t)-\big(\|u_0\|_2^2+\|\nabla u_0\|_2^2\big)^2.\end{aligned}\tag{5.17}$$

Combining (5.16), (5.17) and Hölder's inequality to obtain

$$\begin{aligned}G(t)G''(t)-\frac{q}{2}(G'(t))^2&\ge 2q\int_0^t\|u_s(s)\|_{H_0^1(\Omega)}^2\,\mathrm{d}s\int_0^t\|u(s)\|_{H_0^1(\Omega)}^2\,\mathrm{d}s\\&\quad-2q\left[\int_0^t\int_\Omega (u(s)u_s(s)+\nabla u(s)\cdot\nabla u_s(s))\,\mathrm{d}x\mathrm{d}s\right]^2\\&\ge 2q[M-J(u_0)]G'(t)-q\|u_0\|_{H_0^1(\Omega)}^2G'(t)+\frac{q}{2}\big(\|u_0\|_2^2+\|\nabla u_0\|_2^2\big)^2\\&\ge -q\|u_0\|_{H_0^1(\Omega)}^2G'(t),\end{aligned}$$

which implies that, for all $\rho\in(0,1)$, we have

$$G(t)G''(t)-\frac{q\rho}{2}(G'(t))^2\ge\frac{q(1-\rho)}{2}(G'(t))^2-q\|u_0\|_{H_0^1(\Omega)}^2G'(t).\tag{5.18}$$

Meantime, from (5.5), we obtain

$$\lim_{t\to+\infty}G'(t)=\lim_{t\to+\infty}\|u(t)\|_{H_0^1(\Omega)}^2=+\infty.$$

Thus (5.18) implies that there exists a $t_\rho>0$ such that

$$G(t)G''(t)-\frac{q\rho}{2}(G'(t))^2>0,\quad\forall\,t\ge t_\rho,$$

and

$$\left[G^{1-\frac{q\rho}{2}}(t)\right]'=\left(1-\frac{q\rho}{2}\right)G^{-\frac{q\rho}{2}}(t)G'(t),$$
$$\left[G^{1-\frac{q\rho}{2}}(t)\right]''=\left(1-\frac{q\rho}{2}\right)G^{-1-\frac{q\rho}{2}}(t)\left[G(t)G''(t)-\frac{q\rho}{2}(G'(t))^2\right]>0,\quad\forall\,t\ge t_\rho.$$

Then by $2 - q\rho > 2 - q \geq 0$ and $G(t_\rho) \geq 0$, we can see

$$
\begin{aligned}
G(t) &= \left[G^{1-\frac{q\rho}{2}}(t)\right]^{\frac{2}{2-q\rho}} = \left[G^{1-\frac{q\rho}{2}}(t_\rho) + \int_{t_\rho}^{t} \left(G^{1-\frac{q\rho}{2}}(s)\right)' \mathrm{d}s\right]^{\frac{2}{2-q\rho}} \\
&\geq \left[G^{1-\frac{q\rho}{2}}(t_\rho) + (t - t_\rho)\left(G^{1-\frac{q\rho}{2}}(t_\rho)\right)'\right]^{\frac{2}{2-q\rho}} \\
&= \left[G^{1-\frac{q\rho}{2}}(t_\rho) + (t - t_\rho)\left(1 - \frac{q\rho}{2}\right) G^{-\frac{q\rho}{2}}(t_\rho)G'(t_\rho)\right]^{\frac{2}{2-q\rho}} \\
&\geq \left[(t - t_\rho)\left(1 - \frac{q\rho}{2}\right) G^{-\frac{q\rho}{2}}(t_\rho)G'(t_\rho)\right]^{\frac{2}{2-q\rho}} = C_\rho(t - t_\rho)^{\frac{2}{2-q\rho}},
\end{aligned}
\tag{5.19}
$$

where

$$
C_\rho = \left[\left(1 - \frac{q\rho}{2}\right) G^{-\frac{q\rho}{2}}(t_\rho)G'(t_\rho)\right]^{\frac{2}{2-q\rho}}.
$$

Moreover, using $G''(t) > 0$, $\forall\, t \geq 0$, we get

$$
\int_0^t G'(s)\,\mathrm{d}s \leq tG'(t),
$$

i.e.,

$$
t\|u(t)\|^2_{H_0^1(\Omega)} \geq G(t).
$$

Therefore, combining the inequality above and (5.19), we have

$$
\|u(t)\|^2_{H_0^1(\Omega)} \geq \frac{C_\rho(t - t_\rho)^{\frac{2}{2-q\rho}}}{t}.
$$

Then Theorem 5.1 is proved completely. □

5.2. Finite time blow-up

In this subsection, we give the results of finite time blow-up, life span of blow-up time and blow-up rate for problem (1.1)–(1.3).

To begin with, we recall Levine's convexity lemma, which plays a key role in the proof.

