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A Parallel Algorithm for Generalized Multiple-set Split Feasibility with Application to Optimal Control Problems

Nguyen Thi Thu Thuy* and Nguyen Trung Nghia

Abstract. In this paper, we concentrate on the generalized multiple-set split feasibility problems in Hilbert spaces and propose a new iterative method for this problem. One of the most important of this method is using dynamic step-sizes, in which the information of the previous step is the only requirement to compute the next approximation. The strong convergence result of the suggested algorithm is proven theoretically under some feasible assumptions. When considering the main results in some special cases, we also obtain some applications regarding the solution of the multiple-set split feasibility problem, the split feasibility problem with multiple output sets, and the split feasibility problem as well as the linear optimal control problem. Some numerical experiments on infinite-dimensional spaces and applications in optimal control problems are conducted to demonstrate the advantages and computational efficiency of the proposed algorithms over some existing results.

1. Introduction

Let $\mathcal{H}_1, \mathcal{H}_2^1, \ldots, \mathcal{H}_2^N$ be $(N+1)$ real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The purpose of this paper is to construct several iterative algorithms to solve the following generalized multiple-set split feasibility problem

$$\text{(GSFP)} \qquad \text{find } x^* \in \Omega := \left(\bigcap_{i=1}^{M} \mathcal{C}_i \right) \cap \left(\bigcap_{j=1}^{N} \mathcal{A}_j^{-1} \left(\bigcap_{k=1}^{L} \mathcal{Q}_j^k \right) \right),$$

where M, N and L are positive integer numbers, $\mathcal{C}_i$ is a closed convex subset of $\mathcal{H}_1$, $i = 1, \ldots, M$, $\mathcal{Q}_j^1, \ldots, \mathcal{Q}_j^L$ are L closed convex subsets of $\mathcal{H}_2^j$, and $\mathcal{A}_j \colon \mathcal{H}_1 \to \mathcal{H}_2^j$ is a bounded linear operator, $j = 1, \ldots, N$. A more general form of (GSFP) is (GP) in [20].

Special cases. (a) When $N = 1$, $\mathcal{H}_2 := \mathcal{H}_2^1$, $\mathcal{A} := \mathcal{A}_1$ and $\mathcal{Q}_k := \mathcal{Q}_1^k$, $k = 1, \ldots, L$, we have the multiple-set split feasibility problem (MSFP) that is a general way to characterize the

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*Corresponding author.

inverse problem of intensity-modulated radiation therapy. The MSFP introduced first by Censor et al. [8] is to

$$\text{(MSFP)} \qquad \text{find } x^* \in \Omega_1 = \left(\bigcap_{i=1}^{M} \mathcal{C}_i\right) \cap \left(\mathcal{A}^{-1}\left(\bigcap_{k=1}^{L} \mathcal{Q}_k\right)\right).$$

The CQ–algorithm has been extended by several authors in order to solve (MSFP) (see the papers by Censor and Segal [10], Censor et al. [8], Masad and Reich [17], Xu [25, 26], and others). Many iterative projection methods for solving (MSFP) have been developed (see, for example, [2, 4, 9, 15, 22, 23] and references therein).

To solve (MSFP), Anh [2] proved in 2017 that the following parallel algorithm

$$\text{(1.1)} \quad \begin{cases} x^0 \in \mathcal{H}_1, \ \text{ any element}, \\ u^n = \mathcal{A}x^n, \ \ v^n := P_{\mathcal{Q}_{k_n}} u^n, \ \text{ where } k_n = \arg\max\left\{\|P_{\mathcal{Q}_k} u^n - u^n\| \mid k = 1, \dots, L\right\}, \\ y^n = x^n - \gamma_n \mathcal{A}^*(u^n - v^n), \ \ z^n := P_{\mathcal{C}_{i_n}} y^n, \\ \quad \text{where } i_n = \arg\max\left\{\|P_{\mathcal{C}_i} y^n - y^n\| \mid i = 1, \dots, M\right\}, \\ x^{n+1} = \eta_n x^n + (1-\eta_n) z^n - \alpha_n \mu F(z^n), \ \ n \geq 0, \end{cases}$$

converges strongly to $x^* \in \Omega_1$ which is the unique solution of the following variational inequality problem

$$\text{(VIP)} \qquad \langle (I^{\mathcal{H}_1} - \mathcal{F})x^*, x - x^* \rangle \geq 0, \quad \forall\, x \in \Omega_1,$$

where $I^{\mathcal{H}_1}$ is the identity operator on $\mathcal{H}_1$, $\mathcal{F}\colon \mathcal{H}_1 \to \mathcal{H}_1$, $F := I^{\mathcal{H}_1} - \mathcal{F}$ is a χ-strongly monotone and ξ-Lipschitz continuous mapping, $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$, under the following conditions

$$\text{(C1)} \qquad \{\alpha_n\} \subset (0,1), \ \ \forall\, n \geq 0, \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$$

$$\text{(C2)} \qquad 0 < \mu < \frac{2\chi}{\xi^2},$$

$$\text{(C3)} \qquad 0 < a_1 \leq \gamma_n \leq b_1 < \frac{2}{\|\mathcal{A}\|^2 + 1},$$

$$\text{(C4)} \qquad \{\eta_n\} \subset [0,1) \text{ such that } 0 \leq \eta_n \leq 1 - \alpha_n, \ \forall\, n \geq 0, \quad \lim_{n\to\infty} \eta_n = \eta < 1.$$

At that time, Buong [4] introduced the following algorithm

$$\text{(1.2)} \qquad \begin{cases} x^0 \in \mathcal{H}_1, \quad \text{any element}, \\ x^{n+1} = (I^{\mathcal{H}_1} - \alpha_n \mu F) T_1 \left[I^{\mathcal{H}_1} - \gamma \mathcal{A}^*(I^{\mathcal{H}_2} - T_2)\mathcal{A}\right] x^n, \quad n \geq 0, \end{cases}$$

where $T_1 = \sum_{i=1}^{M} \eta_i P_{\mathcal{C}_i}$ and $T_2 = \sum_{k=1}^{L} \beta_k P_{\mathcal{Q}_k}$ (or $T_1 = P_{\mathcal{C}_M} \cdots P_{\mathcal{C}_1}$ and $T_2 = P_{\mathcal{Q}_L} \cdots P_{\mathcal{Q}_1}$). He proved that the algorithm (1.2) converges strongly to $x^* \in \Omega_1$ which is the unique solution of (VIP), under the conditions (C1), (C2) and

$$\text{(C3}'\text{)} \qquad 0 < \gamma < \frac{1}{\|\mathcal{A}\|^2},$$

$$\text{(C4}'\text{)} \qquad \eta_i > 0 \text{ such that } \sum_{i=1}^{M} \eta_i = 1, \quad \beta_k > 0 \text{ such that } \sum_{k=1}^{L} \beta_k = 1.$$

(b) When $M = 1$, $L = 1$, $\mathcal{C} := \mathcal{C}_1$ and $\mathcal{Q}_j := \mathcal{Q}_j^1$, we have the split feasibility problem with multiple output sets, that introduced by Reich et al. [21] is to

$$\text{(SMOS)} \qquad \text{find } \; x^* \in \Omega_2 := \mathcal{C} \cap \left(\bigcap_{j=1}^{N} \mathcal{A}_j^{-1} \mathcal{Q}_j \right).$$

They introduced a new iterative method by using an optimization approach

$$\text{(1.3)} \quad \begin{cases} x^0 \in \mathcal{H}_1, \quad \text{any element}, \\ x^{n+1} = \alpha_n \mathcal{F}(x^n) + (1-\alpha_n) P_{\mathcal{C}} \left[x^n - \gamma_n \sum_{j=1}^{N} \mathcal{A}_j^* (I^{\mathcal{H}_2^j} - P_{\mathcal{Q}_j}) \mathcal{A}_j x^n \right], \quad n \geq 0, \end{cases}$$

where $\mathcal{F}\colon \mathcal{H}_1 \to \mathcal{H}_1$ is a contraction mapping with the contraction coefficient $\tau \in [0,1)$. They proved that the method (1.3) converges strongly to $x^* \in \Omega_2$ which is the unique solution to (VIP) with Ω_1 replaced by Ω_2 under the conditions (C1) and

$$\text{(C3}''\text{)} \qquad 0 < a_2 \leq \gamma_n \leq b_2 < \frac{2}{N \max_{j=1,\ldots,N} \left\{ \|\mathcal{A}_j\|^2 \right\}} \quad \text{for all } n \geq 0.$$

(c) When $M = N = L = 1$, $\mathcal{C} := \mathcal{C}_1$, $\mathcal{Q} := \mathcal{Q}_1^1$, $\mathcal{H}_2 := \mathcal{H}_2^1$ and $\mathcal{A} := \mathcal{A}_1$, we have the two-set split feasibility problem that introduced first by Censor and Elfving [7] is to

$$\text{(SFP)} \qquad \text{find } \; x^* \in \Omega_3 := \mathcal{C} \cap \mathcal{A}^{-1} \mathcal{Q}.$$

In 2017, Anh et al. [1] considered the following algorithm

$$\text{(1.4)} \qquad x^{n+1} = (1-\alpha_n) P_{\mathcal{C}} \big(x^n + \gamma_n \mathcal{A}^* (P_{\mathcal{Q}} \mathcal{A} x^n - \mathcal{A} x^n) \big), \quad n \geq 0,$$

which converges strongly to the minimum-norm solution of (SFP) with the sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy the conditions (C1) and

$$\text{(C3}'''\text{)} \qquad 0 < a_3 \leq \gamma_n \leq b_3 < \frac{1}{\|\mathcal{A}\|^2 + 1} \quad \text{for all } n \geq 0.$$

In the present article, our aim is to introduce a new iterative algorithm to solve (VIP) with Ω_1 replaced by Ω, the solution-set of (GSFP), by using the viscosity approximation method [18], the selective technique [12, 14], and a modification of the CQ–algorithm. We prove the strong convergence of the presented algorithm which does not need any prior information on the norm of the operators $\mathcal{A}_j$, $j = 1, \ldots, N$ as (C3), (C3′), (C3″), (C3‴). Also, the new algorithms do not require knowing the Lipschitz constant ξ and the strongly monotone constant χ of the involving mapping. Moreover, our algorithm use dynamic step-sizes, chosen based on information of the previous steps.

