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On the Reconstruction of Convection Coefficient in a Semilinear Anomalous Diffusion System

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Abstract. In the paper, we study an inverse problem of recovering a time-dependent convection coefficient from the measured data at an interior/boundary point in a one-dimensional nonlinear subdiffusion model with non-homogeneous boundary conditions. Due to the nonlinearity and non-homogeneous boundary conditions of the system, such an inverse problem is novel and important. We first investigate the unique existence and some regularities of the solution to forward problem by using the transposition method and the fixed point theorem. Then a conditional stability of the inverse problem is obtained based on the regularity of solution for the direct problem and some generalized Gronwall's inequalities. Finally, we transform the inverse problem into a variational problem. The existence and convergence of the regularization solution for the variational problem are proved and we use a modified Levenberg–Marquardt method to find an approximate convection coefficient function. The efficiency and accuracy of the algorithm are illustrated with two numerical examples.

1. Introduction

In recent years the study on anomalous diffusion models has obtained the important achievement. From the viewpoint of statistical physics, normal diffusion is based on the Brownian motion of the particles. Its spatial probability density function is a Gaussian whose mean squared displacement is a linear growth with respect to time. Anomalous diffusion deviates from the standard Fickian description of Brownian motion, the main character of which is that its mean squared displacement is a nonlinear growth with respect to time, such as $\langle x^2(t)\rangle \sim t^\alpha$. One of the popular statistical model of the anomalous diffusion is a continuous time random walk model which incorporates memory effects and under some realistic assumptions leads to a fractional diffusion equation, which is deduced by replacing the standard time derivative with a time fractional derivative and can be used to describe superdiffusion and subdiffusion phenomena. Subdiffusion $(0 < \alpha < 1)$

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and superdiffusion ($1 < \alpha < 2$) have been observed in several applications in engineering, physics and biology, e.g., thermal diffusion in fractal domains [25], and dispersive ion transport in column experiments [11], see the review [21] for physical motivation and an extensive list of physical applications.

In this paper we investigate an inverse problem for recovering a time-dependent convection coefficient in a semilinear time-fractional convection-reaction-diffusion equation with non-homogeneous boundary value. Such an inverse problem we considered has many possible practical applications in modelling of fractional reaction-diffusion processes [4, 7, 28], more precisely in reconstruction of certain parameters of non-homogeneous media. For example, Gafiychuk et al. pointed out in [7] that nonlinear source term can illustrate phenomena including diversity of stationary and spatio-temporary dissipative patterns, oscillations, different types of waves, excitability, bistability, etc., and it gives a great degree of freedom for diversity of self organization phenomena and new nonlinear effects depending on the order of time-space fractional derivatives. Also, Delleur declared in [4] that for example in the transport process of contamination in underground soil, if the pollutants have a convection effect with medium such that the amount of pollutants decreases with respective to time, then the convection term in equation (1.1) is involved where the coefficient $p(t)$ describes the rate of degraded amount to pollutants in unit volume and unit time. However, the time-dependent convection coefficient $p(t)$ is often unknown although time-fractional diffusion equations have many natural advantages to formulate the anomalous subdiffusion phenomena of particles in heterogenous porous media. So we need adopt an inverse problem method to construct the unknown coefficient.

Now we consider the following initial boundary value problem (IBVP) with a convection term for subdiffusion:

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{A}u(x,t) - p(t)q(x,t)u_x(x,t) = F(x,t,u(x,t)), & (x,t) \in Q_T, \\ u_x(0,t) = g_1(t), \quad u_x(1,t) = g_2(t), & t \in (0,T], \\ u(x,0) = 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = (0,1)$ and $Q_T = \Omega \times (0,T]$, and ∂_t^α denotes the Caputo fractional left-sided derivative of order $0 < \alpha < 1$ in time, defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function (see Kilbas et al. [17] and Podlubny [27]) and the differential operator $\mathcal{A}$ is defined by

$$\mathcal{A}u(x,t) = -\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}(x,t)\right).$$

For the convenience of statement we introduce the following notations in this paper. Denote $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(0,T)}$, $(\cdot\,,\cdot)$ as the inner product of $L^2(\Omega)$ and $H^s(\Omega)$, $s \in \mathbb{R}$ are the Sobolev spaces (see Adams [1]). Throughout the paper we always assume the following conditions hold:

$$p(t) \in L^\infty(0,T), \tag{1.2}$$

$$q(x,t) \in L^\infty(0,T; H^1(\Omega)), \tag{1.3}$$

$$g_i(t) \in W_0^{1,\infty}(0,T), \quad i = 1,2, \tag{1.4}$$

$$a(x) \geq \mu, \quad x \in \overline{\Omega},\ a \in C^2(\overline{\Omega}) \tag{1.5}$$

for some $\mu > 0$.

If the order α, all the coefficients $a(x)$, $p(t)$, $q(x,t)$, the nonlinear source term F and the boundary functions $g_1(t)$, $g_2(t)$ are given appropriately, then the problem (1.1) is a direct problem. However, the time-dependent convection coefficient $p(t)$ is unknown in the present paper, and we need to identify it based on an additional data. The inverse problem here is to determine the convection coefficient $p(t)$ in problem (1.1) by additional data

$$u(x_0,t) = h(t), \quad x_0 \in \overline{\Omega},\ 0 < t < T. \tag{1.6}$$

This kind of coefficient inversion problems in fractional diffusion equations have significant application background and have increased extensive attention. As far as we know, there are a good deal of study results in the cases of all coefficients only depending on spatial variable and/or the convection term vanished and/or nonlinear source term vanished in (1.1). This kinds of inverse problems are initiated by the pioneering work [3] for recovering the diffusion coefficient and fractional order from lateral Cauchy data using Sturm–Liouville theory. Henceforth the inverse potential problem for model (1.1) has been analyzed in many works [13–16,22,30,31,34,35]. Jin et al. [13] obtained a uniqueness result in determining the potential from the flux measurements in one dimension. Yamamoto and Zhang [35] gave a conditional stability estimate in determining the potential term in a half-order fractional diffusion equation by the Carleman estimate. Miller et al. [22] discussed an inverse problem of determining the potential and fractional order from the internal data. Tuan [34] proved a uniqueness of determining the potential by only finitely many measurements on the boundary. Sun et al. [30,31] investigated the uniqueness in determining the fractional order(s) and the potential simultaneously for the single-term and multi-term time-fractional diffusion equations, respectively, and gave a valid numerical method. Kaltenbacher and Rundell [15] studied an inverse potential problem from overposed final time data, and recovered numerically the unknown coefficient by an iterative Newton-type method. However, to the best of our knowledge, there are scarce studies for inverse coefficient problems in a nonlinear anomalous convection-reaction-diffusion system with spatio-temporal dependence coefficients.

What's more, we know that the time dependent coefficient inversion problems are also important in engineering. Zhang [36] considered an inverse time-dependent diffusion coefficient problem without a convection term. Fujishiro et al. [5] considered two kinds of inverse time-dependent parameter problems where the unknown parameters act as a source term or a potential term from an interior or a boundary observations, and obtained the stability of inverse problems. Sun et al. [32, 33] studied the stability estimates for recovering a time dependent convection term and a potential term, respectively. Kian and Yamamato [16] derived a stability result for inversion a space-time dependent potential term from lateral Cauchy data.

In this paper, we focus on an inverse time-dependent convection coefficient problem by point measured information. We firstly study the well-posedness of the direct problem and also obtain a conditional stability of the inverse problem based on the regularity of forward problem. Moreover, we propose a variational regularization method for solving the inverse coefficient problem and employ a modification of the traditional Levenberg–Marquardt algorithm to calculate the variational problem. This work is an extension and improvement of [32]: One of the most obvious differences is that the present model is a nonlinear time-fractional diffusion system. Secondly, one of the difficulties is the treatment of non-homogeneous boundary in forward and inverse problems. As is known, a large amount of research on forward and inverse problems of evolution equations depends on the characteristic system whereas the non-homogeneous boundary conditions are the biggest obstacle to solving characteristic systems. What's more, the research on direct problems with non-homogeneous boundary condition could provide a useful tool for considering a boundary optimal control problem and also we find that the conditions for some coefficients can be relaxed under the non-homogeneous boundary condition. Finally, the number of iteration step is obviously shortened in numerical implementation by employing the modified Levenberg–Marquardt algorithm. In addition, we prove the existence and convergence of the solution of regularization for the variational problem.

The main theoretical result in this paper is the following stability result for the inverse convection coefficient problem.

