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Order Cancellation Law in a Semigroup of Closed Convex Sets

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Abstract. In this paper we generalize Robinson's version of an order cancellation law in which some unbounded subsets of a vector space are cancellative elements. We introduce the notion of weakly narrow sets in normed spaces, study their properties and prove the order cancellation law where the canceled set is weakly narrow. Also, we prove the order cancellation law for closed convex subsets of topological vector space where the canceled set has bounded Hausdorff-like distance from its recession cone. We topologically embed the semigroup of closed convex sets sharing a recession cone having bounded Hausdorff-like distance from it into a topological vector space. This result extends Bielawski and Tabor's generalization of Rådström theorem.

1. Introduction

The order cancellation law or Rådström cancellation theorem [23, 25], investigated also in [15, 18, 20], enables an embedding of a semigroup of convex sets into a quotient vector space called a Minkowski–Rådström–Hörmander space [12, 28]. This space plays a crucial role in differentiation of nonsmooth functions (quasidifferential calculus of Demyanov and Rubinov [1, 7, 8, 11, 14, 21]) and in integration and differentiation in the theory of multifunctions [5, 6]. The cancellation by unbounded sets is especially challenging. It is usually impossible. However, in this paper, we significantly extend the range of applicability of this law.

To embed a given commutative semigroup (a set with a binary operator of addition which is associative) of subsets of a vector space into some other vector space, we need the law of cancellation: $A + B = B + C \implies A = C$. This embedding is very important in set-valued analysis. Let us recall that the cancellation law holds in the semigroup $\mathcal{B}(X)$ of nonempty bounded closed convex sets in real Hausdorff topological vector space X, where the addition $\dot{+}$ is defined by $A \dot{+} B = \mathrm{cl}(A + B) = \mathrm{cl}\{a + b : a \in A, b \in B\}$. Minkowski studied the semigroup of compact convex subsets of Euclidean spaces, therefore, the algebraic or vector addition $A + B$ is also called the Minkowski addition.

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In infinite dimentional topological vector spaces, we need to consider $A \dot{+} B$ instead of $A+B$. Rådström [25] applied the cancellation law to embed the convex cone $\mathcal{B}(X)$, where X is a normed vector space, into a normed vector space. Hörmander [19] generalized the Rådström's result to the locally convex spaces and Urbański [28] to the arbitrary topological vector spaces. However, the order law of cancellation which is applied in the aforementioned cased is based on the following theorem.

Theorem 1.1. [28, Proposition 2.1] *Let X be a Hausdorff topological vector space and $A, B, C \subseteq X$. If B is nonempty and bounded and C is closed and convex then* $\mathrm{cl}(A+B) \subseteq \mathrm{cl}(B+C) \implies A \subseteq C$.

In [18] the assumption of closedness of C is weakened. In this paper, we drop the assumption of boundedness of B.

Since the cancellation law does not hold in the semigroup $\mathcal{C}(X)$ of nonempty closed convex sets, this semigroup cannot be embedded into a vector space. However, Robinson [26] proved the cancellation law for the family $\mathcal{C}_V(\mathbb{R}^n)$ of nonempty closed convex subsets of $\mathbb{R}^n$ sharing common recession cone V. He also embedded $\mathcal{C}_V(\mathbb{R}^n)$ into a topological vector space (but not into a normed space). Bielawski and Tabor, [4] restricted the family $\mathcal{C}_V(X)$, where X is a normed vector space to such closed convex sets A that the Hausdorff distance $d_H(A, V)$ is finite. It enabled them to embed restricted $\mathcal{C}_V(X)$ into a normed vector space, generalizing in this case the Rådström's result [25].

In this paper, we investigate the possibility of cancellation

$$A + B \subseteq B + C \quad \implies \quad A \subseteq C$$

for some unbounded sets B. In particular, we generalize in Theorems 4.7 and 4.11 the following Robinson's result.

Theorem 1.2. [26, Lemma 1] *Let $X = \mathbb{R}^n$ and $A, B, C \subseteq X$. If B, C are nonempty closed and convex and* $\mathrm{recc}\, B = \mathrm{recc}\, C$, *then* $A + B \subseteq B + C \implies A \subseteq C$.

A subset V of a vector space X is a *cone* if $tV \subseteq V$ for all $t \geq 0$. In Theorem 5.4 we generalize the following Bielawski–Tabor's order law of cancellation.

Theorem 1.3. [4, Proposition 1] *Let X be a Banach space, V a closed convex cone in X, d_H a Hausdorff metric and C_V be a family $\{A \in \mathcal{C}(X) : d_H(A, V) < \infty\}$. If $A, B, C \in C_V$, then* $\mathrm{cl}(A + B) \subseteq \mathrm{cl}(B + C) \implies A \subseteq C$.

Finally, we generalize topologically in Theorem 5.6, Bielawski–Tabor's embedding theorem.

Theorem 1.4. [4, Theorem 2] *Let X be a Banach space, V a closed convex cone in X, d_H a Hausdorff distance and $C_V = \{A \in \mathcal{C}(X) : d_H(A, V) < \infty\}$. Then the abstract convex cone C_V can be embedded isometrically and isomorphically as a closed convex cone into a Banach space.*

For completeness by abstract convex cone we understand the triple $(S, +, \cdot)$, where the operation of addition of elements of S is associative, commutative and possesses neutral element 0_S and multiplication by non-negative numbers satisfies the following conditions: (1) $1x = x$, (2) $0x = 0_S$, (3) $s(tx) = (st)x$, (4) $t(x+y) = tx+ty$ and (5) $(s+t)x = sx+tx$ for all $x, y \in S$, $s, t \geq 0$.

In Section 2 we prove an order cancellation law in an ordered semigroup with an operator of convergence. This result is applied in Section 5 for convex sets in topological vector spaces having a certain kind of "finite distance" between a set and its recession cone.

In Section 3 we give some properties of asymptotic and recession cones in infinite dimensional spaces. We also present examples of unbounded convex sets with trivial recession cones and of a recession cone of a sum of sets greater than a sum of recession cones of summands.

In Section 4 we introduce the notion of narrow unbounded sets. We prove in Theorem 4.7 that in the inclusion $\mathrm{cl}(A + B) \subseteq \mathrm{cl}(B + C)$ we can cancel a narrow set B with an asymptotic cone contained in a recession cone of closed convex set C. We also present properties of addition of narrow sets that help us in Theorem 4.16 to show that the family of all weakly narrow closed convex subsets of normed vector space sharing a given pointed recession cone is closed with respect to Minkowski addition.

2. Cancellation law in an ordered semigroup with an operator of convergence

In this section, we prove the order cancellation law for, respectively, bounded and convex elements in ordered semigroups. The results of this section will be applied in Section 5 where we give a generalization of the Bielawski–Tabor result [4] to topological vector spaces.

Let $\mathbb{Q}_2^+$ be the set of all non-negative dyadic rationals and let us consider a system $S = (S, +, \cdot, \leq, \lim)$, where '$+$' and '$\cdot$' are binary operations from $S \times S$ and $\mathbb{Q}_2^+ \times S$ into S respectively, $\leq$ is a partial order on S and lim is an limit operator turning S into $\mathcal{L}^*$-space (see [13, p. 63]). Specifically,

(1) if $a_1 = a_2 = a_3 = \cdots = a$ then $\lim_{n\to\infty} a_n = a$,

(2) if $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to\infty} a_{k_n} = a$ for every subsequence (a_{k_n}),

(3) if a_n does not converge to a than there exists a subsequence (a_{k_n}) such that no subsequence of (a_{k_n}) converges to a.

Moreover, let the following conditions be satisfied:

(S1) $a+(b+c)=(a+b)+c$ for all $a,b,c\in S$,

(S2) $a+b=b+a$ for all $a,b\in S$,

(S3) there exists an element $0\in S$ such that $a+0=a$ for all $a\in S$,

(S4) if $a\leq b$ then $a+c\leq b+c$ for all $a,b,c\in S$,

(S5) $1\cdot a=a$ for all $a\in S$,

(S6) if $a\leq b$ then $\alpha\cdot a\leq\alpha\cdot b$ for all $a,b\in S$, $\alpha\in\mathbb{Q}_2^+$,

(S7) $\alpha(\beta\cdot a)=(\alpha\beta)\cdot a$ for all $\alpha,\beta\in\mathbb{Q}_2^+$, $a\in S$,

(S8) $\alpha\cdot(a+b)=\alpha\cdot a+\alpha\cdot b$ for all $\alpha\in\mathbb{Q}_2^+$, $a,b\in S$,

(S9) $(\alpha+\beta)\cdot a\leq\alpha\cdot a+\beta\cdot a$ for all $\alpha,\beta\in\mathbb{Q}_2^+$, $a\in S$,

(S10) for all sequences $(a_n),(b_n)\subseteq S$, if $\lim_{n\to\infty}a_n=a$, $\lim_{n\to\infty}b_n=b$ and $a_n\leq b_n$, $\forall\, n\in\mathbb{N}$, then $a\leq b$,

(S11) for all sequences $(a_n)\subseteq S$ and $b\in S$, if $\lim_{n\to\infty}a_n=a$ then $\lim_{n\to\infty}(a_n+b)=a+b$.