Lemma 5.2. [14] *Let $0 < T \leq +\infty$ and nonnegative function $F \in C^2[0, T)$ satisfy*

$$
F''(t)F(t) - (1 + \lambda)(F'(t))^2 \geq 0,
$$

where $\lambda > 0$ is a constant. If $F(0) > 0$ and $F'(0) > 0$, then

$$
T \leq \frac{F(0)}{\lambda F'(0)} < +\infty \quad \text{and} \quad \lim_{t \to T^-} F(t) = +\infty.
$$

Now, we describe the result of finite time blow-up as follows:

Theorem 5.3 (Finite time blow-up)**.** *Let $u_0 \in X \setminus \{0\}$ and p, q satisfy*

$$\max\left\{1, \frac{2N}{N+4}\right\} < p \le q, \quad 2 < q < p\left(1 + \frac{4}{N}\right). \tag{5.20}$$

(1) *If $J(u_0) \le M$ and $I(u_0) < 0$, then the weak solution $u(t)$ of problem* (1.1)–(1.3) *blows up in finite time, and if $J(u_0) < M$ and $I(u_0) < 0$, the upper bound of blow-up time is given by*

$$T_{\max} \le \frac{4\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)^2(M - J(u_0))}. \tag{5.21}$$

(2) *Furthermore, if p, q satisfy* (5.20) *and q satisfies*

$$\begin{cases} 1 < q < +\infty & \text{if } N = 1, 2, \\ 1 < q < \frac{2N}{N-2} & \text{if } N \ge 3, \end{cases} \tag{5.22}$$

then the lower bound of blow-up time is given by

$$T_{\max} \ge T_L := \frac{\alpha \|u_0\|^{2-(q+\alpha)}_{H_0^1(\Omega)}}{(q + \alpha - 2)B_H^{q+\alpha}}, \tag{5.23}$$

and

$$\|u(t)\|_{H_0^1(\Omega)} \ge \left[\frac{\alpha}{(q+\alpha-2)B_H^{q+\alpha}}\right]^{\frac{1}{q+\alpha-2}} (T_{\max} - t)^{-\frac{1}{q+\alpha-2}}, \tag{5.24}$$

where

$$\alpha = \begin{cases} \frac{p}{2}\left(1 + \frac{4}{N}\right) - \frac{q}{2} > 0 & \text{if } N = 1, 2, \\ \frac{1}{2}\min\left\{\frac{2N}{N-2}, p\left(1 + \frac{4}{N}\right)\right\} - \frac{q}{2} > 0 & \text{if } N \ge 3, \end{cases} \tag{5.25}$$

and B_H is the optimal embedding constant of $H_0^1(\Omega) \subset\subset L^{q+\alpha}(\Omega)$.

Proof. (1) Since $M \le d$, by the similar processes to Theorem 5.1, we obtain $I(u(t)) < 0$, $\forall\, t \in [0, T_{\max})$. Next, we discuss the following two cases:

Case 1*: $J(u_0) < M$.* We prove the solution of problem (1.1)–(1.3) blows up in finite time by contradiction. Assume $T_{\max} = +\infty$, then by the similar processes to Theorem 5.1 (see (5.5)), we know that there exists a $t_0 > 0$ large enough such that

$$\|u(t_0)\|^2_{H_0^1(\Omega)} > \frac{2q}{q-2}\|u_0\|^2_{H_0^1(\Omega)}. \tag{5.26}$$

Now, we define a functional

$$\Gamma(t) := \int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\, \mathrm{d}s + (\widetilde{T} - t)\|u_0\|^2_{H_0^1(\Omega)}, \quad t \in [0, \widetilde{T}], \tag{5.27}$$

where

$$\widetilde{T}=\frac{2(q+2)}{q-2}t_0. \tag{5.28}$$

It is clear that $\Gamma(t)$ is a positive continuous function on $[0,\widetilde{T}]$ and there exist two constants $\eta_1,\eta_2>0$ such that

$$\eta_1\geq\Gamma(t)\geq\eta_2. \tag{5.29}$$

Differentiating directly to obtain

$$\begin{aligned}\Gamma'(t)&=\|u(t)\|^2_{H^1_0(\Omega)}-\|u_0\|^2_{H^1_0(\Omega)}\\&=\|u(t)\|_2^2-\|u_0\|_2^2+\|\nabla u(t)\|_2^2-\|\nabla u_0\|_2^2\\&=\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\|u(s)\|_2^2\,\mathrm{d}s+\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\|\nabla u(s)\|_2^2\,\mathrm{d}s\\&=2\int_0^t\int_\Omega\big(u(s)u_s(s)+\nabla u(s)\cdot\nabla u_s(s)\big)\,\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{5.30}$$

and

$$\Gamma''(t)=2\int_\Omega\big(u(t)u_t(t)+\nabla u(t)\cdot\nabla u_t(t)\big)\,\mathrm{d}x=-2I(u(t)).$$

Combining (1.7) and energy inequality (3.1) to derive

$$\begin{aligned}\Gamma''(t)&=-2qJ(u(t))+\frac{2(q-p)}{p}\|\Delta u(t)\|_p^p+\frac{2}{q}\|u(t)\|_q^q\\&\geq\begin{cases}-2qJ(u_0)+2q\int_0^t\|u_s(s)\|^2_{H^1_0(\Omega)}\,\mathrm{d}s+\frac{2(q-p)}{p}r_*^p & \text{if } p<q,\\-2qJ(u_0)+2q\int_0^t\|u_s(s)\|^2_{H^1_0(\Omega)}\,\mathrm{d}s+\frac{2}{q}R^q & \text{if } p=q\end{cases}\\&=2q(M-J(u_0))+2q\int_0^t\|u_s(s)\|^2_{H^1_0(\Omega)}\,\mathrm{d}s,\end{aligned} \tag{5.31}$$

where we have used Corollary 2.8, Lemma 2.9 and the definition of M in (1.9).