The remaining part of this paper is organized as follows. Section 2 displays some lemmas that will be used for the validity and convergence of the algorithm. Section 3 is devoted to the description of our proposed algorithm and its strong convergence result. In Section 4, we applied our results to the study of finding a solution of (MSFP), (SMOS) and (SFP). In Section 5, we illustrate the proposed method by considering some numerical experiments. Finally, we consider an application of the proposed method to linear discrete optimal control problems.

2. Preliminaries

In this section, we introduce some definitions and facts which can be used in the proof of our main result.

Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. In what follows, we write $x^n \rightharpoonup x$ to indicate that the sequence $\{x^n\}$ converges weakly to x while $x^n \to x$ indicates that the sequence $\{x^n\}$ converges strongly to x. Let $\mathcal{C}$ be a nonempty, closed and convex subset of $\mathcal{H}$. We know that, for each $x \in \mathcal{H}$, there is a unique point $P_{\mathcal{C}}x \in \mathcal{C}$ such that

$$\|x - P_{\mathcal{C}}x\| = \inf_{u \in \mathcal{C}} \|x - u\|.$$

The mapping $P_{\mathcal{C}} \colon \mathcal{H} \to \mathcal{C}$ is called the metric projection from $\mathcal{H}$ onto $\mathcal{C}$ and has the following property

$$\langle x - P_{\mathcal{C}}x, y - P_{\mathcal{C}}x \rangle \leq 0, \quad \forall\, y \in \mathcal{C}, \tag{2.1}$$

(see, for example, [13, Section 3]).

Recall that a mapping $\mathcal{T} \colon \mathcal{C} \to \mathcal{C}$ is said to be ξ-Lipschitz continuous if

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \xi \|x - y\| \quad \text{for all } x, y \in \mathcal{C},$$

where ξ is a positive constant. $\mathcal{T}$ is said to be a contraction operator if $\xi \in [0, 1)$, and nonexpansive if $\xi = 1$. We denote the set of fixed points of $\mathcal{T}$ by $\mathrm{Fix}(\mathcal{T})$, that is, $\mathrm{Fix}(\mathcal{T}) = \{x \in \mathcal{C} \mid \mathcal{T}x = x\}$. It follows from (2.1) that the metric projection $P_{\mathcal{C}}$ is a nonexpansive mapping and that $\mathrm{Fix}(P_{\mathcal{C}}) = \mathcal{C}$.

A mapping $\mathcal{F}\colon \mathcal{H}_1 \to \mathcal{H}_1$ is said to be χ-strongly monotone on $\mathcal{H}_1$ if there exists $\chi > 0$ such that

$$\langle \mathcal{F}x - \mathcal{F}y, x - y\rangle \geq \chi\|x - y\|^2 \quad \text{for all } x, y \in \mathcal{H}_1.$$

It is easy to see that if $\mathcal{F}$ is a contraction mapping with the contraction coefficient $\tau \in [0, 1)$, $I^{\mathcal{H}_1} - \mathcal{F}$ is ξ-Lipschitz continuous and χ-strongly monotone operator on $\mathcal{H}_1$ with $\xi = (1+\tau)$ and $\chi = (1 - \tau)$. So, if Ω is a nonempty closed convex subset of $\mathcal{H}_1$, then (VIP) has a unique solution. Moreover, from (2.1), a point $x^* \in \mathcal{H}_1$ is a solution of (VIP) if and only if $x^* = P_\Omega \mathcal{F} x^*$.

Lemma 2.1. (Opial's lemma, [19]) *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ be a nonexpansive mapping with $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. If $\{x^n\}$ is a sequence in $\mathcal{C}$ converging weakly to x^* and if the sequence $\{(I^{\mathcal{H}} - \mathcal{T})x^n\}$ converges strongly to y, then $(I^{\mathcal{H}} - \mathcal{T})x^* = y$. In particular, if $y = 0$, then $x^* \in \mathrm{Fix}(\mathcal{T})$.*

Lemma 2.2. [3] *Let $\mathcal{H}$ be a real Hilbert space. Then for all $x, y \in \mathcal{H}$,*

$$2\langle x, y\rangle = \|x + y\|^2 - \|x\|^2 - \|y\|^2 = \|x\|^2 + \|y\|^2 - \|x - y\|^2.$$

Lemma 2.3. [11, 24] *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the condition $s_{n+1} \leq (1 - b_n)s_n + b_n c_n$, $n \geq 0$, where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers such that*

(i) $\{b_n\} \subset (0, 1)$ *for all* $n \geq 0$ *and* $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) $\limsup_{n\to\infty} c_n \leq 0$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. (Maingé, [16]) *Let $\{s_n\}$ be a sequence of real numbers. Assume $\{s_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k} \leq s_{n_k+1}$ for all $k \geq 0$. For any $n \geq n_0$, define an integer sequence $\{\nu(n)\}$ as*

$$\nu(n) = \max\{n_0 \leq k \leq n \mid s_k < s_{k+1}\}.$$

Then $\nu(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$ we have

$$\max\{s_{\nu(n)}, s_n\} \leq s_{\nu(n)+1}.$$

3. Algorithm and convergence

In this section, we introduce and investigate a new viscosity approximation method, a selective technique, and a modification of the CQ–algorithm, to solve a class of generalized multiple-set split feasibility problems. This iterative method uses a new adaptive step size criteria making them work well without the prior information about the norm of the operators and the Lipschitz constant. The following conditions need to be satisfied in order to obtain the convergence theorems of the suggested algorithms.

Assumption 3.1. (A1) *$\mathcal{A}_j\colon \mathcal{H}_1 \to \mathcal{H}_2^j$ is a bounded linear mapping, $j = 1, \dots, N$.*

(A2) *$\mathcal{F}\colon \mathcal{H}_1 \to \mathcal{H}_1$ is a contraction mapping with the contraction coefficient $\tau \in [0, 1)$.*

(A3) *$\mathcal{C}_i$ is a closed convex subset of $\mathcal{H}_1$, $i = 1, 2, \dots, M$.*

(A4) *$\mathcal{Q}_j^k$, $k = 1, 2, \dots, L$, are L closed convex subsets of $\mathcal{H}_2^j$, $j = 1, \dots, N$.*

(A5) *The solution set Ω of* (GSFP) *is nonempty.*

From the properties of $\mathcal{C}_i$, $\mathcal{Q}_j^k$ and $\mathcal{A}_j$ for all $i = 1, \dots, M$, $j = 1, \dots, N$ and $k = 1, \dots, L$, we have that Ω is a closed convex subset of $\mathcal{H}_1$.

Our algorithm is expressed as follows.

Algorithm 3.2. Step 0. Select the initial point $x^0 \in \mathcal{H}_1$ and the sequences $\{\alpha_n\}$, $\{\rho_n\}$ and $\{a_n\}$ such that the conditions (C1) and (C4″) are satisfied, where

(C4″) $$0 < a \le \rho_n \le b < 1, \quad 0 < c \le a_n \le d < \infty.$$

Set $n := 0$.

Step 1. Let $\ell_n = \max_{i=1,\dots,M}\{\|P_{\mathcal{C}_i}x^n - x^n\|\}$, $\varpi_n = \{i = 1, \dots, M \mid \|P_{\mathcal{C}_i}x^n - x^n\| = \ell_n\}$.

Step 2. Let $\ell_{j,n} = \max_{k=1,\dots,L}\{\|P_{\mathcal{Q}_j^k}\mathcal{A}_j x^n - \mathcal{A}_j x^n\|\}$, $j = 1, \dots, N$, $\varpi_{j,n} = \{k = 1, \dots, L \mid \|P_{\mathcal{Q}_j^k}\mathcal{A}_j x^n - \mathcal{A}_j x^n\| = \ell_{j,n}\}$, $j = 1, \dots, N$.

Step 3. Let $\Xi_n := \max\left\{\ell_n, \max_{j=1,\dots,N}\{\ell_{j,n}\}\right\}$. If $\Xi_n = \ell_n$ then choose $i_n \in \varpi_n$ and let $v^n := P_{\mathcal{C}_{i_n}}x^n$ and $\mathcal{B} := I^{\mathcal{H}_1}$. Else, choose $k_n \in \varpi_{j_n,n}$ and let $v^n := P_{\mathcal{Q}_{j_n}^{k_n}}\mathcal{A}_{j_n}x^n$ and $\mathcal{B} := \mathcal{A}_{j_n}$, where $\ell_{j_n,n} = \Xi_n$.