Definition 2.1. We say that an element $a\in S$ is *bounded* if $\lim_{n\to\infty}(2^{-n}\cdot a)=0$.

Definition 2.2. We say that an element $a\in S$ is *convex* if $(\alpha+\beta)\cdot a=\alpha\cdot a+\beta\cdot a$ for all $\alpha,\beta\in\mathbb{Q}_2^+$.

Lemma 2.3. *Let S be a semigroup satisfying conditions* (S1)–(S11). *If $a,b,c\in S$, $\lim_{n\to\infty}(2^{-n}b)$ exists, c is convex and $a+b\leq b+c$, then*

$$a+\lim_{n\to\infty}2^{-n}b\leq c+\lim_{n\to\infty}2^{-n}b.$$

Proof. For $k\in\mathbb{N}$, we have

$$\begin{aligned}2^k\cdot a+b&\leq\underbrace{a+a+\cdots+a}_{2^k\text{-times}}+b\leq\underbrace{a+a+\cdots+a}_{(2^k-1)\text{-times}}+b+c\\&\leq\underbrace{a+a+\cdots+a}_{(2^k-2)\text{-times}}+b+c+c\leq\cdots\leq b+\underbrace{c+c+\cdots+c}_{2^k\text{-times}}\\&=b+2^k\cdot c.\end{aligned}$$

Last equality follows from the convexity of an element c. Multiplying both sides of the inequality by 2^{-k} and using (S6) and (S7), we obtain $a + 2^{-k} \cdot b \leq 2^{-k} \cdot b + c$. Now by (S11), we obtain $a + \lim_{n\to\infty}(2^{-n}b) = \lim_{k\to\infty}(a + 2^{-k} \cdot b)$ and $c + \lim_{n\to\infty}(2^{-n}b) = \lim_{k\to\infty}(c + 2^{-k} \cdot b)$. Hence by (S10), we get $a + \lim_{n\to\infty}(2^{-n}b) \leq c + \lim_{n\to\infty}(2^{-n}b)$. □

Theorem 2.4 (Order law of cancellation)**.** *Let S be a semigroup satisfying conditions* (S1)–(S11)*. If $a, b, c \in S$, b is bounded, c is convex and $a + b \leq b + c$, then $a \leq c$.*

Proof. The theorem is a direct corollary from Lemma 2.3. □

Let $S_{bc} \subseteq S$ be a subsemigroup of bounded convex elements. If the topology in S_{bc} is generated by a uniformity in which the addition is strongly uniformly continuous then there exists a topological embedding of the semigroup S_{bc} into a quotient group $\widetilde{S_{bc}} = S_{bc} \times S_{bc}/\sim$, where $(a, b) \sim (c, d) \iff a + d = b + c$ (see [28, Proposition 1.2(i)] and [16, Proposition 3.5]).

Theorem 2.4 will be applied in the last section of this paper. The following trivial example shows a semigroup S satisfying all conditions (S1)–(S11) and containing only convex elements.

Example 2.5. Let S be equal to one of the following sets $\mathbb{Q}_2^+ \cup \{\infty\}$, $\mathbb{Q}_2 \cup \{\infty\}$, $\mathbb{R}_+ \cup \{\infty\}$ or $\mathbb{R} \cup \{\infty\}$. Let addition be the normal addition and S be ordered by natural order induced from the reals and $0 \cdot A = \{0\}$ for all elements A of S. We define a limit $\lim_{n\to\infty} a_n = a$ in usual way. Obviously, the conditions (S1)–(S11) are satisfied. Only the element ∞ is not bounded and all elements of S are convex.

The next example shows a semigroup S satisfying all conditions (S1)–(S11) and containing unbounded and non-convex elements.

Example 2.6. Let S be the family of all nonempty finite subsets of $\mathbb{Q}_2^+ \cup \{\infty\}$, $\mathbb{Q}_2 \cup \{\infty\}$, $\mathbb{R}_+ \cup \{\infty\}$ or $\mathbb{R} \cup \{\infty\}$. Let addition be natural algebraic addition and S be ordered by inclusion and $0 \cdot A = \{0\}$ for all elements A of S. We define a limit $\lim_{n\to\infty} A_n = A$ if and only if every neighborhood of A contains all but finite number of sets A_n and

$$A \subseteq \bigcap_{n_0=1}^{\infty} \operatorname{cl}\left(\bigcup_{n \geq n_0} A_{k_n}\right)$$

for every increasing sequence (k_n) of integers. It is easy to observe that the conditions (S1)–(S11) are satisfied. Bounded elements of S are sets not containing the element ∞. Convex elements of S are all sets containing at most one number other than ∞.

Another example is provided by a family of open domains.

Example 2.7. Let X be a real Hausdorff topological vector space and S^* be the family of all open domains of X, that is of all interiors of closed subsets of X, and $S = S^* \cup \{\{0\}\}$. Let addition be defined by $A \ddot{+} B = \operatorname{int}\operatorname{cl}(A + B)$, the family S be ordered by inclusion and $0 \cdot A = \{0\}$. We define a limit $\lim_{n\to\infty} A_n = A$ if and only if for any neighborhood U of origin there exists n_0 such that $A_n \subseteq A + U$, $A \subseteq A_n + U$ for all $n \geq n_0$. The conditions (S1)–(S11) are satisfied. Bounded elements of S are bounded sets. Convex elements of S are convex sets.

Definition 2.8. A quartet $(S, \mathbb{R}_+, +, \cdot\,)$ satisfying equalities (S1)–(S3), (S5), (S7), (S8), $(\alpha + \beta)a = \alpha a + \beta a$ and $0 \cdot a = 0$ for all $\alpha, \beta \geq 0$, $a, b, c \in S$ is called an *abstract convex cone.*

A family of nonempty compact convex subsets of a topological vector space is a natural example of an abstract convex cone.

3. Assymptotic and recession cones

Our main results are placed in Sections 4 and 5. Since considered sets in those sections are unbounded and convex, here we discuss some properties of the recession cone of convex set.

Definition 3.1. For a Hausdorff topological vector space X and $A \subseteq X$, the set $A_\infty = \{x \in X \mid \exists\,(x_n) \subseteq A, \exists\,(t_n) \subseteq (0,\infty), t_n \to 0, t_n x_n \to x\}$ is called an *assymptotic cone* of A. A set $\operatorname{recc} A = \{x \in X \mid \forall\, a \in A, \forall\, t \in (0, \infty), a + tx \in A\}$ is called a *recession cone* of A.

A recession cone of a set A does not depend on the topology in X but an asymptotic cone does. In case of ambiguity about exact topology, i.e., topology τ, weak topology or norm topology we put, respectively, A^τ_∞, A^*_∞ or $A^{\|\cdot\|}_\infty$. Obviously, a recession cone of a set is contained in an assymptotic cone of this set. We are going to express assymptotic and recession cones in terms of Minkowski subtraction.

Definition 3.2. For a vector space X and subsets $A, B \subseteq X$, the *Minkowski difference* of sets A and B is defined by $A \dot{-} B = \{x \in X : x + B \subseteq A\}$.

The following obvious proposition gives us a useful formula of Minkowski difference of sets.

Proposition 3.3. *If* $A, B \subseteq X$ *then* $A \dot{-} B = \bigcap_{b\in B}(A - b)$.

Let X be a vector space and let $\mathcal{A} \subseteq 2^X$ be a family of subsets of X. We say that $\mathcal{A}$ is *closed under translations* if for every $A \in \mathcal{A}$ and for every $x \in X$, we have $x + A \in \mathcal{A}$.

We also say that $\mathcal{A}$ is *closed under intersections* if for any subfamily $\{A_t\}_{t\in T} \subseteq \mathcal{A}$, the intersection $\bigcap_{t\in T} A_t$ belongs to $\mathcal{A}$.

The following proposition is an intermediate corollary from Proposition 3.3.

