

Uniform Stabilization for a Semilinear Wave Equation with Variable Coefficients and Nonlinear Boundary Conditions

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Abstract. The uniform stabilization of a semilinear wave equation with variable coefficients and nonlinear boundary conditions is considered. The uniform decay rate is established by the Riemannian geometry method.

1. Introduction

Let Ω be an open bounded domain in \mathbb{R}^n with a smooth boundary Γ . We assume that $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed subsets of Γ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. Moreover, we assume $\text{meas}(\Gamma_1) \neq 0$, where meas denotes the $(n-1)$ -dimensional Hausdorff measure.

Consider the following problem

$$(1.1) \quad \begin{cases} u'' + \mathcal{A}u + f_1(u, x) = 0 & \text{in } \Omega \times (0, +\infty), \\ u(x, t) = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}}(x, t) = -f_2(u, x) - c_3(x)u' & \text{on } \Gamma_1 \times (0, +\infty), \\ u(x, 0) = u^0, \quad u'(x, 0) = u^1(x) & \text{in } \Omega, \end{cases}$$

where

$$\mathcal{A}u = -\text{div}(A(x)\nabla u), \quad x = [x_1, \dots, x_n],$$

$A = (a_{ij})$ is a matrix function, $a_{ij} = a_{ji}$ are C^∞ functions in \mathbb{R}^n , ν is the unit normal of Γ pointing toward the exterior of Ω and $\nu_{\mathcal{A}} = A\nu$.

Assumptions.

(A1) We suppose that the second-order differential operator \mathcal{A} satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j > \lambda \sum_{i,j=1}^n \xi_i^2, \quad x \in \overline{\Omega}, \quad 0 \neq \xi = (\xi_1, \xi_2, \dots, \xi_n)^T \in \mathbb{R}^n$$

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for some constant $\lambda > 0$ and assume further that

$$(1.2) \quad \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j > 0, \quad x \in \mathbb{R}^n, \quad 0 \neq \xi = (\xi_1, \xi_2, \dots, \xi_n)^T \in \mathbb{R}^n.$$

(A2) We assume

$$(1.3) \quad c_3(x) \in L^\infty(\Gamma_1), \quad c_3(x) \geq c_0 > 0 \quad \text{a.e. in } \Gamma_1.$$

(A3) Let $f_1: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be a continuous function in $\mathbb{R} \times \Omega$ and differentiable in its first variable such that

$$|f_1(u, y)| \leq c_1(|u|^{\rho+1} + |u| + 1), \quad \left| \frac{\partial f_1}{\partial u}(u, y) \right| \leq c_1(|u|^\rho + 1) \quad \text{and} \quad f_1(0, y) = 0,$$

where

$$c_1 > 0, \quad 0 < \rho < \infty \quad \text{if } n = 2 \quad \text{and} \quad 0 < \rho < \frac{2}{n-2} \quad \text{if } n \geq 3.$$

We also assume that

$$(1.4) \quad F_1(u, y) \geq 0,$$

$$(1.5) \quad \exists \eta_1 > 0 : (1 + \eta_1)F_1(u, y) \leq f_1(u, y)u,$$

where

$$F_1(z, y) = \int_0^z f_1(s, y) ds.$$

(A4) Let $f_2: \mathbb{R} \times \overline{\Omega} \rightarrow \mathbb{R}$ be a continuous function in $\mathbb{R} \times \overline{\Omega}$ and differentiable in its first variable such that

$$|f_2(u, y)| \leq c_2(|u|^{q-1} + |u| + 1) \quad \text{and} \quad f_2(0, y) = 0,$$

where $c_2 > 0$, $3/2 \leq q < \infty$ if $n = 2$, and $3/2 < q < (2n-3)/(n-2)$ if $n \geq 3$. We also assume that

$$(1.6) \quad F_2(u, y) \geq 0,$$

$$(1.7) \quad \exists \eta_2 > 0 : (1 + \eta_2)F_2(u, y) \leq f_2(u, y)u,$$

where

$$F_2(z, y) = \int_0^z f_2(s, y) ds.$$

The problems of control and stabilization of the wave equations have been studied by several authors when $\mathcal{A} \equiv I$. For the internal stabilization, Y. You proved in [28], the energy decay and the exact controllability for the Petrovsky equation with linear

and nonlinear damping. See also [6], where the author studied the decay estimate of the wave equation with a local degenerate linear dissipation in a bounded domain. For the case of nonlinear damping, we have also numerous writing, as [13] of I. Lasiecka and D. Toundykov, the authors proved the energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms, or S. Berrimi and S. A. Messaoudi in [1] about the exponential Decay of solutions to a viscoelastic equation with nonlinear localized damping.

Regarding the boundary stabilization, when the boundary conditions are linear, we may mention among the achieved results, the works [9,25]. When the boundary conditions are nonlinear, many results have also been obtained, see for instance [7,10–12,29].

Concerning the wave equations with memory term, the stabilization and exponential decay were studied by many writers for viscoelastic wave equations, see for instance [1,2,4,8,21].

As regards the controllability and stabilization of the wave equations with variable coefficients by using the Riemannian geometry method, this method is introduced by P.-F. Yao in [27], the author established the observability inequality for exact controllability of wave equations with variable coefficients by the Riemannian geometry method under some geometric conditions for the Dirichlet problem and for the Neumann problem and presented a number of nontrivial examples to verify the observability inequality. Then, many authors followed the study of stabilization using the Riemannian geometry method for more generalized cases, see for instance [2,3,5,16,17,19,20].

Our main goal is to prove the uniform stabilization for a semilinear wave equation with variable coefficients and nonlinear boundary conditions $f_2(u, x)$, by considering a boundary feedback $c_3(x)u'$, i.e., we establish under Assumptions (A1)–(A4) that the energy associated to problem (1.1) is convergent exponentially to 0 when $t \rightarrow \infty$ (see Theorem 3.2). To do this, we combine the Riemannian geometry method, developed by P.-F. Yao [27] and the method developed by I. Lasiecka in [11] and then by I. Lasiecka and D. Tataru in [12], and we finally complete the proof of Theorem 3.2 by using the semi group theory.

This paper is organized as follows: In Section 2, we present the definitions and notations on Riemannian geometry. We state the main result in Section 3 and prove it in Section 4.

2. Notations on Riemannian geometry

All definitions and notations are standard and classical in literature, see [26,27]. Define

$$G(x) = [A(x)]^{-1} = (g_{ij}(x)), \quad i, j = 1, \dots, n, \quad x \in \mathbb{R}^n.$$

For each $x \in \mathbb{R}^n$, we define the inner product and the norm on the tangent space

$\mathbb{R}_x^n = \mathbb{R}^n$ by

$$g(X, Y) = (X, Y)_g = \sum_{i,j=1}^n g_{ij}(x) \alpha_i \beta_j, \quad |X|_g = (X, X)_g^{1/2}, \quad \forall X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i}.$$

It is easily checked from (1.2) that (\mathbb{R}^n, g) is a Riemannian manifold with the Riemannian metric g .