By virtue of (5.30), Hölder's inequality and Schwartz's inequality, we have

$$\begin{aligned}\frac{1}{4}(\Gamma'(t))^2&=\left[\int_0^t\int_\Omega\big(u(s)u_s(s)+\nabla u(s)\cdot\nabla u_s(s)\big)\,\mathrm{d}x\mathrm{d}s\right]^2\\&=\left[\int_0^t\int_\Omega uu_s\,\mathrm{d}x\mathrm{d}s\right]^2+\left[\int_0^t\int_\Omega\nabla u\cdot\nabla u_s\,\mathrm{d}x\mathrm{d}s\right]^2\\&\leq\int_0^t\|u_s\|_2^2\,\mathrm{d}s\int_0^t\|u\|_2^2\,\mathrm{d}s+\int_0^t\|\nabla u_s\|_2^2\,\mathrm{d}s\int_0^t\|\nabla u\|_2^2\,\mathrm{d}s\\&\quad+\int_0^t\|u_s\|_2^2\,\mathrm{d}s\int_0^t\|\nabla u\|_2^2\,\mathrm{d}s+\int_0^t\|u\|_2^2\,\mathrm{d}s\int_0^t\|\nabla u_s\|_2^2\,\mathrm{d}s\\&=\int_0^t\|u_s\|^2_{H^1_0(\Omega)}\,\mathrm{d}s\int_0^t\|u\|^2_{H^1_0(\Omega)}\,\mathrm{d}s,\quad t\in[0,\widetilde{T}].\end{aligned} \tag{5.32}$$

Combining (5.27), (5.31) and (5.32), we can see

$$\Gamma(t)\Gamma''(t) \geq \frac{q}{2}(\Gamma'(t))^2 + 2q(M - J(u_0))\Gamma(t),$$

i.e.,

$$\Gamma(t)\Gamma''(t) - \frac{q}{2}(\Gamma'(t))^2 \geq 2q(M - J(u_0))\Gamma(t) \geq 2q(M - J(u_0))\eta_2 > 0,$$

where we have used (5.29) to derive the last inequality.

On the other hand, it follows from $\Gamma''(t) = -2I(u(t)) > 0$ that

$$\Gamma'(t_0) = \|u(t_0)\|^2_{H^1_0(\Omega)} - \|u_0\|^2_{H^1_0(\Omega)} > \Gamma'(0) = 0.$$

Then by (5.26), (5.28), Lemma 5.2 and the nonincreasing property of $\|u(t)\|^2_{H^1_0(\Omega)}$, we obtain that the maximal existence time $\widehat{T}$ of $\Gamma(t)$ satisfies

$$\begin{aligned}
\widehat{T} &\leq \frac{\int_0^{t_0} \|u(s)\|^2_{H^1_0(\Omega)}\,\mathrm{d}s + (\widetilde{T} - t_0)\|u_0\|^2_{H^1_0(\Omega)}}{\left(\frac{q}{2} - 1\right)\left(\|u(t_0)\|^2_{H^1_0(\Omega)} - \|u_0\|^2_{H^1_0(\Omega)}\right)} + t_0 \\
&\leq \frac{2t_0\|u(t_0)\|^2_{H^1_0(\Omega)} + 2(\widetilde{T} - t_0)\|u_0\|^2_{H^1_0(\Omega)}}{(q-2)\left(\|u(t_0)\|^2_{H^1_0(\Omega)} - \|u_0\|^2_{H^1_0(\Omega)}\right)} + t_0 \\
&< \frac{4qt_0 + 2(q-2)(\widetilde{T} - t_0)}{(q-2)(q+2)} + t_0 < \widetilde{T},
\end{aligned}$$

and

$$\lim_{t \to \widetilde{T}^-} \Gamma(t) = +\infty,$$

which contradicts with (5.29). Therefore, $T_{\max} < +\infty$.