Step 4. Compute $u^n = x^n - \gamma_n \mathcal{B}^*(\mathcal{B}x^n - v^n)$, where the step size γ_n is defined by

$$\gamma_n = \rho_n \frac{\|\mathcal{B}x^n - v^n\|^2}{\|\mathcal{B}^*(\mathcal{B}x^n - v^n)\|^2 + a_n}. \tag{3.1}$$

Step 5. Compute $x^{n+1} = \alpha_n \mathcal{F}(x^n) + (1 - \alpha_n)u^n$.

Step 6. Set $n := n + 1$ and go to Step 1.

Remark 3.3. We note here that the calculation criterion (3.1) is easy to be implemented. Moreover, the condition (C4″) is very simple and easy to check.

We begin our analysis of this algorithm with the following proposition.

Proposition 3.4. *Let $\{x^n\}$ be a sequence generated by Algorithm* 3.2 *and suppose that all conditions in Assumption* 3.1 *are satisfied. Then the following statements hold true:*

(i) *The sequence $\{x^n\}$ is bounded.*

(ii) *For any $u^* \in \Omega$, the following inequality holds:*

$$\|x^{n+1} - u^*\|^2 \le [1 - (1-\tau)\alpha_n]\|x^n - u^*\|^2 + 2\alpha_n \langle \mathcal{F}u^* - u^*, x^{n+1} - u^* \rangle. \tag{3.2}$$

Proof. (i) First, we prove that the sequence $\{x^n\}$ in Algorithm 3.2 is bounded. Indeed, for any $u^* \in \Omega$, the solution set of (GSFP), we consider the following two cases.

Case 1*: $\ell_{j_n,n} = \Xi_n$.* It follows from Steps 2, 3 and 4 in Algorithm 3.2, the property of adjoint operator $\mathcal{A}_{j_n}^*$, $\mathcal{A}_{j_n} u^* \in \mathcal{Q}_{j_n}^{k_n}$, the nonexpansive property of $P_{\mathcal{Q}_{j_n}^{k_n}}$ and Lemma 2.2 that

$$\begin{aligned}
&\|u^n - u^*\|^2 \\
&= \big\|x^n - \gamma_n \mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n - u^*\big\|^2 \\
&= \|x^n - u^*\|^2 + \gamma_n^2 \big\|\mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 - 2\gamma_n \big\langle x^n - u^*, \mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n \big\rangle \\
&= \|x^n - u^*\|^2 + \gamma_n^2 \big\|\mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 - 2\gamma_n \big\langle \mathcal{A}_{j_n} x^n - \mathcal{A}_{j_n} u^*, \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n \big\rangle \\
&= \|x^n - u^*\|^2 + \gamma_n^2 \big\|\mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 \\
&\quad - \gamma_n \Big(\|\mathcal{A}_{j_n} x^n - \mathcal{A}_{j_n} u^*\|^2 + \big\|\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 - \big\|P_{\mathcal{Q}_{j_n}^{k_n}} \mathcal{A}_{j_n} x^n - \mathcal{A}_{j_n} u^*\big\|^2\Big) \\
&= \|x^n - u^*\|^2 + \gamma_n^2 \big\|\mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 \\
&\quad + \gamma_n \Big(\big\|P_{\mathcal{Q}_{j_n}^{k_n}} \mathcal{A}_{j_n} x^n - P_{\mathcal{Q}_{j_n}^{k_n}} \mathcal{A}_{j_n} u^*\big\|^2 - \|\mathcal{A}_{j_n} x^n - \mathcal{A}_{j_n} u^*\|^2 - \big\|\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2\Big) \\
&\le \|x^n - u^*\|^2 + \gamma_n^2 \big\|\mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 \\
&\quad + \gamma_n \Big(\|\mathcal{A}_{j_n} x^n - \mathcal{A}_{j_n} u^*\|^2 - \|\mathcal{A}_{j_n} x^n - \mathcal{A}_{j_n} u^*\|^2 - \big\|\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2\Big) \\
&= \|x^n - u^*\|^2 + \gamma_n^2 \big\|\mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 - \gamma_n \big\|\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2.
\end{aligned}$$

From the last inequality and (3.1), the condition of the step size γ_n and (C4″), we obtain

$$\begin{aligned}
\|u^n - u^*\|^2 \le \|x^n - u^*\|^2 \\
&+ \rho_n^2 \frac{\big\|\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^4}{\Big(\big\|\mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 + a_n\Big)^2} \big\|\mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 \\
&- \rho_n \frac{\big\|\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^4}{\big\|\mathcal{A}_{j_n}^* \big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 + a_n}
\end{aligned}$$

$$
\begin{aligned}
(3.3) \qquad & \leq \|x^n - u^*\|^2 \\
& + \rho_n^2 \frac{\big\|\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^4}{\Big(\big\|\mathcal{A}_{j_n}^*\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 + a_n\Big)^2} \Big(\big\|\mathcal{A}_{j_n}^*\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 + a_n\Big) \\
& - \rho_n \frac{\big\|\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^4}{\big\|\mathcal{A}_{j_n}^*\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 + a_n} \\
& = \|x^n - u^*\|^2 - \rho_n(1-\rho_n) \frac{\big\|\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^4}{\big\|\mathcal{A}_{j_n}^*\big(I^{\mathcal{H}_2^{j_n}} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 + a_n}.
\end{aligned}
$$

Case 2: $\ell_n = \Xi_n$. It follows from Steps 1, 3 and 4 in Algorithm 3.2 that

$$\|u^n - u^*\|^2 = \big\|x^n - \gamma_n (I^{\mathcal{H}_1})^* \big(I^{\mathcal{H}_1} - P_{\mathcal{C}_{i_n}}\big)x^n - u^*\big\|^2,$$

and we also get

$$\|u^n - u^*\|^2 \leq \|x^n - u^*\|^2 - \rho_n(1-\rho_n) \frac{\big\|\big(I^{\mathcal{H}_1} - P_{\mathcal{C}_{i_n}}\big)x^n\big\|^4}{\big\|\big(I^{\mathcal{H}_1} - P_{\mathcal{C}_{i_n}}\big)x^n\big\|^2 + a_n}. \tag{3.4}$$

It follows from the convexity of the norm function on $\mathcal{H}_1$, the contraction property of $\mathcal{F}$ with the contraction coefficient $\tau \in [0,1)$, (3.3), (3.4) and Step 5 in Algorithm 3.2 that

$$
\begin{aligned}
\|x^{n+1} - u^*\| &= \big\|\alpha_n(\mathcal{F}x^n - u^*) + (1-\alpha_n)(u^n - u^*)\big\| \\
&\leq \alpha_n\big(\|\mathcal{F}x^n - \mathcal{F}u^*\| + \|\mathcal{F}u^* - u^*\|\big) + (1-\alpha_n)\|u^n - u^*\| \\
&\leq \alpha_n\big(\|\mathcal{F}x^n - \mathcal{F}u^*\| + \|\mathcal{F}u^* - u^*\|\big) + (1-\alpha_n)\|x^n - u^*\| \\
&\leq \tau\alpha_n\|x^n - u^*\| + \alpha_n\|\mathcal{F}u^* - u^*\| + (1-\alpha_n)\|x^n - u^*\| \\
&= [1 - (1-\tau)\alpha_n]\|x^n - u^*\| + (1-\tau)\alpha_n \frac{\|\mathcal{F}u^* - u^*\|}{1-\tau} \\
&\leq \max\left\{\|x^n - u^*\|, \frac{\|\mathcal{F}u^* - u^*\|}{1-\tau}\right\} \\
&\leq \cdots \\
&\leq \max\left\{\|x^0 - u^*\|, \frac{\|\mathcal{F}u^* - u^*\|}{1-\tau}\right\}.
\end{aligned}
$$

This implies that the sequence $\{x^n\}$ is bounded. Since $P_{\mathcal{C}_i}$ and $P_{\mathcal{Q}_j^k}$ are the nonexpansive mappings, $\mathcal{A}_j$ is the bounded linear operator for all $i = 1, \ldots, M$, $j = 1, \ldots, N$ and $k = 1, \ldots, L$ and the boundedness of the sequence $\{a_n\}$, we get that

$$M := \max\left\{\sup_n \big\|\big(I^{\mathcal{H}_1} - P_{\mathcal{C}_{i_n}}\big)x^n\big\|^2 + a_n,\ \sup_n \big\|\mathcal{A}_{j_n}^*\big(I^{\mathcal{H}_2^j} - P_{\mathcal{Q}_{j_n}^{k_n}}\big)\mathcal{A}_{j_n} x^n\big\|^2 + a_n\right\} < \infty.$$

(ii) Next, we will prove (3.2). Indeed, it follows from Steps 1–3 in Algorithm 3.2, (3.3) and (3.4) that

$$(\Xi_n)^4 \leq \frac{M}{\rho_n(1-\rho_n)}\big(\|x^n - u^*\|^2 - \|u^n - u^*\|^2\big) \tag{3.5}$$

at the nth step iteration. It follows from the convexity of the function $\|\cdot\|^2$, Step 5 in Algorithm 3.2, the condition (C1) and (3.5) that

$$\begin{aligned}
\|x^{n+1} - u^*\|^2 &= \|\alpha_n(\mathcal{F}x^n - u^*) + (1-\alpha_n)(u^n - u^*)\|^2 \\
&\leq \alpha_n\|\mathcal{F}x^n - u^*\|^2 + (1-\alpha_n)\|u^n - u^*\|^2 \\
&\leq \alpha_n\|\mathcal{F}x^n - u^*\|^2 + \|u^n - u^*\|^2 \\
&\leq \alpha_n\|\mathcal{F}x^n - u^*\|^2 + \|x^n - u^*\|^2 - \frac{\rho_n(1-\rho_n)}{M}(\Xi_n)^4,
\end{aligned}$$

which implies that

$$(\Xi_n)^4 \leq \frac{M}{\rho_n(1-\rho_n)}\big(\|x^n - u^*\|^2 - \|x^{n+1} - u^*\|^2 + \alpha_n\|\mathcal{F}x^n - u^*\|^2\big). \tag{3.6}$$