Corollary 3.4. *Let X be a vector space and let $\mathcal{A} \subseteq 2^X$ be a family of subsets of X. If $\mathcal{A}$ is closed under intersections and translations, then $\mathcal{A}$ is closed under the Minkowski difference.*

It is well known that $\operatorname{recc} A = A \dot{-} A$ for any nonempty convex set A.

Lemma 3.5. *Let X be a topological vector space and let $A \subseteq X$ be a closed and convex set, then $A_\infty = A \dot{-} A$.*

Proof. Let $u \in A_\infty$, $x \in A$. By Definition 3.1 we have $u + x = \lim_{n\to\infty}(t_n x_n + x) = \lim_{n\to\infty}(t_n x_n + (1 - t_n)x) \in A$ for some $(x_n) \subseteq A$, $(t_n) \subseteq \mathbb{R}$ such that $t_n \to 0^+$. Hence $u + A \subseteq A$. Therefore, $A_\infty \subseteq A \dot{-} A$.

On the other hand, we have $A \dot{-} A = \operatorname{recc} A \subseteq A_\infty$. □

Corollary 3.6. *Let X be a topological vector space. If a subset $A \subseteq X$ is closed and convex, then the cone A_∞ is also closed and convex.*

The corollary follows from the fact that convexity and closedness are preserved under translation and intersection.

It is well known [27, Theorem 8.4] that if X is finite dimensional normed space and $A \subseteq X$ is closed convex subset, then the condition $A_\infty = \operatorname{recc} A = \{0\}$ implies that the set A is bounded. Now, we show that this is no longer true in infinite dimensional normed space. To do this, we need the following two lemmas.

Lemma 3.7. *Let X be a separable infinite dimensional normed space. Then there exists a sequence $(f_n) \subseteq X^*$ such that*

(i) $\|f_n\| = 1$ *for all* $n \in \mathbb{N}$,

(ii) *for any* $x \in X$, *if* $f_n(x) = 0$ *for every* $n \in \mathbb{N}$, *then* $x = 0$.

Proof. Let $A = \{x_n \mid n \in \mathbb{N}\}$ be a dense and countable set in X. By Hahn–Banach theorem, there exists a sequence $(f_n) \subseteq X^*$ of continuous linear functionals such that $\|f_n\| = 1$ and $f_n(x_n) = \|x_n\|$. Take any $x \in X$ and suppose that $f_n(x) = 0$ for every $n \in \mathbb{N}$. Let $(x_{n_k}) \subseteq A$ be a sequence such that $\|x - x_{n_k}\| < 1/k$. We have $\|x_{n_k}\| = f_{n_k}(x_{n_k} - x) \le \|x - x_{n_k}\| < 1/k$. Hence $x = \lim_{k\to\infty} x_{n_k} = 0$. □

Lemma 3.8. *Let X be infinite dimensional a separable normed space, then there exists $A \subseteq X$ such that A is closed convex unbounded and $A \dot{-} A = \{0\}$.*

Proof. Let $(f_n) \subseteq X^*$ be a sequence such as in the proof of Lemma 3.7. Let $A := \{x \in X \mid |f_n(x)| \leq n, n \in \mathbb{N}\}$. The set A is closed and convex. Now, we show that A is not bounded. Suppose to the contrary that there exists a $\sigma \in \mathbb{R}$ such that for any $x \in A$, we have $\|x\| \leq \sigma$. Let us define a functional $\xi\colon X \to \mathbb{R}$, $\xi(x) := \sup_{n\in\mathbb{N}} |f_n(x)|n^{-1}$. It is easy to see that ξ is some new norm and $A = \{x \in X \mid \xi(x) \leq 1\}$ is a unit ball in this norm. Thus for any $x \in X$, we have $x/\xi(x) \in A$. Hence $\|x/\xi(x)\| \leq \sigma$. Then $\|x\| \leq \sigma\xi(x)$. Now take $m \in \mathbb{N}$ such that $m > \sigma$ and let $u \in \bigcap_{1\leq j\leq m} \ker f_j$, $u \neq 0$, then we have

$$\|u\| \leq \sigma\xi(u) = \sigma \sup_{n>m} |f_n(u)|n^{-1} \leq \sigma m^{-1}\|u\| < \|u\|$$

but this is impossible. Hence A is not bounded.

The set A is convex and contains no ray, since A is bounded in some norm. Then $A \dot{-} A = \{0\}$ because A is convex and contains no ray. □

In the following example, a set A has trivial recession cone, while no linear functional is bounded on A.

Example 3.9. Let c_{00} be a space of all sequences of real numbers with finite amount of nonzero terms. Let τ be any topology in c_{00} in which all projections on axes of coordinates are continuous functionals. Let $A := \operatorname{conv}\bigcup_{n=1}^{\infty}([n, 2n]^n \times \{(0, 0, \ldots)\})$. Let $A_\tau := \operatorname{cl}_\tau A$. Let σ be a product topology in c_{00}. Then σ is weaker than τ. Notice that $A_\sigma \cap (\mathbb{R}^n \times \{(0, 0, \ldots)\}) = \operatorname{conv}\bigcup_{k=1}^{n}([k, 2k]^k \times \{(0, 0, \ldots)\})$. Hence $A_\tau = A$. The set A contains no half-line, and $\operatorname{recc} A = \{0\}$. However, for any functional $f \in c_{00}^*$, the image $f(A)$ is unbounded. Hence A is unbounded for any linear topology in c_{00} in which points can be separated by continuous linear functionals.

Theorem 3.10. *Let X be an infinite dimensional normed space. Then there exists a subset $A \subseteq X$ such that A is closed convex unbounded and $A \dot{-} A = \{0\}$.*

Proof. Consider an infinite countable subset $\Gamma \subset X$ of linearly independent vectors. Let $Y = \operatorname{cl}\operatorname{span}\Gamma$. Since Y is a normed separable infinite dimensional subspace of X, our theorem follows from Lemma 3.8. □

The next theorem shows that in any infinitely dimensional Banach space, the equality $\operatorname{recc} A \dot{+} \operatorname{recc} B = \operatorname{recc}(A \dot{+} B)$ is false for some closed convex subsets A and B.

Theorem 3.11. *Let X be an infinite dimensional Banach space. Then there exists subsets $A, B \in \mathcal{C}(X)$ with trivial reccesion cones such that $\operatorname{recc}(A \dot{+} B)$ is a half-line.*

Proof. Let X be an infinitely dimensional Banach space. By the Mazur theorem [3], there exists a closed subspace Y of X having Schauder basis (e_i), $i = 0, 1, \ldots$. Moreover (see [9]), there exists a constant $C \geq 1$ such that $|x_i| \leq C\|x\|$ for all $x = \sum_{j=0}^{\infty} x_j e_j \in Y$, $i = 0, 1, \ldots$.

Let $A = \operatorname{cl}\operatorname{conv}\{ne_0 + 2^n ne_n \mid n = 0, 1, 2, \ldots\}$, $B = \operatorname{cl}\operatorname{conv}\{ne_0 - 2^n ne_n \mid n = 0, 1, 2, \ldots\}$. Notice that the ray $\{te_0 \mid t \geq 0\}$ is contained in $A + B$. Hence this ray is contained in the recession cone $\operatorname{recc}(A + B)$.

Let us assume that the set $A \dot{+} B$ contains some ray $\{tv \mid t \geq 0\}$, $v \neq 0$, where $v \in X$, $v = \sum_{n=0}^{\infty} v_n e_n$. Let $y \in A+B$. Then $y = \sum_{n=0}^{\infty} \alpha_n (ne_0 + 2^n ne_n) + \sum_{n=0}^{\infty} \beta_n (ne_0 - 2^n ne_n)$, where $\alpha_n, \beta_n \geq 0$, $\sum_{n=0}^{\infty} \alpha_n = \sum_{n=0}^{\infty} \beta_n = 1$ and the set $\{n \mid \max(\alpha_n, \beta_n) > 0\}$ is finite. Notice that

$$C\|tv - y\| \geq \left|tv_n - 2^n n(\alpha_n - \beta_n)\right| \geq t|v_n| - 2^n n$$

for all $n \geq 1$. If $v_n \neq 0$ then $\|tv - y\| > 1$ for all $t > (2^n n + C)/|v_n|$. Hence $\operatorname{dist}(tv, A \dot{+} B) \geq 1$ which implies that $v_n = 0$ for all $n \geq 1$, and we obtain the equality $\operatorname{recc}(A \dot{+} B) = [0, \infty)e_0$.