Next, we give an upper bound of $T_{\max}$. For all $\overline{T} \in (0, T_{\max})$, set

$$F(t) := \int_0^t \|u(s)\|^2_{H^1_0(\Omega)}\,\mathrm{d}s + (\overline{T} - t)\|u_0\|^2_{H^1_0(\Omega)} + a(t+b)^2, \quad t \in [0, \overline{T}],$$

where a and b are positive constants to be determined later. Then by $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^1_0(\Omega)} = -2I(u(t)) > 0$ and direct calculation, we have

$$F(0) = \overline{T}\|u_0\|^2_{H^1_0(\Omega)} + ab^2 > 0,$$
$$F'(t) = \|u(t)\|^2_{H^1_0(\Omega)} - \|u_0\|^2_{H^1_0(\Omega)} + 2a(t+b) > 2a(t+b) > 0,$$

and

$$F'(0) = 2ab > 0,$$

which implies that

$$F(t) > F(0) > 0, \quad t \in [0, \overline{T}].$$

Using the similar processes to obtain (5.31), we have

$$F''(t) \geq 2q(M - J(u_0)) + 2q \int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, \mathrm{d}s > 0.$$

On the other hand, similar to (5.32), by Hölder's inequality and Young's inequality, we can verify

$$\begin{aligned} &\left[\int_0^t \|u(s)\|^2_{H_0^1(\Omega)} \, \mathrm{d}s + a(t+b)^2\right] \left[\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, \mathrm{d}s + a\right] \\ &\geq \left[\frac{1}{2}\big(\|u(t)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)}\big) + a(t+b)\right]^2. \end{aligned}$$

Therefore,

$$\begin{aligned} -(F'(t))^2 &= -4\left[\frac{1}{2}\big(\|u(t)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)}\big) + a(t+b)\right]^2 \\ &= 4\left[\int_0^t \|u(s)\|^2_{H_0^1(\Omega)} \, \mathrm{d}s + a(t+b)^2\right] \left[\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, \mathrm{d}s + a\right] \\ &\quad - 4\left[\frac{1}{2}\big(\|u(t)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)}\big) + a(t+b)\right]^2 \\ &\quad - 4\big[F(t) - (\overline{T} - t)\|u_0\|^2_{H_0^1(\Omega)}\big] \left[\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, \mathrm{d}s + a\right] \\ &\geq -4F(t)\left[\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, \mathrm{d}s + a\right], \end{aligned}$$

from which we can deduce

$$F(t)F''(t) - \frac{q}{2}(F'(t))^2 \geq 2q[(M - J(u_0)) - a]F(t).$$

Now, we choose $a \in (0, M - J(u_0)]$ such that $F(t)F''(t) - \frac{q}{2}(F'(t))^2 \geq 0$, using Lemma 5.2 and taking $\overline{T} \to T_{\max}$, we have

$$T_{\max} \leq \frac{\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)ab} T_{\max} + \frac{b}{q-2}.$$

Choosing $b \in \big(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)a}, +\infty\big)$, we obtain

$$T_{\max} \leq \frac{ab^2}{(q-2)ab - \|u_0\|^2_{H_0^1(\Omega)}}. \tag{5.33}$$

Now, we define

$$\begin{aligned} \Lambda &:= \left\{(a,b) \;\middle|\; a \in (0, M - J(u_0)], b \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)a}, +\infty\right)\right\} \\ &= \left\{(a,b) \;\middle|\; a \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)b}, M - J(u_0)\right], b \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)(M - J(u_0))}, +\infty\right)\right\}. \end{aligned}$$

Then (5.33) can be rewritten as

$$T_{\max} \le \inf_{(a,b)\in\Lambda} \frac{ab^2}{(q-2)ab - \|u_0\|^2_{H_0^1(\Omega)}}.$$

Taking $\varsigma = ab$ and setting

$$f(b,\varsigma) := \frac{\varsigma b}{(q-2)\varsigma - \|u_0\|^2_{H_0^1(\Omega)}},$$

then by the decreasing property of f with respect to ς, we can derive the lower bound of blow-up time

$$\begin{aligned}
T_{\max} &\le \inf_{b\in\left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)(M-J(u_0))},+\infty\right)} f(b, b(M-J(u_0))) \\
&= \inf_{b\in\left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)(M-J(u_0))},+\infty\right)} \frac{b^2(M-J(u_0))}{(q-2)b(M-J(u_0)) - \|u_0\|^2_{H_0^1(\Omega)}} \\
&= \frac{b^2(M-J(u_0))}{(q-2)b(M-J(u_0)) - \|u_0\|^2_{H_0^1(\Omega)}}\Bigg|_{b=\frac{2\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)(M-J(u_0))}} \\
&= \frac{4\|u_0\|^2_{H_0^1(\Omega)}}{(q-2)^2(M-J(u_0))}.
\end{aligned}$$

Case 2: $J(u_0) = M$. From $I(u(t)) < 0$, $\forall\, t \in [0, T_{\max})$, we know that

$$\int_\Omega u(t)u_t(t)\,\mathrm{d}x + \int_\Omega \nabla u(t)\cdot\nabla u_t(t)\,\mathrm{d}x = -I(u(t)) > 0, \quad t\in[0,T_{\max}).$$

Thus $u_t \neq 0$, $\nabla u_t \neq 0$ and $\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,\mathrm{d}s > 0$, $t \in (0, T_{\max})$, from which and energy inequality (3.1) we can take $t_1 \in (0, T_{\max})$ such that

$$J(u(t_1)) \le J(u_0) - \int_0^{t_1} \|u_s(s)\|^2_{H_0^1(\Omega)}\,\mathrm{d}s < d.$$

If we take t_1 as the initial data, then similar to Case 1, we can obtain that the solution u of problem (1.1)–(1.3) blows up in finite time.