From Step 5 in Algorithm 3.2 and the contraction property of $\mathcal{F}$ with the contraction coefficient $\tau \in [0,1)$, we have that

$$\begin{aligned}
\|x^{n+1} - u^*\|^2 &= \big\langle \alpha_n(\mathcal{F}x^n - u^*) + (1-\alpha_n)(u^n - u^*), x^{n+1} - u^*\big\rangle \\
&= (1-\alpha_n)\big\langle u^n - u^*, x^{n+1} - u^*\big\rangle + \alpha_n\big\langle \mathcal{F}x^n - u^*, x^{n+1} - u^*\big\rangle \\
&\leq \frac{1-\alpha_n}{2}\big(\|u^n - u^*\|^2 + \|x^{n+1} - u^*\|^2\big) \\
&\quad + \alpha_n\big\langle \mathcal{F}x^n - \mathcal{F}u^*, x^{n+1} - u^*\big\rangle + \alpha_n\big\langle \mathcal{F}u^* - u^*, x^{n+1} - u^*\big\rangle \\
&\leq \frac{1-\alpha_n}{2}\big(\|u^n - u^*\|^2 + \|x^{n+1} - u^*\|^2\big) \\
&\quad + \frac{\alpha_n}{2}\big(\tau\|x^n - u^*\|^2 + \|x^{n+1} - u^*\|^2\big) + \alpha_n\big\langle \mathcal{F}u^* - u^*, x^{n+1} - u^*\big\rangle.
\end{aligned}$$

This implies that

$$\|x^{n+1} - u^*\|^2 \leq (1-\alpha_n)\|u^n - u^*\|^2 + \alpha_n\tau\|x^n - u^*\|^2 + 2\alpha_n\big\langle \mathcal{F}u^* - u^*, x^{n+1} - u^*\big\rangle.$$

From (3.3), (3.4) and the last inequality, we obtain (3.2). □

The following theorem shows the convergence of Algorithm 3.2.

Theorem 3.5. *Suppose that all conditions in Assumption* 3.1 *are satisfied. Then the sequence* $\{x^n\}$ *generated by Algorithm* 3.2 *converges strongly to the unique solution of* (VIP) *with* Ω_1 *replaced by* Ω.

Proof. Since $\mathcal{F}$ is a contraction mapping, $P_\Omega \mathcal{F}$ is a contraction mapping too. By Banach contraction mapping principle, there exists a unique point $x^* \in \Omega$ such that $P_\Omega \mathcal{F} x^* = x^*$. By (2.1) and the condition (A5), we obtain x^* is the unique solution to the variational inequality (VIP).

We will claim that $\lim_{n\to\infty} \|x^n - x^*\| = 0$. Indeed, it follows from (3.2) with u^* replaced by x^* that

$$\begin{aligned} \|x^{n+1} - x^*\|^2 \le{}& [1 - (1-\tau)\alpha_n]\|x^n - x^*\|^2 \\ &+ (1-\tau)\alpha_n \left[\frac{2}{1-\tau}\big\langle \mathcal{F}x^* - x^*, x^{n+1} - x^*\big\rangle\right], \quad n \ge 0. \end{aligned} \tag{3.7}$$

We consider two cases.

Case 1: There exists an integer $n_0 \ge 0$ such that $\|x^{n+1} - x^*\| \le \|x^n - x^*\|$ for all $n \ge n_0$.

Then $\lim_{n\to\infty} \|x^n - x^*\|$ exists. From the boundedness of the sequence $\{\mathcal{F}x^n\}$, the conditions (C1) and (C4$''$), it follows from (3.6) with u^* replaced by x^* that $\Xi_n \to 0$. Since the definition of Ξ_n, the sequences $\{\ell_n\}$ and $\{\ell_{j,n}\}$ in Algorithm 3.2 also converge to 0. This implies that

$$\lim_{n\to\infty} \big\|\big(I^{\mathcal{H}_1} - P_{\mathcal{C}_i}\big)x^n\big\| = 0 \quad \text{for all } i = 1,\dots,M \tag{3.8}$$

and

$$\lim_{n\to\infty} \big\|\big(I^{\mathcal{H}_2^j} - P_{\mathcal{Q}_j^k}\big)\mathcal{A}_j x^n\big\| = 0 \quad \text{for all } j = 1,\dots,N \text{ and } k = 1,\dots,L. \tag{3.9}$$

From Step 4 in Algorithm 3.2, (3.8) and (3.9), we obtain

$$\|x^n - u^n\| = \gamma_n \|\mathcal{B}^*(\mathcal{B}x^n - v^n)\| \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

From the boundedness of the sequence $\{x^n\}$, Steps 4 and 5 in Algorithm 3.2 and the condition (C1), we also have

$$\|x^{n+1} - u^n\| = \alpha_n \|\mathcal{F}x^n - u^n\| \to 0 \quad \text{as } n \to \infty,$$

combining with (3.10), we have

$$\|x^{n+1} - x^n\| \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Now we show that $\limsup_{n\to\infty} \langle \mathcal{F}x^* - x^*, x^{n+1} - x^*\rangle \le 0$. Indeed, suppose that $\{x^{n_k}\}$ is a subsequence of $\{x^n\}$ such that

$$\limsup_{n\to\infty} \big\langle \mathcal{F}x^* - x^*, x^n - x^*\big\rangle = \lim_{k\to\infty} \big\langle \mathcal{F}x^* - x^*, x^{n_k} - x^*\big\rangle. \tag{3.12}$$

Since $\{x^{n_k}\}$ is bounded, there exists a subsequence $\{x^{n_{k_l}}\}$ of $\{x^{n_k}\}$ which converges weakly to some point $x^\dagger$. Without loss of generality, we may assume that $x^{n_k} \rightharpoonup x^\dagger$. We shall prove that $x^\dagger \in \Omega$. Indeed, from Lemma 2.1 and (3.8) we obtain $x^\dagger \in \mathrm{Fix}(P_{\mathcal{C}_i})$, that is, $x^\dagger \in \mathcal{C}_i$ for all $i = 1, \ldots, M$. Hence $x^\dagger \in \bigcap_{i=1}^M \mathcal{C}_i$. Moreover, since each $\mathcal{A}_j$ is a bounded linear operator, $\mathcal{A}_j x^{n_k} \rightharpoonup \mathcal{A}_j x^\dagger$ for each $j = 1, \ldots, N$. Using Lemma 2.1 and (3.9), we also obtain $\mathcal{A}_j x^\dagger \in \mathcal{Q}_j^k$ for each $j = 1, \ldots, N$ and $k = 1, \ldots, L$. Hence $x^\dagger \in \bigcap_{j=1}^N \mathcal{A}_j^{-1}\big(\bigcap_{k=1}^L \mathcal{Q}_j^k\big)$. Consequently, $x^\dagger \in \Omega$. So, from $x^* = P_\Omega \mathcal{F} x^*$, (3.12) and (2.1), we deduce that

$$\limsup_{n\to\infty} \big\langle \mathcal{F}x^* - x^*, x^n - x^* \big\rangle = \big\langle \mathcal{F}x^* - x^*, x^\dagger - x^* \big\rangle \leq 0,$$

which combined with (3.11) gives

$$\limsup_{n\to\infty} \big\langle \mathcal{F}x^* - x^*, x^{n+1} - x^* \big\rangle \leq 0. \tag{3.13}$$

Now, the inequality (3.7) can be rewritten in the form

$$\|x^{n+1} - x^*\|^2 \leq (1-b_n)\|x^n - x^*\|^2 + b_n c_n, \quad n \geq 0,$$

where

$$b_n = (1-\tau)\alpha_n \quad \text{and} \quad c_n = \frac{2}{1-\tau}\big\langle \mathcal{F}x^* - x^*, x^{n+1} - x^* \big\rangle.$$

Since $\tau \in [0,1)$, $\{\alpha_n\} \subset (0,1)$ and $\sum_{n=0}^\infty \alpha_n = \infty$, $\{b_n\} \subset (0,1)$ and $\sum_{n=0}^\infty b_n = \infty$. Consequently, from $\tau \in [0,1)$ and (3.13), we have that $\limsup_{n\to\infty} c_n \leq 0$. Finally, by Lemma 2.3, $\lim_{n\to\infty} \|x^n - x^*\| = 0$.

Case 2: There exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\|x^{n_k} - x^*\| \leq \|x^{n_k+1} - x^*\|$ for all $k \geq 0$.

Hence, by Lemma 2.4, there exists an integer and nondecreasing sequence $\{\nu(n)\}$ for $n \geq n_0$ (for some n_0 large enough) such that $\nu(n) \to \infty$ as $n \to \infty$,

$$\|x^{\nu(n)} - x^*\| \leq \|x^{\nu(n)+1} - x^*\| \quad \text{and} \quad \|x^n - x^*\| \leq \|x^{\nu(n)+1} - x^*\| \tag{3.14}$$

for each $n \geq 0$.