Theorem 1.1. *Assume $q(x,t) \in L^\infty(0,T;H^1(\Omega))$, $g_i(t) \in W_0^{1,\infty}(0,T)$, $(i=1,2)$, F is a global Lipschitz continuous function in all variables and (1.5) is satisfied. Let u_i be the solution of (1.1) for $p = p_i \in L^\infty(0,T)$ with $\|p_i\|_\infty \le M$ $(i=1,2)$. Assume that there exist $x_0 \in \Omega$ and $\nu > 0$ such that*

$$\left| q(x_0,t)\frac{\partial}{\partial x}u_2(x_0,t) \right| \ge \nu \quad a.e.\ t \in (0,T). \tag{1.7}$$

Then there exists a constant $C > 0$ *depending on* M, T, α, Ω, ν, $\|q\|_{L^\infty(0,T;H^1(\Omega))}$, $\|a\|_{C^2(\overline{\Omega})}$ *and* $\|g_i\|_{W^{1,\infty}(0,T)}$, $i = 1, 2$ *such that*

$$C^{-1}\|\partial_t^\alpha u_1(x_0,\cdot\,) - \partial_t^\alpha u_2(x_0,\cdot\,)\|_\infty \le \|p_1 - p_2\|_\infty \le C\|\partial_t^\alpha u_1(x_0,\cdot\,) - \partial_t^\alpha u_2(x_0,\cdot\,)\|_\infty. \tag{1.8}$$

Moreover, we have an appendant estimate

$$\|u_1(x,t) - u_2(x,t)\|_{L^2(0,T;D(A))} \le C\|p_1 - p_2\|_{L^2(0,T)}. \tag{1.9}$$

Remark 1.2. Here we give some explanation for the condition (1.7). As the convection term describes the diffusion of pollutants with the flow of a medium, and u_x usually describes the speed of the medium, so we have a good understanding of the physical meaning of (1.7) which means that the medium velocity at the measurement point cannot be zero. Otherwise there would be no convection.

The remainder of this paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, we give the existence, uniqueness and some regularities of solution for the direct problem. In Section 4, we give the proof of the main result. In Section 5, we introduce the modified Levenberg–Marquardt method and also give an inversion algorithm. The numerical implementation for two examples are investigated in Section 6. Finally, we give some brief concluding remarks in Section 7.

2. Preliminaries

In this section we give some necessary space and lemmas. Define an operator A in $H^2(\Omega)$ by

$$A\psi = \mathcal{A}\psi + \psi, \quad x \in \Omega,\ \psi \in D(A) = \{\psi \in H^2(\Omega); \psi'(0) = \psi'(1) = 0\}.$$

Notice that A is a self adjoint and positive operator. Let $\{\lambda_k, \phi_k\}_{k=1}^\infty$ be an eigensystem of A in $D(A)$. Then we know $0 < \lambda_1 < \lambda_2 < \cdots$, $\lim_{k\to\infty} \lambda_k = \infty$ satisfying $A\phi_k = \lambda_k\phi_k$, and $\{\phi_k\}_{k=1}^\infty \subset H^2(\Omega)$ forms an orthonormal basis of $L^2(\Omega)$. We can define the Hilbert scale space $D(A^\gamma)$ for $\gamma \ge 0$ (see, e.g., [26]) by

$$D(A^\gamma) = \left\{\psi \in L^2(\Omega); \sum_{k=1}^\infty \lambda_k^{2\gamma}|(\psi,\phi_k)|^2 < \infty\right\},$$

$$A^\gamma\psi = \sum_{k=1}^\infty \lambda_k^\gamma(\psi,\phi_k)\phi_k, \quad \psi \in D(A^\gamma),$$

equipped with the norm $\|\psi\|_{D(A^\gamma)} = \|A^\gamma\psi\|$. We can easily verify $A^{\alpha+\beta}\psi = A^\alpha(A^\beta\psi) = A^\beta(A^\alpha\psi)$ for $\psi \in D(A^{\alpha+\beta})$, $\alpha, \beta \ge 0$. From [6,9], we have

$$D(A^\gamma) \subset H^{2\gamma}(\Omega) \quad \text{and} \quad \|\psi\|_{H^{2\gamma}(\Omega)} \le C\|\psi\|_{D(A^\gamma)}, \quad \psi \in D(A^\gamma) \ \text{ for } 0 \le \gamma \le 1, \tag{2.1}$$

$$D(A^{1/2}) = H^1(\Omega). \tag{2.2}$$

Definition 2.1. The Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Proposition 2.2. [17] *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$. Then there exists a constant $c = c(\alpha, \beta, \mu) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \le \frac{c}{1+|z|}, \quad \mu \le |\arg(z)| \le \pi.$$

Lemma 2.3. [2] *Let $f \in L^p(0,T)$ and $g \in L^q(0,T)$ with $1 \le p, q \le \infty$ and $1/p + 1/q = 1$. Then the function $f * g$ defined by $f * g(t) = \int_0^t f(t-s)g(s)\,ds$ belongs to $C[0,T]$ and satisfies*

$$|f * g(t)| \le \|f\|_{L^p(0,t)} \|g\|_{L^q(0,t)}, \quad t \in [0,T].$$

Lemma 2.4. [2] *Let $u, v \in H^1(\Omega)$. Then $uv \in H^1(\Omega)$ with the estimate*

$$\|uv\|_{H^1(\Omega)} \le C\|v\|_{H^1(\Omega)}$$

with $C > 0$ depending on $\|u\|_{H^1(\Omega)}$.

3. Well-posedness of the direct problem

In this section we will analyze the well-posedness including existence, uniqueness and regularity of solution for the direct problem (1.1).

Theorem 3.1. *Let conditions (1.2)–(1.4) hold and F be a global Lipschitz continuous function in all variables. Then the IBVP (1.1) has a unique solution $u \in C([0,T]; H^2(\Omega))$ satisfying*

$$\partial_t^\alpha u \in L^\infty(0,T; H^{2\gamma}(\Omega))$$

for $0 \le \gamma < 1/2$. Moreover, we have the following estimate

$$\|u\|_{C([0,T];H^2(\Omega))} + \|\partial_t^\alpha u\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \le C\big\{\|g_1\|_{W^{1,\infty}(0,T)} + \|g_2\|_{W^{1,\infty}(0,T)}\big\} \tag{3.1}$$

with $C > 0$ depending on Ω, T, α, γ, $\|p\|_\infty$, $\|q\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|a\|_{C^2(\overline{\Omega})}$.

In order to prove above results, we first employ the transposition method to homogenize the IBVP (1.1).

Assume $\Lambda_g(x,t) = -\frac{1}{2}(1-x)^2 g_1(t) + \frac{1}{2}x^2 g_2(t)$ and let $v(x,t) = u(x,t) - \Lambda_g(x,t)$. Then we know v satisfies the homogeneous Neumann boundary condition. We know from (1.4) that $v(x,0) = u(x,0) - \Lambda_g(x,0) = 0$. So we just have to study the following equations

$$\begin{cases} \partial_t^\alpha v(x,t) + \mathcal{A}v(x,t) - pq(x,t)v_x(x,t) = F(x,t,v+\Lambda_g) + f(x,t), & (x,t) \in Q_T, \\ v_x(0,t) = v_x(1,t) = 0, & t \in (0,T], \\ v(x,0) = 0, & x \in \Omega, \end{cases} \tag{3.2}$$

where

$$f(x,t) = -\partial_t^\alpha \Lambda_g - \mathcal{A}\Lambda_g + pq\partial_x \Lambda_g.$$

By the previous assumptions, we know that

$$f \in L^\infty(0,T;H^1(\Omega)). \tag{3.3}$$

Therefore, it is sufficient to prove the unique existence of IBVP (3.2) in order to investigate the well-posedness of IBVP (1.1).

By the fixed point theorem, we can obtain the following existence, uniqueness and regularity results for problem (3.2).

Lemma 3.2. *Let the conditions* (1.2), (1.3) *and* (3.3) *hold and F be a global Lipschitz continuous function in all variables. Then the IBVP* (3.2) *exists a unique solution $v \in C([0,T];D(A))$ satisfying*

$$Av \in C([0,T];H^{2\gamma}(\Omega)) \quad \textit{and} \quad \partial_t^\alpha v \in L^\infty(0,T;H^{2\gamma}(\Omega))$$

for $0 \le \gamma < 1/2$. Moreover, we have

$$\|v\|_{C([0,T];D(A))} + \|Av\|_{C([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \le C\|f\|_{L^\infty(0,T;H^1(\Omega))}$$

with $C > 0$ depending on Ω, T, α, γ, $\|p\|_\infty$, $\|q\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|\Lambda_g\|_{L^\infty(0,T;H^1(\Omega))}$.

In order to prove above lemma, we define an operator valued function $K(t)$ by

$$K(t)\psi = \sum_{k=1}^\infty (\psi,\phi_k) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k t^\alpha)\phi_k, \quad \psi \in L^2(\Omega),\ t > 0.$$

It is easy to obtain that $K(t) \in L^1(0,T;B(L^2(\Omega)))$, where $B(L^2(\Omega))$ denote the bounded linear operator in $L^2(\Omega)$.

From Proposition 2.2 and $A^\gamma K(t)\psi = \sum_{k=1}^\infty \lambda_k^\gamma (\psi,\phi_k) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k t^\alpha)\phi_k$, we have for $0 \le \gamma \le 1$,

$$
\begin{aligned}
\|A^\gamma K(t)\psi\| &= \left(\sum_{k=1}^\infty \left[\lambda_k^\gamma (\psi,\phi_k) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k t^\alpha)\right]^2 \right)^{1/2} \\
&\le C \left(\sum_{k=1}^\infty \left(\frac{\lambda_k^\gamma t^{\alpha-1}}{1+\lambda_k t^\alpha} |(\psi,\phi_k)| \right)^2 \right)^{1/2} \\
&\le C t^{\alpha-1-\alpha\gamma} \left(\sum_{k=1}^\infty \left(\frac{(\lambda_k t^\alpha)^\gamma}{1+\lambda_k t^\alpha} |(\psi,\phi_k)| \right)^2 \right)^{1/2} \\
&\le C t^{\alpha(1-\gamma)-1} \|\psi\|, \quad \psi \in L^2(\Omega),\ t > 0,
\end{aligned} \tag{3.4}
$$

where we use a fact $\frac{(\lambda_k t^\alpha)^\gamma}{1+\lambda_k t^\alpha}$ is bounded for $0 \le \gamma \le 1$ in the last inequality. In particular, the mapping $t \mapsto A^\gamma K(t)$ belongs to $L^1(0,T;B(L^2(\Omega)))$ for $\gamma < 1$.