(2) It follows from (5.25) that

$$\|u\|_{q+\alpha} \le B_H \|u\|_{H_0^1(\Omega)}, \quad \forall\, u \in H_0^1(\Omega), \tag{5.34}$$

where B_H is the optimal embedding constant of $H_0^1(\Omega) \subset\subset L^{q+\alpha}(\Omega)$. Now, we define

$$\varphi(t) := \|u(t)\|^2_{H_0^1(\Omega)} = \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2. \tag{5.35}$$

Differentiating (5.35) directly and using (5.34), we have

$$\begin{aligned}\varphi'(t) &= -2\|\Delta u(t)\|_p^p + 2\int_\Omega |u(t)|^q \log|u(t)|\,\mathrm{d}x \\ &\le 2\int_\Omega |u(t)|^q \log|u(t)|\,\mathrm{d}x \le \frac{2}{\alpha}\|u(t)\|_{q+\alpha}^{q+\alpha} \\ &\le \frac{2B_H^{q+\alpha}}{\alpha}\|u(t)\|_{H_0^1(\Omega)}^{q+\alpha} = \frac{2B_H^{q+\alpha}}{\alpha}\varphi^{\frac{q+\alpha}{2}}(t) \quad \text{a.e. } t\in[0,T_{\max}).\end{aligned} \tag{5.36}$$

Since $u(t)$ blows up in finite time, we can claim that $\varphi(t) > 0$, $t \in [0, T_{\max})$. In fact, if it is false, then there exists a $t_0 \in [0, T_{\max})$ such that $\varphi(t_0) > 0$. Meantime, by the continuity of $\varphi(t)$ and (5.36), we have $\varphi'(t) \le 0$, $t \in [t_0, T_{\max})$, which contradicts with the fact that the weak solution blows up in finite time. Therefore, we obtain

$$\frac{\varphi'(t)}{\varphi^{\frac{q+\alpha}{2}}(t)} \le \frac{2B_H^{q+\alpha}}{\alpha}. \tag{5.37}$$

Integrating (5.37) on $(0, t)$ to derive

$$\varphi^{1-\frac{q+\alpha}{2}}(0) - \varphi^{1-\frac{q+\alpha}{2}}(t) \le \frac{(q+\alpha-2)B_H^{q+\alpha}}{\alpha}t.$$

Taking $t \to T_{\max}$, we can see

$$T_{\max} \ge \frac{\alpha\|u_0\|_{H_0^1(\Omega)}^{2-(q+\alpha)}}{(q+\alpha-2)B_H^{q+\alpha}}.$$

Moreover, integrating (5.37) on $(t, T_{\max})$ to derive

$$\|u(t)\|_{H_0^1(\Omega)} \ge \left[\frac{\alpha}{(q+\alpha-2)B_H^{q+\alpha}}\right]^{\frac{1}{q+\alpha-2}} (T_{\max}-t)^{-\frac{1}{q+\alpha-2}}.$$

Then Theorem 5.3 is proved completely. □

Corollary 5.4 (Finite time blow-up). *Let $u_0 \in X \setminus \{0\}$ and p, q satisfy* (5.19). *If $J(u_0) < 0$, then the weak solution $u(t)$ of problem* (1.1)–(1.3) *blows up in finite time with the upper bound of blow-up time*

$$T_{\max} \le T_U := \frac{\|u_0\|_{H_0^1(\Omega)}^2}{q(2-q)J(u_0)}, \tag{5.38}$$

and satisfies

$$\|u(t)\|_{H_0^1(\Omega)}^2 \le \left[\frac{\|u_0\|_{H_0^1(\Omega)}^2}{q(2-q)J(u_0)}\right]^{\frac{1}{q-2}} (T_{\max}-t)^{-\frac{1}{q-2}}. \tag{5.39}$$

Proof. We define

$$\psi(t) := -2qJ(u(t)) = -\frac{2q}{p}\|\Delta u(t)\|_p^p - \frac{2}{q}\|u(t)\|_q^q + 2\int_\Omega |u(t)|^q \log|u(t)|\,\mathrm{d}x.$$

Differentiating directly, we have

$$\varphi'(t) = -2\|\Delta u(t)\|_p^p + 2\int_\Omega |u(t)|^q \log|u(t)|\,\mathrm{d}x \geq \psi(t), \tag{5.40}$$

and

$$\psi'(t) = -2q\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) = 2q\|u_t(t)\|_{H_0^1(\Omega)}^2 \geq 0. \tag{5.41}$$

By combining (5.35) with (5.41), and using Hölder's inequality and Schwartz's inequality, we obtain

$$\begin{aligned}\varphi(t)\psi'(t) &\geq 2q\|u(t)\|_{H_0^1(\Omega)}^2\|u_t(t)\|_{H_0^1(\Omega)}^2 \\ &\geq 2q\left[\int_\Omega u(t)u_t(t)\,\mathrm{d}x + \int_\Omega \nabla u(t)\cdot\nabla u_t(t)\,\mathrm{d}x\right]^2 \\ &= \frac{q}{2}(\varphi'(t))^2.\end{aligned} \tag{5.42}$$

It follows from (1.7) that $J(u(t_0)) < 0$ is stronger than $J(u(t_0)) \leq M$ and $I(u(t_0)) < 0$, and by the proof of Theorem 5.3, we can see $\varphi(t) > 0$, $\forall\, t \in [0, T_{\max})$. Meantime, from $\psi(0) = -2qJ(u_0) > 0$ and (5.41) we have $\psi(t) > 0$, $\forall\, t \in [0, T_{\max})$.