We know that $\operatorname{recc} A \cup \operatorname{recc} B \subseteq \operatorname{recc}(A \dot{+} B)$. Does e_0 belong to $\operatorname{recc} A$? If yes, then $e_0 \in A$. Assume that there exists $y \in \operatorname{conv}\{ne_0 + 2^n ne_n \mid n = 0, 1, 2, \ldots\}$ such that $\|e_0 - y\| < 1/(3C)$. Let $y := \sum_{n=1}^{\infty} \alpha_n \cdot (ne_0 + 2^n ne_n)$. Then

$$\max\left(\left|1 - \sum_{n=1}^{\infty} \alpha_n n\right|, \sup\{(\alpha_n 2^n n) \mid n \in \mathbb{N}\}\right) \leq C\|e_0 - y\| < \frac{1}{3}.$$

Hence we obtain the inequalities $\alpha_n n < 1/(3 \cdot 2^n)$, $n \geq 1$. Then $\sum_{n=1}^{\infty} \alpha_n n < 1/3$. But

$$1 - \frac{1}{3} < \left|1 - \sum_{n=1}^{\infty} \alpha_n n\right| \leq C\|e_0 - y\| < \frac{1}{3}.$$

Hence we obtain a contradiction. Then $e_0 \notin A \supset \operatorname{recc} A$. Therefore, the recession cone of A and, similarly, of B is trivial. $\square$

Corollary 3.12. *Let X be an infinite dimensional Banach space. Then there exists subsets $A, B \in \mathcal{C}(X)$ such that* $\operatorname{recc}(A \dot{+} B) \neq \operatorname{recc} A \dot{+} \operatorname{recc} B$.

Remark 3.13. Robinson showed in [26] that in finite dimensional spaces a family of closed convex sets with a fixed common recession cone is closed with respect to Minkowski addition $A \dot{+} B = \operatorname{cl}(A + B)$. Theorem 3.11 shows that in infinite dimensional spaces, even Hilbert spaces, a family of closed convex sets with fixed common recession cone need not to be closed with respect to Minkowski addition. In order to obtain such a family (with common recession cone and closed with respect to Minkowski addition), we need to introduce a condition expressed not in terms of recession cones.

Remark 3.14. Theorem 3.11 shows that in any Banach space X, we cannot cancel by sets with trivial recession cone. For example, let $A, B \subseteq X$ be sets from Theorem 3.11. Then $A \dot{+} B \dot{+} \operatorname{recc}(A \dot{+} B) = A \dot{+} B$. However, $A \dot{+} \operatorname{recc}(A \dot{+} B) \neq A$. Hence the condition for canceling by convex sets in Banach spaces cannot be expressed just in terms of recession cones.

If the space X is not reflexive, then we can always find two bounded closed convex sets, with not closed Minkowski sum. In fact, possibility of finding such two bounded closed convex sets is equivalent to non-reflexivity of the space X, see [24].

Theorem 3.15. *Let X be an infinite dimensional Banach space. Then there exists subsets $A, B \in \mathcal{C}(X)$ with trivial reccesion cones such that the sum $A+B$ is not closed.*

Proof. Let X be an infinitely dimensional Banach space. Again by the Mazur theorem, we have a closed subspace Y of X with Schauder basis (e_i), $i = 0, 1, \ldots$ and a constant $C \geq 1$ such that $|x_i| \leq C\|x\|$ for all $x = \sum_{j=0}^{\infty} x_j e_j \in Y$, $i = 0, 1, \ldots$. Let $A = \operatorname{cl}\operatorname{conv}\{e_0/n + a_n e_n \mid n = 1, 2, \ldots\}$, $B = \operatorname{cl}\operatorname{conv}\{e_0/n - a_n e_n \mid n = 1, 2, \ldots\}$, $a_n > 2^n$. Notice that the origin belongs to the closure of $A+B$. We are going to prove that the origin does not belong to $A+B$. If $x = \sum_{j=0}^{\infty} x_j e_j \in A$, then $x_0 \geq 0$. Also $x_0 \geq 0$ for all $x \in B$. It is enough to show that $x_0 > 0$ for all $x \in A$. If $x \in \operatorname{conv}\{e_0/n + a_n e_n \mid n = 1, 2, \ldots\}$, then there exists a number $k > 0$ such that $x_k \geq 1$. Hence for all $x \in A$, we have $\|x\| \geq 1/C$. Therefore, the origin does not belong to A. Suppose, $y = \sum_{j=0}^{\infty} y_j e_j \in A$ and $y_0 = 0$. Then $y_k > 0$ for some k. There exists a sequence $(x^n) \subseteq \operatorname{conv}\{e_0/n + a_n e_n \mid n = 1, 2, \ldots\}$ such that x^n tends to y. Then x_k^n tends to y_k for all k. We can represent x^n as convex combination $x^n = \sum_{j=0}^{\infty} \alpha_j^n (e_0/j + a_j e_j)$. Hence $x_k^n = \alpha_k^n a_k$ tends to y_k, and α_k^n tends to y_k/a_k. Therefore, $x_0^n \geq \alpha_k^n/k$ where α_k^n/k tends to $y_k/(ka_k) > 0$.

Notice that

$$\operatorname{cl}(A+B) = \operatorname{cl}\left(\operatorname{conv}\left\{\frac{e_0}{n} + a_n e_n, \,\middle|\, n \in \mathbb{N}\right\} + \operatorname{conv}\left\{\frac{e_0}{n} - a_n e_n \,\middle|\, n \in \mathbb{N}\right\}\right).$$

Since for all $x \in \operatorname{conv}\{e_0/n + a_n e_n \mid n = 1, 2, \ldots\} + \operatorname{conv}\{e_0/n - a_n e_n \mid n = 1, 2, \ldots\}$, we have $-a_k \leq x_k \leq a_k$, the same holds true for all $x \in \operatorname{cl}(A+B)$ and $k > 0$. Similarly, $0 \leq x_0 \leq 2$ for all $x \in \operatorname{cl}(A+B)$. Therefore, if v belongs to the $\operatorname{recc}\operatorname{cl}(A+B)$, then $v = 0$. Hence both A and B have a trivial recession cone. □

4. Order cancellation law in normed spaces

In this section, we generalize the order law of cancellation for closed convex sets with common recession cone proved by Robinson in [26] for finite dimensional spaces.

Definition 4.1. Let X be a normed space and let A be a subset of X. Let τ be a linear topology weaker than norm topology. By $\mathfrak{B}_A^\tau$, we define the set $\{\xi \in X \mid$ there exists $(x_n) \subseteq A$ with $\|x_n\| \to \infty$ and $x_n/\|x_n\| \rightharpoonup \xi\}$, where the symbol "$\rightharpoonup$" denotes the τ-convergence in X.

Obviously, the set $\mathfrak{B}_A^\tau$ is a subset of an asymptotic cone A_∞^τ. Also, the cone A_∞^τ is the smallest cone containing the set $\mathfrak{B}_A^\tau$. The definition of the set $\mathfrak{B}_A$ can be found in [22] for the weak topology τ. We denote by $\mathfrak{B}_A^{\|\cdot\|}$, $\mathfrak{B}_A$, $\mathfrak{B}_A^*$ the set $\mathfrak{B}_A^\tau$ for, respectively, norm, weak and *-weak topology in X. We have $\mathfrak{B}_A^s \subseteq \mathfrak{B}_A \subseteq \mathfrak{B}_A^*$. The following proposition is obvious.

Proposition 4.2. *Let X be a normed space and let $A \subseteq X$ and τ be a linear topology weaker than norm topology. Then $\mathfrak{B}_{\operatorname{cl} A}^\tau = \mathfrak{B}_A^\tau$ for the closure $\operatorname{cl} A$ in the norm topology.*

In [22] it was shown that $\operatorname{recc} A = A \dot{-} A = \operatorname{cone} \mathfrak{B}_A$ if the set A is closed and convex. Now we show the following proposition.

Proposition 4.3. *Let X be a normed space and let $A \subseteq X$ be a closed and convex subset of X. If $\operatorname{recc} A \neq \{0\}$, then $\operatorname{recc} A = \operatorname{cone} \mathfrak{B}_A^{\|\cdot\|}$.*

Proof. Let $\xi \in \operatorname{recc} A$, $\xi \neq 0$. By Lemma 3.5, there exist sequences $(x_n) \subseteq A$ and $(t_n) \subseteq \mathbb{R}$, $t_n \to 0^+$ such that $t_n x_n \to \xi$, hence $x_n/\|x_n\| = (t_n x_n)/(t_n\|x_n\|) \to \xi/\|\xi\|$. Therefore, $\xi/\|\xi\| \in \mathfrak{B}_A^{\|\cdot\|}$, so $\operatorname{recc} A \subseteq \operatorname{cone} \mathfrak{B}_A^{\|\cdot\|}$. □

Remark 4.4. Let us notice that $\mathfrak{B}_A^{\|\cdot\|} \neq \emptyset$ implies that $\operatorname{recc} A = A \dot{-} A \neq \{0\}$. Equivalently, $\operatorname{recc} A = \{0\}$ implies that $\mathfrak{B}_A^{\|\cdot\|} = \emptyset$.