Consider the following Cauchy problem in $L^2(\Omega)$:

$$
\begin{cases} \partial_t^\alpha \omega(t) + A\omega(t) = S(t), & t \in (0,T], \\ \omega(0) = 0. \end{cases} \tag{3.5}
$$

By Theorem 2.2 in [29], for $S \in L^\infty(0,T;L^2(\Omega))$, (3.5) admits a unique solution given by

$$
\omega(t) = \int_0^t K(t-s)S(s)\,ds. \tag{3.6}
$$

Notice that $A^{1/2}$ and $K(t)$ can be interchanged, we obtain that for $S \in L^\infty(0,T;H^1(\Omega))$,

$$
A\omega(t) = \int_0^t A^{1/2}K(t-s)A^{1/2}S(s)\,ds.
$$

We know that the map $t \mapsto A^{1/2}K(t)$ belongs to $L^1(0,T;B(L^2(\Omega)))$ from (3.4). Thus by Lemma 2.3, we have $\omega \in C([0,T];D(A))$, and by the generalized Minkowski inequality we get

$$
\begin{aligned}
\|\omega(t)\|_{D(A)} = \|A\omega(t)\| &\le \int_0^t \|A^{1/2}K(t-s)\| \|A^{1/2}S(s)\|\,ds \\
&\le C\int_0^t (t-s)^{\alpha/2-1} \|S(s)\|_{D(A^{1/2})}\,ds \le CT^{\alpha/2} \|S\|_{L^\infty(0,T;D(A^{1/2}))}.
\end{aligned}
$$

Here we define the map $H\colon L^\infty(0,T;D(A^{1/2})) \to C([0,T];D(A))$ by

$$
HS(t) = \int_0^t K(t-s)S(s)\,ds, \quad S \in L^\infty(0,T;D(A^{1/2})),
$$

and we have

$$
\|HS\|_{C([0,T];D(A))} \le CT^{\alpha/2} \|S\|_{L^\infty(0,T;D(A^{1/2}))}. \tag{3.7}
$$

Now we give the proof of Lemma 3.2.

Proof of Lemma 3.2. The IBVP (3.2) could be written as an abstract ODEs

$$\begin{cases} \partial_t^\alpha v(t) + Av(t) = b(t)\partial_x v(t) + v(t) + F(t, v(t) + \Lambda_g(t)) + f(t), & t \in (0,T], \\ v(0) = 0, \end{cases} \tag{3.8}$$

where $v(t) = v(\,\cdot\,,t)$, $b(t) = p(t)q(\,\cdot\,,t)$, $\Lambda_g(t) = \Lambda_g(\,\cdot\,,t)$ and $f(t) = f(\,\cdot\,,t)$. We observe from (3.6) that the solution v of (3.8) can be written as

$$v(t) = \int_0^t K(t-s)\big(b(s)\partial_x v(s) + v(s) + F(s, v(s) + \Lambda_g(s))\big)\, ds + \int_0^t K(t-s) f(s)\, ds.$$

Now we need to look for a fixed point of the nonlinear operator $G\colon C([0,T]; D(A)) \to C([0,T]; D(A))$ defined by

$$G(v)(t) = Q(v)(t) + Hf(t), \quad v \in C([0,T]; D(A)),\ t \in (0,T],$$

where

$$Q(v)(t) = \int_0^t K(t-s) \left(b(s)\frac{\partial}{\partial x} v(s) + v(s) + F\big(s, v(s) + \Lambda_g(s)\big) \right) ds, \quad t \in (0,T].$$

Similarly, we get

$$G^2(v) = G(G(v)) = G(Q(v) + Hf) = Q(Q(v) + Hf) + Hf = Q^2(v) + Q(Hf) + Hf.$$

By induction, we have

$$G^m(v) = Q^m(v) + \sum_{k=0}^{m-1} Q^k(Hf).$$

Here we denote $Q^0 = I$.

By Lemma 2.4, property (2.2) and the Lipschitz continuity of F, we know that $b\frac{\partial}{\partial x}v \in L^\infty(0,T; D(A^{1/2}))$ and

$$\left\| b(s)\frac{\partial}{\partial x} v(s) \right\|_{D(A^{1/2})} \le C\|v(s)\|_{H^2(\Omega)}, \tag{3.9}$$

$$\big\| F(s, v(s) + \Lambda_g(s)) \big\|_{D(A^{1/2})} \le C\|v(s)\|_{H^1(\Omega)} + C, \tag{3.10}$$

where $C > 0$ relies on $\|p\|_\infty$, $\|q\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|\Lambda_g(s)\|_{H^1(\Omega)}$. By (3.4), (3.9) and (3.10), we obtain

$$\begin{aligned} &\|Q(v)(t)\|_{D(A)} \\ &= \left\| \int_0^t A^{1/2} K(t-s) A^{1/2} \left(b(s)\frac{\partial}{\partial x} v(s) + v(s) + F(s, v(s) + \Lambda_g(s)) \right) ds \right\| \\ &\le C \int_0^t (t-s)^{\alpha/2-1} \left\| b(s)\frac{\partial}{\partial x} v(s) + v(s) + F(s, v(s) + \Lambda_g(s)) \right\|_{D(A^{1/2})} ds \\ &\le C \int_0^t (t-s)^{\alpha/2-1} \|v(s)\|_{D(A)}\, ds. \end{aligned} \tag{3.11}$$

By Lemma 2.3, for $v \in C([0,T];D(A))$, we have $Q(v) \in C([0,T];D(A))$ and the estimate

$$\|Q(v)\|_{C([0,T];D(A))} \le CT^{\alpha/2}\|v\|_{C([0,T];D(A))}.$$

Thus we can see that Q maps $C([0,T];D(A))$ into itself. Combining $Hf \in C([0,T];D(A))$, we have that the operator G also maps $C([0,T];D(A))$ into itself. Repeating the similar procedure, we arrive at

$$\begin{aligned}
\|Q^2(v)(t)\|_{D(A)} = \|Q(Q(v))(t)\|_{D(A)} &\le C\int_0^t (t-s)^{\alpha/2-1}\|Q(v)(s)\|_{D(A)}\,ds \\
&\le C^2\int_0^t (t-s)^{\alpha/2-1}\left(\int_0^s (s-\tau)^{\alpha/2-1}\|v(\tau)\|_{D(A)}\,d\tau\right)ds \\
&= C^2\int_0^t \left(\int_\tau^t (t-s)^{\alpha/2-1}(s-\tau)^{\alpha/2-1}\,ds\right)\|v(\tau)\|_{D(A)}\,d\tau \\
&= \frac{(C\Gamma(\alpha/2))^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|v(\tau)\|_{D(A)}\,d\tau.
\end{aligned}$$

By induction, we obtain

$$\|Q^m(v)(t)\|_{D(A)} \le \frac{(C\Gamma(\alpha/2))^m}{\Gamma(m\alpha/2)}\int_0^t (t-\tau)^{m\alpha/2-1}\|v(\tau)\|_{D(A)}\,d\tau, \quad v \in C([0,T];D(A)).$$

By Lemma 2.3, we have $Q^m(v) \in C([0,T];D(A))$ and the estimate

$$\|Q^m(v)\|_{C([0,T];D(A))} \le \rho_m\|v\|_{C([0,T];D(A))}, \quad v \in C([0,T];D(A)), \tag{3.12}$$

where $\rho_m = \frac{(C\Gamma(\alpha/2)T^{\alpha/2})^m}{\Gamma(m\alpha/2+1)}$. Therefore, according to the linearity of integral operator and the Lipschitz continuity of F, similar to (3.11) and (3.12) we can easily obtain that

$$\begin{aligned}
\|G^m(v_1) - G^m(v_2)\|_{C([0,T];D(A))} &= \|Q^m(v_1) - Q^m(v_2)\|_{C([0,T];D(A))} \\
&\le \rho_m\|v_1 - v_2\|_{C([0,T];D(A))}, \quad \forall\, v_1, v_2 \in C([0,T];D(A)).
\end{aligned}$$

We can verify $\rho_m \to 0$ as $m \to \infty$. Therefore, we have $|\rho_m| < 1$ for large $m \in \mathbb{N}$. Therefore, the operator G^m is a contraction mapping from $C([0,T];D(A))$ into itself. Thus the mapping G^m has a unique fixed point also denoted by $v \in C([0,T];D(A))$, that is, $G^m(v) = v$. Since $G^m(G(v)) = G^{m+1}(v) = G(G^m(v)) = G(v)$, the point $G(v)$ is also a fixed point of the mapping G^m. From the uniqueness of the fixed point of G^m, we have $Q(v) + Hf = G(v) = v$, that is, the equation $v = Q(v) + Hf$ has a unique solution v in $C([0,T];D(A))$. In addition, we have

$$v = G(v) = G^m(v) = Q^m(v) + \sum_{k=0}^{m-1} Q^k(Hf).$$

As $Hf \in C([0,T];D(A))$, by (3.12) and (3.7), we have

$$\begin{aligned}\|v\|_{C([0,T];D(A))} &\le \|Q^m(v)\|_{C([0,T];D(A))} + \sum_{k=0}^{m-1} \|Q^k(Hf)\|_{C([0,T];D(A))} \\ &\le \rho_m \|v\|_{C([0,T];D(A))} + \sum_{k=0}^{m-1} \rho_k \|Hf\|_{C([0,T];D(A))} \\ &\le \rho_m \|v\|_{C([0,T];D(A))} + \sum_{k=0}^{m-1} \rho_k C T^{\alpha/2} \|f\|_{L^\infty(0,T;D(A^{1/2}))}.\end{aligned}$$

By taking a sufficiently large $m \in \mathbb{N}$ such that $\rho_m < 1$, we obtain

$$\|v\|_{C([0,T];D(A))} \le C\|f\|_{L^\infty(0,T;H^1(\Omega))} \tag{3.13}$$

with $C > 0$ depending on T, Ω, α, $\|p\|_\infty$, and $\|q\|_{L^\infty(0,T;H^1(\Omega))}$.