From (3.7) with n replaced by $\nu(n)$ and the condition (C1), we have

$$0 < \|x^{\nu(n)+1} - x^*\|^2 - \|x^{\nu(n)} - x^*\|^2 \leq 2\alpha_{\nu(n)}\big\langle \mathcal{F}x^* - x^*, x^{\nu(n)+1} - x^* \big\rangle.$$

Since $\alpha_{\nu(n)} \to 0$ and $\{x^{\nu(n)}\}$ is bounded, we get

$$\lim_{n\to\infty} \big(\|x^{\nu(n)+1} - x^*\|^2 - \|x^{\nu(n)} - x^*\|^2\big) = 0. \tag{3.15}$$

By the similar argument as in Case 1, we obtain

$$\lim_{n\to\infty} \big\|\big(I^{\mathcal{H}_1} - P_{\mathcal{C}_i}\big)x^{\nu(n)}\big\| = 0 \quad \text{for all } i = 1, \ldots, M$$

and

$$\lim_{n\to\infty} \left\| \left(I^{\mathcal{H}_2^j} - P_{\mathcal{Q}_i^k}\right)\mathcal{A}_j x^{\nu(n)} \right\| = 0 \quad \text{for all } j = 1, \ldots, N \text{ and } k = 1, \ldots, L.$$

Also we get

$$\|x^{\nu(n)+1} - x^*\|^2 \leq [1 - (1-\tau)\alpha_{\nu(n)}]\|x^{\nu(n)} - x^*\|^2 + 2\alpha_{\nu(n)}\langle \mathcal{F}x^* - x^*, x^{\nu(n)+1} - x^*\rangle,$$

where $\limsup_{n\to\infty}\langle \mathcal{F}x^* - x^*, x^{\nu(n)+1} - x^*\rangle \leq 0$.

Since the first inequality in (3.14) and $\alpha_{\nu(n)} > 0$, we have that $(1-\tau)\|x^{\nu(n)} - x^*\|^2 \leq 2\langle \mathcal{F}x^* - x^*, x^{\nu(n)+1} - x^*\rangle$. Thus, from $\limsup_{n\to\infty}\langle \mathcal{F}x^* - x^*, x^{\nu(n)+1} - x^*\rangle \leq 0$ and $\tau \in [0,1)$, we get $\lim_{n\to\infty}\|x^{\nu(n)} - x^*\|^2 = 0$. This together with (3.15) implies that $\lim_{n\to\infty}\|x^{\nu(n)+1} - x^*\|^2 = 0$, which together with the second inequality in (3.14) implies that $\lim_{n\to\infty}\|x^n - x^*\| = 0$. This completes the proof. □

4. Corollaries

In this section, we consider some applications for solving (MSFP), (SMOS) and (SFP).

Taking $N = 1$, $\mathcal{H}_2 := \mathcal{H}_2^1$, $\mathcal{A} := \mathcal{A}_1$ and $\mathcal{Q}_k := \mathcal{Q}_1^k$, from Algorithm 3.2 and Theorem 3.5, we have the following result for (MSFP).

Algorithm 4.1. Step 0. Select the initial point $x^0 \in \mathcal{H}_1$ and the sequences $\{\alpha_n\}$, $\{\rho_n\}$ and $\{a_n\}$ such that the conditions (C1) and (C4″) are satisfied. Set $n := 0$.

Step 1. Let $\ell_n = \max_{i=1,\ldots,M}\{\|P_{\mathcal{C}_i}x^n - x^n\|\}$.

Step 2. Let $\widehat{\ell}_n = \max_{k=1,\ldots,L}\{\|P_{\mathcal{Q}_k}\mathcal{A}x^n - \mathcal{A}x^n\|\}$.

Step 3. Let $\Xi_n := \max\{\ell_n, \widehat{\ell}_n\}$. If $\ell_n = \Xi_n$ then choose i_n such that $\ell_n = \|P_{\mathcal{C}_{i_n}}x^n - x^n\|$ and let $v^n := P_{\mathcal{C}_{i_n}}x^n$ and $\mathcal{B} := I^{\mathcal{H}_1}$. Else, choose k_n such that $\widehat{\ell}_n = \|P_{\mathcal{Q}_{k_n}}\mathcal{A}x^n - \mathcal{A}x^n\|$ and let $v^n := P_{\mathcal{Q}_{k_n}}\mathcal{A}x^n$ and $\mathcal{B} := \mathcal{A}$.

Step 4. Compute $u^n = x^n - \gamma_n\mathcal{B}^*\left(\mathcal{B}x^n - v^n\right)$, where the step size γ_n is defined by (3.1).

Step 5. Compute $x^{n+1} = \alpha_n\mathcal{F}x^n + (1-\alpha_n)u^n$.

Step 6. Set $n := n+1$ and go to Step 1.

Corollary 4.2. *Suppose that the conditions* (A1)–(A4) *are satisfied with* $N = 1$. *Then the sequence* $\{x^n\}$ *generated by Algorithm* 4.1 *converges strongly to the unique solution of* (VIP), *provided that the solution set* Ω_1 *of* (MSFP) *is nonempty.*

Take $M = 1$, $L = 1$, $\mathcal{C} := \mathcal{C}_1$ and $\mathcal{Q}_j := \mathcal{Q}_j^1$. From Algorithm 3.2 and Theorem 3.5, we have the following results to study (SMOS) introduced by Reich et al. [21].

Algorithm 4.3. Step 0. Select the initial point $x^0 \in \mathcal{H}_1$ and the sequences $\{\alpha_n\}$, $\{\rho_n\}$ and $\{a_n\}$ such that the conditions (C1) and (C4$''$) are satisfied. Set $n := 0$.

Step 1. Let $\ell_n = \|P_{\mathcal{C}} x^n - x^n\|$.

Step 2. Let $\widehat{\ell}_n = \max_{j=1,\ldots,N}\{\|P_{\mathcal{Q}_j}\mathcal{A}_j x^n - \mathcal{A}_j x^n\|\}$.

Step 3. Let $\Xi_n := \max\{\ell_n, \widehat{\ell}_n\}$. If $\ell_n = \Xi_n$ then $v^n := P_{\mathcal{C}} x^n$ and $\mathcal{B} := I^{\mathcal{H}_1}$. Else, choose j_n such that $\widehat{\ell}_n = \|P_{\mathcal{Q}_{j_n}}\mathcal{A}_{j_n} x^n - \mathcal{A}_{j_n} x^n\|$ and let $v^n := P_{\mathcal{Q}_{j_n}}\mathcal{A}_{j_n} x^n$ and $\mathcal{B} := \mathcal{A}_{j_n}$.

Step 4. Compute $u^n = x^n - \gamma_n \mathcal{B}^*\big(\mathcal{B}x^n - v^n\big)$, where the step size γ_n is defined by (3.1).

Step 5. Compute $x^{n+1} = \alpha_n \mathcal{F} x^n + (1-\alpha_n) u^n$.

Step 6. Set $n := n+1$ and go to Step 1.

Corollary 4.4. *Suppose that the conditions* (A1)–(A4) *are satisfied with* $M = L = 1$. *Then the sequence* $\{x^n\}$ *generated by Algorithm* 4.3 *converges strongly to the unique solution of* (VIP)*, with* Ω_1 *replaced by* Ω_2*, provided the solution set* Ω_2 *of* (SMOS) *is nonempty.*

Taking $M = L = 1$, (MSFP) becomes (SFP). We consider the special case when $\mathcal{F} = 0$ and in this situation, (VIP) becomes the problem of finding the minimum-norm solution of (SFP). From Algorithm 4.1 and Corollary 4.2, we have the following result.

Corollary 4.5. *Let* $\mathcal{C}$ *and* $\mathcal{Q}$ *be two nonempty closed convex subsets of two real Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, respectively. Let* $\mathcal{A}\colon \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator such that the solution set* Ω_3 *of* (SFP) *is nonempty. Let* $\{x^n\}$ *be a sequence generated by*

$$
\begin{cases}
x^0 \in \mathcal{H}_1, \quad \textit{any element}, \\
\ell_n := \|P_{\mathcal{C}} x^n - x^n\|, \quad \widehat{\ell}_n := \|P_{\mathcal{Q}}\mathcal{A}x^n - \mathcal{A}x^n\|, \\
\quad \textit{If } \ell_n > \widehat{\ell}_n,\ v^n := P_{\mathcal{C}} x^n,\ \mathcal{B} := I^{\mathcal{H}_1},\ \textit{else } v^n := P_{\mathcal{Q}}\mathcal{A}x^n,\ \mathcal{B} := \mathcal{A}, \\
x^{n+1} = (1-\alpha_n)\big(x^n - \gamma_n \mathcal{B}^*(\mathcal{B}x^n - v^n)\big), \quad n \geq 0,
\end{cases}
\tag{4.1}
$$

where the step size γ_n *is defined by* (3.1) *and the sequences* $\{\alpha_n\}$*,* $\{\rho_n\}$ *and* $\{a_n\}$ *satisfy the conditions* (C1) *and* (C4$''$)*. Then the sequence* $\{x^n\}$ *converges strongly to the minimum-norm solution of* (SFP)*.*

Remark 4.6. We have the following observations for the offered Algorithms 4.1, 4.3 and the algorithm (4.1).

(1) The CQ–method [5] for (SFP) requiring two projections at each step is only weakly convergent. Yu et al. [27] considered a strongly convergent algorithm for (SFP) and

requires only two projections at each step. However, in this (SFP), the constrained sets are given explicitly rather than the solution sets of (VIP) as in Algorithm 3.2 for (MSFP). Another strongly convergent algorithm for (SFP) proposed in [6] needs four projections at each iteration step.