Denote by $\nabla_0 f$ and $\text{div}_0(X)$ the gradient of f and the divergence of X in the Euclidean metric, respectively, where

$$(2.1) \quad \nabla_0 f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \quad \text{and} \quad \text{div}_0(X) = \sum_{i=1}^n \frac{\partial \alpha_i(x)}{\partial x_i}.$$

Define the gradient $\nabla_g f$ of f in the Riemannian metric g , via the Riesz representation theorem, by

$$(2.2) \quad X(f) = \langle \nabla_g f, X \rangle_g, \quad f \in C^1(\bar{\Omega}),$$

where X is a vector field on the manifold (\mathbb{R}^n, g) . The next lemma (see [27, Lemma 2.1]) will be used to prove many results in this paper.

Lemma 2.1. *Let $X = [x_1, \dots, x_n]$ be the natural coordinate system in \mathbb{R}^n . Let $f, h \in C^1(\bar{\Omega})$. Finally, let H, X be vector fields. Then, with reference to the above notation, we have*

$$(2.3) \quad \langle H(x), A(x)X(x) \rangle_g = H(x) \cdot X(x), \quad x \in \mathbb{R}^n,$$

$$(2.4) \quad \nabla_g f(x) = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x) \frac{\partial f}{\partial x_i} \right) \frac{\partial}{\partial x_j} = A(x) \nabla_0 f, \quad x \in \mathbb{R}^n.$$

If $X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$, then by (2.2) and (2.4),

$$X(f) = \langle \nabla_g f, X \rangle_g = \langle A(x) \nabla_0 f, X \rangle_g = \nabla_0 f \cdot X = \sum_{i=1}^n \xi_i \frac{\partial f}{\partial x_i},$$

by (2.2)–(2.4),

$$(2.5) \quad \begin{aligned} \langle \nabla_g f, \nabla_g h \rangle_g &= \nabla_g f(h) = \langle A(x) \nabla_0 f, \nabla_g h \rangle_g \\ &= \nabla_0 f \cdot \nabla_g h = \nabla_0 f A(x) \nabla_0 h, \quad x \in \mathbb{R}^n. \end{aligned}$$

If H is a vector field on (\mathbb{R}^n, g) ,

$$(2.6) \quad \begin{aligned} \langle \nabla_g f, \nabla_g (H(f)) \rangle_g &= DH \langle \nabla_g f, \nabla_g f \rangle_g + \frac{1}{2} \text{div}_0(|\nabla_g f|_g^2 H)(x) \\ &\quad - \frac{1}{2} (|\nabla_g f|_g^2(x)) (\text{div}_0 H)(x), \quad x \in \mathbb{R}^n, \end{aligned}$$

where DH is the covariant differential. By (1.1), (2.1) and (2.4),

$$\mathcal{A} = - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sum_{j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = -\operatorname{div}_0(A(x)\nabla_0 u) = -\operatorname{div}_0(\nabla_g u), \quad u \in C^2(\Omega).$$

Let H be a vector field on (\mathbb{R}^n, g) . The covariant differential DH of H determines a bilinear form on $\mathbb{R}_x^n \times \mathbb{R}_x^n$, for each $x \in \mathbb{R}^n$, defined by

$$DH(Y, X) = \langle D_X H, Y \rangle_g, \quad \forall X, Y \in \mathbb{R}_x^n,$$

where D and $D_X H$ are the Levi-Civita connection in the Riemannian metric g and the covariant derivative of H with respect to X respectively.

In order to obtain the uniform stabilization of problem (1.1), we need the following assumption.

(A5) There exists a vector field H on the Riemannian manifold (\mathbb{R}^n, g) such that

$$(2.7) \quad DH(X, X) \geq b|X|_g^2, \quad \forall X \in \mathbb{R}_x^n, \quad x \in \bar{\Omega}$$

for some $b > 0$. We also assume that H satisfies

$$H \cdot \nu \begin{cases} \leq 0 & \text{if } x \in \Gamma_0, \\ \geq \delta > 0 & \text{if } x \in \Gamma_1 \end{cases}$$

for some positive δ .

Remark 2.2. The existence of a vector field satisfying Assumption (A5) has been proven in [27], where some examples are given. In particular, for $\mathcal{A} = I$, we can take $H = x - x_0$.

Figure 2.1 represents an example of a domain Ω satisfying the above conditions.

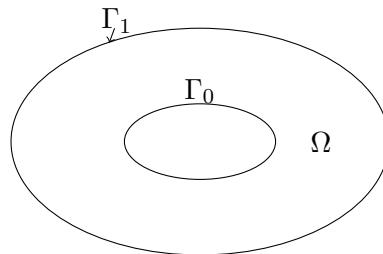


Figure 2.1

Let H be a vector field on \mathbb{R}^n and $f \in C^1(\bar{\Omega})$. Then, from [27], we have the formula for divergence in the Euclidean metric

$$(2.8) \quad \operatorname{div}_0(fH) = f \operatorname{div}_0(H) + H(f)$$

and

$$(2.9) \quad \int_{\Omega} \operatorname{div}_0(H) dx = \int_{\Gamma} H \cdot \nu d\sigma.$$

3. Main result

In the sequel, we denote $Q = \Omega \times]0, T[$, $\Sigma = \Gamma \times]0, T[$, $\Sigma_i = \Gamma_i \times]0, T[$, $i = 0, 1$ and $\|u\|_{L^p(\Omega)}^p = \int_{\Omega} u^p dx$ for $1 \leq p < \infty$. Set

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega); u|_{\Gamma_0} = 0\},$$

and consider the Hilbert space

$$\mathcal{H} = H_{\Gamma_0}^1(\Omega) \times L^2(\Omega)$$

equipped with the inner product

$$\langle U, V \rangle_{\mathcal{H}} = \int_{\Omega} \langle \nabla_g u_1, \nabla_g v_1 \rangle_g dx + \int_{\Omega} u_2 v_2 dx,$$

where $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Multiplying the first equation in problem (1.1) by u' and using Green's formula, we obtain

$$\begin{aligned} (3.1) \quad & \frac{d}{dt} \left\{ \frac{1}{2} \|u'\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} |\nabla_g u|_g^2 dx + \int_{\Omega} F_1(u, x) dx + \int_{\Gamma_1} F_2(u, x) d\sigma \right\} \\ & = -c_3(x) \|u'\|_{L^2(\Gamma_1)}^2 = E'(t). \end{aligned}$$

Then the energy E is defined by

$$E(t) = \frac{1}{2} \left\{ \|u'\|_{L^2(\Omega)}^2 + \int_{\Omega} |\nabla_g u|_g^2 dx d\sigma \right\} + \int_{\Omega} F_1(u, x) dx + \int_{\Gamma_1} F_2(u, x) dx.$$

Integrating (3.1) over $(0, t)$, we have

$$(3.2) \quad E(t) - E(0) = -c_3(x) \int_0^t \|u'\|_{L^2(\Gamma_1)}^2 ds \leq 0, \quad \forall t > 0.$$

Therefore, the energy is a decreasing function of time.