Therefore, combining (5.40) with (5.42), we can see

$$\frac{\psi'(t)}{\psi(t)} \geq \frac{q}{2}\frac{\varphi'(t)}{\varphi(t)}.$$

Integrating the inequality above on $(0, t)$ and using (5.40) to derive

$$\frac{\varphi'(t)}{\varphi^{\frac{q}{2}}(t)} \geq \frac{\psi(0)}{\varphi^{\frac{q}{2}}(0)}, \tag{5.43}$$

then integrating (5.43) on $(0, t)$, we have

$$\frac{1}{\varphi^{\frac{q}{2}-1}(t)} \leq \frac{1}{\varphi^{\frac{q}{2}-1}(0)} - \frac{q-2}{2}\frac{\psi(0)}{\varphi^{\frac{q}{2}}(0)}t.$$

Taking $t \to T_{\max}$ to obtain

$$T_{\max} \leq \frac{\|u_0\|_{H_0^1(\Omega)}^2}{q(2-q)J(u_0)},$$

and integrating (5.43) on $(t, T_{\max})$ to derive

$$\|u(t)\|_{H_0^1(\Omega)}^2 \leq \left[\frac{\|u_0\|_{H_0^1(\Omega)}^2}{q(2-q)J(u_0)}\right]^{\frac{1}{q-2}} (T_{\max} - t)^{-\frac{1}{q-2}}.$$

Then Corollary 5.4 is proved completely. □

Remark 5.5. From (1.7) we can see that $J(u_0) < 0$ implies $I(u_0) < 0$. Hence, if $J(u_0) < 0$, p, q satisfy (5.20) and q satisfies (5.22), we also obtain the lower bound of blow-up time T_L such that $T_L \le T_U$. In fact, it follows from $J(u_0) < 0$ that

$$\begin{aligned} -J(u_0) &= -\frac{1}{p}\|\Delta u_0\|_p^p + \frac{1}{q}\int_\Omega |u_0|^q \log|u_0|\,\mathrm{d}x - \frac{1}{q^2}\|u_0\|_q^q \\ &\le \frac{1}{q\alpha}\|u_0\|_{q+\alpha}^{q+\alpha} \le \frac{B_H^{q+\alpha}}{q\alpha}\|u_0\|_{H_0^1(\Omega)}^{q+\alpha}, \end{aligned}$$

which implies

$$\frac{\alpha\|u_0\|_{H_0^1(\Omega)}^{2-(q+\alpha)}}{B_H^{q+\alpha}} \le \frac{\|u_0\|_{H_0^1(\Omega)}^2}{-qJ(u_0)}.$$

Therefore, we have

$$\frac{\alpha\|u_0\|_{H_0^1(\Omega)}^{2-(q+\alpha)}}{(q+\alpha-2)B_H^{q+\alpha}} \le \frac{\|u_0\|_{H_0^1(\Omega)}^2}{-q(q-2)J(u_0)},$$

i.e., $T_L \le T_U$.

For all $t_0 \in [0, T_{\max})$, if we take t_0 as the initial time, then we can obtain the following corollary by Corollary 5.4.

Corollary 5.6 (Finite time blow-up)**.** *Let* $u_0 \in X \setminus \{0\}$ *and* p, q *satisfy* (5.20). *If* $J(u(t_0)) < 0$, $\forall\, t_0 \in [0, T_{\max})$, *then the weak solution* $u(t)$ *of problem* (1.1)–(1.3) *blows up in finite time.*

5.3. Blow-up with arbitrary initial energy

The blow-up results studied in Sections 5.1 and 5.2 are closely dependent on the depth of potential well d, but the value of d is small and difficult to calculate exactly. Therefore, we establish a blow-up condition independent of d in this subsection.

Theorem 5.7 (Blow-up with arbitrary initial energy)**.** *Let* $u_0 \in X \setminus \{0\}$ *and* p, q *satisfy*

$$2 < p < q < p\left(1 + \frac{4}{N}\right).$$

If

$$J(u_0) \le \frac{q-p}{2q\kappa_p^p(1+\overline{B}^2)}\|u_0\|_{H_0^1(\Omega)}^2 - \frac{(p-2)(q-p)}{2pq}|\Omega|, \tag{5.44}$$

where κ_p *and* $\overline{B}$ *are the optimal embedding constants of* $W_0^{2,p}(\Omega) \subset\subset W_0^{1,p}(\Omega)$, $\forall\, p > 1$ *and* $H_0^1(\Omega) \subset\subset L^2(\Omega)$, *respectively, then the weak solution* $u(t)$ *of problem* (1.1)–(1.3) *blows up in finite time.*