For a fixed $0 \le \gamma < 1/2$, similar to the treatment of (3.11), we obtain

$$\begin{aligned}&\|Av(t)\|_{D(A^\gamma)} \\ &\le \left\| \int_0^t A^{\gamma+1/2} K(t-s) A^{1/2} \left(b(s)\frac{\partial}{\partial x} v(s) + v(s) + F(s, v(s) + \Lambda_g(s)) \right) ds \right\| \\ &\quad + \left\| \int_0^t A^{\gamma+1/2} K(t-s) A^{1/2} f(s)\, ds \right\| \\ &\le C \int_0^t (t-s)^{\alpha(1/2-\gamma)-1} \left(\|v(s)\|_{D(A)} + \|f(s)\|_{D(A^{1/2})} \right) ds.\end{aligned} \tag{3.14}$$

By Lemma 2.3, we have $Av \in C([0,T];D(A^\gamma))$ and the following estimate from (3.13) and (3.14),

$$\begin{aligned}\|Av\|_{C([0,T];D(A^\gamma))} &\le C T^{\alpha(1/2-\gamma)} \left(\|v\|_{C([0,T];D(A))} + \|f\|_{L^\infty(0,T;D(A^{1/2}))} \right) \\ &\le C\|f\|_{L^\infty(0,T;D(A^{1/2}))}.\end{aligned}$$

Therefore, we have $Av \in C([0,T];H^{2\gamma}(\Omega))$ with $0 \le \gamma < 1/2$ from (2.1), and

$$\|Av\|_{C([0,T];H^{2\gamma}(\Omega))} \le C\|f\|_{L^\infty(0,T;H^1(\Omega))} \tag{3.15}$$

from (2.2). By the original equation $\partial_t^\alpha v = -Av + b\frac{\partial}{\partial x} v + v + F + f$, combining (3.9), (3.10), (3.13) and (3.15), we see that $\partial_t^\alpha v \in L^\infty(0,T;H^{2\gamma}(\Omega))$ with the estimate

$$\begin{aligned}&\|\partial_t^\alpha v\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \\ &\le C\|f\|_{L^\infty(0,T;H^1(\Omega))} + \|b\partial_x v + v\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} + \|F\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} + \|f\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \\ &\le C\|f\|_{L^\infty(0,T;H^1(\Omega))}.\end{aligned}$$

Thus we complete the proof. $\square$

Now we prove our main result for the forward problem in this section.

Proof of Theorem 3.1. From (1.2)–(1.5), we have $a' \in C^1(\overline{\Omega})$ (also $a' \in H^1(\Omega)$), and $\partial_t^\alpha g_i(t) \in C[0,T]$, $i = 1,2$ from Lemma 2.3. So we have

$$\begin{aligned}
\Lambda_g(x,t) &= -\frac{1}{2}(1-x)^2 g_1(t) + \frac{1}{2}x^2 g_2(t) \in C([0,T];H^2(\Omega)),\\
\partial_x \Lambda_g &= (1-x)g_1(t) + x g_2(t) \in C([0,T];H^2(\Omega)),\\
\partial_{xx}\Lambda_g &= g_2(t) - g_1(t) \in C([0,T];H^2(\Omega)),\\
\mathcal{A}\Lambda_g &= -a'(x)\partial_x\Lambda_g - a\partial_{xx}\Lambda_g \in C([0,T];H^1(\Omega)),\\
\partial_t^\alpha \Lambda_g &= -\frac{1}{2}(1-x)^2\partial_t^\alpha g_1(t) + \frac{1}{2}x^2\partial_t^\alpha g_2(t) \in C([0,T];H^2(\Omega)).
\end{aligned}$$

Thus we have that $f = -\partial_t^\alpha\Lambda_g - \mathcal{A}\Lambda_g + pq\partial_x\Lambda_g \in L^\infty(0,T;H^1(\Omega))$ by Lemma 2.4, and

$$\|f\|_{L^\infty(0,T;H^1(\Omega))} \le C\big(\|g_1\|_{W^{1,\infty}(0,T)} + \|g_2\|_{W^{1,\infty}(0,T)}\big), \tag{3.16}$$

where C depends on $\|p\|_\infty$, $\|q\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|a\|_{C^2(\overline{\Omega})}$. By Lemma 3.2, we obtain that the IBVP (3.2) has a unique solution $v \in C([0,T];D(A))$, and $Av \in C([0,T];H^{2\gamma}(\Omega))$, $\partial_t^\alpha v \in L^\infty(0,T;H^{2\gamma}(\Omega))$ for $0 \le \gamma < 1/2$. Therefore, we obtain the IBVP (1.1) has a unique solution $u = v + \Lambda_g \in C([0,T];H^2(\Omega))$. Moreover, we get $\partial_t^\alpha u \in L^\infty(0,T;H^{2\gamma}(\Omega))$ for $0 \le \gamma < 1/2$, and from (3.16) the following estimates hold:

$$\begin{aligned}
\|u\|_{C([0,T];H^2(\Omega))} &\le C\big(\|f\|_{L^\infty(0,T;H^1(\Omega))} + \|\Lambda_g\|_{C([0,T];H^2(\Omega))}\big)\\
&\le C\big\{\|g_1\|_{W^{1,\infty}(0,T)} + \|g_2\|_{W^{1,\infty}(0,T)}\big\},\\
\|\partial_t^\alpha u\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} &\le \big(\|f\|_{L^\infty(0,T;H^1(\Omega))} + \|\partial_t^\alpha\Lambda_g\|_{L^\infty(0,T;H^{2\gamma}(\Omega))}\big)\\
&\le C\big\{\|g_1\|_{W^{1,\infty}(0,T)} + \|g_2\|_{W^{1,\infty}(0,T)}\big\},
\end{aligned}$$

where C depends on Ω, T, α, γ, $\|p\|_\infty$, $\|q\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|a\|_{C^2(\overline{\Omega})}$. □

4. Proof of the stability for inverse problem

In this section, we give the proof of Theorem 1.1. To accomplish this objective, we first introduce the following two Gronwall's type inequalities. And then we obtain the conditional stability of inverse problem based on above analysis of well-posedness of the direct problem.

Lemma 4.1. *Let $C, \alpha > 0$ and $u, d \in L^1(0,T)$ be nonnegative functions satisfying*

$$u(t) \le Cd(t) + C\int_0^t (t-s)^{\alpha-1}u(s)\,ds, \quad t \in (0,T).$$

Then we have

$$u(t) \le Cd(t) + C\int_0^t (t-s)^{\alpha-1}d(s)\,ds, \quad t \in (0,T).$$

Lemma 4.2. *Let $a, b, \alpha > 0$ and $u \in L^1(0,T)$ be nonnegative functions satisfying*

$$u(t) \le a + b\int_0^t (t-s)^{\alpha-1} u(s)\,ds \quad a.e.\ t \in (0,T).$$

Then we have

$$u(t) \le aE_{\alpha,1}\big((b\Gamma(\alpha))^{1/\alpha} t^\alpha\big) \quad a.e.\ t \in (0,T).$$

For the proofs of above two lemmas, see also Lemmas 7.1.1 and 7.1.2 on pp. 188–189 of [12], here we omit the details.