Definition 4.5. Let X be a normed space, $A \subseteq X$ and τ be a linear topology weaker than norm topology. We say that A is τ-narrow if for any sequence $(x_n) \subseteq A$ with $\|x_n\| \to \infty$, we can choose a subsequence (x_{n_k}) with $x_{n_k}/\|x_{n_k}\|$ converging in the topology τ to a point other than the origin.

Every norm-bounded set and every subset of finite dimensional space X is narrow. On the other hand, some subsets of infinite dimensional space having trivial recession cone are not narrow, for example, the set A from the proof of Lemma 3.8 and the sets A and B from the proof of Theorem 3.11.

Let X be infinite dimensional and $A \subset X$. Denote $A' := \{x/\|x\| \mid x \in A\}$. If the closure of A' is sequentially compact in the topology τ (e.g., $\operatorname{cl} A'$ is compact in the weak topology) and A' can be strictly separated from the origin (by τ-closed hyperplane) then the set A is τ-narrow.

Consider $X = l^2$, $A = \{ne_n \mid n \in \mathbb{N}\}$, then the set $\operatorname{cl} A' = A' \cup \{0\}$ is weakly sequentially compact but A' cannot be separated from the origin. Any subsequence of $(e_n) \subset A'$ converges weakly to 0. On the other hand, if $X = l^1$, $A = \{ne_1 + ne_n \mid n \in \mathbb{N}\}$, then the set A' is separated from the origin but $\operatorname{cl} A'$ is not weakly compact. Notice that no subsequence of $\big((e_1 + e_n)/2\big)_n \subset A'$ is convergent. In these two examples, the set A is not weakly narrow.

The following proposition is an obvious implication of Proposition 4.2.

Proposition 4.6. *Let X be a normed space and let $A \subseteq X$ and τ be a linear topology weaker than norm topology. Let the set A be τ-narrow. Then the closure* $\operatorname{cl} A$ *in the norm topology is also τ-narrow.*

Proof. Let $(x_k) \subseteq \operatorname{cl} A$ be a sequence such that $\|x_k\|$ tends to ∞. Then $x_k = a_k + u_k$ for some sequences $(a_k) \subseteq A$ and $(u_k) \subseteq X$ such that $\|u_k\|$ tends to 0. Hence $\|a_k\| = \|x_k - u_k\|$ tends to ∞. There exists a subsequence (a_{k_l}) such that $a_{k_l}/\|a_{k_l}\|$ tends to some point $a \neq 0$ in the topology τ. We have the difference

$$\frac{x_{k_l}}{\|x_{k_l}\|} - \frac{a_{k_l}}{\|a_{k_l}\|} = \frac{u_{k_l}}{\|x_{k_l}\|} + \frac{a_{k_l}}{\|a_{k_l}\|}\frac{\|a_{k_l}\| - \|x_{k_l}\|}{\|x_{k_l}\|}$$

tending to the origin. Therefore, $x_{k_l}/\|x_{k_l}\|$ tends to $a \neq 0$. □

From Definition 4.5, it follows that if a unit ball in X is τ-sequentially compact, then a set A is τ-narrow if and only if $0 \notin \mathfrak{B}^{\tau}_{A}$.

Theorem 4.7. *Let X be a normed space and τ be a linear topology in X weaker than the norm topology. Let $A, B, C \subseteq X$, C be closed in τ and convex,* $\operatorname{recc} C$ *be pointed, i.e.,* $\operatorname{recc} C \cap (-\operatorname{recc} C) = \{0\}$, *$B$ be τ-narrow and $B^{\tau}_{\infty} \subseteq \operatorname{recc} C$. Then the inclusion $A + B \subseteq B + C$ implies that $A \subseteq C$.*

Proof. Assume that $0 \in B$ and $A = \{0\}$. Then by induction, we can prove that for every $n \in \mathbb{N}$ we have $0 \in B \subseteq B + \underbrace{C + \cdots + C}_{n} = B + nC$, hence $0 \in \bigcap_{n\in\mathbb{N}}(B/n + C)$. Then there exist $b_n \in B$ and $c_n \in C$ such that $0 = b_n/n + c_n$.

Case 1*: The sequence b_n/n is norm-bounded.* If (b_n) is norm-bounded than b_n/n tends to 0 and 0 is the limit of the sequence (c_n). If (b_n) is not norm-bounded, then applying the assumption that the set B is τ-narrow, for some subsequence (b_{n_k}) the sequence $b_{n_k}/\|b_{n_k}\|$ tends in the topology τ to some $b_0 \in B^{\tau}_{\infty} \setminus \{0\}$. The number sequence $\|b_{n_k}\|/n_k$ is bounded, and contains a subsequence $\|b_{n_{k_l}}\|/n_{k_l}$ converging to $t \geq 0$. Hence $b_{n_{k_l}}/n_{k_l} = b_{n_{k_l}}/\|b_{n_{k_l}}\| \cdot \|b_{n_{k_l}}\|/n_{k_l}$ converges, in the topology τ, to tb_0. Then $\lim_{l\to\infty} c_{n_{k_l}} = -tb_0 \in C$. Notice that $tb_0 \in B^{\tau}_{\infty} \subseteq \operatorname{recc} C$, and $0 = -tb_0 + tb_0 \in C$.

Case 2*: The sequence b_n/n is not bounded.* We may assume that $\|b_n/n\| \to \infty$ and write $0 = b_n/\|b_n\| + nc_n/\|b_n\|$. By assumption there exists a subsequence $b_{n_k}/\|b_{n_k}\|$ tending in the topology τ to some $b_0 \neq 0$ from B^{τ}_{∞} and, consequently, $\lim_{k\to\infty}(n_k c_{n_k})/\|b_{n_k}\| = -b_0$. We obtain $b_0 \in B^{\tau}_{\infty} \subseteq \operatorname{recc} C$ and, by Lemma 3.5, also $-b_0 \in C^{\tau}_{\infty} = \operatorname{recc} C$ which is impossible, since $\operatorname{recc} C$ is pointed.

We just have proved that the theorem holds in the case of $0 \in B$ and $A = \{0\}$. In general case let $A + B \subseteq B + C$ and $b \in B$. Consider any $x \in A$. Then $x + (B - b) \subseteq (B - b) + C$, and $\{0\} + (B - b) \subseteq (B - b) + (C - x)$. The sets $\{0\}$, $(B - b)$ and $(C - x)$

satisfy assumptions of the theorem in its proved case. Hence $0 \in (C - x)$, and $x \in C$. Therefore, $A \subseteq C$. $\square$

Lemma 4.8. *Let X be a locally convex space. Let $B \subseteq X$ and any $A, C \subseteq X$ such that C is closed and convex, the inclusion $A+B \subseteq B+C$ implies that $A \subseteq C$. Then the inclusion $A + B \subseteq \mathrm{cl}(B + C)$ implies that $A \subseteq C$.*

Proof. Notice that the implication $A + B \subseteq B + \mathrm{cl}(C + U) \implies A \subseteq \mathrm{cl}(C + U)$ holds true for any convex neighborhood U of the origin. Assume that $A + B \subseteq \mathrm{cl}(B + C)$. We obtain that $A+B \subseteq B+C+U \subseteq B+\mathrm{cl}(C+U)$ and, by the assumptions of the theorem, $A \subseteq \mathrm{cl}(C+U) \subseteq C+U+U$. Then $A+B \subseteq \mathrm{cl}(B+C)$ implies that A is contained in the intersection of all $C + 2U$. Therefore, $A \subseteq C$. $\square$

Corollary 4.9. *Let X be a normed space. Let τ be a locally convex topology which is equal to or weaker than the norm topology. Let $A, B, C \subseteq X$, C be closed in τ and convex, B be τ-narrow, $\operatorname{recc} C$ be pointed and $B^{\tau}_{\infty} \subseteq \operatorname{recc} C$. Then the inclusion $A+B \subseteq \mathrm{cl}_{\tau}(B+C)$ implies that $A \subseteq C$.*

A normed space X is said to be a WCF space (see [2]) if it contains a weakly compact subset K such that $X = \mathrm{cl}\,\mathrm{lin}\, K$.