Proof. We prove the result by contradiction. We have known that $J(u(t_0)) < 0$, $\forall\, t_0 \in [0, T_{\max})$ leads to blow-up in finite time on the basis of Corollary 5.6. Thus we suppose that $u(t)$ exists globally and $J(u(t)) \geq 0$, $\forall\, t \in [0, +\infty)$. By Bochner Theorem, we have

$$\int_0^t \|u_s(s)\|_{H_0^1(\Omega)}\,\mathrm{d}s \geq \left\|\int_0^t u_s(s)\,\mathrm{d}s\right\|_{H_0^1(\Omega)} = \|u(t) - u_0\|_{H_0^1(\Omega)} \geq \|u(t)\|_{H_0^1(\Omega)} - \|u_0\|_{H_0^1(\Omega)}.$$

Then relying on the fact of $J(u_0) \geq J(u(t)) \geq 0$ and Hölder's inequality, we obtain

$$\begin{aligned}\|u(t)\|_{H_0^1(\Omega)} &\leq \|u_0\|_{H_0^1(\Omega)} + \int_0^t \|u_s(s)\|_{H_0^1(\Omega)}\,\mathrm{d}s \\ &\leq \|u_0\|_{H_0^1(\Omega)} + t^{\frac{1}{2}}\left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,\mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq \|u_0\|_{H_0^1(\Omega)} + t^{\frac{1}{2}}(J(u_0) - J(u(t)))^{\frac{1}{2}} \\ &\leq \|u_0\|_{H_0^1(\Omega)} + t^{\frac{1}{2}} J(u_0)^{\frac{1}{2}}.\end{aligned} \tag{5.45}$$

On the other hand, it follows from Young's inequality and (2.3) ($\gamma = p > 2$) that

$$\begin{aligned}\frac{1}{1+\overline{B}^2}\|u(t)\|^2_{H_0^1(\Omega)} \leq \|\nabla u(t)\|_2^2 &\leq \frac{2}{p}\|\nabla u(t)\|_p^p + \frac{p-2}{p}|\Omega| \\ &\leq \frac{2\kappa_p^p}{p}\|\Delta u(t)\|_p^p + \frac{p-2}{p}|\Omega|,\end{aligned} \tag{5.46}$$

where κ_p is the optimal embedding constant of $H_0^1(\Omega) \subset\subset L^2(\Omega)$, i.e., $\|u\|_2 \leq \overline{B}\|u\|_{H_0^1(\Omega)}$. Then by (5.46), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H_0^1(\Omega)} \\ &= -\|\Delta u(t)\|_p^p + \int_\Omega |u|^q \log|u|\,\mathrm{d}x \\ &= \left(\frac{q}{p} - 1\right)\|\Delta u(t)\|_p^p + \frac{1}{q}\|u(t)\|_q^q - qJ(u(t)) \\ &\geq \frac{q-p}{\kappa_p^p(1+\overline{B}^2)}\left[\frac{1}{2}\|u(t)\|^2_{H_0^1(\Omega)} - \frac{\kappa_p^p(1+\overline{B}^2)(p-2)}{2p}|\Omega| - \frac{q\kappa_p^p(1+\overline{B}^2)}{q-p}J(u(t))\right],\end{aligned}$$

and by the nonincreasing property of $J(u(t))$, we have

$$y'(t) \geq \frac{q-p}{\kappa_p^p(1+\overline{B}^2)}y(t),$$

where

$$y(t) = \frac{1}{2}\|u(t)\|^2_{H_0^1(\Omega)} - \frac{\kappa_p^p(1+\overline{B}^2)(p-2)}{2p}|\Omega| - \frac{q\kappa_p^p(1+\overline{B}^2)}{q-p}J(u(t)).$$

The Gronwall's inequality further indicates

$$\|u(t)\|^2_{H_0^1(\Omega)} \geq 2y(0)e^{\frac{q-p}{\kappa_p^p(1+\overline{B}^2)}t} + \frac{q\kappa_p^p(1+\overline{B}^2)}{q-p}J(u(t)) + \frac{\kappa_p^p(1+\overline{B}^2)(p-2)}{2p}|\Omega|.$$

Now, we can deduce $\|u(t)\|^2_{H^1_0(\Omega)} > 0$ and $y(0) > 0$ by $u_0 \in X\setminus\{0\}$ and (5.44), respectively. Recalling the assumption that $J(u(t)) \geq 0$, $t \in [0, +\infty)$, we obtain

$$\|u(t)\|_{H^1_0(\Omega)} \geq [2y(0)]^{\frac{1}{2}} e^{\frac{q-p}{2\kappa_p^p(1+\overline{B}^2)}t},$$

which contradicts (5.45) for sufficiently large $t > 0$. Therefore, $u(t)$ blows up in finite time. Then Theorem 5.7 is proved completely. □

6. Extinction phenomenon

In this section, we present the result of extinction for problem (1.1)–(1.3).

We recall a lemma playing the key role in the proof.