Proof of Theorem 1.1. Let u_i be the solutions to (1.1) with respect to $p = p_i$ $(i = 1, 2)$. Setting $u = u_1 - u_2$ and $p = p_1 - p_2$, then u solves

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{A}u(x,t) - p_1(t)q(x,t)u_x(x,t) & \\ \quad = p(t)q(x,t)\partial_x u_2(x,t) + F(x,t,u_1) - F(x,t,u_2), & (x,t) \in Q_T, \\ u_x(0,t) = u_x(1,t) = 0, & t \in (0,T], \\ u(x,0) = 0, & x \in \Omega. \end{cases} \tag{4.1}$$

Assume $b(x,t) = p_1(t)q(x,t)$ and $R(x,t) = q(x,t)\frac{\partial}{\partial x}u_2(x,t)$, then $u(x,t)$ is given by

$$\begin{aligned} u(t) = &\int_0^t K(t-s)\big[b(s)u_x(s) + u(s) + \big(F(s,u_1(s)) - F(s,u_2(s))\big)\big]\,ds \\ &+ \int_0^t p(s)K(t-s)R(s)\,ds. \end{aligned}$$

We first estimate $\|u(t)\|_{D(A)}$. By Lemma 2.4, we see that $R = q\frac{\partial}{\partial x}u_2 \in L^\infty(0,T;H^1(\Omega))$ and the following estimate (from (3.9) and (3.1))

$$\|R\|_{L^\infty(0,T;H^1(\Omega))} \le C\|u_2\|_{C([0,T];H^2(\Omega))} \le C\big\{\|g_1\|_{W^{1,\infty}(0,T)} + \|g_2\|_{W^{1,\infty}(0,T)}\big\}, \tag{4.2}$$

where $C > 0$ depends on Ω, T, $\|p_2\|_\infty$, $\|q\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|a\|_{C^2(\overline{\Omega})}$.

By the Lipschitz continuity of F, we have

$$\big\|F(s,u_1(s)) - F(s,u_2(s))\big\|_{D(A^{1/2})} \le C\|u(s)\|_{H^1(\Omega)}. \tag{4.3}$$

Similar to the procedure of (3.11), we acquire from (4.2) and (4.3) that

$$\begin{aligned} \|u(t)\|_{D(A)} \le &\left\|\int_0^t A^{1/2}K(t-s)A^{1/2}\big[b(s)u_x(s) + u(s) + \big(F(s,u_1(s)) - F(s,u_2(s))\big)\big]\,ds\right\| \\ &+ \left\|\int_0^t p(s)A^{1/2}K(t-s)A^{1/2}R(s)\,ds\right\| \\ \le\, & C\int_0^t (t-s)^{\alpha/2-1}\|u(s)\|_{D(A)}\,ds + C\int_0^t (t-s)^{\alpha/2-1}|p(s)|\,ds, \end{aligned}$$

where $C>0$ relies on Ω, T, M, $\|q\|_{L^\infty(0,T;H^1(\Omega))}$, $\|a\|_{C^2(\overline{\Omega})}$ and $\|g_i\|_{W^{1,\infty}(0,T)}$, $i=1,2$.

Denote $d(t)=\int_0^t (t-s)^{\alpha/2-1}|p(s)|\,ds$, then we obtain from Lemma 4.1 that

$$\|u(t)\|_{D(A)} \le Cd(t) + \int_0^t (t-s)^{\alpha/2-1} d(s)\,ds, \quad t\in(0,T).$$

Because

$$\begin{aligned}
\int_0^t (t-s)^{\alpha/2-1} d(s)\,ds &= \int_0^t (t-s)^{\alpha/2-1}\left(\int_0^s (s-\tau)^{\alpha/2-1}|p(\tau)|\,d\tau\right) ds \\
&= \int_0^t \left(\int_\tau^t (t-s)^{\alpha/2-1}(s-\tau)^{\alpha/2-1}\,ds\right)|p(\tau)|\,d\tau \\
&= B(\alpha/2,\alpha/2)\int_0^t (t-\tau)^{\alpha-1}|p(\tau)|\,d\tau \\
&\le T^{\alpha/2}B(\alpha/2,\alpha/2)\int_0^t (t-\tau)^{\alpha/2-1}|p(\tau)|\,d\tau \\
&\le Cd(t),
\end{aligned}$$

we obtain $\|u(t)\|_{D(A)}\le Cd(t)$, $t\in(0,T)$, that is,

$$\|u(t)\|_{D(A)} \le C\int_0^t (t-s)^{\alpha/2-1}|p(s)|\,ds, \quad t\in(0,T).$$

By Young's inequality for the convolution, we arrive at

$$\int_0^T \|u(t)\|_{D(A)}^2\,dt \le C\int_0^T \left(\int_0^t (t-s)^{\alpha/2-1}|p(s)|\,ds\right)^2 dt \le CT^\alpha \|p\|_{L^2(0,T)}^2.$$

Then (1.9) is obtained. Likewise, we have that for $0\le\gamma<1/2$,

$$\|Au(t)\|_{D(A^\gamma)} \le C\int_0^t (t-s)^{\alpha(1/2-\gamma)-1}|p(s)|\,ds, \quad t\in(0,T).$$

Let $x=x_0$ in the first equation of (4.1), we have

$$\begin{aligned}
p(t)R(x_0,t) = \partial_t^\alpha u(x_0,t) &+ Au(x_0,t) - b(x_0,t)u_x(x_0,t) - u(x_0,t) \\
&- \big(F(t,u_1(t)) - F(t,u_2(t))\big) \quad \text{a.e. } t\in(0,T).
\end{aligned} \tag{4.4}$$

Let $1/4<\gamma<1/2$. By the Sobolev embedding, we obtain

$$\begin{aligned}
&\big|Au(x_0,t) - b(x_0,t)u_x(x_0,t) - u(x_0,t) - \big(F(t,u_1(t)) - F(t,u_2(t))\big)\big| \\
&\le C\big\|Au(\cdot,t) - b(\cdot,t)u_x(\cdot,t) - u(\cdot,t) - \big(F(t,u_1(t)) - F(t,u_2(t))\big)\big\|_{H^{2\gamma}(\Omega)} \\
&\le C\|Au(\cdot,t)\|_{H^{2\gamma}(\Omega)} + C\|u_x(\cdot,t)\|_{H^{2\gamma}(\Omega)} + C\|u(\cdot,t)\|_{H^{2\gamma}(\Omega)} \\
&\le C\|Au(\cdot,t)\|_{D(A^\gamma)} \le C\int_0^t (t-s)^{\alpha(1/2-\gamma)-1}|p(s)|\,ds, \quad t\in(0,T),
\end{aligned} \tag{4.5}$$

where we use the Sobolev embedding $H^{2\gamma} \hookrightarrow C[\overline{\Omega}]$ for $\gamma > 1/4$ in the first inequality, and the Lipschitz continuity on F in the second inequality, and the fact $\|u_x\|_{H^{2\gamma}} \lesssim \|u\|_{H^2}$, $\|u\|_{H^{2\gamma}} \lesssim \|u\|_{H^1}$ for $\gamma < 1/2$, and $\|Au\|_{D(A^\gamma)} \sim \|u\|_{D(A^{\gamma+1})} \gtrsim \|u\|_{H^2}$ for $\gamma > 0$ in the third inequality. From condition (1.7), we have

$$|R(x_0,t)| = \left| q(x_0,t)\frac{\partial}{\partial x}u_2(x_0,t)\right| > \nu > 0.$$

As a consequence, combining (4.5) and (4.4), we arrive at

$$\begin{aligned}|p(t)| &\leq C|\partial_t^\alpha u(x_0,t)| \\ &\quad + C\big|Au(x_0,t) - b(x_0,t)u_x(x_0,t) - u(x_0,t) - \big(F(t,u_1(t)) - F(t,u_2(t))\big)\big| \\ &\leq C\|\partial_t^\alpha u(x_0,t)\|_{L^\infty(0,T)} + C\int_0^t (t-s)^{\alpha(1/2-\gamma)-1}|p(s)|\,ds \quad \text{a.e. } t\in(0,T).\end{aligned}$$

Employing Lemma 4.2, we have

$$|p(t)| \leq C\|\partial_t^\alpha u(x_0,t)\|_{L^\infty(0,T)}.$$

Therefore we have proved the right-hand side of (1.8). On the other hand, from (4.5), (4.4), combining the Sobolev embedding, we acquire

$$\begin{aligned}|\partial_t^\alpha u(x_0,t)| &\leq |p(t)R(x_0,t)| \\ &\quad + \big|Au(x_0,t) - b(x_0,t)u_x(x_0,t) - u(x_0,t) - \big(F(t,u_1(t)) - F(t,u_2(t))\big)\big| \\ &\leq C|p(t)|\|R(\,\cdot\,,t)\|_{D(A^{1/2})} + C\int_0^t (t-s)^{\alpha(1/2-\gamma)-1}|p(s)|\,ds \\ &\leq C\big(\|R\|_{L^\infty(0,T;D(A^{1/2}))} + T^{\alpha(1/2-\gamma)}\big)\|p\|_{L^\infty(0,T)}.\end{aligned}$$

Thus we complete the proof. □

Remark 4.3. Under the assumptions of Theorem 1.1, we know from Theorem 3.1 that the IBVP (1.1) admits a unique solution $u \in C([0,T];H^2(\Omega))$ with $\partial_t^\alpha u \in L^\infty(0,T;H^{2\gamma}(\Omega))$ for some $0 \leq \gamma < 1/2$. By the Sobolev embedding theorem, we have $\partial_t^\alpha u(x_0,\cdot) \in L^\infty(0,T)$ and $\frac{\partial}{\partial x}u(x_0,t) \in C[0,T]$ for $d/4 < \gamma < 1/2$. Therefore, our stability result only holds in one-dimensional case and also the formulation of condition (1.7) makes sense.