Corollary 4.10. *Let A, B, C be subsets of a dual X^* to a WCF space X, C be *-weakly closed and convex, B be nonempty and B^*_{∞} be asymptotic cone with respect to *-weak topology. Assume that $0 \notin \mathfrak{B}^*_B$, and $B^*_{\infty} \subseteq \operatorname{recc} C$. If the cone $\operatorname{recc} C$ is pointed, then the inclusion $A + B \subseteq \mathrm{cl}^*(B + C)$ implies that $A \subseteq C$.*

Proof. Since X is WCF, by [2, Corollary 2] and Eberlein–Šmulian theorem [29], the unit ball in X^* is *-weakly sequentially compact, and B is *-weakly-narrow. Applying Corollary 4.9, we obtain our corollary. $\square$

Since every reflexive Banach space X is a dual of a WCF space and its *-weak topology coincides with the weak topology, the next theorem follows from Corollary 4.10.

Theorem 4.11. *Let A, B, C be subsets of a reflexive Banach space X, C be closed and convex, B be nonempty and B_{∞} be asymptotic cone with respect to weak topology. Assume that $0 \notin \mathfrak{B}_B$, $B_{\infty} \subseteq \operatorname{recc} C = V$ and V is a pointed cone. Then the inclusion $A + B \subseteq \mathrm{cl}_{\mathrm{weak}}(B + C)$ implies that $A \subseteq C$.*

The theorem generalizes Robinson's result [26, Lemma 1] in $\mathbb{R}^n$. The condition $0 \notin \mathfrak{B}_B$ in the theorem is crucial. Without it the theorem is false in every infinite dimensional Banach space. In Theorem 3.11, we have $\mathrm{recc}(A \dot{+} B) = [0, \infty)e_0$. Then $([0, \infty)e_0 + B) + A \subseteq A \dot{+} B$. Yet, $([0, \infty)e_0 + B) \not\subseteq B$. The reason is that even though $\operatorname{recc} A = \operatorname{recc} B = \{0\}$ and $\mathfrak{B}^s_A = \emptyset$, we have $0 \in \mathfrak{B}_A$.

Proposition 4.12. *Let X be a reflexive Banach space. Let $B, C \subseteq X$, B, C be closed and convex,* $(\operatorname{recc} B) \cap (-\operatorname{recc} C) = \{0\}$ *and* $0 \notin \mathfrak{B}_B$ *(B be narrow in the weak topology). Then the sum $B + C$ is closed.*

Proof. Let $x \in \operatorname{cl}(B + C)$. Then there exist sequences $(b_n) \subseteq B$, $(c_n) \subseteq C$ and $(u_n) \subseteq X$, $\|u_n\| \to 0^+$ such that $b_n + c_n + u_n = x$. First, if $\|b_n\|$ is bounded then some subsequence (b_{n_k}) weakly tends to $b \in B$. Hence c_{n_k} tends to $x - b \in C$, and $x \in B + C$. Second, if $\|b_n\|$ tends to infinity then some subsequence $b_{n_k}/\|b_{n_k}\|$ weakly tends to some $b \in \operatorname{recc} B$, $b \neq 0$. Hence $c_{n_k}/\|c_{n_k}\|$ tends to some $-b \in \operatorname{recc} C$, which contradicts the assumption that the cone V is pointed. □

Proposition 4.12 is also a consequence of Dieudonné's theorem [10].

Proposition 4.13. *Let X be a normed vector space. Let $B, C \subseteq X$, B, C be closed and convex,* $\operatorname{recc} B = \operatorname{recc} C$ *be pointed, and B be weakly-narrow. Then* $\operatorname{recc} \operatorname{cl}(B+C) = \operatorname{recc} C$.

Proof. Let $x \in \operatorname{recc} \operatorname{cl}(B+C), x \neq 0$. Then $b_0 + c_0 + [0, \infty)x \subseteq \operatorname{cl}(B+C)$ for some $b_0 \in B$, $c_0 \in C$. There exist sequences $(b_n) \subseteq B$, $(c_n) \subseteq C$ and $(u_n) \subseteq X$, $\|u_n\| \to 0^+$ such that $b_n + c_n + u_n = b_0 + c_0 + nx$. Notice that

$$x = \frac{b_n}{n} + \frac{c_n}{n} + \frac{u_n}{n} - \frac{b_0}{n} - \frac{c_0}{n}.$$

First, if $\|b_n\|/n$ tends to 0, then c_n/n tends to x, and $x \in C_\infty = \operatorname{recc} C$. Second, if $\|b_n\|/n$ tends to $\beta \in (0, \infty)$, then some subsequence b_{n_k}/n_k tends to $\beta b \in \operatorname{recc} B$, where $b_{n_k}/\|b_{n_k}\|$ weakly tends to some point $b \neq 0$. Hence c_{n_k}/n_k tends to $x - \beta b \in \operatorname{recc} C$, and $x \in \operatorname{recc} B + \operatorname{recc} C = \operatorname{recc} C$.

Third, if $\|b_n\|/n$ tends to infinity, then again some subsequence $b_{n_k}/\|b_{n_k}\|$ weakly tends to a nonzero element $b \in \operatorname{recc} B$. Notice that also $(b_0 + c_0 - u_{n_k} + n_k x - b_{n_k})/\|b_{n_k}\|$ tends to $-b$ while the fraction $\|b_0 + c_0 - u_{n_k} + n_k x - b_{n_k}\|/\|b_{n_k}\|$ tends to 1. Then $c_{n_k}/\|c_{n_k}\| = (b_0 + c_0 - u_{n_k} + n_k x - b_{n_k})/\|b_0 + c_0 - u_{n_k} + n_k x - b_{n_k}\|$ tends to $-b$ and $-b \in \operatorname{recc} C$, which contradicts the assumption that the cone $\operatorname{recc} C$ is pointed. □

Obviously, the assumption of weak-narrowness of one of the sets in the last proposition is essential.

Theorem 4.14. *Let X be a normed space and τ be a linear topology weaker than the norm topology. Let subsets $B, C \subseteq X$ be τ-narrow and $B^\tau_\infty \cap (-C^\tau_\infty) = \{0\}$. Then the set $\operatorname{cl}_{\|\cdot\|}(B + C)$ is τ-narrow.*

Proof. Let a $(x_k) \subseteq B + C$, $\|x_k\|$ tends to ∞. For some sequences $(b_k) \subseteq B$ and $(c_k) \subseteq C$, we have $x_k = b_k + c_k$.

First, assume that the sequence $(\|b_k\|)$ is bounded. Hence $\|c_k\| \to \infty$. For some subsequence (c_{k_l}) terms $c_{k_l}/\|c_{k_l}\|$ tend to some $c \neq 0$ in the topology τ. Also $x_{k_l}/\|x_{k_l}\| - c_{k_l}/\|c_{k_l}\| = x_{k_l}/\|x_{k_l}\| - c_{k_l}/\|x_{k_l}\| + c_{k_l}/\|x_{k_l}\| - c_{k_l}/\|c_{k_l}\| = b_{k_l}/\|x_{k_l}\| + (\|c_{k_l}\| - \|x_{k_l}\|)/\|x_{k_l}\| \cdot c_{k_l}/\|c_{k_l}\|$. Since the sequence $(\|x_{k_l} - c_{k_l}\| = \|b_{k_l}\|)$ is bounded, a term $(\|c_{k_l}\| - \|x_{k_l}\|)/\|x_{k_l}\|$ tends to 0. Since $c_{k_l}/\|c_{k_l}\|$ is convergent in the topology τ, the sequence $x_{k_l}/\|x_{k_l}\| - c_{k_l}/\|c_{k_l}\|$ tends to 0. Then the sequence $x_{k_l}/\|x_{k_l}\|$ tends to $c \neq 0$.

Second, let $\|b_k\| \to \infty$ and $\|c_k\| \to \infty$. By assumption there exists a sequence (k_l) such that $b_{k_l}/\|b_{k_l}\| \to b \neq 0$, $c_{k_l}/\|c_{k_l}\| \to c \neq 0$ and $\|b_{k_l}\|/(\|b_{k_l}\| + \|c_{k_l}\|) \to t \in [0,1]$. If $(\|b_{k_l}\| + \|c_{k_l}\|)/\|x_{k_l}\| \to \infty$, then

$$\frac{x_{k_l}}{\|b_{k_l}\| + \|c_{k_l}\|} = \frac{b_{k_l}}{\|b_{k_l}\|}\frac{\|b_{k_l}\|}{\|b_{k_l}\| + \|c_{k_l}\|} + \frac{c_{k_l}}{\|c_{k_l}\|}\frac{\|c_{k_l}\|}{\|b_{k_l}\| + \|c_{k_l}\|}$$

is convergent and tends to $0 = tb + (1-t)c$. Then $b = 0$ or $c = 0$ or $0 \neq tb = (t-1)c \in B_\infty^\tau \cap (-C_\infty^\tau)$ which contradicts the assumptions.