(2) Our iterative schemes are embedded with inertial effects, which allows them to accelerate the convergence speed of the algorithms. Furthermore, the Lipschitz constant and norm of operator do not need to be known. Therefore, the convergence conditions of algorithms obtained in this section are weaker than those in (1.1) of Anh [2], (1.2) of Buong [4], (1.3) of Reich et al. [21] and (1.4) of Anh et al. [1], which makes them more widespread and useful in practical applications.

5. Numerical experiment

In this section, we perform some computational tests that occur in infinite-dimensional spaces and compare the offered iterative schemes with several previously known strongly convergent algorithms, which including the algorithm (1.1) introduced by Anh [2], the algorithm (1.2) presented by Buong [4] and the algorithm (1.3) suggested by Reich et al. [21]. All the programs were implemented in Python 3.7 running on a laptop with Intel(R) Core(TM) i5-5200U CPU @ 2.20GHz, 12 GB RAM.

First, we illustrate the convergence of Algorithm 3.2.

Example 5.1. Take $\mathcal{H}_1 = \mathcal{H}_2 = L^2[0,1]$ with the inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)\,dt$ and norm $\|x\| = \left(\int_0^1 x^2(t)\,dt\right)^{1/2}$ for all $x, y \in L^2[0,1]$. Consider (GSFP) with

$$\Omega := \left(\bigcap_{i=1}^{M} \mathcal{C}_i\right) \cap \left(\bigcap_{j=1}^{3} \mathcal{A}_j^{-1}\left(\bigcap_{k=1}^{L} \mathcal{Q}_j^k\right)\right),$$

where the closed convex sets $\mathcal{C}_i \subset L^2[0,1]$ and $\mathcal{Q}_j^k \subset L^2[0,1]$, $j = 1, 2, 3$, are given by

$$\begin{aligned}
\mathcal{C}_i &= \left\{x \in L^2[0,1] \mid \langle a_i, x\rangle = b_i\right\} \quad \text{where } a_i(t) = t^{i+1} \text{ and } b_i = \frac{1}{2(4+i)},\\
\mathcal{Q}_1^k &= \left\{y \in L^2[0,1] \mid \langle c_1^k, y\rangle \ge d_1^k\right\} \quad \text{where } c_1^k(t) = t + k \text{ and } d_1^k = \frac{7}{72},\\
\mathcal{Q}_2^k &= \left\{y \in L^2[0,1] \mid \langle c_2^k, y\rangle \ge d_2^k\right\} \quad \text{where } c_2^k(t) = 2t + k \text{ and } d_2^k = \frac{5}{48},\\
\mathcal{Q}_3^k &= \left\{y \in L^2[0,1] \mid \langle c_3^k, y\rangle \ge d_3^k\right\} \quad \text{where } c_3^k(t) = 3t + k \text{ and } d_3^k = \frac{13}{120}
\end{aligned}$$

for all $i = 1, \dots, M$, $k = 1, \dots, L$, $t \in [0,1]$, the operators $\mathcal{A}_j \colon L^2[0,1] \to L^2[0,1]$, $j = 1, 2, 3$, are defined by $(\mathcal{A}_1 x)(t) = x(t)/3$, $(\mathcal{A}_2 x)(t) = x(t)/4$ and $(\mathcal{A}_3 x)(t) = x(t)/5$. It is easy to see that $\mathcal{A}_j$, $j = 1, 2, 3$, are bounded linear operators. Since $x(t) = t^2/2 \in \Omega$, $\Omega \neq \emptyset$.

In this example, we define the function $\varepsilon_1(n)$ as follows:

$$\varepsilon_1(n) = \frac{1}{4}\Bigg[\frac{1}{M}\sum_{i=1}^{M}\|x^n - P_{\mathcal{C}_i}x^n\|^2 + \frac{1}{L}\sum_{k=1}^{L}\Big(\big\|\mathcal{A}_1x^n - P_{\mathcal{Q}_1^k}\mathcal{A}_1x^n\big\|^2 + \big\|\mathcal{A}_2x^n - P_{\mathcal{Q}_2^k}\mathcal{A}_2x^n\big\|^2 + \big\|\mathcal{A}_3x^n - P_{\mathcal{Q}_3^k}\mathcal{A}_3x^n\big\|^2\Big)\Bigg].$$

It can be seen that if at the nth step, $\varepsilon_1(n) = 0$ then $x^n \in \Omega$, that is, x^n is a solution of (GSFP).

We now consider the convergence of Algorithm 3.2 with $M = 30$, $L = 50$, $\rho_n = 0.75$, $a_n = 0.25$ for all $n \geq 0$ and obtain Table 5.1 of numerical results with stop condition is $\varepsilon_1(n) < 10^{-5}$.

α_n		$(n+1)^{-0.95}$	$(n+1)^{-1.0}$	$(n+2)^{-0.95}$	$(n+2)^{-1.0}$
$\mathcal{F}x = 0.5x$	Time (s)	770.42439	473.76078	811.52912	479.57265
	Iter. (n)	79613	46197	79665	46460
$\mathcal{F}x = 0.75x$	Time (s)	464.73435	271.84241	438.13586	258.63001
	Iter. (n)	42132	25803	42070	25094
$\mathcal{F}x = 0.99x$	Time (s)	19.87261	15.62950	16.93058	15.35156
	Iter. (n)	1592	1448	1582	1439

Table 5.1: Performance of Algorithm 3.2 with initial point $x^0(t) = \frac{1}{2(10+t)}$ for all $t \in [0, 1]$.

Table 5.2 is the performance of Algorithm 3.2 with different starting points.

$x^0(t) = \sin 2t$		$x^0(t) = \cos 2t$		$x^0(t) = \frac{1}{2(1+t)}$		$x^0(t) = \frac{1}{2(10+t)}$	
Time (s)	Iter. (n)	Time (s)	Iter. (n)	Time (s)	Iter. (n)	Time (s)	Iter. (n)
89.32911	8403	84.18712	5263	131.65541	12859	250.88909	25183

Table 5.2: Performance of Algorithm 3.2 with $\mathcal{F}x = 0.75x$, $\alpha_n = (n+1)^{-1.0}$ and $\varepsilon_1(n) < 10^{-5}$.

The behavior of the approximation solution $x^n(t)$ with $\varepsilon_1(n) < 10^{-5}$ is presented in Figure 5.1.

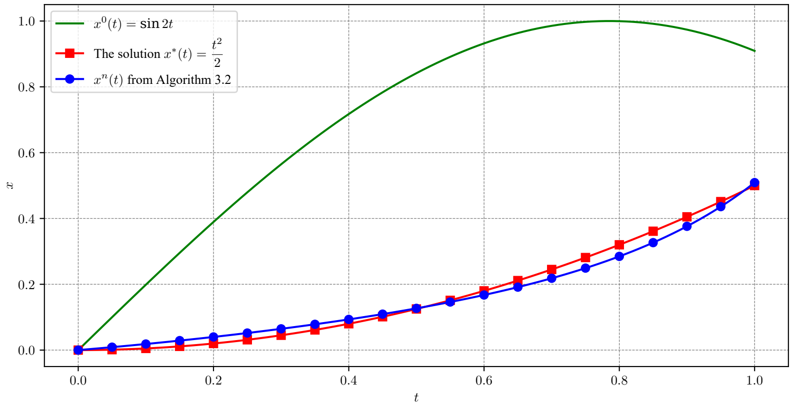

Figure 5.1: The behavior of $x^n(t)$ with $\mathcal{F}(x) = 0.75x$ and $x^0(t) = \sin(2t)$ for all $t \in [0,1]$ and $\alpha_n = (n+1)^{-1.0}$.

Example 5.2. In Example 5.1, take $\mathcal{A} := \mathcal{A}_1$, $\mathcal{Q}_k := \mathcal{Q}_1^k$, $k = 1, \ldots, L$, and consider (MSFP) with $\Omega_1 := \big(\bigcap_{i=1}^M \mathcal{C}_i\big) \cap \big(\mathcal{A}^{-1}\big(\bigcap_{k=1}^L \mathcal{Q}_k\big)\big)$. Since $x(t) = t^2/2 \in \Omega_1$, $\Omega_1 \neq \emptyset$. Now, we compare Algorithm 4.1 with two algorithms (1.1) and (1.2). The first one was introduced by Anh [2] and the second one was introduced by Buong [4]. The operator and parameters of these methods are chosen as follows:

- In our algorithm, we choose $\mathcal{F}x = 0.5x$, $\alpha_n = \frac{1}{n+1}$, $\rho_n = 0.8$, $a_n = 10^{-3}$ for all $n \geq 0$.
- In (1.1), $Fx = 0.5x$, $\alpha_n = \frac{1}{n+1}$, $\gamma_n = 0.8$, $\mu = 10^{-3}$ and $\eta_n = \frac{80n+1}{81n+572}$ for all $n \geq 0$.
- In (1.2), $Fx = 0.5x$, $\alpha_n = \frac{1}{n+1}$ for all $n \geq 0$, $\gamma = 0.8$, $\mu = 10^{-3}$, $\eta_i = \frac{2i}{M(M+1)}$, $i = 1, \ldots, M$, and $\beta_k = \frac{2k}{L(L+1)}$, $k = 1, \ldots, L$.