5. The Levenberg–Marquardt method and inversion algorithm

According to the previous discussion, the convection term $p(t)$ can be determined uniquely from Theorem 1.1 by the measured data at a point in mathematical theory. Here we propose a numerical method to find an approximate solution by using additional condition (1.6). We know that most of inversion algorithms are based on regularization strategies so as to overcome ill-posedness of inverse problems, and different kinds of inverse problems may need different approximate methods on the basis of conditional well-posedness analysis. Here we introduce the Levenberg–Marquardt method and give an inversion algorithm to obtain an approximate solution of the convection coefficient in this section.

Based on Theorem 3.1, we can define a forward operator

$$\mathcal{F}\colon p(t) \in H^1(0,T) \to u(x_0,t;p) \in L^2(0,T). \tag{5.1}$$

Thus the inverse problem is translated into solving the following abstract operator equation

$$\mathcal{F}(p) = h(t) \triangleq u(x_0,t;p).$$

We have $\partial_t^\alpha u(x_0,t;p) \in L^\infty(0,T)$ from Theorem 3.1. Assume $\partial_t^\alpha u(x_0,t;p) = \psi(t)$, then we know $u(x_0,t;p) = I_{0+}^\alpha \psi(t) + u(x_0,0;p)$ for $u(x_0,t;p) \in C[0,T]$ by the monograph [17], where $\psi(t) \in L^\infty(0,T)$. Therefore, we have $u(x_0,t;p) \in H^\alpha(0,T)$ by Theorem 2.1 in the paper [8]. As $H^\alpha(0,T) \hookrightarrow L^2(0,T)$ compactly, so the operator $\mathcal{F}\colon H^1(0,T) \to L^2(0,T)$ is compact. Thus the inverse convection coefficient problem is ill-posed. Let $p^* \in H^1(0,T)$ be a suitable guess of p. In order to ensure a stable numerical reconstruction of $p(t)$, we give the following minimization problem with a high order Tikhonov regularization term

$$\min_{p\in H^1(0,T)} J_\mu(p), \tag{5.2}$$

where $J_\mu(p) = \|u(x_0,t;p) - h^\delta(t)\|^2_{L^2(0,T)} + \mu\|p-p^*\|^2_{H^1(0,T)}$, $\mu > 0$ is a regularization parameter, and h^δ is the noisy function of h.

Lemma 5.1. [2] *Assume that E is a uniformly convex Banach space. Let $\{x_n\}$ be a sequence in E such that $x_n \rightharpoonup x$ weakly in $\sigma(E,E')$ and*

$$\limsup \|x_n\| \le \|x\|.$$

Then $x_n \to x$ strongly.

Proposition 5.2. *Under the conditions of Theorem* 1.1*, there exists at least one minimizer $p_\mu^\delta \in H^1(0,T)$ for the variational problem* (5.2)*.*

Proof. Since the functional J_μ is nonnegative, there exists a constant $d = \inf_{p\in H^1(0,T)} J_\mu(p)$. Thus, there exists a sequence $p_n \in H^1(0,T)$ such that $J_\mu(p_n) \to d$. Therefore, we obtain $\mu\|p_n - p^*\|^2_{H^1(0,T)}$ is bounded. This illustrates that $\{p_n\}$ is bounded in $H^1(0,T)$, then there exists a subsequence, still denoted by p_n, such that $p_n \rightharpoonup p_\mu^\delta$ in $H^1(0,T)$ and $p_n \to p_\mu^\delta$ in $L^2(0,T)$. Based on (1.9) and the Sobolev embedding, we have $u(x_0,t;p_n) \to u(x_0,t;p_\mu^\delta)$ in $L^2(0,T)$. That is,

$$\|u(x_0,t;p_n) - h^\delta(t)\|^2_{L^2(0,T)} \to \|u(x_0,t;p_\mu^\delta) - h^\delta(t)\|^2_{L^2(0,T)}, \quad n \to \infty.$$

According to weak lower semicontinuity of H^1-norm, we have

$$\mu\|p_\mu^\delta - p^*\|^2_{H^1(0,T)} \le \liminf_{n\to\infty} \mu\|p_n - p^*\|^2_{H^1(0,1)}.$$

Therefore, we have

$$d \le J_\mu(p_\mu^\delta) \le \liminf_{n\to\infty} J_\mu(p_n) = d.$$

This shows that p_μ^δ is a minimizer. $\square$

Proposition 5.3. *Under the conditions of Theorem* 1.1*, assume that* $\mathcal{F}(p) = h$ *and the noisy data* $h^\delta \in L^2(0,T)$ *satisfying* $\|h^\delta - h\|_{L^2(0,T)} \le \delta$ *and let* $\mu(\delta)$ *satisfy* $\mu(\delta) \to 0$ *and* $\delta^2/\mu(\delta) \to 0$ *as* $\delta \to 0$*, then the minimizer* p_μ^δ *of variational problem* (5.2) *is convergent, i.e.,* $p^\delta_{\mu(\delta)} \to p$ *in* $H^1(0,T)$ *as* $\delta \to 0$.

Proof. Let δ_k be any sequence such that $\delta_k \to 0$, and denote $\mu_k = \mu(\delta_k)$. By the definition of the minimizer p_μ^δ, we have

$$\begin{aligned}&\|\mathcal{F}(p^{\delta_k}_{\mu_k}) - h^{\delta_k}\|^2_{L^2(0,T)} + \mu_k\|p^{\delta_k}_{\mu_k} - p^*\|^2_{H^1(0,T)}\\ &\le \|\mathcal{F}(p) - h^{\delta_k}\|^2_{L^2(0,T)} + \mu_k\|p - p^*\|^2_{H^1(0,T)} \le \delta_k^2 + \mu_k\|p - p^*\|^2_{H^1(0,T)}.\end{aligned} \tag{5.3}$$

So we know $\|p^{\delta_k}_{\mu_k} - p^*\|^2_{H^1(0,T)} \le \delta_k^2/\mu_k + \|p - p^*\|^2_{H^1(0,T)}$ and $\limsup_{\delta_k\to 0}\|p^{\delta_k}_{\mu_k} - p^*\|_{H^1(0,T)} \le \|p - p^*\|_{H^1(0,T)}$ by the condition for μ. This illustrates that $\|p^{\delta_k}_{\mu_k}\|_{H^1(0,T)}$ is bounded and has a weak convergent subsequence by the reflexivity of $H^1(0,T)$ and also denotes $p^{\delta_k}_{\mu_k}$ such that

$$p^{\delta_k}_{\mu_k} \rightharpoonup z \quad \text{in } H^1(0,T).$$

Since $H^1(0,T) \hookrightarrow L^2(0,T)$ compactly, there exists the subsequence still denoted by $p^{\delta_k}_{\mu_k}$ such that

$$p^{\delta_k}_{\mu_k} \to z \quad \text{in } L^2(0,T).$$

By the estimate (1.9), we have that

$$\mathcal{F}(p^{\delta_k}_{\mu_k}) \to \mathcal{F}(z) \quad \text{in } L^2(0,T).$$

From (5.3), we have

$$\|\mathcal{F}(p^{\delta_k}_{\mu_k}) - h^{\delta_k}\|_{L^2(0,T)} \le \delta_k^2 + \mu_k\|p - p^*\|^2_{H^1(0,T)},$$

and thus $\lim_{\delta_k\to 0}\mathcal{F}(p^{\delta_k}_{\mu_k}) = h$ and $\mathcal{F}(z) = h$. By the uniqueness of $\mathcal{F}(p) = h$, we know that $z = p$. According to the weak lower semicontinuity of the norm in Hilbert space, we have

$$\|p - p^*\|_{H^1(0,T)} \le \liminf_{\delta_k\to 0}\|p^{\delta_k}_{\mu_k} - p^*\|_{H^1(0,T)} \le \limsup_{\delta_k\to 0}\|p^{\delta_k}_{\mu_k} - p^*\|_{H^1(0,T)} \le \|p - p^*\|_{H^1(0,T)},$$

and hence we know that $\lim_{\delta_k\to 0}\|p_{\mu_k}^{\delta_k}-p^*\|_{H^1(0,T)}=\|p-p^*\|_{H^1(0,T)}$. Combining with the weak convergence of $p_{\mu_k}^{\delta_k}$ in $H^1(0,T)$, we obtain by Lemma 5.1 that $p_{\mu_k}^{\delta_k}\to p$. From the uniqueness of $\mathcal{F}(p)=h$, we arrive at $p_{\mu(\delta)}^{\delta}\to p$ in $H^1(0,T)$. □

In the following, we use the modified Levenberg–Marquardt method to minimize problem (5.2). The Levenberg–Marquardt method which is a kind of the Newton-type method was first introduced by [18, 20]. In recent years, it has been well applied to fractional equations, for example, see [19, 32, 33]. From physical considerations we known that h^δ is a reasonably close approximation of some ideal $h=\mathcal{F}(p)$ in the range of $\mathcal{F}$. Let p^* is an approximation of p. Then the nonlinear mapping $\mathcal{F}(p)$ in (5.1) can be replaced approximatively by its linearization around p^*. We can obtain

$$\mathcal{F}(p)=\mathcal{F}(p^*)+\mathcal{F}_p'(p^*)(p-p^*).$$

Then the inverse problem $\mathcal{F}(p)=h^\delta$ can be transformed into a linear inverse problem

$$\mathcal{F}_p'(p^*)(p-p^*)=h^\delta-\mathcal{F}(p^*).$$

Therefore, it is easily seen that the variational problem (5.2) is equivalent to minimizing

$$\min_{\delta p\in H^1(0,T)} J_\mu(p)=\left\|\mathcal{F}_p'(p^*)\delta p-(h^\delta-\mathcal{F}(p^*))\right\|_{L^2(0,T)}^2+\mu\|\delta p\|_{H^1(0,T)}^2,$$

where $\delta p=p-p^*$.