Therefore, we have $(\|b_{k_l}\| + \|c_{k_l}\|)/\|x_{k_l}\| \to r \in [1, \infty)$. Then the term

$$\frac{x_{k_l}}{\|x_{k_l}\|} = \left(\frac{b_{k_l}}{\|b_{k_l}\|}\frac{\|b_{k_l}\|}{\|b_{k_l}\| + \|c_{k_l}\|} + \frac{c_{k_l}}{\|c_{k_l}\|}\frac{\|c_{k_l}\|}{\|b_{k_l}\| + \|c_{k_l}\|}\right)\frac{\|b_{k_l}\| + \|c_{k_l}\|}{\|x_{k_l}\|}$$

tends to $r(tb + (1-t)c) \neq 0$.

We have just proved that $B + C$ is τ-narrow. By Proposition 4.6, the set $\mathrm{cl}_{\|\cdot\|}(B + C)$ is τ-narrow. $\square$

Corollary 4.15. *Let X be a reflexive Banach space. Let $B, C \subseteq X$ be closed and convex sets sharing a pointed recession cone* $\operatorname{recc} B = \operatorname{recc} C$. *Assume that $0 \notin \mathfrak{B}_B \cup \mathfrak{B}_C$. Then $0 \notin \mathfrak{B}_{\mathrm{cl}(B+C)}$, where* cl *is the weak closure.*

Proof. Notice that for reflexive spaces the unit ball in X is weakly sequentially compact. Hence the condition $0 \notin \mathfrak{B}_B \cup \mathfrak{B}_C$ implies that B and C are weakly narrow. Since their asymptotic cones are contained in one pointed recession cone, the assumptions of Theorem 4.14 are fulfilled for the weak topology τ. Therefore, the set $\mathrm{cl}_{\|\cdot\|}(B + C) = \mathrm{cl}(B + C)$ is weakly narrow and $0 \notin \mathfrak{B}_{\mathrm{cl}(B+C)}$. $\square$

Theorem 4.16. *Let X be a normed vector space and $V \subseteq X$ be a closed convex pointed cone. Then the family of all weakly narrow closed convex subsets of X sharing the recession cone V with modified Minkowski addition $\dot{+}$ defined by $A \dot{+} B := \mathrm{cl}(A + B)$ and multiplication by non-negative numbers (we assume $0 \cdot A := V$) is an abstract convex cone. Moreover, this abstract convex cone has a neutral, with respect to $\dot{+}$, element V and satisfies the cancellation law.*

Proof. First, notice that for any two weakly narrow closed convex sets A and B with a common pointed recession cone V the set $\mathrm{cl}(A+B)$ is weakly narrow by Theorem 4.14 and the recession cone of $\mathrm{cl}(A+B)$ is equal to V by Proposition 4.13. Then the family of all weakly narrow closed convex sets is closed with respect to the addition $\dot{+}$. Checking conditions for an abstract convex cone is straightforward. The cancellation law in the family of all weakly narrow closed convex sets follows from Theorem 4.11. $\square$

In view of Proposition 4.12, we can formulate the following corollary.

Corollary 4.17. *Let X be a reflexive Banach space and V be a closed convex pointed cone. Then the family of all subsets $A \subseteq X$ sharing the recession cone V and satisfying the condition $0 \notin \mathfrak{B}_A$ with Minkowski addition and multiplication by non-negative numbers (we assume $0 \cdot A := V$) is an abstract convex cone having a neutral element V and satisfying cancellation law.*

Let X be a normed vector space and $V \subseteq X$ be a closed convex pointed cone. Then by Theorem 4.16, we can embed the family $\mathcal{C}_V^n(X)$ of all weakly narrow closed convex subsets of X sharing the recession cone V in a Minkowski–Rådström–Hörmander vector space $(\mathcal{C}_V^n(X))^2/\sim$. Moreover, by [17, Theorem 2.5] if, additionally, X is a reflexive Banach space and the set $\{x \in V \mid \|x\| = 1\}$ can be strictly separated from the origin then every quotient class $[A, B] \in (\mathcal{C}_V^n(X))^2/\sim$ has a minimal representative, i.e., an inclusion-minimal pair in the family of all pairs $(C, D) \in [A, B]$ such that $0 \in C$.

5. Order cancellation law in topological vector spaces

In this section we prove order cancellation law for topological vector space where we cancel by closed and convex sets. These results generalize theorems obtained by Tabor and Bielawski in [4].

Let X be a topological vector space and let $V \subseteq X$ be a closed and convex cone. By $\mathcal{C}(X)$ we denote the family of al nonempty, closed and convex subsets of X and by $\mathcal{B}(X)$ the family of all nonempty, bounded, closed and convex subsets of X. Now let $\mathcal{C}_V^0(X) = \{A \in \mathcal{C}(X) \mid V \subseteq \operatorname{recc} A\}$ and

$$\mathcal{C}_V(X) = \{A \in \mathcal{C}(X) \mid A \subseteq V \dot{+} B, V \subseteq A \dot{+} B \text{ for some } B \in \mathcal{B}(X)\}.$$

We may look at elements of $\mathcal{C}_V(X)$ as sets of bounded Hausdorff-like distance from the cone V. The family $\mathcal{C}_V(X)$ is a subfamily of all 'bounded' elements of $\mathcal{C}_V^0(X)$. For a sequence $(A_n) \subseteq \mathcal{C}_V^0(X)$, we define the limit operator as follows: $\lim_{n\to\infty}(A_n) := A \in \mathcal{C}_V^0(X)$ if and only if for every neighborhood U of zero in X there exists $k \in \mathbb{N}$ such that $A \subseteq A_n + U$ and $A_n \subseteq A + U$ for $n \geq k$.

Moreover, the family $\mathcal{C}_V^0(X)$ with the addition $A \dot{+} B = \mathrm{cl}(A+B)$ and the multiplication by non-negative numbers defined by $\lambda A := \{\lambda a \mid a \in A\}$, $0 \cdot A := V$ is an abstract convex cone.

Remark 5.1. The algebraic system $(\mathcal{C}_V^0(X), \cdot, \dot{+}, \subseteq, \lim)$ satisfies the axioms (S1)–(S11).

Now, we prove two lemmas concerning the family $\mathcal{C}_V(X)$.

Lemma 5.2. *For any $A \in \mathcal{C}_V(X)$, we have* $\mathrm{recc}\, A = V$.

Proof. Take any $A \in \mathcal{C}_V(X)$. First, we prove that $V \subseteq \mathrm{recc}\, A$. Let us observe that for every $n \in \mathbb{N}$, we have $V = \frac{1}{n}V \subseteq \frac{1}{n}A \dot{+} \frac{1}{n}B$ for some $B \in \mathcal{B}(X)$. Now, take arbitrary $a \in A$, $v \in V$. For any neighborhood U of the origin, let W be another neighborhood of the origin such that $W + W + W \subseteq U$. Let also $m \in \mathbb{N}$ be large enough that $\frac{a}{m} \in W$ and $\frac{1}{m}B \subseteq W$. Then for some $a_1 \in A$ and $b_1 \in B$, we have

$$\begin{aligned} a + v \in a + \frac{1}{m}a_1 + \frac{1}{m}b_1 + W &= \left[\frac{1}{m}a_1 + \left(1 - \frac{1}{m}\right)a\right] + \frac{1}{m}a + \frac{1}{m}b_1 + W \\ &\subseteq A + W + W + W \subseteq A + U. \end{aligned}$$

Hence $A + V \subseteq \mathrm{cl}\, A = A$. Then $V \subseteq \mathrm{recc}\, A$.

To prove the inverse inclusion, observe that $A \subseteq V \dot{+} B$ for some $B \in \mathcal{B}(X)$. Take any $a \in A$. We have

$$\mathrm{recc}\, A = \mathrm{recc}(A - a) = \bigcap_n \frac{A - a}{n} \subseteq \bigcap_n \frac{V \dot{+} B - a}{n} = \bigcap_n \left(V \dot{+} \frac{B - a}{n}\right) \subseteq \mathrm{cl}\, V = V,$$

and the proof is complete. □

Lemma 5.3. *For any $A \in \mathcal{C}_V(X)$, we have* $\lim_{n\to\infty} 2^{-n}A = V$.