	Alg. 4.1		Anh's Alg. (1.1)		Buong's Alg. (1.2)	
	Time (s)	Iter. (n)	Time (s)	Iter. (n)	Time (s)	Iter. (n)
$x^0(t) = \frac{1}{2(1+t)}$	1.45803	297	12.25931	2379	6.24796	952
$x^0(t) = \frac{1}{t^2+1}$	0.64271	131	107.20125	23396	56.02806	8398
$x^0(t) = \cos 10t$	1.69369	336	53.12184	9991	8.16611	1166

Table 5.3: Performance of the three algorithms in Example 5.2 with different starting points.

In this example, the defined function $\varepsilon_2(n)$ as follows:

$$\varepsilon_2(n) = \frac{1}{2}\left(\frac{1}{M}\sum_{i=1}^{M}\left\|x^n - P_{\mathcal{C}_i}x^n\right\|^2 + \frac{1}{L}\sum_{k=1}^{L}\left\|\mathcal{A}x^n - P_{\mathcal{Q}_k}\mathcal{A}x^n\right\|^2\right).$$

Note that, if at the nth step, $\varepsilon_2(n) = 0$ then $x^n \in \Omega_1$, that is, x^n is a solution of (MSFP). In all the methods, we use the same stoping rule $\varepsilon_2(n) < 10^{-5}$. We have tested the algorithms with different starting points, the results are presented in Table 5.3 and Figure 5.2.

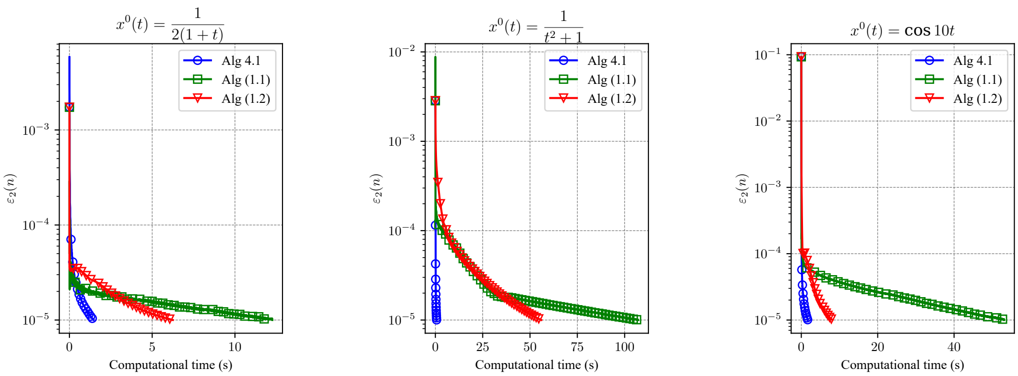

Figure 5.2: Performance of the three algorithms in Example 5.2 with different starting points.

Table 5.4 and Figure 5.3 show the performance of the three algorithms with $x^0(t) = \frac{1}{t^2+1}$ and different M and L.

	Alg. 4.1		Anh's Alg. (1.1)		Buong's Alg. (1.2)	
	Time (s)	Iter. (n)	Time (s)	Iter. (n)	Time (s)	Iter. (n)
$M = 5$, $L = 7$	1.19461	1176	61.67297	63015	2.43713	2153
$M = 15$, $L = 25$	1.07309	379	57.49268	18795	17.22387	4959
$M = 200$, $L = 100$	7.56092	248	397.34452	18233	1978.76401	60392

Table 5.4: Performance of the three algorithms in Example 5.2 with different M and L.

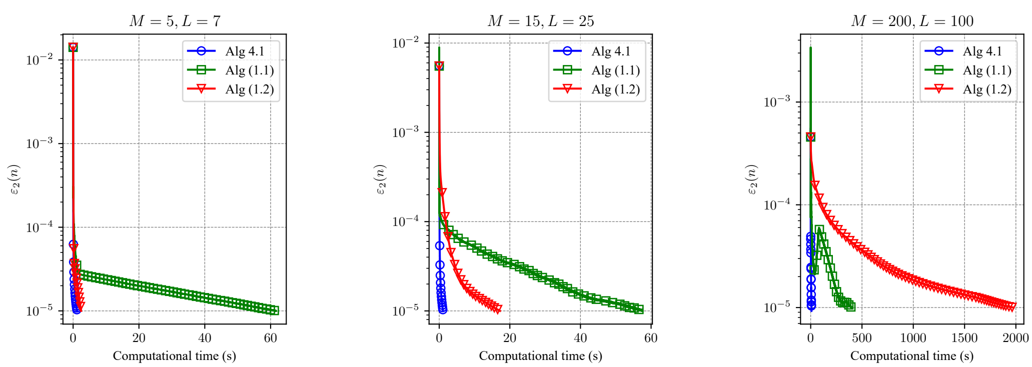

Figure 5.3: Performance of the three algorithms in Example 5.2 with different M and L.

From Tables 5.3, 5.4 and Figures 5.2, 5.3, we can see that the our algorithm shows a better behavior in most cases.

Example 5.3. Finally, we illustrate the convergence of Algorithm 4.3, without computing $\|\mathcal{A}_j\|$, $j = 1, \ldots, N$, like the algorithm (1.3) which was introduced by Reich et al. [21].

In Example 5.1, take $\mathcal{C} := \mathcal{C}_1$. Let $\mathcal{Q}_j = \{y \in L^2[0,1] \mid \langle c_j, y\rangle \geq d_j\}$, where $c_j(t) = jt + 2$ and $d_j = 13/120$ and $(\mathcal{A}_j x)(t) = x(t)/(j+2)$ for all $j = 1, \ldots, N$, $t \in [0,1]$. We consider (SMOS) with

$$\Omega_2 := \mathcal{C} \cap \left(\bigcap_{j=1}^{N} \mathcal{A}_j^{-1}(\mathcal{Q}_j) \right).$$

Since $x^*(t) = t^2/2$, Ω_2 is nonempty. We implement Algorithm 4.3 with $N = 100$, $\mathcal{F}x = 0.9x$, $\rho_n = 0.99$ and $a_n = 10^{-9}$. In this case, the defined function $\varepsilon_3(n)$ becomes

$$\varepsilon_3(n) = \frac{1}{2}\left(\left\|x^n - P_{\mathcal{C}} x^n\right\|^2 + \frac{1}{N}\sum_{j=1}^{N} \left\|\mathcal{A}_j x^n - P_{\mathcal{Q}_j}\mathcal{A}_j x^n\right\|^2 \right).$$

Note that, if at the nth step, $\varepsilon_3(n) = 0$ then $x^n \in \Omega_2$, that is, x^n is a solution of (SMOS). In Table 5.5, we use the starting point $x^0(t) = \frac{1}{2(1+t)}$ and the stopping rule $\varepsilon_3(n) < 10^{-6}$.

	$\alpha_n = \frac{1}{n+1}$		$\alpha_n = \frac{1}{\sqrt{n}+1}$		$\alpha_n = \frac{1}{\sqrt[4]{n}+1}$	
	Time (s)	Iter. (n)	Time (s)	Iter. (n)	Time (s)	Iter. (n)
Alg. 4.3	0.12290	17	1.41318	254	376.63804	63541

Table 5.5: Performance of Algorithm 4.3 in Example 5.3.

6. An application to discrete optimal control problems

In the section, we consider an application to discrete optimal control problems. Also, some comparisons of our algorithm (4.1) with (1.4) of Anh et al. [1] are reported. All Python codes were run on a PC with Intel(R) Core(TM) i5-5200U CPU @ 2.20GHz, 12 GB RAM under version Python 3.7.

Let A_i and B_i, $i = 0, 1, \ldots, \mathcal{N} - 1$, be $n \times n$ and $n \times m$ real matrices, respectively. Consider a linear discrete optimal control problem

$$\begin{aligned} &x_{i+1} = A_{i+1}x_i + B_{i+1}u_i, \quad u_i \in \mathcal{C}_i, \ \ i = 0, 1, \ldots, \mathcal{N} - 1, \quad x_0 = 0, \ \ x_{\mathcal{N}} \in \mathcal{Q}, \\ &J(x, u) := \sum_{i=0}^{\mathcal{N}-1} \|u_i\|^2 \to \min_{u_i}, \end{aligned} \tag{6.1}$$

where $\mathcal{C}_i \subset \mathbb{R}^m$, $i = 0, 1, \ldots, \mathcal{N} - 1$, and $\mathcal{Q} \subset \mathbb{R}^n$ are closed convex subsets, which describe the control and the state constraints, respectively. Constructing a matrix $\boldsymbol{\mathcal{A}} := \begin{bmatrix} \mathcal{A}_0 & \mathcal{A}_1 & \ldots & \mathcal{A}_{\mathcal{N}-1} \end{bmatrix}$ of dimension $n \times \mathcal{N}m$, in which $\mathcal{A}_i := A_{\mathcal{N}} A_{\mathcal{N}-1} \cdots A_{i+2} B_{i+1}$, $i = 0, 1, \ldots, \mathcal{N}-2$, and $\mathcal{A}_{\mathcal{N}-1} = B_{\mathcal{N}}$, and defining the set $\boldsymbol{\mathcal{C}} := \mathcal{C}_0 \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_{\mathcal{N}-1} \subset \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$. Let $\mathbf{u} := (u_0, u_1, \ldots, u_{\mathcal{N}-1})$ be the vector of controls u_i, $i = 0, 1, \ldots, \mathcal{N}-1$, and $\|\mathbf{u}\|^2 := \sum_{i=0}^{\mathcal{N}-1} \|u_i\|^2$. Then, the problem (6.1) becomes the following SFP

$$\text{find } \mathbf{u}^* = \arg\min \big\{ \|\mathbf{u}\| \mid \mathbf{u} \in \boldsymbol{\mathcal{C}},\ \boldsymbol{\mathcal{A}}\mathbf{u} \in \mathcal{Q} \big\}.$$