Theorem 3.1. *Under Assumptions (A1)–(A4), we suppose that $f_1(u, x)$ and $f_2(u, x)$ are Lipschitz continuous on u . Then, for each initial data $(u^0, u^1) \in \mathcal{H}$, problem (1.1) has a unique solution u such that*

$$(u, u') \in C(0, T; H_{\Gamma_0}^1(\Omega)) \times C(0, T; L^2(\Omega)).$$

Theorem 3.1 can be proven by using Faedo Galerkin method (cf. J. L. Lions [18]). Our stability result is as follows.

Theorem 3.2. *Assume that Assumptions (A1)–(A4) are fulfilled. Let u be the solution of problem (1.1). Let H be a vector field on (\mathbb{R}^n, g) satisfying Assumption (A5). Then, for every constant $M > 1$, there exists a constant $w > 0$ such that*

$$E(t) \leq M e^{-wt} E(0), \quad \forall t \geq 0.$$

4. Proof of main result

In this section we will prove Theorem 3.2 by several steps, to do this, we need to state some results.

Lemma 4.1. [27, Proposition 2.1, Part 1] *Let u be a solution of the following problem*

$$(4.1) \quad u'' + \mathcal{A}u + f_1(u, x) = 0 \quad \text{in } \Omega \times (0, +\infty).$$

Let H be a vector field in $\bar{\Omega}$. Then

$$(4.2) \quad \begin{aligned} & \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) d\sigma dt + \frac{1}{2} \int_0^T \int_{\Gamma} (|u'|^2 - |\nabla_g u|_g^2) H \cdot \nu d\sigma dt \\ &= \int_{\Omega} u' H(u) \Big|_0^T + \int_0^T \int_{\Omega} DH(\nabla_g u, \nabla_g u) dx dt \\ & \quad + \frac{1}{2} \int_0^T \int_{\Omega} (|u'|^2 - |\nabla_g u|_g^2) \operatorname{div}_0 H dx dt + \int_0^T \int_{\Omega} f_1(u, x) H(u) dx dt. \end{aligned}$$

Proof. We multiply (4.2) by $H(u)$ and integrate in Ω , we obtain

$$\int_{\Omega} u'' H(u) dx + \int_{\Omega} \mathcal{A}u H(u) dx + \int_{\Omega} f_1(u, x) H(u) dx = 0.$$

Using Green's formula and Lemma 2.1, parts (2.4), (2.5) and (2.6), we obtain

$$(4.3) \quad \begin{aligned} \int_{\Omega} \mathcal{A}u H(u) dx &= \int_{\Omega} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i} (H(u)) dx - \int_{\Gamma} \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) d\sigma \\ &= \int_{\Omega} \nabla_g u (H(u)) dx - \int_{\Gamma} \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) d\sigma \\ &= \int_{\Omega} \langle \nabla_g u, \nabla_g (H(u)) \rangle_g dx - \int_{\Gamma} \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) d\sigma \\ &= \int_0^T \int_{\Omega} DH(\nabla_g u, \nabla_g u) dx dt + \frac{1}{2} \int_0^T \int_{\Gamma} |\nabla_g u|_g^2 H \cdot \nu d\sigma dt \\ & \quad - \int_{\Gamma} \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) d\sigma - \frac{1}{2} \int_0^T \int_{\Omega} |\nabla_g u|_g^2 \operatorname{div}_0(H) dx dt. \end{aligned}$$

On the other hand, integrating by parts and taking into account (2.8), we deduce

$$(4.4) \quad \begin{aligned} \int_{\Omega} u'' H(u) dx &= \int_{\Omega} u' H(u) \Big|_0^T dx - \int_{\Omega} \int_0^T u' H(u') dt dx \\ &= (u', H(u)) \Big|_0^T dx - \frac{1}{2} \int_{\Omega} \int_0^T H((u')^2) dx dt \\ &= (u', H(u)) \Big|_0^T dx + \frac{1}{2} \int_0^T \int_{\Omega} (u')^2 \operatorname{div}_0(H) dx dt \\ & \quad - \frac{1}{2} \int_0^T \int_{\Gamma} (u')^2 H \cdot \nu d\sigma dt. \end{aligned}$$

Equations (4.4) and (4.3), together with (4.1), yield (4.2). \square

Lemma 4.2. *Let u be a solution of the problem (1.1) and $P \in C^2(\overline{\Omega})$. Then*

$$\begin{aligned}
 (4.5) \quad & \int_0^T \int_{\Omega} P(|u'|^2 - |\nabla_g u|_g^2) dx dt \\
 &= (u', uP)|_0^T + \frac{1}{2} \int_0^T \int_{\Omega} u^2 \mathcal{A}P dx dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} u^2 \nabla_g P \cdot \nu d\sigma dt \\
 &+ \int_0^T \int_{\Gamma_1} f_2(u, x) uP d\sigma dt - \int_0^T \int_{\Gamma_1} c_3(x) u' uP d\sigma dt + \int_0^T \int_{\Omega} f_1(u, x) uP dx dt.
 \end{aligned}$$

Proof. [27, Proposition 2.1, Part 2] From Lemma 2.1, we have

$$(4.6) \quad \mathcal{A}P = - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sum_{j=1}^n a_{ij}(x) \frac{\partial P}{\partial x_j} \right) = \operatorname{div}_0(A(x) \nabla_0 P).$$

From (4.6) and formula (2.8), we obtain

$$\begin{aligned}
 (4.7) \quad & \langle \nabla_g u, \nabla_g(Pu) \rangle_g(x) = P |\nabla_g u|_g^2(x) + u \langle \nabla_g u, \nabla_g P \rangle_g(x) \\
 &= P |\nabla_g u|_g^2 + \frac{1}{2} \nabla_g P(u^2) \\
 &= P |\nabla_g u|_g^2 + \frac{1}{2} \operatorname{div}_0(u^2 \nabla_g P) + \frac{1}{2} u^2 \mathcal{A}P.
 \end{aligned}$$