Lemma 6.1. [12] *Suppose that $0 < l < r \leq 1$ and $\sigma, \beta \geq 0$ are positive constants. If nonnegative and absolutely continuous function $h(t)$ satisfies*

$$h'(t) + \sigma h^l(t) \leq \beta h^r(t), \quad t \geq 0,$$
$$h(0) > 0, \quad \beta h^{r-l}(0) < \sigma,$$

then we have

$$h(t) \leq \left[-\sigma_0(1-l)t + h^{1-l}(0)\right]^{\frac{1}{1-l}}, \quad 0 < t < T_0,$$

and

$$h(t) \equiv 0, \quad t \geq T_0,$$

where $\sigma_0 = \sigma - \beta h^{r-l}(0)$ and $T_0 = \frac{h^{1-l}(0)}{\sigma_0(1-l)}$.

Theorem 6.2 (Extinction). *Assume $\max\left\{1, \frac{2N}{N+2}\right\} < p < q < 2$ and $0 < \|u_0\|_{H^1_0(\Omega)} < \frac{|\Omega|^{\frac{q+\alpha-2}{2}}}{B_p^p}$, then the weak solution of problem (1.1)–(1.3) becomes extinct in finite time. Furthermore, we have the following estimates:*

(6.1)

$$\|u(t)\|_{H^1_0(\Omega)} \leq \left[\|u_0\|^{2-p}_{H^1_0(\Omega)} - \sigma_0(2-p)\left(2^{\frac{p}{2}-1}B_p^{-p} - \frac{2^{\frac{q+\alpha}{2}-1}}{\alpha}|\Omega|^{1-\frac{q+\alpha}{2}}\|u_0\|^{q+\alpha-p}_{H^1_0(\Omega)}\right)t\right]^{\frac{1}{2-p}}, \quad 0 < t < T_*,$$

and

$$\|u(t)\|_{H^1_0(\Omega)} \equiv 0, \quad t \geq T_*,$$

where

$$T_* = \frac{\|u_0\|^{2-p}_{H^1_0(\Omega)}}{(2-p)\left(2^{\frac{p}{2}-1}B_p^{-p} - \frac{1}{\alpha}2^{\frac{q+\alpha}{2}-1}|\Omega|^{1-\frac{q+\alpha}{2}}\|u_0\|^{q+\alpha-p}_{H^1_0(\Omega)}\right)} \tag{6.2}$$

and $\alpha > 0$ is sufficiently small such that $q + \alpha < 2$.

Proof. We define

$$M(t) := \frac{1}{2}\|u(t)\|^2_{H^1_0(\Omega)}.$$

Multiplying (1.1) by u and integrating over Ω, we have

$$M'(t) + \|\Delta u(t)\|_p^p = \int_\Omega |u|^q \log|u| \, \mathrm{d}x. \tag{6.3}$$

Now, we use Rellich–Kondrachov Theorem (see [1, p. 168]) to derive

$$W^{2,p}_0(\Omega) \subset\subset H^1_0(\Omega), \quad p > \frac{2N}{N+2},$$

i.e.,

$$\|u\|_{H^1_0(\Omega)} \le B_p \|\Delta u\|_p, \quad \forall\, u \in W^{2,p}_0(\Omega). \tag{6.4}$$

Combining (2.8), (6.3), (6.4) and using Hölder's inequality, we deduce that there exists a $\alpha > 0$ such that

$$\begin{aligned} M'(t) + 2^{\frac{p}{2}} B_p^{-p} M^{\frac{p}{2}}(t) &\le \frac{1}{\alpha}\|u\|^{q+\alpha}_{q+\alpha} \le \frac{1}{\alpha}|\Omega|^{1-\frac{q+\alpha}{2}}\|u\|_2^{q+\alpha} \\ &\le \frac{1}{\alpha} 2^{\frac{q+\alpha}{2}} |\Omega|^{1-\frac{q+\alpha}{2}} M^{\frac{q+\alpha}{2}}(t). \end{aligned}$$

Then by Lemma 6.1 and assumption $0 < \|u_0\|_{H^1_0(\Omega)} < \frac{|\Omega|^{\frac{q+\alpha-2}{2}}}{B_p^{\frac{p}{p}}}$, we can see that

$$M(t) \le \left[-\sigma_0\left(1 - \frac{p}{2}\right)t + M^{1-\frac{p}{2}}(0)\right]^{\frac{2}{2-p}}, \quad 0 < t < T_*,$$

and

$$M(t) \equiv 0, \quad t \ge T_*,$$

where $\sigma_0 = 2^{\frac{p}{2}} B_p^{-p} - \frac{1}{\alpha} 2^{\frac{q+\alpha}{2}} |\Omega|^{1-\frac{q+\alpha}{2}} M^{\frac{q+\alpha-p}{2}}(0)$ and $T_* = \frac{2M^{\frac{2-p}{2}}(0)}{(2-p)\sigma_0}$. Therefore, the conclusion follows by $\|u(t)\|_{H^1_0(\Omega)} = \sqrt{2M(t)}$. Then Theorem 6.2 is proved completely. □

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