Now we consider the discretization of above variational problem. Suppose that $\{\varphi_s(t), s=1,2,\dots,\infty\}$ is a basis in $H^1(0,T)$, let

$$p(t)\approx p^S(t)=\sum_{s=1}^{S}a_s\varphi_s(t)\quad\text{and}\quad p^*(t)\approx p^{*S}(t)=\sum_{s=1}^{S}a_s^*\varphi_s(t),$$

where p^S is the S-dimensional approximation solution to $p(t)$ and $S\in\mathbb{N}$ is a truncated level of $p(t)$, and a_s, $s=1,2,\dots,S$ are the expansion coefficients. We set a finite dimensional space as

$$\Phi^S=\operatorname{span}\{\varphi_1,\varphi_2,\dots,\varphi_S\},$$

and S-dimensional vector $\mathbf{a}=(a_1,a_2,\dots,a_S)\in\mathbb{R}^S$, so we can identify an approximation $p^S(t)\in\Phi^S$ with a vector $\mathbf{a}\in\mathbb{R}^S$.

According to above discussions, by setting $u(x_0,t;\mathbf{a})=u(x_0,t;p^S)=\mathcal{F}(p^S)$ as a unique solution of the forward problem, a feasible way for numerical solution to solve the following finite-dimensional minimization problem

$$\min_{\delta\mathbf{a}\in\mathbb{R}^S}\left\{\left\|\nabla_{\mathbf{a}}u(x_0,t;\mathbf{a}^*)\delta\mathbf{a}^T-(h^\delta-u(x,t;\mathbf{a}^*))\right\|_{L^2(0,T)}^2+\mu\delta\mathbf{a}A\delta\mathbf{a}^T\right\}, \tag{5.4}$$

where $A = \big((\varphi_i, \varphi_j)_{H^1}\big)_{S\times S}$, $\mathbf{a}^* = (a_1^*, a_2^*, \ldots, a_S^*)$, $\delta\mathbf{a} = \mathbf{a} - \mathbf{a}^*$ and $\mathbf{a}^T$ denotes the transpose of $\mathbf{a}$.

In the following, we give an iterative inversion algorithm for determining the coefficient $\mathbf{a}$. Given the initial $\mathbf{a}^0$ and let $\mathbf{a}^* = \mathbf{a}^0$. Suppose $\mathbf{a}^k \in \mathbb{R}^S$ is the kth step iteration value, then we obtain $(k+1)$th step approximation by solving

$$\mathbf{a}^{k+1} = \mathbf{a}^k + \delta\mathbf{a}^k, \quad k = 0, 1, \ldots, \tag{5.5}$$

where $\delta\mathbf{a}^k$ is called a perturbation for given $\mathbf{a}_k$ and k is the number of iterations.

Now we discretize the time domain $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_K = T$, where K denotes the number of grid points. Then above L^2 norm can be reduced to the discrete Euclidean norm and the variational problem (5.4) at the kth step becomes

$$\min_{\delta\mathbf{a}^k\in\mathbb{R}^S} \left\{ \frac{T}{K} \|\delta\mathbf{a}^k B^T - (\eta - \xi)\|_2^2 + \mu(\delta\mathbf{a}^k) A (\delta\mathbf{a}^k)^T \right\}, \tag{5.6}$$

where

$$B = (b_{ks})_{K\times S}, \quad b_{ks} = \frac{u(x_0, t_k; a_1^k, \ldots, a_s^k + \tau, \ldots, a_S^k) - u(x_0, t_k; \mathbf{a}^k)}{\tau}, \quad k = 1, 2, \ldots, K,$$

and τ is the numerical differentiation step, and

$$\xi = \big(u(x_0, t_1; \mathbf{a}^k), u(x_0, t_2; \mathbf{a}^k), \ldots, u(x_0, t_K; \mathbf{a}^k)\big), \quad \eta = \big(h^\delta(t_1), h^\delta(t_2), \ldots, h^\delta(t_K)\big).$$

By the variational theory, the minimization of (5.6) is reduced to the following normal equation

$$\left(\frac{\mu K}{T} A + B^T B\right) \delta\mathbf{a}^k = B^T(\eta - \xi).$$

Hence, the perturbation can be obtained via the formula

$$\delta\mathbf{a}^k = \left(\frac{\mu K}{T} A + B^T B\right)^{-1} B^T(\eta - \xi).$$

Substitute it into the iterative scheme (5.5) until the stop criterion is satisfied.

6. Numerical experiments

In this section, we present two examples to verify the effectiveness of the algorithm. In numerical computations, we always set $T = 1$ without loss of generality. The grid points on $[0, 1]$ and $[0, T]$ are both 101 when solving the direct problem by finite difference method in [24] see also [30]. Here we point out that the elliptic operator A is the Laplace operator in the following two examples and we choose the measured point $x_0 = 0$. The noisy is generated by adding a random perturbation, i.e.,

$$h^\delta = h + \epsilon h \cdot (2 \cdot \mathrm{rand}(\mathrm{size}(h)) - 1).$$

The corresponding noise level is calculated by $\delta = \|h^\delta - h\|_{L^2(0,T)}$.

To show the accuracy of the numerical solution we compute the approximate error by

$$re_k = \frac{\|p_k(t) - p(t)\|_{L^2(0,T)}}{\|p(t)\|_{L^2(0,T)}},$$

where $p_k(t)$ is the reconstructed coefficients at the kth iteration, and $p(t)$ is the exact solution.

The residual E_k at the kth iteration is given by

$$E_k = \left\|u(x_0, t; p_k) - h^\delta(t)\right\|_{L^2(0,T)}.$$

In an iteration algorithm, the most important work is to find a suitable stopping rule. In this study we use the well-known discrepancy principle [23], i.e., we choose k satisfying the following inequality

$$E_k \le \zeta\delta < E_{k-1},$$

where $\zeta > 1$ is a constant and can be taken heuristically to be 1.01, as suggested by Hanke and Hansen [10].

In this case we take $x_0 = 0$ and Φ_S is chosen as a subspace of eigenfunctions

$$\Phi_S = \operatorname{span}\left\{1, \sqrt{2}\cos(\pi x), \ldots, \sqrt{2}\cos((S-1)\pi x)\right\}.$$

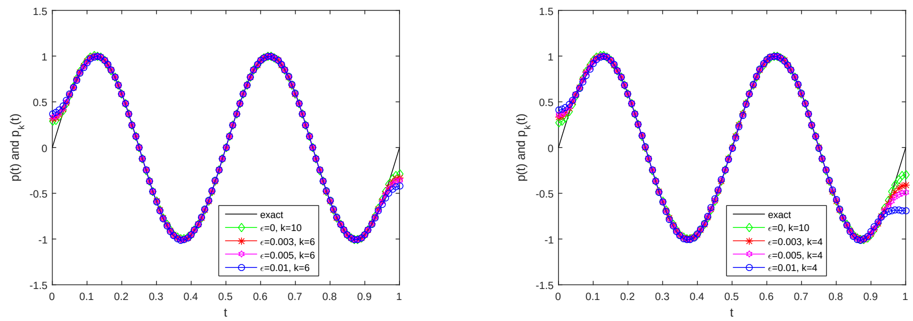
Example 6.1. We first test a smooth solution. Suppose the unknown convection coefficient $p(t) = \sin(4\pi t)$, $q(x,t) = \exp(t)\sin(\pi x)$, $F(x,t,u(x,t)) = 0$, initial value $\varphi(x) = x^2(1-x)^2$ and the boundary $u_x(0,t) = u_x(1,t) = b(t)$, where

$$b(t) = \begin{cases} 0, & 0 \le t < 0.1, \\ 2.5(t-0.1), & 0.1 \le t < 0.5, \\ 1 + 2.5(0.5-t), & 0.5 \le t < 0.9, \\ 0, & 0.9 \le t \le 1. \end{cases}$$

The boundary data $u(0,t)$ is obtained by solving the direct problem (1.1) by using the finite difference method. The numerical results for Example 6.1 by using the discrepancy principle for various noise levels in the cases of $\alpha = 0.4$, $\alpha = 0.7$ are shown in Figures 6.1(a) and 6.1(b) respectively and a numerical differentiation step size $\tau = 0.002$. We choose the initial guess as $p_0 = p^* = 0$, the truncated level $S = 18$.