It follows from (4.1), (2.9), (4.7) and Green's formula that

$$\begin{aligned}
 (4.8) \quad & (u', uP)|_0^T = \int_0^T [(u_{tt}, uP) + (u', u'P)] dt \\
 &= \int_0^T [(-\mathcal{A} - f_1(u, x), uP) + (u', u'P)] dt \\
 &= \int_0^T \int_{\Omega} [-\langle \nabla_g u, \nabla_g(uP) \rangle_g - f_1(u, x) uP + |u'|^2 P] dx dt \\
 &- \int_0^T \int_{\Gamma_1} f_2(u, x) uP d\sigma dt + \int_0^T \int_{\Gamma_1} c_3(x) u' uP d\sigma dt \\
 &= \int_0^T \int_{\Omega} P(|u'|^2 - |\nabla_g u|_g^2) dx dt - \int_0^T \int_{\Gamma_1} f_2(u, x) uP d\sigma dt \\
 &- \frac{1}{2} \int_0^T \int_{\Omega} u^2 \mathcal{A}P dx dt - \frac{1}{2} \int_0^T \int_{\Gamma_1} u^2 \nabla_g P \cdot \nu d\sigma dt \\
 &- \int_0^T \int_{\Omega} f_1(u, x) uP dx dt + \int_0^T \int_{\Gamma_1} c_3(x) u' uP d\sigma dt.
 \end{aligned}$$

Equation (4.5) follows from (4.8). □

First, we deal with the value of $|\nabla_g u|_g^2$ and $H(u)$ on the boundary Γ (see [15, 27]). For $x \in \Gamma$, the vector $\nabla_g u$ has the decomposition into a direct sum in (\mathbb{R}_x^n, g) as

$$(4.9) \quad \nabla_g u(x) = \left\langle \nabla_g u(x), \frac{\nu_{\mathcal{A}}(x)}{|\nu_{\mathcal{A}}|_g} \right\rangle_g \frac{\nu_{\mathcal{A}}(x)}{|\nu_{\mathcal{A}}|_g} + \nabla_{g\tau} u,$$

where $\nabla_{g\tau}u$ is the tangential gradient. It follows from (4.9) that

$$\begin{aligned} |\nabla_g u|_g^2 &= \nabla_g u(u) = \frac{1}{|\nu_{\mathcal{A}}(x)|_g^2} \langle \nabla_g u(x), \nu_{\mathcal{A}}(x) \rangle_g^2 + |\nabla_{g\tau} u|_g^2 \\ (4.10) \quad &= \frac{1}{|\nu_{\mathcal{A}}|_g^2} \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 + |\nabla_{g\tau} u|_g^2. \end{aligned}$$

Similarly, H can be written as

$$H(u) = \langle \nabla_g u, H \rangle_g = \frac{H(x) \cdot \nu(x)}{|\nu_{\mathcal{A}}(x)|_g^2} \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right) + \langle \nabla_{g\tau} u, H \rangle_g.$$

Now, substituting $P = \frac{1}{2} \operatorname{div}_0 H$ in (4.5), we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^T \int_{\Omega} \operatorname{div}_0 H (|u'|^2 - |\nabla_g u|_g^2) dx dt \\ (4.11) \quad &= \frac{1}{2} (u', u \operatorname{div}_0 H)|_0^T + \frac{1}{4} \int_0^T \int_{\Omega} u^2 \mathcal{A} \operatorname{div}_0 H dx dt \\ &+ \frac{1}{4} \int_0^T \int_{\Gamma_1} u^2 \nabla_g \operatorname{div}_0 H \cdot \nu d\sigma dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} f_2(u, x) u \operatorname{div}_0 H d\sigma dt \\ &- \frac{1}{2} \int_0^T \int_{\Gamma_1} c_3(x) u' u \operatorname{div}_0 H d\sigma dt + \frac{1}{2} \int_0^T \int_{\Omega} f_1(u, x) u \operatorname{div}_0 H dx dt. \end{aligned}$$

Combining (4.11) with Lemma 4.1, we infer

$$\begin{aligned} &\frac{1}{2} \int_0^T \int_{\Gamma_1} (|u'|^2 - |\nabla_g u|_g^2) H \cdot \nu d\sigma dt \\ (4.12) \quad &= \int_{\Omega} u' H(u)|_0^T + \frac{1}{2} (u', u \operatorname{div}_0 H)|_0^T + \int_0^T \int_{\Omega} f_1(u, x) H(u) dx dt \\ &+ \frac{1}{2} \int_0^T \int_{\Omega} f_1(u, x) u \operatorname{div}_0 H dx dt + \frac{1}{4} \int_0^T \int_{\Omega} u^2 \mathcal{A} \operatorname{div}_0 H dx dt \\ &+ \frac{1}{4} \int_0^T \int_{\Gamma_1} u^2 \nabla_g \operatorname{div}_0 H \cdot \nu d\sigma dt + \int_0^T \int_{\Gamma_1} f_2(u, x) H(u) d\sigma dt \\ &+ \frac{1}{2} \int_0^T \int_{\Gamma_1} f_2(u, x) u \operatorname{div}_0 H d\sigma dt - \int_0^T \int_{\Gamma_1} c_3(x) u' H(u) d\sigma dt \\ &- \frac{1}{2} \int_0^T \int_{\Gamma_1} c_3(x) u' u \operatorname{div}_0 H d\sigma dt + \int_0^T \int_{\Omega} DH(\nabla_g u, \nabla_g u) dx dt. \end{aligned}$$

Now, we estimate the terms of the right-hand side of (4.12), by Cauchy–Schwarz inequality, we have

$$(4.13) \quad \int_{\Omega} u' H(u)|_0^T + \frac{1}{2} (u', u \operatorname{div}_0 H)|_0^T \leq C[E(0) + E(T)]$$

and

$$(4.14) \quad \left| \frac{1}{4} \int_0^T \int_{\Omega} u^2 \mathcal{A} \operatorname{div}_0 H \, dxdt + \frac{1}{4} \int_0^T \int_{\Gamma_1} u^2 \nabla_g \operatorname{div}_0 H \cdot \nu \, d\sigma dt \right| \\ \leq C \int_0^T \int_{\Omega} u^2 \, dxdt + C \int_0^T \int_{\Gamma_1} u^2 \, d\sigma dt,$$

where C will denote various positive constant which may be different at different occurrences.