According to algorithms (4.1) and (1.4), choose $\mathbf{u}^0$ arbitrarily, the next approximation is determined by

$$\text{(6.2)} \qquad \begin{cases} \ell_n := \|P_{\boldsymbol{\mathcal{C}}}\mathbf{u}^n - \mathbf{u}^n\|, \quad \widehat{\ell}_n := \|P_{\mathcal{Q}}\boldsymbol{\mathcal{A}}\mathbf{u}^n - \boldsymbol{\mathcal{A}}\mathbf{u}^n\|, \\ \quad \text{If } \ell_n > \widehat{\ell}_n,\ v^n := P_{\boldsymbol{\mathcal{C}}}\mathbf{u}^n,\ \mathcal{B} := I^{\mathbb{R}^{\mathcal{N}m}}, \text{ else } v^n := P_{\mathcal{Q}}\boldsymbol{\mathcal{A}}\mathbf{u}^n,\ \mathcal{B} := \boldsymbol{\mathcal{A}}, \\ \mathbf{u}^{n+1} = (1-\alpha_n)\big(\mathbf{u}^n - \gamma_n \mathcal{B}^\top\big(\mathcal{B}\mathbf{u}^n - v^n\big)\big), \quad n \geq 0, \end{cases}$$

and

$$\text{(6.3)} \qquad \mathbf{u}^{n+1} = (1-\alpha_n) P_{\boldsymbol{\mathcal{C}}}\big(\mathbf{u}^n + \gamma_n \boldsymbol{\mathcal{A}}^\top\big(P_{\mathcal{Q}}\boldsymbol{\mathcal{A}}\mathbf{u}^n - \boldsymbol{\mathcal{A}}\mathbf{u}^n\big)\big), \quad n \geq 0,$$

where $P_{\boldsymbol{\mathcal{C}}}\mathbf{u} = \big(P_{\mathcal{C}_0}u_0, P_{\mathcal{C}_1}u_1, \ldots, P_{\mathcal{C}_{\mathcal{N}-1}}u_{\mathcal{N}-1}\big)$ and $\boldsymbol{\mathcal{A}}\mathbf{u} = \sum_{i=0}^{\mathcal{N}-1} \mathcal{A}_i u_i$ for $\mathbf{u} = (u_0, u_1, \ldots, u_{\mathcal{N}-1})$.

Now considering a numerical experiment. Let us consider the following optimal control problem

$$\dot{x} = x + 4u,\ t \in (0,1),\ \ |u(t)| \leq 0.5,\ \ x(0) = 0,\ \ x(1) = 2,$$
$$J(x,u) := \int_0^1 u^2(t)\,dt \to \min_{u(t)}.$$

It can be seen that the optimal control is $u_{\text{opt}}(t) = \frac{e^{-t}}{2\sinh(1)}$. Divide the interval $(0,1)$ into $\mathcal{N}$ parts with the step size $h = 1/\mathcal{N}$ and replacing $\dot{x}(t_i) \approx (x_{i+1} - x_i)/h$, $t_i = ih$, we come to problem (6.1) with $A_{i+1} = 1 + h$, $B_{i+1} = 4h$, $\mathcal{A}_i = 4h(1+h)^{\mathcal{N}-i-1}$, $\mathcal{C}_i = [-0.5, 0.5]$, $i = 0, 1, \ldots, \mathcal{N}-1$, and $\mathcal{Q} = [2-\epsilon, 2+\epsilon]$, where $\epsilon = 10^{-7}$. It is well known that this discretization has the error estimate $O(h)$. This indicates that the difference between the discretized solution $\mathbf{u}^n(t)$ and the original solution $u_{\text{opt}}(t)$ is proportional to the mesh size h.

The computation shows that the iterations $\{\mathbf{u}^n\}$ defined by (6.2) are good approximations to the optimal control $u_{\text{opt}}(t)$ at grid points $t_i = ih$, $i = 0, 1, \ldots, \mathcal{N}-1$. Indeed, defining the approximation error function $\varepsilon(n) := \big(\sum_{i=0}^{\mathcal{N}-1}\big(u_i^n - u_{\text{opt}}(t_i)\big)^2\big)^{1/2}$, choosing $\alpha_n = 1/(10^4 n^{0.75} + 1)$ and $\rho_n = 0.75$ for all $n \geq 0$, we find that after $n = 69$ iterations, the error $\varepsilon(n) = 0.001162$ if $h = 10^{-3}$ and $a_n = 10^{-6}$; and after $n = 1017$ iterations, the error $\varepsilon(n) = 0.000367$ if $h = 10^{-4}$ and $a_n = 10^{-9}$, which are acceptable compared to the discretization error $O(h)$.

Now we compare iterative methods (6.2) and (6.3). We take

- $h = 10^{-3}$, $\alpha_n = \frac{1}{10n+1}$ and $a_n = 10^{-8}$ for all $n \geq 0$,
- $h = 10^{-4}$, $\alpha_n = \frac{1}{10^3 n+1}$ and $a_n = 10^{-9}$ for all $n \geq 0$,

and obtain Tables 6.1, 6.2 and Figure 6.1 of numerical results.

	Our algorithm (6.2)				Anh et al.'s algorithm (6.3)		
ρ_n	$\varepsilon(n)$	Iter. (n)	Time (s)	γ_n	$\varepsilon(n)$	Iter. (n)	Time (s)
0.5	0.0024952	398	3.4280	0.5	0.0024999	6817	76.6170
0.6	0.0024897	332	3.1551	0.6	0.0024997	5681	64.9117
0.7	0.0024952	284	2.7654	0.7	0.0024999	4869	57.8918
0.8	0.0024861	249	2.2786	0.8	0.0024991	4261	46.2954
0.9	0.0024897	221	1.9618	0.9	0.0024997	3787	41.5421

Table 6.1: Performance of two algorithms with $h = 10^{-3}$ and the stopping rule $\varepsilon(n) < 2.5 \times 10^{-3}$.

	Our algorithm (6.2)				Anh et al.'s algorithm (6.3)		
ρ_n	$\varepsilon(n)$	Iter. (n)	Time (s)	γ_n	$\varepsilon(n)$	Iter. (n)	Time (s)
0.5	0.00049977	1143	95.3601	0.5	0.00049995	9127	756.1902
0.6	0.00049987	952	80.5837	0.6	0.00049991	7606	628.6861
0.7	0.00049982	816	74.0353	0.7	0.00049995	6519	546.1119
0.8	0.00049977	714	57.4974	0.8	0.00049995	5704	544.7699
0.9	0.00049956	635	42.8769	0.9	0.00049997	5070	422.5159

Table 6.2: Performance of two algorithms with $h = 10^{-4}$ and the stopping rule $\varepsilon(n) < 5 \times 10^{-4}$.

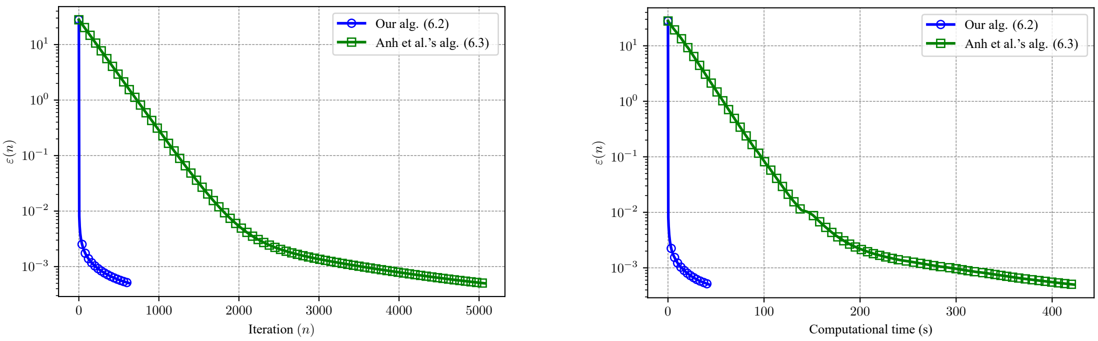

Figure 6.1: Performance of algorithms (6.2) and (6.3) with $h = 10^{-4}$ and $\rho_n = \gamma_n = 0.9$.

In these cases, we can see that our algorithm (6.2) has better performance than (6.3) of Anh et al. [1] does.

7. Conclusion

In this paper, we based on the viscosity approximation method, the selective technique and the CQ–method to approach the solution of the generalized multiple-set split feasibility problem in real Hilbert spaces and obtained a new self-adaptive algorithm for this problem. The self-adaptive feature is expressed via the way to compute the step-sizes, in which the only requirement is the information of the previous step, and the norm of the operator involved is no more necessary. The strong convergence theorem of the proposed algorithm is proven under some suitable conditions. We also obtained some strong convergence results for the solution of (MSFP), (SMOS) and (SFP) as special cases of our main results. The advantages and computational efficiency of the suggested iterative algorithms over some existing ones were confirmed by several numerical experiments.

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Nguyen Thi Thu Thuy and Nguyen Trung Nghia
School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Hanoi, Vietnam
E-mail addresses: thuy.nguyenthithu2@hust.edu.vn,
nghia.nt173561@sis.hust.edu.vn