Example 6.2. In the second example, we test a non-smooth solution with a cusp. Suppose the unknown convection coefficient

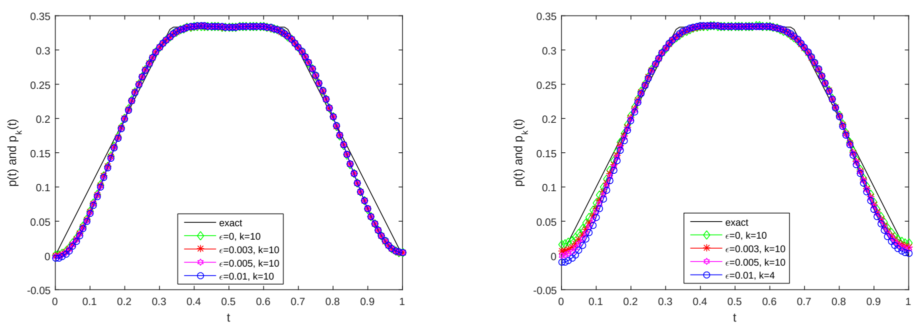
$$p(t) = \begin{cases} t, & 0 \le t < 1/3, \\ 1/3, & 1/3 \le t < 2/3, \\ 1-t, & 2/3 \le t \le 1 \end{cases}$$



(a) $\alpha = 0.4$ (b) $\alpha = 0.7$

Figure 6.1: The numerical results for Example 6.1 for various noise levels with $\mu = 10^{-5}\delta^{1/3}$.

and the rest of definite conditions are the same as in Example 6.1. The boundary data $u(0,t)$ is obtained by solving the direct problem (1.1) by using the finite difference method. The numerical results for Example 6.2 by using the discrepancy principle for various noise levels in the cases of $\alpha = 0.4$, $\alpha = 0.8$ are shown in Figures 6.2(a) and 6.2(b) respectively and a numerical differentiation step size $\tau = 0.002$. We choose the initial guess as $p_0 = p^* = 0$, the truncated level $S = 5$.



(a) $\alpha = 0.4$ (b) $\alpha = 0.8$

Figure 6.2: The numerical results for Example 6.2 for various noise levels with $\mu = 10^{-3}\delta^{1/3}$.

From Figures 6.1, 6.2 and Table 6.1, we can see that the numerical results of the convection coefficients for Examples 6.1 and 6.2 match the exact ones quite well up to 1% noise added in the "exact" measured data $u(x_0,t)$, except around the two endpoints. On the other hand, it can be seen that the numerical results become a little worse when the relative noise levels increase and are not sensitive to the fractional order α. Finally, we find that the number of iteration steps is significantly reduced by using the modified Levenberg–Marquardt algorithm in combination with the Morozov discrepancy principle.

$\alpha \setminus \epsilon$	0	0.003	0.005	0.01
$\alpha = 0.9$	0.0546(10)	0.0941(4)	0.1215(4)	0.2719(3)
$\alpha = 0.7$	0.0566(10)	0.0848(4)	0.1033(4)	0.1513(4)
$\alpha = 0.5$	0.0582(10)	0.0717(5)	0.0810(5)	0.1054(5)
$\alpha = 0.3$	0.0574(10)	0.0659(8)	0.0719(8)	0.0876(8)

Table 6.1: The error e_k and stop steps with different α and ϵ in Example 6.1.

7. Concluding remarks

In this paper, we devote to determine a time-dependent convection coefficient $p(t)$ in a semilinear time-fractional diffusion model with non-homogeneous boundary value for one dimensional case. Because the regularity of a nonlinear anomalous diffusion system with non-homogeneous boundary condition has important applications in optimal control problems, we first obtained the unique existence and regularity of the solution for the direct problem in this paper (see e.g., Theorem 3.1). Then the stability of the solution for the inverse problem was provided by using the regularity of the corresponding direct problem and some generalized Gronwall's inequalities (see e.g., Theorem 1.1). Finally, we transformed the inverse problem into a variational problem, and the variational problem was solved by employing the modified Levenberg–Marquardt method.

However, there are still several questions deserving further investigation in this work. Because of the choice of the working space of the solution of the forward problem and the higher order of inverse problem (convection coefficient inversion), the conditional stability of the inverse problem is only true in one dimension case. So we will consider the conditional stability of higher dimensional problem in the future. Second, the condition (1.7) in Theorem 1.1 seems unreasonable from an engineering applications point of view, but it is still open without it from point of the mathematics. So this is another question for us to consider in the future.

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References

[1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics **65**, Academic Press, New York, 1975.

[2] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[3] J. Cheng, J. Nakagawa, M. Yamamoto and T. Yamazaki, *Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation*, Inverse Problems **25** (2009), no. 11, 115002, 16 pp.

[4] J. W. Delleur, *The Handbook of Groundwater Engineering*, Springer, CRC Press, 1998.

[5] K. Fujishiro and Y. Kian, *Determination of time dependent factors of coefficients in fractional diffusion equations*, Math. Control Relat. Fields **6** (2016), no. 2, 251–269.

[6] D. Fujiwara, *Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order*, Proc. Japan Acad. **43** (1967), 82–86.

[7] V. Gafiychuk, B. Datsko and V. Meleshko, *Mathematical modeling of time fractional reaction-diffusion systems*, J. Comput. Appl. Math. **220** (2008), no. 1-2, 215–225.

[8] R. Gorenflo, Y. Luchko and M. Yamamoto, *Time-fractional diffusion equation in the fractional Sobolev spaces*, Fract. Calc. Appl. Anal. **18** (2015), no. 3, 799–820.

[9] R. Gorenflo and M. Yamamoto, *Operator-theoretic treatment of linear Abel integral equations of first kind*, Japan J. Indust. Appl. Math. **16** (1999), no. 1, 137–161.

[10] M. Hanke and P. C. Hansen, *Regularization methods for large-scale problems*, Surveys Math. Indust. **3** (1993), no. 4, 253–315.

[11] Y. Hatano and N. Hatano, *Dispersive transport of ions in column experiments: An explanation of long-tailed profiles*, Water Resour. Res. **34** (1998), no. 5, 1027–1033.

[12] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, Berlin, 1981.

[13] B. Jin and W. Rundell, *An inverse problem for a one-dimensional time-fractional diffusion problem*, Inverse Problems **28** (2012), no. 7, 075010, 19 pp.

[14] ———, *A tutorial on inverse problems for anomalous diffusion processes*, Inverse Problems **31** (2015), no. 3, 035003, 40 pp.

[15] B. Kaltenbacher and W. Rundell, *On an inverse potential problem for a fractional reaction-diffusion equation*, Inverse Problems **35** (2019), no. 6, 065004, 31 pp.

[16] Y. Kian and M. Yamamoto, *Reconstruction and stable recovery of source terms and coefficients appearing in diffusion equations*, Inverse Problems **35** (2019), no. 11, 115006, 24 pp.

[17] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies **204**, Elsevier Science B.V., Amsterdam, 2006.

[18] K. Levenberg, *A method for the solution of certain non-linear problems in least squares*, Quart. Appl. Math. **2** (1944), 164–168.

[19] K. Liao and T. Wei, *Identifying a fractional order and a space source term in a time-fractional diffusion-wave equation simultaneously*, Inverse Problems **35** (2019), no. 11, 115002, 23 pp.

[20] D. W. Marquardt, *An algorithm for least-squares estimation of nonlinear parameters*, J. Soc. Indust. Appl. Math. **11** (1963), 431–441.

[21] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 77 pp.

[22] L. Miller and M. Yamamoto, *Coefficient inverse problem for a fractional diffusion equation*, Inverse Problems **29** (2013), no. 7, 075013, 8 pp.

[23] V. A. Morozov, *Methods for Solving Incorrectly Posed Problems*, Springer-Verlag, New York, 1984.

[24] D. A. Murio, *Implicit finite difference approximation for time fractional diffusion equations*, Comput. Math. Appl. **56** (2008), no. 4, 1138–1145.

[25] R. R. Nigmatullin, *The realization of the generalized transfer equation in a medium with fractal geometry*, Phys. Stat. Sol. B. **133** (1986), no. 1, 425–430.

[26] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences **44**, Springer-Verlag, New York, 1983.

[27] I. Podlubny, *Fractional Differential Equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering **198**, Academic Press, San Diego, CA, 1999.

[28] S. Z. Rida, A. M. A. El-Sayed and A. A. M. Arafa, *On the solutions of time-fractional reaction-diffusion equations*, Commun. Nonlinear Sci. Numer. Simul. **15** (2010), no. 12, 3847–3854.

[29] K. Sakamoto and M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl. **382** (2011), no. 1, 426–447.

[30] L. L. Sun, Y. S. Li and Y. Zhang, *Simultaneous inversion of the potential term and the fractional orders in a multi-term time-fractional diffusion equation*, Inverse Problems **37** (2021), no. 5, Paper No. 055007, 26 pp.

[31] L. Sun and T. Wei, *Identification of the zeroth-order coefficient in a time fractional diffusion equation*, Appl. Numer. Math. **111** (2017), 160–180.

[32] L. Sun, X. Yan and T. Wei, *Identification of time-dependent convection coefficient in a time-fractional diffusion equation*, J. Comput. Appl. Math. **346** (2019), 505–517.

[33] L. Sun, Y. Zhang and T. Wei, *Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation*, Appl. Numer. Math. **135** (2019), 228–245.

[34] V. K. Tuan, *Inverse problem for fractional diffusion equation*, Fract. Calc. Appl. Anal. **14** (2011), no. 1, 31–55.

[35] M. Yamamoto and Y. Zhang, *Conditional stability in determining a zeroth-order coefficient in a half-order fractional diffusion equation by a Carleman estimate*, Inverse Problems **28** (2012), no. 10, 105010, 10 pp.

[36] Z. Zhang, *An undetermined coefficient problem for a fractional diffusion equation*, Inverse Problems **32** (2016), no. 1, 015011, 21 pp.

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