Using Cauchy–Schwartz inequality and Assumption (A4), we have

$$(4.15) \quad \left| \int_0^T \int_{\Gamma_1} c_3(x) u' H(u) \, d\sigma dt \right| \\ \leq C \left[\int_0^T \int_{\Gamma_1} c_3(x) |\nabla_g u|_g^2 \, d\sigma dt + \int_0^T \int_{\Gamma_1} c_3(x) |u'|^2 \, d\sigma dt \right],$$

$$(4.16) \quad \left| \frac{1}{2} \int_0^T \int_{\Gamma_1} c_3(x) u' u \operatorname{div}_0 H \, d\sigma dt \right| \\ \leq C \left[\int_0^T \int_{\Gamma_1} c_3(x) |u|^2 \, d\sigma dt + \int_0^T \int_{\Gamma_1} c_3(x) |u'|^2 \, d\sigma dt \right],$$

$$(4.17) \quad \left| \int_0^T \int_{\Gamma_1} f_2(u, x) H(u) \, d\sigma dt \right| \\ \leq C \left[\int_0^T \int_{\Gamma_1} |\nabla_g u|_g^2 \, d\sigma dt + \int_0^T \int_{\Gamma_1} (|u|^{2(q-1)} + |u|^2 + 1) \, d\sigma dt \right]$$

and

$$(4.18) \quad \left| \frac{1}{2} \int_0^T \int_{\Gamma_1} f_2(u, x) u \operatorname{div}_0 H \, d\sigma dt \right| \leq C \left[\int_0^T \int_{\Gamma_1} (|u|^{2(q-1)} + |u|^2 + 1) \, d\sigma dt \right].$$

Using Cauchy–Schwartz inequality, the inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, $a, b, \varepsilon > 0$ and Assumption (A3), we infer

$$(4.19) \quad \left| \int_0^T \int_{\Omega} f_1(u, x) H(u) \, dxdt \right| \\ \leq \varepsilon \int_0^T \int_{\Omega} |\nabla_g u|_g^2 \, dxdt + \frac{C}{4\varepsilon} \int_0^T \int_{\Omega} (|u|^{2(\rho+1)} + |u|^2 + 1) \, dxdt$$

and

$$(4.20) \quad \left| \frac{1}{2} \int_0^T \int_{\Omega} f_1(u, x) u \operatorname{div}_0 H \, dxdt \right| \\ \leq \varepsilon \int_0^T \int_{\Omega} |\nabla_g u|_g^2 \, dxdt + \frac{C}{4\varepsilon} \int_0^T \int_{\Omega} (|u|^{2(\rho+1)} + |u|^2 + 1) \, dxdt$$

for any ε . Inserting (4.13)–(4.19) and (4.20) into (4.12), by (2.7), using the fact that $H \cdot \nu \geq 0$ on Γ_1 and $u = 0$ on Γ_0 , we deduce

$$(4.21) \quad \begin{aligned} & b \int_0^T \int_{\Omega} |\nabla_g u|_g^2 dx dt \\ & \leq C \left[\int_0^T \int_{\Omega} 2\varepsilon |\nabla_g u|_g^2 dx dt + \int_0^T \int_{\Gamma_1} \{c_3(x)|u'|^2 + [c_3(x) + 1]|\nabla_g u|_g^2\} d\sigma dt + E(0) + E(T) \right] \\ & \quad + C \left[\int_0^T \int_{\Omega} \{|u|^{2(\rho+1)} + [c_3(x) + 1]|u|^2 + 1\} dx dt + \int_0^T \int_{\Gamma_1} \{|u|^2 + |u|^{2(q-1)} + 1\} d\sigma dt \right]. \end{aligned}$$

From [14, Lemma 7.2], we have

$$(4.22) \quad \begin{aligned} & \int_{\tau}^{T-\tau} \int_{\Gamma_1} |\nabla_{g\tau} u|_g^2 d\sigma dt \\ & \leq C_{\tau, \ell, \rho} \left[\int_0^T \int_{\Gamma_1} \left\{ \left| \frac{\partial u}{\partial \nu_{\mathcal{A}}} \right|^2 + |u'|^2 \right\} d\sigma dt + C_T \|u\|_{L^2(0, T; H^{1/2+\ell}(\Omega))} + \int_0^T \int_{\Omega} |f_1(u, x)|^2 dx dt \right], \end{aligned}$$

where $\tau, \ell > 0$ are arbitrarily small and ρ is as in the hypothesis. Applying (4.21) with $(0, T)$ replaced by $(\tau, T - \tau)$, taking into account the decomposition in (4.10) and using the regularity result in (4.22), we obtain

$$\begin{aligned} & b \int_{\tau}^{T-\tau} \int_{\Omega} |\nabla_g u|_g^2 dx dt \\ & \leq C \left[\int_0^T \int_{\Gamma_1} \{[c_3^2(x) + c_3(x) + 1]|u'|^2\} d\sigma dt + E(0) + E(T) + R(u) \right], \end{aligned}$$

where

$$\begin{aligned} R(u) &= \int_0^T \int_{\Omega} \{|u|^{2(\rho+1)} + |u|^2 + 1\} dx dt \\ & \quad + \int_0^T \int_{\Gamma_1} \{|u|^2 + |u|^{2(q-1)} + 1\} d\sigma dt + \|u\|_{L^2(0, T; H^{1/2+\ell}(\Omega))}. \end{aligned}$$

On the other hand, for a fixed τ ,

$$b \left\{ \int_0^{\tau} \int_{\Omega} |\nabla_g u|_g^2 dx dt + \int_{T-\tau}^T \int_{\Omega} |\nabla_g u|_g^2 dx dt \right\} \leq 2b\tau E(0).$$

Hence

$$(4.23) \quad b \int_0^T \int_{\Omega} |\nabla_g u|_g^2 dx dt \leq C \left[\int_0^T \int_{\Gamma_1} c(x)|u'|^2 d\sigma dt + E(0) + E(T) + R(u) \right],$$

where $c(x)$ is a positive function defined by

$$(4.24) \quad c(x) = c_3^2(x) + c_3(x) + 1.$$

Choose $P = \frac{1}{2}b$, where b is a positive constant given in (2.7). By Lemma 4.2, we have

$$\begin{aligned}
 & \frac{1}{2}b \int_0^T \int_{\Omega} (|u'|^2 - |\nabla_g u|_g^2) dx dt \\
 &= \frac{1}{2}b(u', u)|_0^T + \frac{1}{2}b \int_0^T \int_{\Omega} f_1(u, x)u dx dt \\
 (4.25) \quad &+ \frac{1}{2}b \int_0^T \int_{\Gamma_1} f_2(u, x)u d\sigma dt + \frac{1}{2}b \int_0^T \int_{\Gamma_1} c_3(x)u'u d\sigma dt \\
 &\leq C \left\{ \int_0^T \int_{\Gamma_1} c_3(x)|u'|^2 d\sigma dt + E(0) + E(T) + R(u) \right\}.
 \end{aligned}$$

Combining (4.12)–(4.25), we obtain

$$(4.26) \quad \int_0^T E(t) dt \leq C \left\{ \int_0^T \int_{\Gamma_1} c(x)|u'|^2 d\sigma dt + E(0) + E(T) + R(u) \right\}.$$

We now estimate $R(u)$ in terms of

$$\int_0^T \int_{\Omega} |u|^2 dx dt + \int_0^T \int_{\Gamma_1} |u|^2 d\sigma dt + E(0).$$

Proposition 4.3. *For every solution u of (1.1) and time T large enough, the following estimate holds true,*

$$(4.27) \quad R(u) \leq C_T(E(0)) \left\{ \int_0^T \int_{\Omega} |u|^2 dx dt + \int_0^T \int_{\Gamma_1} |u|^2 d\sigma dt + E(0) \right\},$$

where the constant $C_T(E(0))$ remains bounded for bounded values of $E(0)$.