Proof. Let $B \in \mathcal{B}(X)$ be such that $A \subseteq V \dot{+} B$, $V \subseteq A \dot{+} B$. Take any neighborhood U of zero in X and let W be a balanced neighborhood of zero such that $W + W \subseteq U$ and let $k \in \mathbb{N}$ be such that $2^{-n}B \subseteq W$ for $n \geq k$. Then we have $2^{-n}A \subseteq 2^{-n}(V + B + W) \subseteq V + W + W \subseteq V + U$ and, analogously, $V = 2^{-n}V \subseteq 2^{-n}A + U$. Thus $\lim_{n\to\infty} 2^{-n}A = V$. □

The following theorems follow from the above considerations and Section 2.

Theorem 5.4. *Let $A, B, C \in \mathcal{C}_V^0(X)$ and suppose that $\lim_{n\to\infty} 2^{-n}B$ exists and is contained in V then the inclusion $A \dot{+} B \subseteq B \dot{+} C$ implies $A \subseteq C$.*

Proof. In order to apply Lemma 2.3, notice that convexity of a closed convex set C implies the convexity in the sense of Definition 2.2. □

The next corollary from Theorem 5.4 and Lemma 5.3 is straightforward.

Corollary 5.5. *Let $A, B, C \in \mathcal{C}_V(X)$. Then the inclusion $A \dot{+} B \subseteq B \dot{+} C$ implies $A \subseteq C$.*

Tabor and Bielawski [4] proved a version of order cancellation law for closed convex sets having finite Hausdorff distance from a fixed convex cone V in a normed space X. Corollary 5.5 is another version of order cancellation law which generalizes to topological spaces a result obtained by Tabor and Bielawski.

In the Cartesian product $\mathcal{C}_V(X) \times \mathcal{C}_V(X)$, due to cancellation law we can introduce an equivalence relation $\sim$ in an usual way:

$$(A, B) \sim (C, D) \iff A \dot{+} D = B \dot{+} C.$$

Let $\widetilde{\mathcal{C}}_V(X) = \mathcal{C}_V(X) \times \mathcal{C}_V(X)/\sim$ be the quotient set of the equivalence classes with respect to $\sim$. By $[A, B] \in \widetilde{\mathcal{C}}_V(X)$, we denote the equivalence class of an element $(A, B) \in \mathcal{C}_V(X) \times \mathcal{C}_V(X)$.

The semigroup $\mathcal{C}_V(X)$ with the addition $\dot{+}$ and scalar multiplication defined by $\lambda A := \{\lambda a : a \in A\}$ for $\lambda > 0$ and $0 \cdot A = V$ is an abstract convex cone. Thanks to Corollary 5.5, it can be proved that the quotient set $\widetilde{\mathcal{C}}_V(X)$ with addition

$$[A, B] + [C, D] := [A \dot{+} C, B \dot{+} D]$$

and scalar multiplication defined by

$$\lambda[A, B] := \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0, \\ [(-\lambda)B, (-\lambda)A] & \text{if } \lambda \leq 0 \end{cases}$$

is a real vector space.

Moreover, the function $j \colon \mathcal{C}_V(X) \to \widetilde{\mathcal{C}}_V(X)$ defined by $j(A) = [A, V]$ is a canonical embedding which satisfies $j(A \dot{+} B) = j(A) + j(B)$ and $j(\lambda A) = \lambda j(A)$ for all $A, B \in \mathcal{C}_V(X)$ and $\lambda \geq 0$.

Now, let $\mathcal{W}$ be a basis of balanced neighbourhoods of 0 in topological vector space X. For $U \in \mathcal{W}$, we define by W_U the set

$$\{(A, B) \in (\mathcal{C}_V(X))^2 : A \subseteq B \dot{+} K, B \subseteq A \dot{+} K \text{ for some } K \in \mathcal{C}(X), K \subseteq U\}.$$

The family $\{W_U\}_{U \in \mathcal{W}}$ forms a basis of uniformity on $\mathcal{C}_V(X)$. The uniformity induces a topology in $\mathcal{C}_V(X)$. Let us denote this topology by τ_H. The scalar multiplication $\cdot$ and the addition $\dot{+}$ in $\mathcal{C}_V(X)$ are continuous in the topology τ_H.

We define a basis of neighbourhoods of 0 in the space $\widetilde{\mathcal{C}}_V(X)$ as follows. Let $U \in \mathcal{W}$ and let

$$\mathcal{T}_U = \{[A, B] \in \widetilde{\mathcal{C}}_V(X) : A \subseteq B \dot{+} K, B \subseteq A \dot{+} K \text{ for some } K \in \mathcal{C}(X), K \subseteq U\}.$$

The family $\{\mathcal{T}_U\}_{U\in\mathcal{W}}$ forms a basis of neighbourhoods of 0 in the space $\widetilde{\mathcal{C}}_V(X)$ and defines a linear topology on the space the space $\widetilde{\mathcal{C}}_V(X)$. Denote this topology by σ_H.

Now, we can express the following theorem which generalizes embedding theorem obtained by Tabor and Bielawski for normed spaces [4].

Theorem 5.6. *The canonical mapping $j\colon \mathcal{C}_V(X) \to j(\mathcal{C}_V(X)) \subseteq \widetilde{\mathcal{C}}_V(X)$ is an isomorphic embedding of a topological abstract convex cone $(\mathcal{C}_V(X), \dot{+}, \cdot, \tau_H)$ into a real topological vector space $(\widetilde{\mathcal{C}}_V(X), +, \cdot, \sigma_H)$. Moreover, the canonical mapping $j\colon \mathcal{C}_V(X) \to j(\mathcal{C}_V(X)) \subseteq \widetilde{\mathcal{C}}_V(X)$ is a homeomorphism.*

The results from this section apply not only to locally convex spaces but also to topological vector spaces like l^p, L^p, $0 < p < 1$. In the case of locally convex vector spaces the introduced topology in $\mathcal{C}_V(X)$ coincides with 'Hausdorff' topology, where sets of families $\mathcal{T}_{A,U} := \{B \in \mathcal{C}_V(X) \mid A \subseteq B \dot{+} U, B \subseteq A \dot{+} U\}$, $U \in \mathcal{W}$ form basis of neighborhoods of $A \in \mathcal{C}_V(X)$. In spaces which are not locally convex a problem arises because belonging $B \in \mathcal{T}_{A,U}$ does not imply that $B_1 \in \mathcal{T}_{A_1,U}$ for all $(A_1, B_1) \in [A, B]$. Notice that $A \subseteq B \dot{+} U \implies A \dot{+} C \subseteq B \dot{+} C \dot{+} U$, $C \in \mathcal{C}_V(X)$ but having the inclusion $A \dot{+} C \subseteq B \dot{+} C \dot{+} U$ we may not be able to cancel C, since the set $B \dot{+} C \dot{+} U$ may not be convex. Then the Minkowski–Rådström–Hörmander space $\mathcal{C}_V(X) \times \mathcal{C}_V(X)/\sim$ does not posses a topology compatible with the topology of neighborhoods $\mathcal{T}_{A,U}$.

6. Conclusions

Rådström, Hörmander and Urbański embedded a semigroup of bounded closed convex subsets of a topological vector space into a quotient vector space called a Minkowski–Rådström–Hörmander space. Robinson (for all subsets of Euclidean space) and Bielawski and Tabor (for subsets with finite Hausdorff distance from their recession cone in normed space) obtained similar results in the case of a semigroup of unbounded closed convex sets sharing recession cone. The main difficulty was to prove an order cancellation law, because one needs to cancel unbounded sets.

Robinson's result cannot be extended directly to infinite dimensional spaces, since the recession cone of Minkowski sum of two sets need not to be equal to the sum of their recession cones (see Example 3.9, Corollary 3.12, Remark 3.14). In order to overcome the difficulty, we introduced the notion of narrow sets which allowed us to extend an order cancellation law (Theorem 4.7) beyond a semigroup of sets having finite Hausdorff distance from their recession cone.

In the definition of narrow sets the existence of a norm is essential. We still do not know whether the notion of narrow sets can be extended to locally convex or metrizable spaces. Can the property of narrowness be weakened without loosing canceling property?

Is it possible to introduce a metric or topology in the Minkowski–Rådström–Hörmander space obtained by Robinson?

We also extended Bielawski and Tabor result beyond normed space by extending the idea of finite Hausdorff distance to subsets of any topological vector space.

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