Proof. Recall that

$$\begin{aligned}
 (4.28) \quad R(u) &= \int_0^T \int_{\Omega} \{|u|^{2(\rho+1)} + |u|^2 + 1\} dx dt \\
 &+ \int_0^T \int_{\Gamma_1} \{|u|^2 + |u|^{2(q-1)} + 1\} d\sigma dt + \|u\|_{L^2(0,T;H^{1/2+\ell}(\Omega))}.
 \end{aligned}$$

Noting that $H_{\Gamma_0}^1(\Omega) \subset L^{2(\rho+1)}(\Omega)$ and using hypothesis (1.4), (1.6), we obtain

$$(4.29) \quad \int_0^T \int_{\Omega} |u|^{2(\rho+1)} dx dt \leq C \int_0^T \int_{\Omega} |\nabla_g u|_g^2 dx dt \leq CTE(0).$$

By the trace theorem and Assumption (A4), we have $H^1(\Omega) \subset L^{2(q-1)}(\Gamma_1)$, therefore

$$\begin{aligned}
 & \int_0^T \int_{\Gamma_1} \{|u|^2 + |u|^{2(q-1)}\} d\sigma dt \\
 (4.30) \quad & \leq C \int_0^T \int_{\Gamma_1} |u|^2 d\sigma dt + C \int_0^T \int_{\Omega} \{|u|^2 + |\nabla_g u|_g^2\} dx dt \\
 & \leq C \left\{ \int_0^T \int_{\Gamma_1} |u|^2 d\sigma dt + \int_0^T \int_{\Omega} |u|^2 dx dt + E(0) \right\}.
 \end{aligned}$$

Since $E(0) > 0$, it follows that

$$(4.31) \quad \int_0^T \int_{\Omega} 1 \, dxdt + \int_0^T \int_{\Gamma_1} 1 \, d\sigma dt = \frac{(|\Omega| + |\Gamma_1|)}{E(0)} \cdot TE(0) \leq C_T E(0).$$

From [12], we have

$$(4.32) \quad \|u\|_{L^2(0,T;H^{1/2+\ell}(\Omega))} \leq \epsilon \int_0^T \int_{\Omega} |\nabla_g u|_g^2 \, dxdt + \frac{C}{4\epsilon} \int_0^T \int_{\Omega} |u|^2 \, dxdt.$$

Combining (4.28)–(4.32) and choosing ε small enough, we obtain (4.27). \square

Hence, (4.26) becomes

$$\begin{aligned} & \int_0^T E(t) \, dt \\ & \leq C \left\{ \int_0^T \int_{\Gamma_1} c(x) |u'|^2 \, d\sigma dt + E(0) + E(T) + \int_0^T \int_{\Omega} |u|^2 \, dxdt + \int_0^T \int_{\Gamma_1} |u|^2 \, d\sigma dt \right\}. \end{aligned}$$

Applying the dissipative property inherent in the relation (3.2), i.e., $\forall T \geq 0$,

$$(4.33) \quad E(0) = E(T) + \int_0^T \int_{\Gamma_1} c_3(x) |u'|^2 \, d\sigma dt,$$

we obtain

Proposition 4.4. *For time T large enough, the following estimate holds for the solution u of (1.1):*

$$(4.34) \quad E(T) \leq C_T(E(0)) \left\{ \int_0^T \int_{\Gamma_1} c(x) |u'|^2 \, d\sigma dt + \int_0^T \int_{\Omega} |u|^2 \, dxdt + \int_0^T \int_{\Gamma_1} |u|^2 \, d\sigma dt \right\},$$

where the constant $C_T(E(0))$ remains bounded for bounded values of $E(0)$.

4.1. Absorption of the lower order terms

Now, we are going to eliminate the lower order terms on the right-hand side of (4.34) by applying a ‘nonlinear’ compactness-uniqueness argument as in I. Lasiecka [11] and Lasiecka et al. [12].

Lemma 4.5. *For time T large enough, the following estimate holds for the solution u of (1.1):*

$$\int_0^T \int_{\Omega} |u|^2 \, dxdt + \int_0^T \int_{\Gamma_1} |u|^2 \, d\sigma dt \leq C(E(0)) \int_0^T \int_{\Gamma_1} c(x) |u'|^2 \, d\sigma dt,$$

where the constant $C(E(0))$ remains bounded for bounded values of $E(0)$.

Proof. We use the proof by contradiction. If Lemma 4.5 is false, there exists a sequence $\{(u_l(0), u'_l(0))\}_{l=1}^\infty$ and a corresponding sequence $\{(u_l(t), u'_l(t))\}_{l=1}^\infty$ which satisfies for all l ,

$$(4.35) \quad \begin{cases} u_l'' + \mathcal{A}u_l + f_1(u_l, x) = 0 & \text{in } \Omega \times (0, +\infty), \\ u_l(x, t) = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\ \frac{\partial u_l}{\partial \nu_{\mathcal{A}}}(x, t) = -f_2(u_l, x) - c_3(x)u'_l & \text{on } \Gamma_1 \times (0, +\infty), \\ u_l(x, 0) = u_l^0, \quad u'_l(x, 0) = u_l^1(x) & \text{in } \Omega \end{cases}$$

with

$$(4.36) \quad \lim_{l \rightarrow \infty} \frac{\int_0^T \int_{\Omega} |u_l|^2 dx dt + \int_0^T \int_{\Gamma_1} |u_l|^2 d\sigma dt}{\int_0^T \int_{\Gamma_1} c(x) |u'_l|^2 d\sigma dt} = \infty$$

while the energy of the initial data $\{(u_l(0), u'_l(0))\}_{l=1}^\infty$ denoted by $E(u_l(0))$ is uniformly bounded in l .

By the energy relation (3.2), the sequence $E(u_l(t))$ is also uniformly bounded for $0 \leq t \leq T$. Hence, there exists a subsequence u_l , such that

$$(4.37) \quad \begin{cases} u_l \rightarrow u & \text{weakly in } H^1(Q), \\ u_l \rightarrow u & \text{strongly in } L^2(Q), \\ u_l \rightarrow u & \text{strongly in } L^2(\Sigma). \end{cases}$$

We consider two possibilities

Case 1: $u \neq 0$. By a compactness result [24, Corollary 4], Assumptions (A3) and (A4), using Sobolev embedding and the convergence in (4.37), it follows that

$$(4.38) \quad f_1(u_l, x) \rightarrow f_1(u, x) \quad \text{strongly in } L^\infty(0, T; L^2(\Omega)),$$

$$(4.39) \quad f_2(u_l, x) \rightarrow f_2(u, x) \quad \text{strongly in } L^\infty(0, T; L^2(\Gamma)).$$

From (4.36), we deduce that $c(x)u'_l \rightarrow 0$ in $L^2(\Sigma_1)$. Then, using convergences (4.38), (4.39) and passing to the limit in the problem (4.35), we have

$$(4.40) \quad \begin{cases} u'' + \mathcal{A}u + f_1(u, x) = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}}(x, t) = -f_2(u, x), \quad u' = 0 & \text{on } \Gamma_1 \times (0, +\infty). \end{cases}$$

Moreover, taking the derivative of (4.40) with respect to t , we have, for $u' = v$,

$$\begin{cases} v'' + \mathcal{A}v + \frac{\partial f_1}{\partial u}(u, x)v = 0 & \text{in } \Omega \times (0, +\infty), \\ v = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\ \frac{\partial v}{\partial \nu_{\mathcal{A}}}(x, t) = v = 0 & \text{on } \Gamma_1 \times (0, +\infty). \end{cases}$$

By Assumption (A3), there exists a constant $C > 0$ such that

$$\left| \frac{\partial f_1}{\partial u} \right|^n \leq C(|u|^{\frac{2n}{n-2}} + 1).$$

Since $H^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$, we get $\frac{\partial f_1}{\partial u} \in L^\infty(0, T; L^n(\Omega))$, then, for $T > 2 \operatorname{diam} \Omega$, we may apply the uniqueness continuation result of A. Ruiz [23] adapted to our case, which yields $v = u' = 0$. Then, from (4.40), we get the elliptic equation

$$(4.41) \quad \begin{cases} \mathcal{A}u = -f_1(u, x) & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}}(x, t) = -f_2(u, x) & \text{on } \Gamma_1 \times (0, +\infty). \end{cases}$$

Multiplying the first equation in (4.41) by u and using Green's formula, we obtain

$$\int_{\Omega} \{ |\nabla_g u|_g^2 + f_1(u, x)u \} dx + \int_{\Gamma_1} f_2(u, x)u d\sigma = 0.$$

By (1.4), (1.5), (1.6) and (1.7), we have $u = 0$, which contradicts our assumption that $u \neq 0$.

Case 2: $u = 0$. Denote

$$(4.42) \quad \lambda_l = (\|u_l\|_{L^2(Q)}^2 + \|u_l\|_{L^2(\Sigma)}^2)^{1/2},$$

$$(4.43) \quad \tilde{u}_l = \frac{1}{\lambda_l} \cdot u_l.$$

Then

$$(4.44) \quad \|\tilde{u}_l\|_{L^2(Q)}^2 + \|\tilde{u}_l\|_{L^2(\Sigma)}^2 = 1.$$

Since $u = 0$, from (4.36), we have

$$(4.45) \quad \lambda_l \rightarrow 0 \quad \text{as } l \rightarrow \infty.$$

Also, we see that \tilde{u}_l satisfies

$$(4.46) \quad \begin{cases} \tilde{u}_l'' + \mathcal{A}\tilde{u}_l + \frac{f_1(u_l, x)}{\lambda_l} = 0 & \text{in } \Omega \times (0, +\infty), \\ \tilde{u}_l(x, t) = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\ \frac{\partial \tilde{u}_l}{\partial \nu_{\mathcal{A}}}(x, t) = -\frac{f_2(u_l, x)}{\lambda_l} - c_3(x)\tilde{u}_l' & \text{on } \Gamma_1 \times (0, +\infty). \end{cases}$$

Using the dissipative relation (4.33) (applied to u_l), followed by the estimate (4.34), we have for all $t \in (0, T]$,

$$(4.47) \quad E_l(t) \leq C_T(E_l(0)) \left\{ \int_0^T \int_{\Gamma} c(x)|u_l'|^2 dx dt + \int_0^T \int_{\Omega} |u_l|^2 d\sigma dt + \int_0^T \int_{\Gamma_1} |u_l|^2 dx dt \right\},$$

where $E_l(0)$ is obtained by replacing u^0 and u^1 by u_l^0 and u_l^1 respectively in $E(0)$.

Dividing both sides of (4.47) by λ_l , we deduce that the sequence $E(\tilde{u}_l(t))$ is uniformly bounded for $0 \leq t \leq T$. Hence, there exists a subsequence \tilde{u}_l , such that

$$\begin{cases} \tilde{u}_l \rightarrow u & \text{weakly in } H^1(Q), \\ \tilde{u}_l \rightarrow u & \text{strongly in } L^2(Q), \\ \tilde{u}_l \rightarrow u & \text{strongly in } L^2(\Sigma). \end{cases}$$

In order to pass to the limit in problem (4.46), we need the following

Proposition 4.6.

$$(4.48) \quad c_3(x)\tilde{u}_l' \rightarrow 0 \quad \text{strongly in } L^2(\Sigma_1),$$

$$(4.49) \quad \frac{f_2(u_l, x)}{\lambda_l} \rightarrow \frac{\partial f_2}{\partial u}(0, x)\tilde{u} \quad \text{strongly in } L^2(\Sigma),$$

$$(4.50) \quad \frac{f_1(u_l, x)}{\lambda_l} \rightarrow \frac{\partial f_1}{\partial u}(0, x)\tilde{u} \quad \text{strongly in } L^2(Q),$$

where λ_l is given by (4.42).

Proof. (4.48) follows directly from (4.36) and (4.43). For the second convergence, we have

$$\begin{aligned} \Delta_l &= \left\| \frac{f_2(u_l, x)}{\lambda_l} - \frac{\partial f_2}{\partial u}(0, x)\tilde{u}_l \right\|_{L^2(\Sigma)}^2 \\ &= \int_{|u_l| \leq \varepsilon} \left| \frac{f_2(u_l, x)}{\lambda_l} - \frac{\partial f_2}{\partial u}(0, x)\tilde{u}_l \right|^2 dxdt + \int_{|u_l| > \varepsilon} \left| \frac{f_2(u_l, x)}{\lambda_l} - \frac{\partial f_2}{\partial u}(0, x)\tilde{u}_l \right|^2 dxdt. \end{aligned}$$

Using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and (4.43), we obtain

$$\begin{aligned} \Delta_l &\leq \int_{|u_l| \leq \varepsilon} \tilde{u}_l^2 \left| \frac{f_2(u_l, x)}{u_l} - \frac{\partial f_2}{\partial u}(0, x) \right|^2 dxdt \\ &\quad + 2 \int_{|u_l| > \varepsilon} \frac{f_2^2(u_l, x)}{\lambda_l^2} dxdt + 2 \left| \frac{\partial f_2}{\partial u}(0, x) \right|^2 \int_{|u_l| > \varepsilon} |\tilde{u}_l|^2 dxdt. \end{aligned}$$

Then, from Assumption (A4), we get

$$(4.51) \quad \Delta_l \leq \|\tilde{u}_l\|_{L^2(\Sigma)}^2 \zeta_\varepsilon^2 + C \int_{|u_l| > \varepsilon} \left[\frac{|u_l|^{2(q-1)}}{\lambda_l^2} + \frac{|u_l|^2}{\lambda_l^2} + \frac{1}{\lambda_l^2} \right] dxdt,$$

where $\zeta_\varepsilon = \sup_{|y| \leq \varepsilon} \left| \frac{f_2(y, x)}{y} - \frac{\partial f_2}{\partial y}(0, x) \right|$, $\zeta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Since $|u_l| > \varepsilon$ in the second member of the right-hand side of (4.51), it follows that

$$\begin{aligned} (4.52) \quad \Delta_l &\leq \|\tilde{u}_l\|_{L^2(\Sigma)}^2 \zeta_\varepsilon^2 + C \int_{|u_l| > \varepsilon} \frac{|u_l|^{2(q-1)}}{\lambda_l^2} \left[1 + \frac{1}{\varepsilon^{2(q-1)-2}} + \frac{1}{\varepsilon^{2(q-1)}} \right] dxdt \\ &\leq \|\tilde{u}_l\|_{L^2(\Sigma)}^2 \zeta_\varepsilon^2 + C_\varepsilon \lambda_l^{2(q-1)-2} \cdot \|\tilde{u}_l\|_{L^{2(q-1)}(\Sigma)}^{2(q-1)}. \end{aligned}$$

By the trace theorem and taking into account Assumption (A4), we have

$$H^1(\Omega) \subset L^{2(q-1)}(\Sigma) \quad \text{and} \quad H^1(\Omega) \subset L^2(\Sigma).$$

Using the above injections for the last inequality of the equation (4.52), we deduce

$$(4.53) \quad \Delta_l \leq C \|\tilde{u}_l\|_{H^1(\Omega)}^2 \zeta_\varepsilon^2 + C_\varepsilon \lambda_l^{2(q-1)-2} \cdot \|\tilde{u}_l\|_{H^1(\Omega)}^{2(q-1)}.$$

Since \tilde{u}_l is bounded in $L^\infty(0, T; H^1(\Omega))$ and $2(q-1) - 2 > 0$, using the limit in (4.45), we see that the second member of the right-hand side of (4.53) satisfies

$$\begin{aligned} \lim_{l \rightarrow \infty} \sup_l \left[C_\varepsilon \lambda_l^{2(q-1)-2} \cdot \|\tilde{u}_l\|_{H^1(\Omega)}^{2(q-1)} \right] &= \lim_{l \rightarrow \infty} \sup_l C_\varepsilon \lambda_l^{2(q-1)-2} \cdot \lim_{l \rightarrow \infty} \sup_l \|\tilde{u}_l\|_{H^1(\Omega)}^{2(q-1)} \\ &\leq \lim_{l \rightarrow \infty} C_\varepsilon \lambda_l^{2(q-1)-2} \cdot \sup_l \|\tilde{u}_l\|_{H^1(\Omega)}^{2(q-1)} = 0, \end{aligned}$$

consequently

$$\lim_{l \rightarrow \infty} \sup_l \Delta_l \leq \sup_l \|\tilde{u}_l\|_{H^1(\Omega)}^2 \zeta_\varepsilon^2 \quad \text{and} \quad \lim_{l \rightarrow \infty} \Delta_l = 0$$

as $\varepsilon \rightarrow 0$. Here we have used the fact that \tilde{u}_l is bounded in $L^\infty(0, T; H^1(\Omega))$ and $\zeta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. That is, we get (4.49).

Also, (4.50) may be proven in the same way. \square

Applying Proposition 4.6 and passing to the limit in the problem (4.46), it follows that

$$(4.54) \quad \begin{cases} \tilde{u}'' + \mathcal{A}\tilde{u} + \frac{\partial f_1}{\partial u}(0, x)\tilde{u} = 0 & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \tilde{u}}{\partial \nu_{\mathcal{A}}}(x, t) = -\frac{\partial f_2}{\partial u}(0, x)\tilde{u} & \text{on } \Gamma_1 \times (0, +\infty), \\ \tilde{u} = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\ \tilde{u}' = 0 & \text{on } \Gamma_1 \times (0, +\infty). \end{cases}$$

Moreover, for $\tilde{u}' = v$,

$$\begin{cases} v'' + \mathcal{A}v + \frac{\partial f_1}{\partial u}(0, x)v = 0 & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial v}{\partial \nu_{\mathcal{A}}}(x, t) = 0 & \text{on } \Gamma_1 \times (0, +\infty), \\ v = 0 & \text{on } \Gamma_0 \times (0, +\infty). \end{cases}$$

Using the uniqueness continuation result of A. Ruiz [23], we have $v = \tilde{u}' = 0$. Then, by (4.54), we get

$$(4.55) \quad \begin{cases} \mathcal{A}\tilde{u} + \frac{\partial f_1}{\partial u}(0, x)\tilde{u} = 0 & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \tilde{u}}{\partial \nu_{\mathcal{A}}}(x, t) = -\frac{\partial f_2}{\partial u}(0, x)\tilde{u} & \text{on } \Gamma_1 \times (0, +\infty), \\ \tilde{u} = 0 & \text{on } \Gamma_0 \times (0, +\infty). \end{cases}$$

As in Case 1, multiplying the first equation in (4.55) by \tilde{u} , we obtain $\tilde{u} = 0$, which contradicts (4.44). Hence, Lemma 4.5 is proved. \square

Combining Proposition 4.4 and Lemma 4.5, we conclude, for time T large enough that

$$(4.56) \quad E(T) \leq C \int_0^T \int_{\Gamma_1} c(x) |u'|^2 d\sigma dt,$$

where $c(x)$ is a positive function given in (4.24).

From (4.56), using hypothesis (1.3) and applying the dissipative relation (3.2), we easily deduce

$$(4.57) \quad E(T) \leq \frac{C}{1+C} E(0).$$

Estimate (4.57) combined with the semigroup property (cf. J. Rauch and M. Taylor [22]) implies that

$$E(t) \leq M e^{-wt} E(0), \quad \forall t \geq 0$$

with

$$M = \frac{1}{1 + \frac{1}{C}} \quad \text{and} \quad w = \frac{1}{T} \log M.$$

This completes the proof of Theorem 3.2.

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